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► **To cite this version:**

Harald Luschgy, Gilles Pagès. Functional quantization rate and mean pathwise regularity of processes with an application to Lévy processes. 2007. hal-00018341v3

HAL Id: hal-00018341

<https://hal.science/hal-00018341v3>

Preprint submitted on 3 Apr 2007 (v3), last revised 3 Apr 2008 (v4)

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Functional quantization rate and mean regularity of processes with an application to Lévy processes

HARALD LUSCHGY* AND GILLES PAGÈS †

April 3, 2007

Abstract

We investigate the connections between the mean pathwise regularity of stochastic processes and their $L^r(\mathbb{P})$ -functional quantization rate as random variables taking values in some $L^p([0, T], dt)$ -spaces ($0 < p \leq r$). Our main tool is the Haar basis. We then emphasize that the derived functional quantization rate may be optimal (like *e.g.* for the Brownian motion or the symmetric stables processes) so that the rate is optimal as universal upper-bound. As a first application we establish the $O((\log N)^{-\frac{1}{2}})$ -upper bound for general Itô processes which include multi-dimensional diffusions. Then, we focus on the specific family of Lévy processes for which we derive a general quantization rate based on the regular variation properties of its Lévy measure at 0. The case of compound Poisson processes which appears as degenerate in the former approach, is studied specifically: one observes some rates which are in-between finite dimensional and infinite dimensional “usual” rates.

Key words: Functional quantization, Gaussian process, Haar basis, Lévy process, Poisson process

2000 Mathematics Subject Classification: 60E99, 60G51, 60G15, 60G52, 60J60.

1 Introduction

In this paper, we investigate the connection between the functional $L^r(\mathbb{P})$ -quantization rate for a process $X = (X_t)_{t \in [0, T]}$ and the $L^r(\mathbb{P})$ -mean pathwise regularity of the mapping $t \mapsto X_t$ from $[0, T] \rightarrow L^r(\mathbb{P})$ in an abstract setting with a constructive approach (we mean that all the rates are established using some explicit sequences of quantizers).

First let us briefly recall what functional quantization is and how it was introduced. Let $(E, \|\cdot\|)$ denote a finite-dimensional ($E = \mathbb{R}$ or \mathbb{R}^d) or infinite-dimensional ($E = L^p([0, T], dt)$, $1 \leq p < \infty$, $\mathcal{C}([0, T]), \dots$) separable Banach space (or complete quasi-normed space like $E = L^p([0, T], dt)$, $0 < p < 1$) and let $\alpha \subset E$ be a finite subset of size $\text{card}(\alpha) \leq N$, $N \geq 1$. The *Voronoi quantization* of an E -valued random vector $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow E$ with respect to the codebook α is simply the projection of X onto α following the nearest neighbour rule *i.e.*

$$\hat{X}^\alpha = \pi_\alpha(X)$$

where

$$\pi_\alpha = \sum_{a \in \alpha} \mathbf{1}_{C_a(\alpha)},$$

*Universität Trier, FB IV-Mathematik, D-54286 Trier, BR Deutschland. luschgy@uni-trier.de

†Laboratoire de Probabilités et Modèles aléatoires, UMR 7599, Université Paris 6, case 188, 4, pl. Jussieu, F-75252 Paris Cedex 5. E-mail: gpa@ccr.jussieu.fr

$(C_a(\alpha))_{a \in \alpha}$ Borel partition of E satisfying for every $a \in \alpha$

$$C_a(\alpha) \subset \left\{ u \in E : \|u - a\| \leq \min_{b \in \alpha \setminus \{a\}} \|u - b\| \right\}.$$

Then, the L^r -mean quantization error ($0 < r < \infty$) is defined by

$$\|X - \widehat{X}^\alpha\|_{L^r_E(\mathbb{P})} = \left(\mathbb{E} \min_{a \in \alpha} \|X - a\|^r \right)^{\frac{1}{r}}.$$

This quantity is finite as soon as $X \in L^r_E(\mathbb{P})$. The set α is called N -codebook or N -quantizer. One shows that such random vectors \widehat{X}^α are the best approximation of X among all α -valued random vectors. The minimal N^{th} quantization error of X is then defined by

$$e_{N,r}(X, E) := \inf \{ (\mathbb{E} \min_{a \in \alpha} \|X - a\|^r)^{1/r} : \alpha \subset E, \text{card}(\alpha) \leq N \}. \quad (1.1)$$

When $E = L^p([0, T], dt)$ (with its usual norm or quasi-norm denoted $\|\cdot\|_{L^p_T}$ from now on), an E -valued random variable X is a (bi-measurable) stochastic process $X = (X_t)_{t \in [0, T]}$ defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ whose trajectories $(X_t(\omega))_{0 \leq t \leq T}$ (almost) all belong to $L^p([0, T], dt)$. The L^r -integrability assumption then reads

$$\mathbb{E} \left(\left(\int_0^T |X_t|^p dt \right)^{\frac{r}{p}} \right) < +\infty.$$

It is still an open question whether L^r -optimal N -quantizers for Gaussian random vectors always exist in an abstract Banach setting (see [15]). However in many situations of interest for processes, including all the $L^p([0, T], dt)$ -spaces, $1 \leq p < +\infty$, the existence of at least one such L^r -optimal codebook has been established (provided $\mathbb{E}\|X\|^r < +\infty$). Note however that this is not the case for the space $\mathcal{C}([0, T])$ of continuous functions. For more details on the existence problem for optimal quantizers we refer to [15].

On the other hand, optimal L^r -quantizers always exist when $E = \mathbb{R}^d$, $d \geq 1$. In this finite dimensional setting this problem is known as *optimal vector quantization* and has been extensively investigated since the early 1950' with some applications to Signal processing and transmission (see [11] or [12]). In d -dimension the convergence rate of $e_{N,r}$ is ruled by the so-called Zador theorem

$$\lim_N N^{\frac{1}{d}} e_{N,r}(X, \mathbb{R}^d) = J_{r,d} \left(\int_{\mathbb{R}^d} g^{\frac{d}{d+r}}(\xi) d\xi \right)^{1+\frac{r}{d}} \quad (1.2)$$

where g denotes the density of the absolutely continuous part of the distribution \mathbb{P}_X of X (see [13]).

Since the early 2000's much attention has been paid to the infinite dimensional case. This is the so-called *functional quantization* problem for stochastic processes: the aim is to quantize some processes viewed as random vectors taking values in their path-spaces, supposed to be an $L^p([0, T], dt)$ space, $1 \leq p < +\infty$. Many results have been obtained for several families of processes with a special attention paid to Gaussian processes and (Brownian) diffusion processes by several authors. Thus, in the purely Hilbert setting ($r = 2$, $E = L^2([0, T], dt)$) the sharp rate of quantization of the Brownian motion $(W_t)_{t \in [0, T]}$ is given (see (3.6) in [20]) by

$$e_{N,2}(W, L^2([0, T], dt)) \sim \frac{\sqrt{2}T}{\pi(\log N)^{\frac{1}{2}}} \quad (1.3)$$

The existence of such a *sharp rate* for Brownian motion has been extended to $L^p([0, T], dt)$ spaces for $1 \leq p \leq \infty$ (see [8]). Similar sharp rates (with an explicit constant) hold for a wide class of Gaussian processes including the fractional Brownian motions for which we have

$$e_{N,2}(W^H, L^2([0, T], dt)) \sim \frac{c(H, T)}{(\log N)^H}$$

where H denotes the Hurst parameter of the fractional Brownian motion W^H , the Ornstein-Uhlenbeck process, the Brownian sheet, etc, in the purely Hilbert setting (see [20]). The exact rate has also been established in [19] (Section 3) for a wider class of Gaussian processes. In [19, 20], these results are based on the (sharp or exact) asymptotic behaviour of the eigenvalues of high order of the Karhunen-Loève expansion of the Gaussian process. As a by-product, this approach provides very simple explicit sequences of rate optimal asymptotic quantizers (provided the Karhunen-Loève expansion of the process itself is accessible). Their numerical implementation has lead to some unexpectedly promising numerical applications in Finance, especially for the pricing of path-dependent options like Asian options in several popular models of asset dynamics (Black-Scholes, stochastic volatility Heston and SABR models, etc). For these aspects we refer to [23] or [30]. We also mention a applications of quantization to statistical clustering of data functional (see *e.g.* [24]) and some more recent developments concerning functional data investigated in [28] and [29].

Still for Gaussian processes an important connection with small ball probability problem has been made (see [6, 14]). Some exact or sharp rates of convergence for different classes of Brownian diffusions have also been proved recently (see [21], [7]) with a rate driven by $(\log N)^{-\frac{1}{2}}$.

The common feature shared by all these results is that there is a one-to-one correspondence between the exponent a that rules the $(L^r(\mathbb{P}), L^p(dt))$ -quantization rate of these processes in the $\log(N)^{-a}$ scale and their mean pathwise regularity *i.e.* the largest exponent a that satisfies

$$\forall s, t \in [0, T], \quad \|X_t - X_s\|_{L^r(\mathbb{P})} \leq C_r |t - s|^a. \quad (1.4)$$

Although such a correspondence is not really surprising given the connection between quantization rate and small ball probabilities in the Gaussian setting, this leads naturally to try deriving a general abstract result that connects these two features of a process. This is the aim of the Section 2 of this paper in which we show that the mean pathwise regularity always provide a universal upper-bound for the $(L^r(\mathbb{P}), L^p(dt))$ -quantization rate ($0 < p \leq r$). We retrieve then the rate obtained by more specific approaches for all the processes mentioned above. We also extend, to general Brownian diffusion processes and even general Itô processes the rate formerly obtained for specific classes of diffusions in [7, 21]. We also obtain some first quantization rates for some classes of Lévy processes. The main tool is to expand a process on the simplest wavelet basis – the Haar basis (known to be unconditional when $p > 1$) – and to use a non-asymptotic version of the Zador theorem (coming out as a slight improvement of the Pierce Lemma, see [13])

At this point, the next question is to wonder conversely whether this always provide the true quantization rate. In this naive form, the answer to this question is clearly no because equation (1.4) only takes into account the mean-pathwise Hölder regularity of a process and one can trivially build (see [19]) some processes with smoother mean regularity (like processes with \mathcal{C}^k , $k \geq 1$, trajectories). We did not extend our approach in that direction for the sake of simplicity but there is no doubt that developing similar techniques as those used in Section 2 one can connect higher order mean pathwise regularity and quantization rate like in the Hölder setting. This would need to consider an appropriate wavelet basis. In fact we point out in section 4 devoted to general Lévy processes that the answer maybe negative – the quantization rate can be infinitely faster

than the mean pathwise regularity – for different reasons in connection with the dimensionality of process: a Poisson process is in some sense an almost finite dimensional random vector which induces a very fast quantization rate which does not take place in the $(\log N)^{-a}$ scale although the mean-pathwise $L^r(\mathbb{P})$ -regularity of a Poisson process is Hölder (and depends on r , see *e.g.* (3.19) and (3.20)). Conversely, we emphasize on several classes of examples that the upper bound derived from mean regularity provides the true rate of quantization. This follows from a comparison with the lower bound that can be derived from small deviation results (see *e.g.* [14] or the remark below Theorem 1 which enlightens the connection between functional quantization and small deviation theory). Thus we prove that our approach yields the exact rate for a wide class of subordinated Lévy processes (including symmetric α -stable processes).

The main result of Section 4 is Theorem 2 which provides a functional quantization rate for a general Lévy process X having no Brownian component: this rate is ruled by the behaviour of the Lévy measure ν around 0 (*e.g.* the index of X for a stable process). As an example for Lévy processes which do have infinitely many small jumps, if the (infinite) Lévy measure ν (is locally absolutely continuous around 0) and satisfies

$$\exists c > 0, \quad \mathbf{1}_{\{0 < |x| \leq c\}} \nu(dx) \leq \frac{C}{|x|^{\underline{\theta}+1}} \mathbf{1}_{\{0 < |x| \leq c\}} dx$$

for some $\underline{\theta} \in (0, 2]$, then, for every $p, r \in (0, \underline{\theta}]$ such that $0 < p \leq r$ and $X_1 \in L^r(\mathbb{P})$

$$e_{N,r}(X, L^p([0, T], dt)) = O((\log N)^{-\frac{1}{\underline{\theta}}}).$$

This makes a connection between quantization rate and the Blumenthal-Gettoor index β of X when ν satisfies the above upper-bound with $\underline{\theta} = \beta$. In fact, a more general result is established in Theorem 2: when the “0-tail function” $\underline{\nu} : x \mapsto \nu([-x, x]^c)$ has regular variation as x goes to 0, with index $-\underline{\theta}$, then $\underline{\theta} = \beta$ (see [5]) and we establish a close connection between the quantization rate of X and $\underline{\nu}$ and $\underline{\theta}$. In many cases of interest, including α -stable processes and other classes of subordinated Lévy processes, we show that this general upper-bound provides the exact rate of quantization: it matches with the lower bound estimates derived from the connection between quantization rate and small deviation estimates (see *e.g.* [14]). When the Lévy process does have a Brownian component, its exact quantization rate is $(\log N)^{-\frac{1}{2}}$ like the Brownian motion (when $0 < p < r < 2$, $X_1 \in L^r(\mathbb{P})$).

When the Lévy measure is finite (then $\underline{\theta} = 0$), we also establish some quantization rates for the compound Poisson processes and show they are infinitely faster than the above ones. To this end, we design an explicit sequence of quantizers which can clearly be implemented for numerical purposes. In fact the whole proof is constructive provided the Lévy measure is “tractable” enough.*

The paper is organized as follows: Section 2 is devoted to the abstract connection between mean regularity and quantization rate of processes. Section 3 is devoted to some first applications to various families of processes. As far as we know, some of this rates are new. In several cases of interest, these rates are shown to be optimal. Its main result is Theorem 1. Section 4 provides an upper-bound for the quantization rate of general Lévy process in connection with the behaviour of the Lévy measure around 0. The main results are Theorem 2 and Proposition 3. In Section 5.1, we provide the exact rate for a Lévy process having a Brownian component. Finally, in Section 5.2, we derive the exact quantization rate for subordinated Lévy processes.

NOTATIONS: • $L_T^p := L^p([0, T], dt)$ and $|f|_{L_T^p} = (\int_0^T |f(t)|^p dt)^{\frac{1}{p}}$.

• Let $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ be two sequences of positive real numbers. The symbol $a_n \sim b_n$ means $a_n = b_n + o(b_n)$ and $a_n \approx b_n$ means $a_n = O(b_n)$ and $b_n = O(a_n)$.

- $[x]$ denotes the integral part of the real number x and $x_+ = \max(x, 0)$ its positive part. • $\log_m(x)$ is for the m times iterated logarithm function.
- $\|Y\|_r := \|Y\|_{L^r(\mathbb{P})}$ for any random variable Y defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.
- Throughout the paper, the letter C (possibly with subscripts) will denote a positive real constant that may vary from line to line.
- For a càdlàg continuous time process $X = (X_t)_{t \geq 0}$, X_{t-} will denote its left limit and $\Delta X_t := X_t - X_{t-}$ its jump at time t .

2 Mean pathwise regularity and quantization error rate: an upper bound

In this section, we derive in full generality an upper-bound for the $(L^r(\mathbb{P}), L^p_T)$ -quantization error $e_{N,r}(X, L^p_T)$ based on the path regularity of the mapping $t \mapsto X_t$ from $[0, T]$ to $L^p(\mathbb{P})$. The main result of this section is Theorem 1 below. Then we will illustrate on several examples that this rate may be optimal or not.

As a first step we will reformulate the so-called Pierce Lemma (see [13], p.82) which is the main step of the proof of Zador's Theorem for unbounded random variables. Note that the proof of its original formulation (see below) relies on random quantization.

Lemma 1 (*Extended Pierce Lemma*) *Let $r, \delta > 0$. There exists a real constant $C_{r,\delta}$ such that, for every random variable $X : (\Omega, \mathcal{A}) \rightarrow (\mathbb{R}, \mathbb{B}(\mathbb{R}))$,*

$$\forall N \geq 1, \quad e_{N,r}(X, \mathbb{R}) = \inf_{\text{card}(\alpha) \leq N} \|X - \widehat{X}^\alpha\|_r \leq C_{r,\delta} \|X\|_{r+\delta} N^{-1}.$$

Proof: It follows from the original Pierce Lemma (see [13], p.82) that there exists a universal real constant $C_{r,\delta}^0 > 0$ and an integer $N_{r,\delta} \geq 1$ such that, for any random variable $X : (\Omega, \mathcal{A}) \rightarrow (\mathbb{R}, \mathbb{B}(\mathbb{R}))$,

$$\forall N \geq N_{r,\delta}, \quad \inf_{\text{card}(\alpha) \leq N} \mathbb{E}|X - \widehat{X}^\alpha|^r \leq C_{r,\delta}^0 (1 + \mathbb{E}|X|^{r+\delta}) N^{-r}.$$

Using the scaling property of quantization: for every $\lambda > 0$,

$$\|X - \widehat{X}^\alpha\|_r = \frac{1}{\lambda} \|(\lambda X) - \widehat{\lambda X}^{\lambda\alpha}\|_r$$

where $\lambda\alpha = \{\lambda a, a \in \alpha\}$, one derives from Pierce Lemma by considering $X/\|X\|_{r+\delta}$ and setting $\lambda := 1/\|X\|_{r+\delta}$ that

$$\forall N \geq N_{r,\delta}, \quad \inf_{\text{card}(\alpha) \leq N} \|X - \widehat{X}^\alpha\|_r \leq (2C_{r,\delta}^0)^{\frac{1}{r}} \|X\|_{r+\delta} N^{-1}.$$

Now, for every $N \in \{1, \dots, N_{r,\delta} - 1\}$, setting $\alpha := \{0\}$ yields

$$\inf_{\text{card}(\alpha) \leq N} \|X - \widehat{X}^\alpha\|_r \leq \|X\|_r \leq N_{r,\delta} \|X\|_{r+\delta} N^{-1}.$$

Combining the last two inequalities and setting $C_{r,\delta} = \max((2C_{r,\delta}^0)^{\frac{1}{r}}, N_{r,\delta})$ completes the proof. \diamond

Let $(e_n)_{n \geq 0}$ denote the Haar basis defined as the restrictions on $[0, T]$ of the following functions $e_0 := T^{-\frac{1}{2}} \mathbf{1}_{[0,T]}$, $e_1 := T^{-\frac{1}{2}} (\mathbf{1}_{[0,T/2]} - \mathbf{1}_{[T/2,T]})$, $e_{2^n+k} := 2^{\frac{n}{2}} e_1(2^n \cdot - kT)$, $n \geq 0$, $k \in \{0, \dots, 2^n - 1\}$.

With this normalization, it makes up an orthonormal basis of the Hilbert space $(L_T^2, (\cdot|\cdot))$ where $(f|g) = \int_0^T fg(t) dt$ and a (monotone) Schauder basis of L_T^p , $p \in [1, +\infty)$ i.e. $(f|e_0)e_0 + \sum_{n \geq 0} \sum_{0 \leq k \leq 2^n - 1} (f|e_{2^n+k})e_{2^n+k}$ converges to f in L_T^p , for every $f \in L_T^p$ (see [27]). Furthermore, it clearly satisfies for every $f \in L_T^1$ and every $p > 0$,

$$\forall n \geq 0, \quad \int_0^T \left| \sum_{k=0}^{2^n-1} (f|e_{2^n+k})e_{2^n+k}(t) \right|^p dt = 2^{n(\frac{p}{2}-1)} T^{1-\frac{p}{2}} \sum_{k=0}^{2^n-1} |(f|e_{2^n+k})|^p. \quad (2.1)$$

The second key to establish a general connection between quantization rate and mean pathwise regularity is the following standard properties of the Haar basis: for every $f \in L_T^1$,

$$\begin{aligned} (f|e_{2^n+k}) &= 2^{\frac{n}{2}} T^{-\frac{1}{2}} \left(\int_{kT2^{-n}}^{(2k+1)T2^{-(n+1)}} f(u) du - \int_{(2k+1)T2^{-(n+1)}}^{(k+1)T2^{-n}} f(u) du \right) \\ &= 2^{\frac{n}{2}} T^{-\frac{1}{2}} \int_0^{T2^{-(n+1)}} \left(f(kT2^{-n} + u) - f((2k+1)T2^{-(n+1)} + u) \right) du. \end{aligned} \quad (2.2)$$

Let $(X_t)_{t \in [0, T]}$ be a bi-measurable process defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with \mathbb{P} -almost all paths lying in L_T^1 such that $X_t \in L^\rho(\mathbb{P})$ for every $t \in [0, T]$ for some positive real exponent $\rho > 0$. When $\rho \in (0, 1)$, we assume that X has càdlàg paths (right continuous, left limited) to ensure the measurability of the supremum in Assumption (2.3) below.

We make the following φ -Lipschitz assumption on the map $t \mapsto X_t$ from $[0, T]$ into $L^\rho(\mathbb{P})$: there is a non-decreasing function $\varphi: \mathbb{R}_+ \rightarrow [0, +\infty]$, continuous at 0 with $\varphi(0) = 0$ such that

$$(L_{\varphi, \rho}) \equiv \begin{cases} (i) \quad \forall s, t \in [0, T], & \mathbb{E} |X_t - X_s|^\rho \leq (\varphi(|t-s|))^\rho & \text{if } \rho \geq 1, \\ (ii) \quad \forall t \in [0, T], \forall h \in (0, T], & \mathbb{E} (\sup_{t \leq s \leq (t+h) \wedge T} |X_s - X_t|^\rho) \leq (\varphi(h))^\rho & \text{if } 0 < \rho < 1. \end{cases} \quad (2.3)$$

(One may assume without loss of generality that φ is always finite but that (i) and (ii) are only true for $|t-s|$ or h small enough respectively). Note that this assumption implies that $\mathbb{E}(|X|_{L_T^\rho}^\rho) < +\infty$ so that, in particular, $\mathbb{P}(d\omega)$ -a.s., $t \mapsto X_t(\omega)$ lies in L_T^ρ (which in turn implies that the paths lie in L_T^1 if $\rho \geq 1$).

We make a *regularly varying assumption* on φ at 0 with index $b \geq 0$ i.e. for every $t > 0$

$$\lim_{x \rightarrow 0} \frac{\varphi(tx)}{\varphi(x)} = t^b. \quad (2.4)$$

In accordance with the literature (see [3]) this means that $x \mapsto \varphi(1/x)$ is regularly varying at infinity with index $-b$ (which is a more usual notion in that field). When $b = 0$, φ is said to be *slowly varying* at 0.

Let $r, p \in (0, \rho)$. Our aim is to evaluate the $L^r(\mathbb{P})$ -quantization rate of the process X viewed as an L_T^p -valued random variable induced by the ‘‘Haar product quantizations’’ of X defined by

$$\widehat{X} = \widehat{\xi}_0^{N_0} e_0 + \sum_{n \geq 0} \sum_{k=0}^{2^n-1} \widehat{\xi}_{2^n+k}^{N_{2^n+k}} e_{2^n+k} \quad (2.5)$$

where $\xi_k := (X|e_k) \in L^\rho(\mathbb{P})$, $k \geq 0$, and $\widehat{\xi}^N$ denotes an N -quantization ($N \geq 1$) of the (real-valued) random variable ξ i.e. a quantization of ξ by a codebook α^N having N elements. A quantization taking finitely many values, we set $N_{2^n+k} = 1$ and $\widehat{\xi}_{2^n+k}^{N_{2^n+k}} = 0$ for large enough n (which may be a non optimal 1-quantizer for $\xi_{2^n+k}^{N_{2^n+k}}$).

We will see that this local behaviour of φ at 0 induces an upper-bound for the functional quantization error rate of X (regardless of the values of r and p except for constants).

Theorem 1 Let $X = (X_t)_{t \in [0, T]}$ be a (bi-measurable) process defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that $X_t \in L^\rho(\mathbb{P})$ for an exponent $\rho > 0$. Assume that X satisfies (2.3) (the φ -Lipschitz assumption $(L_{\varphi, \rho})$) for this exponent ρ where φ is regularly varying (in the sense of (2.4)) with index $b \geq 0$ at 0 (then $|X|_{L_T^\rho} \in L^1(\mathbb{P})$). Then

$$\forall r, p \in (0, \rho), \quad e_{N, r}(X, L_T^p) \leq C_{r, p} \begin{cases} \varphi(1/\log N) & \text{if } b > 0, \\ \psi(1/\log N) & \text{if } b = 0, \end{cases}$$

with $\psi(x) = (\int_0^x (\varphi(\xi))^{r \wedge 1} d\xi / \xi)^{1/(r \wedge 1)}$, assuming moreover $\int_0^1 (\varphi(\xi))^{r \wedge 1} d\xi / \xi < +\infty$ if $b = 0$. In particular if $\varphi(u) = c u^b$, $b > 0$, then

$$e_{N, r}(X, L_T^p) = O((\log N)^{-b}). \quad (2.6)$$

Proof. Relying on the following two obvious inequalities:

$$|f|_{L_T^p} \leq T^{1/p-1/p'} |f|_{L_T^{p'}}, \quad p \leq p'$$

for every Borel functions $f : [0, T] \rightarrow \mathbb{R}$ and

$$\|Z\|_r \leq \|Z\|_{r'}, \quad r \leq r'$$

for every random variables $Z : \Omega \rightarrow \mathbb{R}$, one may assume without loss of generality that, either

$$1 \leq p = r < \rho \quad \text{or} \quad 0 < p = r < \rho \leq 1.$$

CASE 1 ($1 \leq p = r < \rho$): Let $N \geq 1$ be a fixed integer. One considers a Haar product quantization \widehat{X} of X with a (product) codebook having at most N elements *i.e.* such that $N_0 \times \prod_{n, k} N_{2^n+k} \leq N$. Its characteristics will be specified further on. Then, using (2.1)

$$\begin{aligned} |X - \widehat{X}|_{L_T^r} &\leq T^{\frac{1}{r}-\frac{1}{2}} |\xi_0 - \widehat{\xi}_0^{N_0}| + \sum_{n \geq 0} \left| \sum_{k=0}^{2^n-1} (\xi_{2^n+k} - \widehat{\xi}_{2^n+k}^{N_{2^n+k}}) e_{2^n+k} \right|_{L_T^r} \\ &= T^{\frac{1}{r}-\frac{1}{2}} |\xi_0 - \widehat{\xi}_0^{N_0}| + T^{\frac{1}{r}-\frac{1}{2}} \sum_{n \geq 0} 2^{n(\frac{1}{2}-\frac{1}{r})} \left(\sum_{k=0}^{2^n-1} |\xi_{2^n+k} - \widehat{\xi}_{2^n+k}^{N_{2^n+k}}|^r \right)^{\frac{1}{r}} \end{aligned}$$

so that, both $\|\cdot\|_r$ and $\|\cdot\|_1$ being norms,

$$\begin{aligned} \| |X - \widehat{X}|_{L_T^r} \|_r &\leq T^{\frac{1}{r}-\frac{1}{2}} \| |\xi_0 - \widehat{\xi}_0^{N_0}| \|_r + T^{\frac{1}{r}-\frac{1}{2}} \sum_{n \geq 0} 2^{n(\frac{1}{2}-\frac{1}{r})} \left\| \left(\sum_{k=0}^{2^n-1} |\xi_{2^n+k} - \widehat{\xi}_{2^n+k}^{N_{2^n+k}}|^r \right)^{\frac{1}{r}} \right\|_r \\ &= T^{\frac{1}{r}-\frac{1}{2}} \| \xi_0 - \widehat{\xi}_0^{N_0} \|_r + T^{\frac{1}{r}-\frac{1}{2}} \sum_{n \geq 0} 2^{n(\frac{1}{2}-\frac{1}{r})} \left\| \sum_{k=0}^{2^n-1} |\xi_{2^n+k} - \widehat{\xi}_{2^n+k}^{N_{2^n+k}}|^r \right\|_1^{\frac{1}{r}} \\ &\leq T^{\frac{1}{r}-\frac{1}{2}} \| \xi_0 - \widehat{\xi}_0^{N_0} \|_r + T^{\frac{1}{r}-\frac{1}{2}} \sum_{n \geq 0} 2^{n(\frac{1}{2}-\frac{1}{r})} \left(2^n \max_{0 \leq k \leq 2^n-1} \| |\xi_{2^n+k} - \widehat{\xi}_{2^n+k}^{N_{2^n+k}}|^r \|_1 \right)^{\frac{1}{r}} \\ &= T^{\frac{1}{r}-\frac{1}{2}} \| \xi_0 - \widehat{\xi}_0^{N_0} \|_r + T^{\frac{1}{r}-\frac{1}{2}} \sum_{n \geq 0} 2^{\frac{n}{2}} \max_{0 \leq k \leq 2^n-1} \| |\xi_{2^n+k} - \widehat{\xi}_{2^n+k}^{N_{2^n+k}}|^r \|_1^{\frac{1}{r}} \\ &= T^{\frac{1}{r}-\frac{1}{2}} \| \xi_0 - \widehat{\xi}_0^{N_0} \|_r + T^{\frac{1}{r}-\frac{1}{2}} \sum_{n \geq 0} 2^{\frac{n}{2}} \max_{0 \leq k \leq 2^n-1} \| \xi_{2^n+k} - \widehat{\xi}_{2^n+k}^{N_{2^n+k}} \|_r. \quad (2.7) \end{aligned}$$

Let $\delta := \rho - r$. It follows from Lemma 1 (Pierce Lemma) that, for every $N \geq 1$, and every r.v. $\xi \in L^r(\mathbb{P})$,

$$\inf_{\text{card}(\alpha) \leq N} \|\xi - \widehat{\xi}^\alpha\|_r \leq C_{r,\rho} \|\xi\|_\rho N^{-1}. \quad (2.8)$$

Now, using the monotony in p of the L^p -norms with respect to the probability measure $2^{n+1} \mathbf{1}_{[0, 2^{-(n+1)T}]}(t) dt/T$, Fubini's Theorem, the $(L_{r,\varphi})$ -Lipschitz continuity assumption (2.3)(i) and (2.2), yields

$$\begin{aligned} \mathbb{E} |\xi_{2^n+k}|^\rho &= \mathbb{E} |(X|e_{2^n+k})|^\rho \\ &\leq 2^{\frac{n}{2}\rho} T^{-\rho/2} \mathbb{E} \left(\int_0^{2^{-(n+1)T}} |X_{\frac{k}{2^n}T+u} - X_{\frac{2k+1}{2^{n+1}}T+u}| du \right)^\rho \\ &\leq 2^{\frac{n}{2}\rho} 2^{-(n+1)\rho} T^{\rho/2} \mathbb{E} \left(\int_0^{2^{-(n+1)T}} |X_{\frac{k}{2^n}T+u} - X_{\frac{2k+1}{2^{n+1}}T+u}|^\rho 2^{n+1} du/T \right) \\ &\leq 2^{-\rho} 2^{-\frac{n}{2}\rho+n+1} T^{\rho/2-1} \int_0^{2^{-(n+1)T}} \mathbb{E} |X_{\frac{k}{2^n}T+u} - X_{\frac{2k+1}{2^{n+1}}T+u}|^\rho du \\ &\leq 2^{-\frac{n}{2}\rho+n+1-\rho} T^{\rho/2-1} \int_0^{2^{-(n+1)T}} (\varphi(T/2^{n+1}))^\rho du \\ &\leq C_{X,T,r,\rho} 2^{-\frac{n}{2}\rho} (\varphi(T/2^{n+1}))^\rho. \end{aligned} \quad (2.9)$$

At this stage, we assume *a priori* that that the size sequence $(N_{2^n+k})_{n \geq 0, k=0, \dots, 2^n-1}$ of the marginal codebooks is nonincreasing as $2^n + k$ increases and satisfies

$$1 \leq \prod_{k \geq 0} N_k \leq N.$$

We assume that all the quantizations induced by these codebooks are L^r -optimal up to $n \leq m$ i.e.

$$\|\xi_{2^n+k} - \widehat{\xi}_{2^n+k}\|_r = \inf_{\text{card}(\alpha) \leq N_{2^n+k}} \|\xi_{2^n+k} - \widehat{\xi}_{2^n+k}^\alpha\|_r$$

and that $\widehat{\xi}_{2^n+k} = 0$ otherwise. Then combining (2.7), (2.9) and (2.8) (Pierce Lemma) yields

$$\begin{aligned} \| |X - \widehat{X}|_{L_T^r} \|_r &\leq C_{X,T,r,\rho} \left(\frac{1}{N_0} + \sum_{n \geq 0} \frac{\varphi(T2^{-(n+1)})}{N_{2^{n+1}}} \right) \\ &\leq C_{X,T,r,\rho} \left(\frac{1}{N_0} + \frac{1}{T} \sum_{n \geq 0} \sum_{k=0}^{2^{n+1}-1} \frac{\Phi(2T/(2^{n+1}+k))}{N_{2^{n+1}+k}} \right) \\ &= C_{X,T,r,\rho} \left(\frac{1}{N_0} + \frac{1}{T} \sum_{k \geq 2} \frac{\Phi(2T/k)}{N_k} \right) \end{aligned}$$

where $\Phi(x) := x \varphi(x)$, $x \in (0, +\infty)$. This function Φ is regularly varying (at 0) with index $b+1$. This implies in particular that there is a real constant $c > 0$ such that $\Phi(T/k) \leq c \Phi(1/(k+1))$ for every $k \geq 2$. Hence, inserting for convenience the term $\Phi(1/2)/N_1$ and modifying the real constant $C_{X,T,r,\rho}$ in an appropriate way finally yields

$$\| |X - \widehat{X}|_{L_T^r} \|_r \leq C_{X,T,r,\rho} \sum_{k \geq 1} \frac{\Phi(1/k)}{N_{k-1}}.$$

Now set for convenience $\nu_k = \Phi(1/k)$, $k \geq 1$. Note that in case $b = 0$, the integrability condition $\int_0^1 \varphi(\xi)/\xi d\xi < +\infty$ implies $\sum_k \nu_k < +\infty$. Consequently an upper-bound for the quantization rate is given by the solution of the following optimal allocation problem

$$\begin{aligned} e_{N,r}(X, L_T^r) &\leq C_{X,T,r,\rho} \min \left\{ \sum_{k \geq 1} \frac{\nu_k}{N_{k-1}}, \prod_{k \geq 0} N_k \leq N, N_0 \geq \dots \geq N_k \geq \dots \geq 1 \right\} \\ &= C_{X,T,r,\rho} \min \left\{ \sum_{k=1}^m \frac{\nu_k}{N_{k-1}} + \sum_{k \geq m+1} \nu_k, m \geq 1, \prod_{0 \leq k \leq m-1} N_k \leq N, N_0 \geq \dots \geq N_{m-1} \geq 1 \right\}. \end{aligned} \quad (2.10)$$

The rest of the proof follows the approach developed in [19] (Section 4.1, especially Lemma 4.2, Theorem 4.6 (i)-(iii) and its proof) and [20]. However, one must be aware that we had to modify some notations.

Proposition 1 (See [19] for a proof) Assume $\nu_k = \Phi(1/k)$, $k \geq 1$, where $\Phi(x) = x\varphi(x)$, $\varphi : (0, +\infty)$ is an non-decreasing, regularly varying function at 0 with index $b \geq 0$ (with $\int_0^1 \varphi(\xi) \frac{d\xi}{\xi} < +\infty$ when $b = 0$). Then

(i) $\lim_k \nu_k / \nu_{k+1} = 1$,

(ii) $\left(\prod_{k=1}^n \nu_k \right)^{\frac{1}{n}} \sim e^{b+1} \nu_n$,

(iii) $\sum_{k=n+1}^{\infty} \nu_k + n \nu_k \sim c \psi(1/n)$ where $c = 1 + 1/b$ if $b > 0$ and $c = 1$ if $b = 0$ and

$$\psi(x) = \varphi(x) \quad \text{if } b > 0 \quad \text{and} \quad \psi(x) := \int_0^x \varphi(\xi) \frac{d\xi}{\xi} \quad \text{if } b = 0.$$

Proof.

Set $m = m^*(N) = \max \left\{ m \geq 1 : N^{\frac{1}{m}} \nu_m \left(\prod_{j=1}^m \nu_j \right)^{-1/m} \geq 1 \right\}$,

and $N_{k-1} = N_{k-1}(N) := \left[N^{\frac{1}{m}} \nu_k \left(\prod_{j=1}^m \nu_j \right)^{-1/m} \right] \geq 1, \quad k = 1, \dots, m.$

It follows from Proposition 1 (ii) that

$$m = m^*(N) \sim \frac{\log N}{b+1} \quad \text{as } N \rightarrow \infty.$$

Then,

$$\begin{aligned} \sum_{k=1}^m \frac{\nu_k}{N_{k-1}} &\leq \max_{k \geq 1} (1 + 1/N_{k-1}) m N^{-\frac{1}{m}} \left(\prod_{j=1}^m \nu_j \right)^{\frac{1}{m}} \\ &\leq 2 m N^{-\frac{1}{m}} \left(\prod_{j=1}^m \nu_j \right)^{\frac{1}{m}} \\ &\leq 2 m \nu_m. \end{aligned}$$

Consequently using this time (iii) in Proposition 1

$$\begin{aligned} \sum_{k=1}^m \frac{\nu_k}{N^{k-1}} + \sum_{k \geq m+1} \nu_k &\leq 2 \left(m \nu_m + \sum_{k \geq m+1} \nu_k \right) \\ &= O(\psi(1/\log N)) \end{aligned}$$

so that

$$\| |X - \widehat{X}|_{L_T^p} \|_r = O(\psi(1/\log N)).$$

CASE 2 ($\rho \leq 1$). One relies this time on the pseudo-triangular inequality

$$|f + g|_{L_T^r}^r \leq |f|_{L_T^r}^r + |g|_{L_T^r}^r$$

which follows from the elementary inequality $(u + v)^r \leq u^r + v^r$.

$$\begin{aligned} |X - \widehat{X}|_{L_T^r}^r &\leq T^{1-\frac{r}{2}} |\xi_0 - \widehat{\xi}_0^{N_0}|^r + \sum_{n \geq 0} \left| \sum_{k=0}^{2^n-1} (\xi_{2^{n+k}} - \widehat{\xi}_{2^{n+k}}^{N_{2^n+k}}) e_{2^{n+k}} \right|_{L_T^r}^r \\ &= T^{1-\frac{r}{2}} |\xi_0 - \widehat{\xi}_0^{N_0}|^r + T^{1-\frac{r}{2}} \sum_{n \geq 0} 2^{n(\frac{r}{2}-1)} \sum_{k=0}^{2^n-1} |\xi_{2^{n+k}} - \widehat{\xi}_{2^{n+k}}^{N_{2^n+k}}|^r \end{aligned}$$

so that,

$$\begin{aligned} \| |X - \widehat{X}|_{L_T^r} \|_r^r &= \| |X - \widehat{X}|_{L_T^r}^r \|_1 \\ &\leq T^{1-\frac{r}{2}} \| |\xi_0 - \widehat{\xi}_0^{N_0}|^r \|_1 + T^{1-\frac{r}{2}} \sum_{n \geq 0} 2^{n(\frac{r}{2}-1)} \left\| \sum_{k=0}^{2^n-1} |\xi_{2^{n+k}} - \widehat{\xi}_{2^{n+k}}^{N_{2^n+k}}|^r \right\|_1 \\ &\leq T^{1-\frac{r}{2}} \| |\xi_0 - \widehat{\xi}_0^{N_0}|^r \|_r + T^{1-\frac{r}{2}} \sum_{n \geq 0} 2^{n(\frac{r}{2}-1)} 2^n \max_{0 \leq k \leq 2^n-1} \| |\xi_{2^{n+k}} - \widehat{\xi}_{2^{n+k}}^{N_{2^n+k}}|^r \|_1 \\ &= T^{1-\frac{r}{2}} \| |\xi_0 - \widehat{\xi}_0^{N_0}|^r \|_r + T^{1-\frac{r}{2}} \sum_{n \geq 0} 2^{\frac{nr}{2}} \max_{0 \leq k \leq 2^n-1} \| |\xi_{2^{n+k}} - \widehat{\xi}_{2^{n+k}}^{N_{2^n+k}}|^r \|_1 \\ &= T^{1-\frac{r}{2}} \| |\xi_0 - \widehat{\xi}_0^{N_0}|^r \|_r + T^{1-\frac{r}{2}} \sum_{n \geq 0} 2^{\frac{nr}{2}} \max_{0 \leq k \leq 2^n-1} \| \xi_{2^{n+k}} - \widehat{\xi}_{2^{n+k}}^{N_{2^n+k}} \|_r^r. \quad (2.11) \end{aligned}$$

This inequality replaces (2.7). Then, one notes that

$$\begin{aligned} \mathbb{E} |\xi_{2^{n+k}}|^\rho &\leq 2^{\frac{n}{2}\rho} T^{-\rho/2} \left(2^{-(n+1)} T \varphi(T/2^{n+1}) \right)^\rho \\ &= C_{X,T,r,\rho} 2^{-\frac{n}{2}\rho} \left(\varphi(T/2^{n+1}) \right)^\rho \end{aligned}$$

so that

$$\| |X - \widehat{X}|_{L_T^r} \|_r^r \leq C_{X,T,r,\rho} \left(\frac{1}{N_0^r} + \sum_{n \geq 0} \frac{\varphi(T2^{-(n+1)})^r}{N_{2^{n+1}}^r} \right).$$

Then, set $\widetilde{\varphi}(u) = (\varphi(u))^r$, $\widetilde{N}_k = N_k^r$ and $\widetilde{N} := N^r$. One proceeds for $\| |X - \widehat{X}|_{L_T^r} \|_r^r$ with these “tilded” parameters like for $\| |X - \widehat{X}|_{L_T^r} \|_r$ in the case $\rho > 1$. \diamond

Remarks. • ABOUT THE CASE $p > r$. When $p \geq \rho > r$, the $(L^r(\mathbb{P}), L_T^p)$ -quantization problem remains consistent. But there is a price to be paid for considering a p exponent greater than ρ .

Thus, if φ in $(L_{(\rho,\varphi)})$ has regular variations with exponents $b > 0$ at 0 and if $b + \frac{1}{p} - \frac{1}{r} > 0$, the same approach yields the rate

$$e_{N,r}(X, L_T^p) \leq C_{X,r,\delta,T,p} \varphi(1/\log N) (\log N)^{\frac{1}{r} - \frac{1}{p}}.$$

We do not know whether it comes from our approach or if it is the best possible universal rate.

• **ABOUT LOWER BOUNDS.** In several situations, when the assumption $(L_{\rho,\varphi})$ is optimal in terms of mean regularity of a process, the upper-bound for the functional quantization rate turns out to be the true rate. We have no general result in that direction so far since most lower bound results rely on a different technology, namely the small ball deviation theory. Thus, in [14], a connection is established between (functional) quantization and small ball deviation for Gaussian processes. In particular this approach provides a method to derive a lower bound for the $(L^r(\mathbb{P}), L_T^p)$ quantization rate from some upper bound for the small ball problem. A careful reading of the paper (see the proof of Theorem 1.2 in [14]) shows that this small deviation lower bound holds for any *unimodal* (w.r.t. 0) non zero process. To be precise: let $p \in (0, \infty)$ and assume that \mathbb{P}_X is L_T^p -unimodal in the following sense: there exists a real $\varepsilon_0 > 0$ such that

$$\forall x \in L_T^p, \forall \varepsilon \in (0, \varepsilon_0], \quad \mathbb{P}(|X - x|_{L_T^p} \leq \varepsilon) \leq \mathbb{P}(|X|_{L_T^p} \leq \varepsilon).$$

(For centered Gaussian processes this follows for $p \geq 1$ from Anderson's inequality). If

$$G(-\log(\mathbb{P}(|X|_{L_T^p} \leq \varepsilon))) = \Omega(1/\varepsilon) \quad \text{as } \varepsilon \rightarrow 0$$

for some increasing unbounded function $G : (0, \infty) \rightarrow (0, \infty)$, then

$$\forall r \in (0, \infty), \quad \forall c > 1, \quad e_{N,r}(X, L_T^p) = \Omega\left(\frac{1}{G(\log(cN))}\right). \quad (2.12)$$

3 Applications and examples

In this section, we give some examples which illustrate that the upper-bound derived from the mean pathwise regularity may be optimal or not.

3.1 Application to Itô processes and d -dimensional diffusion processes

• Let W denote an \mathbb{R}^d -valued standard Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $(\mathcal{F}_t^W)_{t \in [0, T]}$ denote its natural filtration (completed with all the \mathbb{P} -negligible sets). Let X be a 1-dimensional Itô process defined by

$$dX_t = G_t dt + H_t \cdot dW_t, \quad X_0 = x_0 \in \mathbb{R}$$

where $(G_t)_{t \in [0, T]}$ is a real-valued process and $(H_t)_{t \in [0, T]}$ is an \mathbb{R}^d -valued process, both assumed $(\mathcal{F}_t^W)_{t \in [0, T]}$ -progressively measurable. Assume there exists a real number $\rho \geq 2$ such that

$$\sup_{t \in [0, T]} \mathbb{E}|G_t|^\rho + \sup_{t \in [0, T]} \mathbb{E}|H_t|^\rho < +\infty. \quad (3.13)$$

where $|\cdot|$ denotes any norm on \mathbb{R}^d . Then, it is classical background, see *e.g.* [4], that the φ -Lipschitz Assumption $(L_{\varphi,\rho})(i)$ (*i.e.* (2.3)(i)) is satisfied with $\varphi(u) = cu^{\frac{1}{2}}$. It follows from Theorem 1

$$\forall r, p \in (0, \rho), \quad e_{N,r}(X, L_T^p) = O\left((\log N)^{-1/2}\right).$$

- Let $X = (X^1, \dots, X^d)$ be an \mathbb{R}^d -valued diffusion process defined by

$$dX_t = b(t, X_t) dt + \sigma(t, X_t) dW_t, \quad X_0 = x_0 \in \mathbb{R}^d$$

where $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathcal{M}(d \times q, \mathbb{R})$ are Borel functions satisfying

$$\forall t \in [0, T], \forall x \in \mathbb{R}^d, \quad |b(t, x)| + \|\sigma(t, x)\| \leq C(1 + |x|)$$

and W is an \mathbb{R}^q -valued standard Brownian motion⁽¹⁾. Then, every component X^i is an Itô process (with $G_t = b^i(t, X_t)$ and $H_t := \sigma^{i \cdot}(t, X_t)$) for which Assumption (3.13) is satisfied for every $\rho > 0$ (see *e.g.* [4]). On the other hand, if (u^1, \dots, u^d) denotes the canonical basis of \mathbb{R}^d and $|\cdot|$ denotes any norm on \mathbb{R}^d , then for every $p \geq 1$ and every $f := \sum_{1 \leq i \leq d} f^i u^i : [0, T] \rightarrow \mathbb{R}^d$,

$$|f|_{L_{\mathbb{R}^d}^p([0, T], dt)} \leq \sum_{i=1}^d |f^i|_{L_T^p} |u^i|.$$

Now, one can quantize each Itô process $(X_t^i)_{t \in [0, T]}$, $i = 1, \dots, d$ using an (L^r, L_T^p) -optimal quantizer $\alpha^{(i)}$ of size $\lceil \sqrt[d]{N} \rceil$. It is clear that the resulting product quantizer $\prod_{i=1}^d \alpha^{(i)}$ of size $\lceil \sqrt[d]{N} \rceil^d \leq N$ induces an $(L^r, L_{\mathbb{R}^d}^p([0, T], dt))$ -quantization error $O((\log N)^{-1/2})$ (see *e.g.* [21]). Combining these obvious remarks finally yields

$$\forall r, p > 0, \quad e_{N,r}(X, L_{\mathbb{R}^d}^p([0, T], dt)) = O((\log N)^{-1/2}).$$

In the “smooth” case $H \equiv 0$, the regularity assumption $(L_{\varphi, \rho})$ is satisfied with $\varphi(u) = cu$. One gets the universal upper bound

$$\forall r, p \in (0, \rho), \quad e_{N,r}(X, L_T^p) = O((\log N)^{-1}).$$

Both rates are optimal as universal rates for $p \geq 1$ as can be seen from $X = W$ and $X = \int_0^\cdot G_s ds$ with $G_t = \int_0^t (t-s)^{\beta-\frac{1}{2}} dW_s$ ($\beta > 0$ and $d = 1$) respectively (see [14]).

As far as quantization rate is concerned, this extends to general d -dimensional diffusions a first result obtained in [21] by stochastic calculus techniques for a more restricted class of Brownian diffusions (which included 1-dimensional ones). This also extends (the upper bound part of the) the result obtained in [7] for another class of (essentially 1-dimensional) Brownian diffusions. For the class investigated in [21] it is shown that under an ellipticity assumption on σ , this rate is optimal in case $r, p \geq 1$. In [7], still with a (mild) ellipticity assumption, the rate is sharp for $p \geq 1$. This leads us to conjecture that this rate is optimal for non too degenerate Brownian diffusions.

3.2 Application to fractional Brownian motion

The Fractional Brownian Motion W^H with Hurst constant $H \in (0, 1]$ is a Gaussian process satisfying for every $\rho > 0$,

$$\mathbb{E} |W_t^H - W_s^H|^\rho = C_{H,p} |t - s|^{\rho H} \quad \text{and} \quad (W_s^H)_{0 \leq s \leq t} \stackrel{\mathcal{L}}{\sim} t^H (W_{s/t}^H)_{0 \leq s \leq t}.$$

So, using Theorem 1, we obtain $e_{N,r}(W^H, L_T^p) = O((\log N)^{-H})$ as an $(L^r(\mathbb{P}), |\cdot|_{L_T^p})$ -quantization rate for every $r, p > 0$. This rate is known to be optimal for $p \geq 1$: in fact a sharp rate is established (see [20] when $p = r = 2$ or [8]) (*i.e.* the computation of the exact value of $\lim_N N(\log N)^H e_{N,r}(W^H, L_T^p)$).

¹The above assumption does not imply that such a diffusion process X exists. The existence holds as soon as b and σ are Lipschitz in x uniformly with respect to $t \in [0, T]$

3.3 Stationary processes

Let X be a centered weakly (square integrable) stationary process. Then

$$\mathbb{E}|X_t - X_s|^2 = \mathbb{E}|X_{t-s} - X_0|^2 = 2 \operatorname{Var}(X_0)(1 - c(|t - s|))$$

where $c(t)$ denotes the correlation between X_t and X_0 . Hence if

$$c(u) = 1 - \kappa u^{2a} + o(u^{2a}) \quad \text{as } u \rightarrow 0,$$

then the $L^r(\mathbb{P})$ -rate for L_T^p -quantization, $0 < p, r < 2$, will be $O((\log(N))^{-a})$. If furthermore, X is a Gaussian process (like the Ornstein-Uhlenbeck process with $a = 1/2$), then this $O((\log N)^{-a})$ rate holds for any $r, p > 0$ since for every $\rho \in \mathbb{N}^*$,

$$\mathbb{E}|X_t - X_s|^\rho = \mathbb{E}|X_{t-s} - X_0|^\rho = C_\rho(\operatorname{Var}(X_0)(1 - c(|t - s|)))^{\rho/2}.$$

3.4 Self-similar processes with stationary increments

Let $X = (X_t)_{t \in [0, T]}$ be an H -self-similar process with stationary increments ($H \in (0, \infty)$). Assume $X_1 \in L^\rho(\mathbb{P})$ for some $\rho \geq 1$. Then

$$\mathbb{E}|X_t - X_s|^\rho = C_\rho |t - s|^{\rho H}$$

for every $s, t \in [0, T]$. Since X is stochastically continuous, it has a bi-measurable modification. Then, Theorem 1 gives

$$\forall r, p \in (0, \rho), \quad e_{N,r}(X, L_T^p) = O((\log N)^{-H}).$$

If, furthermore, X is α -stable, $\alpha \in (1, 2)$, then $X_1 \in L^\rho(\mathbb{P})$ for every $\rho \in [1, \alpha)$ so that

$$\forall r, p \in (0, \alpha), \quad e_{N,r}(X, L_T^p) = O((\log N)^{-H}).$$

This class of examples comprises *e.g.* the linear H -fractional α -motions with $\alpha \in (1, 2)$, $H \in (0, 1)$ and the log-fractional α -stable motions with $\alpha \in (1, 2)$, where $H = 1/\alpha$ (see [26], [10]).

3.5 Lévy processes: a first approach

A (càdlàg) Lévy process $X = (X_t)_{t \in \mathbb{R}_+}$ – or Process with Stationary Independent Increments (*PSII*) – is characterized by its so-called local characteristics appearing in the Lévy-Khintchine formula (for an introduction to Lévy processes we refer to [2, 16, 25]). These characteristics depend on the way the “big” jumps are truncated. We will adopt in the following the convention that the truncation occurs at size 1. So that, for every $t \in \mathbb{R}_+$,

$$\mathbb{E}(e^{iuX_t}) = e^{-t\psi(u)} \quad \text{where} \quad \psi(u) = -iua + \frac{1}{2}\sigma^2 u^2 - \int_{\mathbb{R} \setminus \{0\}} (e^{iux} - 1 - iux \mathbf{1}_{\{|x| \leq 1\}}) \nu(dx)$$

where $a, \sigma \in \mathbb{R}$, ν is a non-negative measure on $\mathbb{R} \setminus \{0\}$ such that $\nu(x^2 \wedge 1) < +\infty$. The measure ν is called the *Lévy measure* of the process. One shows that a Lévy process is a compound Poisson process if and only if ν is a finite measure and has finite variation if and only if $\int_{\{|x| \leq 1\}} |x| \nu(dx) < +\infty$. Furthermore

$$X_t \in L^\rho(\mathbb{P}) \quad \text{if and only if} \quad \int_{\{|x| \geq 1\}} |x|^\rho \nu(dx) < +\infty.$$

We will extensively use the following Compensation Formula (see *e.g.* [2] p.7)

$$\mathbb{E} \sum_{s \geq 0} F(s, X_{s-}, \Delta X_s) \mathbf{1}_{\{\Delta X_s \neq 0\}} = \mathbb{E} \int_{\mathbb{R}_+} ds \int_{\mathbb{R} \setminus \{0\}} F(s, X_{s-}, \xi) \nu(d\xi) \quad (3.14)$$

where $F : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}_+$ is a Borel function. As concerns Assumption (2.3) note that the very definition of a Lévy process implies that

$$\mathbb{E} |X_t - X_s|^\rho = \mathbb{E} |X_{t-s}|^\rho \quad \text{and} \quad \mathbb{E} \sup_{s \in [t, t+h]} |X_t - X_s|^\rho = \mathbb{E} \sup_{s \in [0, h]} |X_s|^\rho$$

so that we may focus on the distribution of X_t and $X_t^* := \sup_{s \in [0, t]} |X_s|$. Finally, note that it follows from the usual symmetry principle (see [25]) that for any Lévy process, $\mathbb{P}(X_t^* > u + v) \leq \mathbb{P}(|X_t| > u) / \mathbb{P}(X_t^* \leq v/2)$ so that $\mathbb{E}|X_t|^r$ and $\mathbb{E}|X_t^*|^r$ are simultaneously finite or infinite when $r > 0$.

The following result is established in [22].

Lemma 2 (*Millar's Lemma*) *Assume $\sigma = 0$. If there is a real number $\rho \in (0, 2]$ such that $\int_{\mathbb{R} \setminus \{0\}} |x|^\rho \nu(dx) < +\infty$, then there exists some real constants $a_\rho \in \mathbb{R}$ and $C_\rho > 0$ such that*

$$\forall t \geq 0, \quad \mathbb{E} \left(\sup_{s \in [0, t]} |X_s - a_\rho s|^\rho \right) \leq C_\rho t. \quad (3.15)$$

Furthermore, one may set $a_\rho = 0$ if $\rho \geq 1$.

Hence, it follows as a consequence of Theorem 1 that

$$\forall r, p \in (0, \rho), \quad e_{N,r}(X, L_T^p) = O \left((\log N)^{-\frac{1}{\rho}} \right). \quad (3.16)$$

This follows from the following straightforward remark: let $\beta \subset L_T^p$ be an N -quantizer and let $\xi \in L_T^p$ (here $\xi(t) = a_\rho t$); then

$$\| |X - \widehat{X}^\beta|_{L_T^p} \|_r = \| |(X + \xi) - (\widehat{X} + \xi)^{\xi + \beta}|_{L_T^p} \|_r \quad \text{with} \quad \xi + \beta = \{\xi + f, f \in \beta\}.$$

However rate (3.16) may be sub-optimal as illustrated below with α -stable processes and Poisson processes. In Section 4 we establish two improvements of this rate under some natural hypothesis (see Theorem 2 for a broad class of Lévy processes with an infinite Lévy measure and Proposition 3 for compound Poisson processes).

• **The α -stable processes** The (strictly) α -stable processes are families of Lévy processes indexed by $\alpha \in (0, 2)$ satisfying a self-similarity property, namely

$$\forall t \in \mathbb{R}_+, \quad X_t \stackrel{\mathcal{L}}{\sim} t^{1/\alpha} X_1 \quad \text{and} \quad \sup_{0 \leq s \leq t} |X_s| \stackrel{\mathcal{L}}{\sim} t^{1/\alpha} \sup_{0 \leq s \leq 1} |X_s|.$$

Furthermore,

$$\sup\{r : \mathbb{E}(\sup_{0 \leq s \leq 1} |X_s|^r) < +\infty\} = \alpha \quad \text{and} \quad \mathbb{E}|X_1|^\alpha = +\infty.$$

Consequently it follows from Theorem 1 applied with $\varphi(u) := u^{\frac{1}{\alpha}}$ that

$$\forall p, r \in (0, \alpha), \quad e_{N,r}(X, L_T^p) = O \left(\frac{1}{(\log N)^{\frac{1}{\alpha}}} \right). \quad (3.17)$$

In the symmetric case, an α -stable process X being subordinated to a Brownian motion ($X_t = W_{A_t}$ with A one-sided $\alpha/2$ -stable process) has a unimodal distribution by the Anderson inequality (see further on Section 5.2 entirely devoted to subordinated Lévy processes). Plugging into (2.12) the small deviation estimates established in [18] shows the rate optimality of our upper bound for $e_{N,r}$ when $p \geq 1$, *i.e.*

$$\forall r \in (0, \alpha), \forall p \in [1, \alpha), \quad e_{N,r}(X, L_T^p) \approx (\log N)^{-\frac{1}{\alpha}}. \quad (3.18)$$

• **The Γ -processes** These are subordinators (non-decreasing Lévy processes) whose distribution \mathbb{P}_{X_t} at time t is a $\gamma(\alpha, t)$ -distribution

$$\mathbb{P}_{X_t}(dx) = \frac{\alpha^t}{\Gamma(t)} \mathbf{1}_{(0, \infty)}(x) x^{t-1} e^{-\alpha x} dx.$$

So, easy computations show that for every $\rho > 0$,

$$\mathbb{E}|X_t|^\rho = \frac{\Gamma(t + \rho)}{\alpha^\rho \Gamma(t + 1)} t \sim \frac{\Gamma(\rho)}{\alpha^\rho \Gamma(1)} t \quad \text{as } t \rightarrow 0.$$

Consequently it follows from Theorem 1 that

$$\forall p \in (0, +\infty), \forall r \in (0, p], \quad e_{N,r}(X, L_T^p) = O\left(\frac{1}{(\log(N))^{\frac{1}{p}-\varepsilon}}\right), \quad \forall \varepsilon > 0.$$

• **Compound Poisson processes from the mean regularity viewpoint** One considers a compound Poisson process

$$X_t = \sum_{k=1}^{K_t} U_k,$$

where $K = (K_t)_{t \in [0, T]}$ denotes a standard Poisson process with intensity $\lambda = 1$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $(U_k)_{k \geq 1}$ an i.i.d. sequence of random variables defined on the same probability space, with $U_1 \in L^\rho(\mathbb{P})$ for some $\rho > 0$. Then, standard computations show that,

$$\mathbb{E} \sup_{0 \leq s \leq t} \left| \sum_{k=1}^{K_s} U_k \right|^\rho \leq \mathbb{E} \sum_{k=1}^{K_t} |U_k|^\rho = t \|U_1\|_\rho^\rho \quad \text{if } 0 < \rho \leq 1, \quad (3.19)$$

$$\mathbb{E} \left| \sum_{k=1}^{K_t} U_k \right|^\rho \leq t \|U_1\|_\rho^\rho \times \left[e^{-t} \sum_{k \geq 1} \frac{t^{k-1} k^\rho}{k!} \right] \quad \text{if } \rho > 1, \quad (3.20)$$

Consequently, Assumption (2.3) is fulfilled with $\varphi(u) = cu^b$, $b = 1/\rho$, c positive real constant. Theorem 2 then yields

$$\forall r, p \in (0, \rho), \quad e_{N,r}(X, L_T^p) = O((\log N)^{-1/\rho}).$$

Note that when $\rho \leq 2$, this is a special case of (3.15). These rates are very far from optimality as it will be seen further on (in Section 4 some faster rates are established by a completely different approach based on the almost finite-dimensional feature of the paths of such elementary jump processes). This will emphasize that the mean regularity of $t \mapsto X_t$ does not always rule the quantization rate.

4 A quantization rate for general Lévy processes without Brownian component

The aim of this section is to provide a general result for Lévy processes without Brownian component, with a special attention paid to compound Poisson processes which appears as a critical case of the main theorem. Before stating the main results, we need some further notations related to Lévy processes. Set

$$\underline{\theta} := \inf \left\{ \theta > 0 : \int_{\{|x| \leq 1\}} |x|^\theta \nu(dx) < +\infty \right\} \in [0, 2], \quad (4.21)$$

$$r^* := \sup \left\{ r > 0 : \int_{\{|x| > 1\}} |x|^r \nu(dx) < +\infty \right\} \leq +\infty. \quad (4.22)$$

The exponent $\underline{\theta}$ is known as the Blumenthal-Gettoor index of X (and is often denoted $\beta(X)$ in the literature). One defines on $(0, \infty)$ the tail function of the Lévy measure $\nu : u \mapsto \underline{\nu}(u) := \nu([-u, u]^c)$. Finally we set for every $\underline{\theta} > 0$, $\underline{\ell}(t) := t \underline{\nu}(t^{\frac{1}{\underline{\theta}}})$ and for every $\rho > 0$

$$\Lambda_\rho(t) := (\underline{\ell}(t))^{\frac{1}{2}} + (\underline{\ell}(t))^{\frac{1}{\rho}} + (\underline{\ell}(t))^{\frac{2}{\rho}} \mathbf{1}_{\underline{\theta} \in (1, 2] \cup IV(1)}$$

where $IV(1) = \emptyset$ if $\underline{\theta} = 1$ and $\nu(|x|) < +\infty$ and $IV(1) = \{1\}$ if $\underline{\theta} = 1$ and $\nu(|x|) = +\infty$.

Theorem 2 *Let $X = (X_t)_{t \in [0, T]}$ be a Lévy process with Lévy measure ν and without Brownian component. Assume $r^*, \underline{\theta} > 0$.*

(a) *Assume $\underline{\theta} \in (0, 2] \setminus \{1\}$. If $\int_{\{|x| \leq 1\}} |x|^{\underline{\theta}} \nu(dx) < +\infty$ (i.e. $\underline{\theta}$ holds as a minimum) or if the Lévy measure satisfies*

$$\exists c \in (0, 1], \exists C > 0, \quad \mathbf{1}_{\{0 < |x| \leq c\}} \nu(dx) \leq \frac{C}{|x|^{\underline{\theta}+1}} \mathbf{1}_{\{0 < |x| \leq c\}} dx \quad (4.23)$$

then

$$\forall r, p \in (0, \underline{\theta} \wedge r^*), \quad e_{N,r}(X, L_T^p) = O((\log N)^{-\frac{1}{\underline{\theta}}}). \quad (4.24)$$

(b) *Assume $\underline{\theta} \in (0, 2) \setminus \{1\}$. If the tail function of the Lévy measure ν has regular variation with index $-b$ at 0, then $b = \underline{\theta}$ and the function $\underline{\ell}$ is slowly varying at 0. If furthermore, the functions $t \mapsto t^{\frac{1}{\underline{\theta}}} \Lambda_\rho(t)$ are non-decreasing in a neighbourhood of 0, then*

$$\forall r, p \in (0, \underline{\theta} \wedge r^*), \quad e_{N,r}(X, L_T^p) = O\left((\log N)^{-\frac{1}{\underline{\theta}}} \Lambda_\rho((\log N)^{-1})\right), \quad \forall \rho \in (r \vee p, \underline{\theta}). \quad (4.25)$$

(c) *Assume $\underline{\theta} < r^*$. For every $r \in [\underline{\theta}, r^*)$ and every $p \in (0, r]$,*

$$e_{N,r}(X, L_T^p) = O((\log N)^{-\frac{1}{r} + \eta}), \quad \forall \eta > 0. \quad (4.26)$$

(d) *When $\underline{\theta} = 1$, if ν is symmetric or $\nu(|x|) < +\infty$ then the above rates (4.24) and (4.25) are still valid.*

Remarks. • The conclusion in (a) remains valid for any $\underline{\theta} \in (0, 2]$ satisfying $\int_{\{|x| \leq 1\}} |x|^{\underline{\theta}} \nu(dx) < +\infty$ or (4.23), not only for the Blumenthal-Gettoor index. In particular, we obtain with $\underline{\theta} = 2$,

$$\forall r, p \in (0, 2 \wedge r^*), \quad e_{N,r}(X, L_T^p) = O((\log N)^{-\frac{1}{2}}).$$

• When $\underline{\theta} \in \{1, 2\}$, some rates can also be derived (even when ν is not symmetric and $\nu(|x|) = +\infty$): thus in item (a), if $\underline{\theta} = 1$, one can show by adapting the proof of case $\underline{\theta} \in (1, 2)$ in Proposition 2 below that

$$e_{N,r}(X, L_T^p) = O\left(\frac{\log \log N}{\log N}\right).$$

• In most natural settings, there is a dominating term in the definition of the function Λ_ρ . Thus, one may set in (4.25)

$$\Lambda_\rho(t) = \begin{cases} (\underline{\ell}(t))^{\frac{1}{\rho}} \mathbf{1}_{\{\underline{\theta} \in (0,1] \setminus IV(1)\}} + (\underline{\ell}(t))^{\frac{2}{\rho}} \mathbf{1}_{\{\underline{\theta} \in (1,2] \cup IV(1)\}} & \text{when } \lim_{t \rightarrow 0} \underline{\ell}(t) = +\infty \\ (\underline{\ell}(t))^{\frac{1}{2}} & \text{when } \lim_{t \rightarrow 0} \underline{\ell}(t) = 0. \end{cases}$$

• Note that this theorem provides no rate when $\underline{\theta} = 0$ which is the case of an important class of Lévy processes including compound Poisson processes. In fact for these processes the quantization rate is not ruled by the mean regularity of their paths, as emphasized in Section 4.1.

• The proof of this theorem relies on Theorem 1, *i.e.* on the mean pathwise regularity of X , hence the critical value $\underline{\theta}$ for ρ cannot be overcome by such an approach since Assumption $(L_{\varphi,\rho})$ for $\rho > \underline{\theta}$ would imply that X has a pathwise continuous modification by the Kolmogorov criterion.

EXAMPLES. • Note that for α -stable processes, $r^* = \underline{\theta} = \alpha$, ν satisfies (4.23) and $\lim_{u \rightarrow 0} \underline{\ell}(u) \in (0, \infty)$ so that both rates obtained from (4.24) and (4.25) coincide with that obtained in Section 3.5, *i.e.* $O((\log N)^{-\frac{1}{\alpha}})$. This rate is most likely optimal.

• Let $\nu_{a,\underline{\theta}}^1(dx) := \kappa |x|^{-\underline{\theta}-1} (-\log |x|)^{-a} \mathbf{1}_{(0,c]}(|x|) dx$, with $0 < c < 1$, $\kappa > 0$, $a > 0$ if $\underline{\theta} \in (0, 2)$, then $\underline{\ell}(u) \sim \underline{\theta}^{a-1} (-\log u)^{-a}$ as $u \rightarrow 0$. If a Lévy process X has $\nu_{a,\underline{\theta}}^1$ as a (symmetric) Lévy measure, then $r^* = +\infty$ and

$$\forall r, p \in (0, \underline{\theta}), \quad e_{N,r}(X, L_T^p) = O\left((\log N)^{-\frac{1}{\underline{\theta}}} (\log \log N)^{-\frac{a}{2}}\right).$$

Such a rate improves the one provided by (4.24)

• Let $\nu_{a,\underline{\theta}}^2(dx) = \kappa |x|^{-\underline{\theta}-1} (-\log |x|)^a \mathbf{1}_{(0,c]}(|x|) dx$, $\kappa, a > 0$, $0 < c < 1$, $\underline{\theta} \in (0, 2)$. Then $\underline{\ell}(u) \sim \underline{\theta}^{-a-1} (-\log u)^a$ as $u \rightarrow 0$. Note that $\nu_{a,\underline{\theta}}^2$ does not satisfy (4.23). If a Lévy process X has $\nu_{a,\underline{\theta}}^2$ as a (symmetric) Lévy measure, then $r^* = +\infty$ and

$$\forall r, p \in (0, \underline{\theta}), \quad e_{N,r}(X, L_T^p) = \begin{cases} O\left((\log N)^{-\frac{1}{\underline{\theta}}} (\log \log N)^{\frac{a}{\underline{\theta}-\eta}}\right), & \eta \in (0, \underline{\theta}) \text{ if } \underline{\theta} < 1 \\ O\left((\log N)^{-\frac{1}{\underline{\theta}}} (\log \log N)^{\frac{2a}{\underline{\theta}-\eta}}\right), & \eta \in (0, \underline{\theta}) \text{ if } \underline{\theta} \in [1, 2). \end{cases}$$

• *Hyperbolic Lévy motions* have been applied to option pricing in Finance (see [9]). These processes are Lévy processes whose distribution \mathbb{P}_{X_1} at time 1 is a symmetric (centered) hyperbolic distribution

$$\mathbb{P}_{X_1} = C e^{-\delta \sqrt{1+(x/\gamma)^2}} dx, \quad \gamma, \delta > 0.$$

Hyperbolic Lévy processes are martingales with no Brownian component, satisfy $r^* = +\infty$. Their symmetric Lévy measure has a Lebesgue density that behaves like $C x^{-2}$ as $x \rightarrow 0$ (so that (4.23) is satisfied with $\underline{\theta} = 1$). Hence one obtains for every $r, p \in (0, 1)$,

$$e_{N,r}(X, L_T^p) = O((\log N)^{-1})$$

and, for every $r \geq 1$ and every $p \in (0, r]$, $e_{N,r}(X, L_T^p) = O((\log N)^{-\frac{1}{r}+\eta})$, $\eta > 0$.

The proof of this theorem is divided in several steps and is deferred to Section 4.3. The reason is that it relies on the decomposition of X as the sum of a “bounded” jump and a “big” jump Lévy processes which are treated successively in the following two sections.

4.1 Lévy processes with bounded jumps

We consider in this section a Lévy process X *without Brownian component* ($\sigma = 0$), with jumps bounded by a real constant $c > 0$. This means in term of the Lévy measure ν of X that

$$\nu([-c, c]^c) = 0. \quad (4.27)$$

Then for every $\rho > 0$ and every $t \geq 0$, $X_t \in L^\rho(\mathbb{P})$ i.e. $r^* = +\infty$. In Proposition 2 below we establish Theorem 2 in that setting.

Proposition 2 *Let $(X_t)_{t \in [0, T]}$ be a Lévy process satisfying (4.27) and $\underline{\theta} > 0$. Then claims (a)-(b)-(c)-(d) in Theorem 2 hold true with $r^* = \infty$.*

Proof. The proof of this proposition is decomposed into several steps. We consider $\underline{\theta}$ as defined in Theorem 1. Note that in the present setting $\underline{\theta} = \inf\{\theta > 0 : \int |x|^\theta \nu(dx) < +\infty\}$ and that $\int |x|^\theta \nu(dx) < +\infty$ for every $\theta > \underline{\theta}$. The starting idea is to part the “small” and the “big” jumps of X in a non homogeneous way with respect to the function $s \mapsto s^{\frac{1}{\underline{\theta}}}$. We will inspect successively the cases $\underline{\theta} \in (0, 1)$ (or $\underline{\theta} = 1$ holds as a minimum) and $\underline{\theta} \in [1, 2]$.

STEP 1 (*Decompositions of X*): When $\underline{\theta} \in (0, 1)$ or $\underline{\theta} = 1$ holds as a minimum, then

$$\mathbb{E} \left| \sum_{0 < s \leq T} \Delta X_s \right| \leq \mathbb{E} \sum_{0 < s \leq T} |\Delta X_s| = T \int |x| \nu(dx) < +\infty$$

consequently X \mathbb{P} -a.s. has finite variation and one can decompose X as

$$X_t = \xi(t) + \sum_{0 < s \leq t} \Delta X_s. \quad (4.28)$$

where $\xi(t) = at$ is a linear function.

Assume now $\underline{\theta} \in [1, 2]$. One may decompose X as follows

$$\begin{aligned} X_t &= \xi(t) + X_t^{(\underline{\theta})} + M_t^{(\underline{\theta})} \\ \text{with } \xi(t) &:= t \mathbb{E}(X_1), \\ X_t^{(\underline{\theta})} &:= \sum_{0 < s \leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s| > s^{\frac{1}{\underline{\theta}}}\}} - \int_0^t ds \int_{\{s^{\frac{1}{\underline{\theta}} < |x| \leq c\}} x \nu(dx). \end{aligned} \quad (4.29)$$

Note that $X^{(\underline{\theta})}$ has finite variations on $[0, T]$ since

$$\int_0^t ds \int_{\{s^{\frac{1}{\underline{\theta}} < |x| \leq c\}} |x| \nu(dx) = \int_{\{|x| \leq c\}} |x| (|x|^{\underline{\theta}} \wedge t) \nu(dx) \leq \int_{\{|x| \leq c\}} |x|^{1+\underline{\theta}} \nu(dx) < +\infty.$$

Both $X^{(\underline{\theta})}$ and $M^{(\underline{\theta})}$ are martingales with (non-homogeneous) independent increments. Their increasing predictable “bracket” processes are given by

$$\langle X^{(\underline{\theta})} \rangle_t = \int_0^t ds \int_{\{|x| > s^{\frac{1}{\underline{\theta}}}\}} x^2 \nu(dx) \quad \text{and} \quad \langle M^{(\underline{\theta})} \rangle_t = \int_0^t ds \int_{\{|x| \leq s^{\frac{1}{\underline{\theta}}}\}} x^2 \nu(dx).$$

We may consider from now on the (supremum process of the) Lévy process

$$\tilde{X}_t := X_t - \xi(t) \quad (4.30)$$

where ξ is the the linear function defined by (4.28) and (4.29) respectively. Since the linear function ξ lies in L_T^ρ , it does not affect the quantization rate which is invariant by translation.

STEP 2 (*Increments estimates in $L^\rho(\mathbb{P})$*): In this step we evaluate $\sup_{0 \leq s \leq t} |\tilde{X}_s|$ in $L^\rho(\mathbb{P})$, $\rho \in (0, 2]$. Throughout this step, letter c comes from (4.27).

Lemma 3 (a) Assume $\underline{\theta} \in (0, 1)$ or $\underline{\theta} = 1$ holds as a minimum. For every $\rho \in (0, 1]$ and $t \in [0, T]$,

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq s \leq t} |\tilde{X}_s|^\rho \right) &\leq C_\rho \left(\left(\int_0^t ds \int_{\{|x| \leq s^{\frac{1}{2}}\}} x^2 \nu(dx) \right)^{\frac{\rho}{2}} + \int_0^t ds \int_{\{s^{\frac{1}{2}} < |x| \leq c\}} |x|^\rho \nu(dx) \right. \\ &\quad \left. + \sup_{0 \leq s \leq t} \left| \int_0^s du \int_{\{|x| \leq u^{\frac{1}{2}}\}} x \nu(dx) \right|^\rho \right). \end{aligned} \quad (4.31)$$

(b) Assume $\underline{\theta} \in [1, 2]$. For every $\rho \in (0, 2]$ and every $t \in [0, T]$,

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq s \leq t} |\tilde{X}_s|^\rho \right) &\leq C_\rho \left(\left(\int_0^t ds \int_{\{|x| \leq s^{\frac{1}{2}}\}} x^2 \nu(dx) \right)^{\frac{\rho}{2}} + \int_0^t ds \int_{\{s^{\frac{1}{2}} < |x| \leq c\}} |x|^\rho \nu(dx) \right. \\ &\quad \left. + \left(\int_0^t ds \int_{\{s^{\frac{1}{2}} < |x| \leq c\}} |x|^{\frac{\rho}{2}} \nu(dx) \right)^2 \right) + \sup_{0 \leq s \leq t} \left| \int_0^s du \int_{\{u^{\frac{1}{2}} < |x| \leq c\}} x \nu(dx) \right|^\rho \end{aligned} \quad (4.32)$$

Proof: (a) Then \tilde{X} is a pure jump process (with finite variations). Using $\rho \in (0, 1]$ and Doob's inequality yield

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} |\tilde{X}_s|^\rho &\leq \mathbb{E} \sup_{0 \leq s \leq t} \left| \sum_{0 \leq u \leq s} \Delta X_u \mathbf{1}_{\{|\Delta X_u| \leq u^{\frac{1}{2}}\}} \right|^\rho + \mathbb{E} \sup_{0 \leq s \leq t} \left| \sum_{0 \leq u \leq s} \Delta X_u \mathbf{1}_{\{|\Delta X_u| > u^{\frac{1}{2}}\}} \right|^\rho \\ &\leq \left(\mathbb{E} \sup_{0 \leq s \leq t} \left(\sum_{0 \leq u \leq s} \Delta X_u \mathbf{1}_{\{|\Delta X_u| \leq u^{\frac{1}{2}}\}} \right)^2 \right)^{\frac{\rho}{2}} + \mathbb{E} \sum_{0 \leq s \leq t} |\Delta X_s|^\rho \mathbf{1}_{\{|\Delta X_s| > s^{\frac{1}{2}}\}} \\ &\leq C_\rho \left(\left(\mathbb{E} \sup_{0 \leq s \leq t} \left(\sum_{0 \leq u \leq s} \Delta X_u \mathbf{1}_{\{|\Delta X_u| \leq u^{\frac{1}{2}}\}} - \int_0^s du \int_{\{|x| \leq u^{\frac{1}{2}}\}} x \nu(dx) \right)^2 \right)^{\frac{\rho}{2}} \right. \\ &\quad \left. + \sup_{0 \leq s \leq t} \left| \int_0^s du \int_{\{|x| \leq u^{\frac{1}{2}}\}} x \nu(dx) \right|^\rho + \int_0^t ds \int_{\{s^{\frac{1}{2}} < |x| \leq c\}} |x|^\rho \nu(dx) \right) \\ &\leq C_\rho \left(\left(\int_0^t ds \int_{\{|x| \leq s^{\frac{1}{2}}\}} x^2 \nu(dx) \right)^{\frac{\rho}{2}} + \sup_{0 \leq s \leq t} \left| \int_0^s du \int_{\{|x| \leq u^{\frac{1}{2}}\}} x \nu(dx) \right|^\rho + \int_0^t ds \int_{\{s^{\frac{1}{2}} < |x| \leq c\}} |x|^\rho \nu(dx) \right). \end{aligned}$$

(b) It follows from Doob's inequality (and $0 < \rho/2 \leq 1$)

$$\mathbb{E} \left(\sup_{0 \leq s \leq t} |M_s^{(\underline{\theta})}|^\rho \right) \leq \left[\mathbb{E} \sup_{0 \leq s \leq t} (M_s^{(\underline{\theta})})^2 \right]^{\frac{\rho}{2}} \leq \left(4 \int_0^t ds \int_{\{|x| \leq s^{\frac{1}{2}}\}} x^2 \nu(dx) \right)^{\frac{\rho}{2}}.$$

On the other hand, since $\rho \in (0, 2]$, on has

$$\begin{aligned} \sup_{0 \leq s \leq t} |X_s^{(\underline{\theta})}|^\rho &\leq C_\rho \left(\left(\sum_{0 \leq s \leq t} |\Delta X_s|^{\frac{\rho}{2}} \mathbf{1}_{\{|\Delta X_s| > s^{\frac{1}{2}}\}} \right)^2 + \sup_{0 \leq s \leq t} \left| \int_0^s du \int_{\{u^{\frac{1}{2}} < |x| \leq c\}} x \nu(dx) \right|^\rho \right) \\ &\leq C_\rho \left(\left(\sum_{0 \leq s \leq t} |\Delta X_s|^{\frac{\rho}{2}} \mathbf{1}_{\{|\Delta X_s| > s^{\frac{1}{2}}\}} - \int_0^t ds \int_{\{|x| > s^{\frac{1}{2}}\}} |x|^{\frac{\rho}{2}} \nu(dx) \right)^2 + \left(\int_0^t ds \int_{\{|x| > s^{\frac{1}{2}}\}} |x|^{\frac{\rho}{2}} \nu(dx) \right)^2 \right. \\ &\quad \left. + \sup_{0 \leq s \leq t} \left| \int_0^s du \int_{\{u^{\frac{1}{2}} < |x| \leq c\}} x \nu(dx) \right|^\rho \right). \end{aligned}$$

Hence, using again Doob's inequality

$$\mathbb{E} \sup_{0 \leq s \leq t} |X_s^{(\underline{\theta})}|^\rho \leq C_\rho \left(\int_0^t ds \int_{\{s^{\frac{1}{\underline{\theta}} < |x| \leq c\}} |x|^\rho \nu(dx) + \left(\int_0^t ds \int_{\{s^{\frac{1}{\underline{\theta}} < |x| \leq c\}} |x|^{\frac{\rho}{2}} \nu(dx) \right)^2 + \sup_{0 < s \leq t} \left| \int_0^s du \int_{\{u^{\frac{1}{\underline{\theta}} < |x| \leq c\}} x \nu(dx) \right|^\rho \right). \quad \diamond$$

Lemma 4 (First extended Millar's Lemma) (a) Assume $\underline{\theta} \in (0, 2] \setminus \{1\}$. If the Lévy measure satisfies Assumption (4.23) then

$$\forall \rho \in (0, \underline{\theta}), \forall t \in [0, T], \quad \mathbb{E} \sup_{0 \leq s \leq t} |\tilde{X}_s|^\rho \leq C_\rho t^{\frac{\rho}{\underline{\theta}}}. \quad (4.33)$$

(b) Assume $\underline{\theta} \in (0, 2) \setminus \{1\}$ and that the function $u \mapsto \underline{\nu}(u)$ has regular variation with index $-b$ at 0. Then $b = \underline{\theta}$ and, for every $\rho \in (0, \underline{\theta})$, there exists $T_\rho \in (0, T]$ such that

$$\forall t \in [0, T_\rho], \quad \mathbb{E} \sup_{0 \leq s \leq t} |\tilde{X}_s|^\rho \leq C_\rho (t^{\frac{1}{\underline{\theta}}} \Lambda_\rho(t))^\rho. \quad (4.34)$$

(c) When $\underline{\theta} = 1$, the above upper-bounds still hold provided ν is symmetric or $\nu(|x|) < +\infty$.

Proof: (a) One just needs to investigate all the integrals appearing in the right hand side of Inequalities (4.31) and (4.32) in Lemma 3. Let $\rho \in (0, \underline{\theta})$ and let $t \in [0, c^{\underline{\theta}} \wedge T]$. Then, if $\underline{\theta} \in (0, 2)$,

$$\int_0^t ds \int_{\{0 < |x| \leq s^{\frac{1}{\underline{\theta}}\}} x^2 \nu(dx) \leq C \int_0^t ds \int_{\{0 < |x| \leq s^{\frac{1}{\underline{\theta}}\}} |x|^{1-\underline{\theta}} dx \leq C \int_0^t s^{\frac{2}{\underline{\theta}}-1} ds = Ct^{\frac{2}{\underline{\theta}}}$$

where the real constant C comes from (4.23). If $\underline{\theta} = 2$,

$$\int_0^t ds \int_{\{0 < |x| \leq s^{\frac{1}{\underline{\theta}}\}} x^2 \nu(dx) \leq \int_{[-c, c]} x^2 \nu(dx) t = \int_{[-c, c]} x^2 \nu(dx) t^{\frac{2}{\underline{\theta}}}.$$

Then, still for every $t \in [0, c^{\underline{\theta}} \wedge T]$,

$$\int_0^t ds \int_{\{s^{\frac{1}{\underline{\theta}} < |x| \leq c\}} |x|^\rho \nu(dx) \leq C \int_0^t ds \int_{\{s^{\frac{1}{\underline{\theta}} < |x| \leq c\}} |x|^{\rho-\underline{\theta}-1} dx \leq \frac{C}{\underline{\theta}-\rho} \int_0^t s^{\frac{\rho}{\underline{\theta}}-1} ds = Ct^{\frac{\rho}{\underline{\theta}}}.$$

When $\underline{\theta} \in (0, 1)$,

$$\sup_{0 \leq s \leq t} \left| \int_0^s du \int_{\{|x| \leq u^{\frac{1}{\underline{\theta}}\}} x \nu(dx) \right| \leq \int_0^t ds \int_{\{|x| \leq s^{\frac{1}{\underline{\theta}}\}} |x| \nu(dx) \leq C \int_0^t \frac{s^{\frac{1}{\underline{\theta}}-1}}{1-\underline{\theta}} ds = \frac{C}{1-\underline{\theta}} t^{\frac{1}{\underline{\theta}}}.$$

When $\underline{\theta} = 1$ and $\int |x| \nu(dx) < +\infty$, then this term is trivially upper-bounded by $t \int |x| \nu(dx)$. It is 0 when ν is symmetric. Similarly when $\underline{\theta} \in (1, 2]$, still for every $t \in [0, c^{\underline{\theta}} \wedge T]$,

$$\sup_{0 \leq s \leq t} \left| \int_0^s du \int_{\{u^{\frac{1}{\underline{\theta}} < |x| \leq c\}} x \nu(dx) \right| \leq \int_0^t ds \int_{\{|x| > s^{\frac{1}{\underline{\theta}}\}} |x| \nu(dx) \leq C \int_0^t \frac{s^{\frac{1}{\underline{\theta}}-1}}{\underline{\theta}-1} ds = \frac{C}{\underline{\theta}-1} t^{\frac{1}{\underline{\theta}}}$$

and

$$\int_0^t ds \int_{\{s^{\frac{1}{\underline{\theta}} < |x| \leq c\}} |x|^{\frac{\rho}{2}} \nu(dx) \leq C \int_0^t ds \int_{\{s^{\frac{1}{\underline{\theta}} < |x| \leq c\}} |x|^{\frac{\rho}{2}-\underline{\theta}-1} dx \leq \frac{C}{\underline{\theta}-\frac{\rho}{2}} \int_0^t s^{\frac{\rho}{2\underline{\theta}}-1} ds = Ct^{\frac{\rho}{2\underline{\theta}}}.$$

One derives from (4.31) and (4.32) that there exists a positive real constant C_ρ such that

$$\forall t \in [0, c^{\underline{\theta}} \wedge T], \quad \mathbb{E} \sup_{0 \leq s \leq t} |\tilde{X}_s|^\rho \leq C_\rho t^{\frac{\rho}{\underline{\theta}}}.$$

The inequality holds for every $t \in [0, T]$ by simply adjusting the constant C_ρ .

(b) The fact that $b = \underline{\theta}$ was first established in [5]. We provide below a short proof, on the way to our main result, for the reader's convenience. It follows from Theorem 1.4.1 in [3] that $\underline{\nu}(u) = u^{-b}\ell(u)$ where ℓ is a (non-negative) slowly varying function. Consequently, one clearly has that, for every $\rho > 0$ and every $u > 0$

$$u^{\rho-b}\ell(u) \leq \int_{\{|x|>u\}} |x|^\rho \nu(dx).$$

Now, the left hand side of the above inequality goes to ∞ as $u \rightarrow 0$ as soon as $\rho < b$ since ℓ has slow variations (see Proposition 1.3.6 in [3]). Consequently $\rho \leq \underline{\theta}$. Letting θ go to b implies that $b \leq \underline{\theta}$.

We will make use of the following easy equality which follows from the very definition of $\underline{\nu}$: for every non-negative Borel function $f: \mathbb{R}_+ \rightarrow \mathbb{R}$,

$$\int_{\mathbb{R}} f(|x|) \nu(dx) = - \int_{\mathbb{R}_+} f(x) d\underline{\nu}(x). \quad (4.35)$$

In particular, for every $x \in (0, c]$ and every $a > 0$,

$$\int_{\{|u|\geq x\}} |u|^a \nu(du) = - \int_x^c u^a d\underline{\nu}(u)$$

Assume $b < \underline{\theta}$. Then, it follows from Theorem 1.6.4 in [3] that for every $a \in (b, \underline{\theta})$,

$$\int_x^c u^a d\underline{\nu}(u) \sim \frac{b}{b-a} x^a \underline{\nu}(x) = \frac{b}{b-a} x^{a-b} \ell(x) \rightarrow 0 \quad \text{as } x \rightarrow 0,$$

since ℓ is slowly varying. This contradicts $\int |u|^a \nu(du) = +\infty$. Consequently $b = \underline{\theta}$.

Now, Theorem 1.6.5 in [3] implies that for any $a > \underline{\theta}$

$$\int_{\{|u|\leq x\}} |u|^a \nu(du) = - \int_{(0,x]} u^a d\underline{\nu}(u) \sim \frac{\underline{\theta}}{a-\underline{\theta}} x^a \underline{\nu}(x) \quad \text{as } x \rightarrow 0.$$

Since $\underline{\theta} \neq 2$, this yields

$$\int_{\{|x|\leq s^{\frac{1}{2}}\}} x^2 \nu(dx) \sim \frac{\underline{\theta}}{2-\underline{\theta}} s^{\frac{2}{2-\underline{\theta}}} \underline{\nu}(s^{\frac{1}{2}}) \quad \text{as } s \rightarrow 0$$

which in turn implies that

$$\int_0^t ds \int_{\{|x|\leq s^{\frac{1}{2}}\}} x^2 \nu(dx) \sim \frac{\underline{\theta}}{2-\underline{\theta}} \int_0^t s^{\frac{2}{2-\underline{\theta}}} \underline{\nu}(s^{\frac{1}{2}}) ds \quad \text{as } t \rightarrow 0.$$

The function $s \mapsto \underline{\nu}(s^{\frac{1}{2}})$ has regular variation (at 0) with index -1 , hence Theorem 1.6.1 in [3] implies

$$\int_0^t ds \int_{\{|x|\leq s^{\frac{1}{2}}\}} x^2 \nu(dx) \sim C_{\underline{\theta}} t^{\frac{2}{2-\underline{\theta}}+1} \underline{\nu}(t^{\frac{1}{2}}) \quad \text{as } t \rightarrow 0.$$

Finally,

$$\left(\int_0^t ds \int_{\{|x|\leq s^{\frac{1}{2}}\}} x^2 \nu(dx) \right)^{\frac{\rho}{2}} \sim C_{\rho, \underline{\theta}} \left(t^{\frac{1}{2}} (\underline{\nu}(t))^{\frac{1}{2}} \right)^{\rho} \quad \text{as } t \rightarrow 0. \quad (4.36)$$

When $\underline{\theta} \in (0, 1)$ and $\rho \in (0, \underline{\theta})$, the same approach leads to

$$\sup_{0 < s \leq t} \left| \int_0^s du \int_{\{|x|\leq u^{\frac{1}{2}}\}} x \nu(dx) \right| \leq \int_0^t ds \int_{\{|x|\leq s^{\frac{1}{2}}\}} |x| \nu(dx) \sim C_{\underline{\theta}} t^{\frac{1}{2}} \underline{\nu}(t) \quad \text{as } t \rightarrow 0.$$

Then, it follows from Theorem 1.6.4 in [3] that, for every $\rho \in (0, \underline{\theta})$,

$$\int_{\{s^{\frac{1}{\underline{\theta}}} \leq |x| \leq c\}} |x|^\rho \nu(dx) = - \int_{s^{\frac{1}{\underline{\theta}}} \leq |x| \leq c} x^\rho d\underline{\nu}(x) \sim \frac{\underline{\theta}}{\underline{\theta} - \rho} s^{\frac{\underline{\theta}}{\underline{\theta} - \rho}} \underline{\nu}(s^{\frac{1}{\underline{\theta}}}) \quad \text{as } s \rightarrow 0$$

so that
$$\int_0^t \int_{\{s^{\frac{1}{\underline{\theta}}} \leq |x| \leq c\}} |x|^\rho \nu(dx) \sim \frac{\underline{\theta}}{\underline{\theta} - \rho} \int_0^t s^{\frac{\underline{\theta}}{\underline{\theta} - \rho}} \underline{\nu}(s^{\frac{1}{\underline{\theta}}}) ds \sim C_{\rho, \underline{\theta}} \left(t^{\frac{1}{\underline{\theta}}} (\underline{\ell}(t))^{\frac{1}{\rho}} \right)^\rho \quad \text{as } t \rightarrow 0.$$

Similarly (by formally setting $\rho = 1$ in the former Equation) one shows that if $\underline{\theta} \in (1, 2]$,

$$\sup_{0 < s \leq t} \left| \int_0^s du \int_{\{u^{\frac{1}{\underline{\theta}}} \leq |x| \leq c\}} x \nu(dx) \right| \leq \int_0^t ds \int_{\{s^{\frac{1}{\underline{\theta}}} \leq |x| \leq c\}} |x| \nu(dx) \sim C_{\underline{\theta}} t^{\frac{1}{\underline{\theta}}} \underline{\ell}(t) \quad \text{as } t \rightarrow 0. \quad (4.37)$$

Finally, one shows similarly for the last term in (4.32) that when $\rho \in (0, \underline{\theta})$,

$$\left(\int_0^t \int_{\{s^{\frac{1}{\underline{\theta}}} \leq |x| \leq c\}} |x|^{\frac{\rho}{2}} \nu(dx) \right)^2 \sim C_{\rho, \underline{\theta}} \left(t^{\frac{1}{\underline{\theta}}} (\underline{\ell}(t))^{\frac{2}{\rho}} \right)^\rho \quad \text{as } t \rightarrow 0.$$

Plugging these estimates in (4.31) and (4.32) and noting that, by Young's inequality,

$$\underline{\ell}(t) \leq C_\rho \left((\underline{\ell}(t))^{\frac{1}{2}} + (\underline{\ell}(t))^{\frac{1}{\rho}} \mathbf{1}_{\{\rho \leq 1\}} + (\underline{\ell}(t))^{\frac{2}{\rho}} \mathbf{1}_{\{1 < \rho \leq 2\}} \right)$$

finally yields that \tilde{X} satisfies Assumption $(L_{\varphi, \rho})$ with the announced function φ_ρ .

(c) When ν is symmetric (and $\underline{\theta} \in (1, 2]$), for every $s \in [0, T]$,

$$\int_0^s du \int_{\{u^{\frac{1}{\underline{\theta}}} \leq |x| \leq c\}} x \nu(dx) = 0$$

so that the condition $\underline{\theta} \neq 1$ induced by (4.37) is no longer necessary. Similarly, when $\underline{\theta} \in (0, 1]$,

$$\int_0^s du \int_{\{|x| \leq u^{\frac{1}{\underline{\theta}}}\}} x \nu(dx) = 0. \quad \diamond$$

STEP 3 (Higher moments and end of the proof): Claims (a), when $\underline{\theta}$ holds as a minimum, and (c), when $r < 2$, straightforwardly follow from Millar's Inequality (3.15) by applying Theorem 1 to the function $\varphi(u) = u^{\frac{1}{2}}$ with $\rho = \underline{\theta}$ for item (a) and $\varphi(u) = u^{\frac{1}{\rho}}$ with $\rho \in (r, 2]$ for item (c).

Claim (a) when Assumption (4.23) is fulfilled follows from Lemma 4(a) and Theorem 1 applied with the function $\varphi(u) = u^{\frac{1}{2}}$. Finally item (b) follows from Lemma 4(b) and Theorem 1.

Item (d) follows from Lemma 4(c) and Theorem 1. At this stage it remains to prove Item (c) when $r \geq 2$. This follows (when $r > 2$) from the extension of Millar's upper-bound established in the lemma below.

Lemma 5 (Second Extended Millar's Lemma) *Let $(X_t)_{t \in [0, T]}$ be a Lévy process without Brownian part such that $\nu([-c, c]^c) = 0$. For every $\rho \geq 2$, there exists a real constant $C_{\rho, T} > 0$ such that*

$$\forall t \in [0, T], \quad \mathbb{E} \left(\sup_{0 \leq s \leq t} |X_s|^\rho \right) \leq C_{\rho, T} t.$$

Proof. One considers again $\tilde{X}_t = X_t - t \mathbb{E} X_1$ which is a martingale Lévy process. Let $k_\rho := \max\{l : 2^l < \rho\}$. For every $k = 1, \dots, k_\rho$, one defines the martingales

$$N_t^{(k)} := \sum_{0 < s \leq t} |\Delta X_s|^{2^k} - t \int |x|^{2^k} \nu(dx).$$

The key of the proof is to apply the B.D.G. Inequality in cascade. It follows from the B.D.G. Inequality that

$$\begin{aligned}\mathbb{E} \sup_{0 \leq s \leq t} |\tilde{X}_s|^\rho &\leq C_\rho \mathbb{E} \left(\sum_{0 < s \leq t} (\Delta X_s)^2 \right)^{\rho/2} \\ &\leq C_\rho \left(\mathbb{E} (N_t^{(1)})^{\rho/2} + \left(t \int x^2 \nu(dx) \right)^{\rho/2} \right).\end{aligned}$$

Now, for every $k \in \{1, \dots, k_\rho - 1\}$, still using the B.D.G. Inequality yields

$$\begin{aligned}\mathbb{E} (N_t^{(k)})^{\rho/2^k} &\leq C_{\rho,k} \mathbb{E} \left(\sum_{0 < s \leq t} |\Delta X_s|^{2^{k+1}} \right)^{\rho/2^{k+1}} \\ &\leq C_{\rho,k} \left(\mathbb{E} (N_t^{(k+1)})^{\rho/2^{k+1}} + \left(t \int |x|^{2^{k+1}} \nu(dx) \right)^{\rho/2^{k+1}} \right).\end{aligned}$$

Finally, one gets

$$\begin{aligned}\mathbb{E} \sup_{0 \leq s \leq t} |\tilde{X}_s|^\rho &\leq C_\rho \left(\sum_{k=1}^{k_\rho} \left(t \int |x|^{2^k} \nu(dx) \right)^{\rho/2^k} + \mathbb{E} \left(\sum_{0 < s \leq t} |\Delta X_s|^{2^{k_\rho+1}} \right)^{\rho/2^{k_\rho+1}} \right) \\ &\leq C_\rho \left(\sum_{k=1}^{k_\rho} \left(t \int |x|^{2^k} \nu(dx) \right)^{\rho/2^k} + \mathbb{E} \sum_{0 < s \leq t} |\Delta X_s|^\rho \right) \\ &= C_\rho \left(\sum_{k=1}^{k_\rho} \left(t \int |x|^{2^k} \nu(dx) \right)^{\rho/2^k} + t \int |x|^\rho \nu(dx) \right).\end{aligned}$$

since $\rho/2^{k_\rho+1} \leq 1$. The conclusion follows from the fact that $t^{\rho/2^k} = o(t)$. \diamond

4.2 Compound Poisson process

In this section we consider a compound Poisson process $(X_t)_t$ defined by

$$X_t := \sum_{n \geq 1} U_n \mathbf{1}_{\{S_n \leq \lambda T\}}, \quad t \geq 0.$$

where $S_n = Z_1 + \dots + Z_n$, $(Z_n)_{n \geq 1}$ is an i.i.d. sequence of $\mathcal{Exp}(1)$ distributed random variables, $(U_n)_{n \geq 1}$ is an i.i.d. sequence of random variables, independent of $(Z_n)_{n \geq 1}$ with $U_1 \in L^\rho$, $\rho > 0$ and $\lambda > 0$ is the the jump intensity. For convenience we also introduce the underlying standard Poisson process $(K_t)_{t \geq 0}$ defined by

$$K_t := \sum_{n \geq 1} \mathbf{1}_{\{S_n \leq \lambda T\}}, \quad t \geq 0.$$

so that (with the convention that $\sum_\emptyset = 0$)

$$X_t = \sum_{k=1}^{K_t} U_k. \tag{4.38}$$

Proposition 3 *Let X be a compound Poisson process. Then for every $p, r \in (0, r^*)$, $p \leq r$,*

$$\forall \varepsilon > 0, \quad e_{N,r}(X, L_T^p) = O\left(\exp\left(-\frac{1}{\sqrt{r(p+1+\varepsilon)}}\sqrt{\log(N)\log_2(N)}\right)\right). \quad (4.39)$$

Furthermore when X is a standard Poisson process, then, one can replace $p+1+\varepsilon$ by $p+\varepsilon$ in (4.39).

Remarks. • Note that (4.39) implies that

$$\forall a > 0, \quad e_{N,r}(X, L_T^p) = o((\log N)^{-a}).$$

• In fact the rate obtained in the above proposition holds as soon as X has the form (4.38) where (Z_n) is as above and (U_n) is $L^r(\mathbb{P})$ -bounded for every $r < r^*$, independent of $(Z_n)_{n \geq 1}$.

Proof. We divide the proof in two steps, one is devoted to the standard Poisson process, the second one to the general case. We will assume that $r^* > 1$ throughout the proof so that, so that as it was already emphasized in the proof of Theorem 1 we may assume without loss of generality that $r, p \in (0, r^*) \cap [1, +\infty)$. The case $r^* \leq 1$ is left to the reader but can be treated by replacing the “triangular” Minkowski inequality by the pseudo-triangular inequalities $|f+g|_{L_T^p}^p \leq |f|_{L_T^p}^p + |g|_{L_T^p}^p$ and $\|U+V\|_r^r \leq \|U\|_r^r + \|V\|_r^r$.

STEP 1 (*Standard case*): One quantizes the standard Poisson K in a very natural way by setting

$$\widehat{K}_t := \sum_{n \geq 1} \mathbf{1}_{\{\widehat{S}_n \leq \lambda t\}}, \quad t \geq 0,$$

with

$$\widehat{S}_n := \widehat{S}_n^{\alpha_n},$$

where $\alpha_n = \alpha'_n \cup \{\lambda T\}$, α'_n is an $L^{r'}$ -optimal $(N_n - 1)$ -quantization of $S_n^{tr} := S_n \mathbf{1}_{\{S_n \leq \lambda T\}}$ and $r' = \frac{r}{p}$. Furthermore, one assume that the sequence (N_n) is non-increasing and satisfies $\prod_n N_n \leq N$ (so that $N_n = 1$ for large enough n). Then, for every $p \geq 1$, it follows from the (extended) Minkowski inequality

$$|K - \widehat{K}|_{L_T^p} \leq \sum_{n \geq 1} |\mathbf{1}_{\{S_n \leq \lambda\}} - \mathbf{1}_{\{\widehat{S}_n \leq \lambda\}}|_{L_T^p}.$$

Now

$$\begin{aligned} |\mathbf{1}_{\{S_n \leq \lambda\}} - \mathbf{1}_{\{\widehat{S}_n \leq \lambda\}}|_{L_T^p}^p &= \int_0^T |\mathbf{1}_{\{S_n \leq \lambda t\}} - \mathbf{1}_{\{\widehat{S}_n \leq \lambda t\}}|^p dt \\ &= \frac{1}{\lambda} |S_n \wedge (\lambda T) - \widehat{S}_n \wedge (\lambda T)| = \frac{1}{\lambda} |S_n \wedge (\lambda T) - \widehat{S}_n|. \end{aligned}$$

Now $\{S_n > \lambda T\} \subset \{\widehat{S}_n = \lambda T\}$ since $\max \alpha_n = \lambda T$. On the other hand $S_n = S_n^{tr}$ on $\{S_n \leq \lambda T\}$ so that

$$|S_n \wedge (\lambda T) - \widehat{S}_n| = |S_n \wedge (\lambda T) - \widehat{S}_n \mathbf{1}_{\{S_n \leq \lambda T\}}| = |S_n^{tr} - \widehat{S}_n^{tr} \mathbf{1}_{\{S_n \leq \lambda T\}}| \leq |S_n^{tr} - \widehat{S}_n^{tr}|.$$

Also note that when $N_n = 1$, $\widehat{S}_n = \lambda T$ so that $|S_n \wedge (\lambda T) - \widehat{S}_n| = (\lambda T - S_n)_+$. Consequently, for every $r \geq 1$,

$$\begin{aligned} \| |K - \widehat{K}|_{L_T^p} \|_r &\leq \sum_{n \geq 1} \| |\mathbf{1}_{\{S_n \leq \lambda\}} - \mathbf{1}_{\{\widehat{S}_n \leq \lambda\}}|_{L_T^p} \|_r \\ &\leq \frac{1}{\lambda^{\frac{1}{p}}} \sum_{n \geq 1} \| |S_n \wedge (\lambda T) - \widehat{S}_n|_{L_T^p} \|_{r'} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\lambda^{\frac{1}{p}}} \left(\sum_{n, N_n \geq 2} \|S_n^{tr} - \widehat{S}_n^{tr \alpha_n}\|_{r'}^{\frac{1}{p}} + \sum_{n, N_n=1} \|(\lambda T - S_n)_+\|_{r'}^{\frac{1}{p}} \right) \\
&\leq \frac{1}{\lambda^{\frac{1}{p}}} \left(\sum_{n, N_n \geq 2} \|S_n^{tr} - \widehat{S}_n^{tr \alpha'_n}\|_{r'}^{\frac{1}{p}} + \sum_{n, N_n=1} \|(\lambda T - S_n)_+\|_{r'}^{\frac{1}{p}} \right).
\end{aligned}$$

The extended Pierce Lemma (Lemma 1) yields that, for every $n \geq 1$ such that $N_n \geq 2$, for every $\delta > 0$,

$$\begin{aligned}
\|S_n^{tr} - \widehat{S}_n^{tr \alpha'_n}\|_{r'} &\leq \|S_n^{tr}\|_{r'+\delta/p} C_{r,p,\delta} |N_n - 1|^{-1} \\
&\leq 2 \|S_n \mathbf{1}_{\{S_n \leq \lambda T\}}\|_{\frac{r+\delta}{p}} C_{r,p,\delta} N_n^{-1}.
\end{aligned}$$

Set $\mu := r' + \delta/p = \frac{r+\delta}{p}$ so that $\mu p = r + \delta$.

$$\begin{aligned}
\| |K - \widehat{K}|_{L_T^p} \|_r &\leq C_{p,r,\delta} \frac{1}{\lambda^{\frac{1}{p}}} \left(\sum_{n, N_n \geq 2} \|S_n \mathbf{1}_{\{S_n \leq \lambda T\}}\|_{\mu}^{\frac{1}{p}} \frac{1}{N_n^{\frac{1}{p}}} + \sum_{n, N_n=1} \|(\lambda T - S_n)_+\|_{\mu}^{\frac{1}{p}} \right) \\
&\leq C_{p,r,\delta} T^{\frac{1}{p}} \left(\sum_{n \geq 1} (\mathbb{P}(S_n \leq \lambda T))^{\frac{1}{\mu p}} \frac{1}{N_n^{\frac{1}{p}}} \right). \tag{4.40}
\end{aligned}$$

Now, standard computations show that

$$\mathbb{P}(\{S_n \leq \lambda T\}) = \frac{(\lambda T)^n}{(n-1)!} \int_0^1 u^{n-1} e^{-\lambda T u} du \leq \frac{(\lambda T)^n}{n!}.$$

Hence, setting $A = (\lambda T)^{\frac{1}{\mu p}}$, yields

$$(\mathbb{P}(S_n \leq \lambda T))^{\frac{1}{\mu p}} \leq \frac{(\lambda T)^{\frac{n}{\mu p}}}{(n!)^{\frac{1}{\mu p}}} \leq \frac{A^n}{(n!)^{\frac{1}{\mu p}}}.$$

Set for every $x \geq 0$, $a(x) := \frac{A^x}{\Gamma(x+1)^{\frac{1}{\mu p}}}$. This function reaches a unique maximum at some $x_0 \geq 0$ and then decreases to 0 as $x \rightarrow \infty$. We modify the function a by setting $a_0(x) := a(x) \vee a(x_0)$ so that the function a_0 becomes non-increasing and log-concave since Γ is log-convex. Now let

$$a_n := a_0(n), \quad n \geq 1.$$

Finally, the quantization problem (4.40) for the standard Poisson K is “upper-bounded” by the following optimal integral “bit allocation” problem

$$\min \left\{ \sum_{n \geq 1} \frac{a_n}{N_n^{1/p}}, \quad N_n \geq 1, \quad \prod_{n \geq 1} N_n \leq N \right\}. \tag{4.41}$$

Then let $m \geq 2x_0 + 1$ be a temporarily fixed integer. We set for $N \geq 1$,

$$N_n = \left\lceil \frac{a_n^p N^{\frac{1}{m}}}{(\prod_{1 \leq k \leq m} a_k)^{\frac{p}{m}}} \right\rceil, \quad 1 \leq n \leq m, \quad N_n = 1, \quad n \geq m+1.$$

The sequence $N_n, 1 \leq n \leq m$, is nonincreasing. This will ensure that

$$N_n \geq 1, \quad 1 \leq n \leq m.$$

We wish to choose m as a function of N so that

$$a_m N^{\frac{1}{pm}} \geq \left(\prod_{1 \leq k \leq m} a_k \right)^{\frac{1}{m}}.$$

Using log-concavity, this is clearly satisfied provided

$$a_m N^{\frac{1}{pm}} \geq a_0((m+1)/2) = a((m+1)/2) \quad (4.42)$$

(since $(m+1)/2 \geq x_0$). Inequality (4.42) reads by taking log, to

$$\frac{m-1}{2} \log A + \frac{1}{pm} \log N \geq \frac{1}{\mu p} (\log(\Gamma(m+1)) - \log(\Gamma(1+(m+1)/2))). \quad (4.43)$$

We will make use of the following classical inequalities: for every $t \geq 1/2$,

$$0 \leq \log(\Gamma(t+1)) - \log(\sqrt{2\pi}) - (t+1/2) \log t + t \leq 1$$

Then, after some easy computations, one shows that Inequality (4.43) is satisfied as soon as

$$\frac{m-1}{2} \log A + \frac{1}{pm} \log N \geq \frac{1}{\mu p} \left(\frac{m}{2} \log m - \frac{m}{8} - \frac{1}{2} \log m + \frac{5}{2} \right).$$

If one sets (this is probably optimal)

$$m = m(N) := \left\lceil 2 \sqrt{\mu \frac{\log N}{\log_2 N}} \right\rceil$$

then the above inequality is satisfied as well as $m(N) \geq 2x_0 + 1$, for every large enough N , provided that one increases the value of A . With N_n and m settled as above and using that $\frac{x}{[x]} \leq 2$ for every $x \geq 1$, one gets

$$\sum_{n \geq 1} \frac{a_n}{N_n^{\frac{1}{p}}} \leq 2^{\frac{1}{p}} m N^{-\frac{1}{pm}} \left(\prod_{k=1}^m a_k \right)^{1/m} + \sum_{n \geq m+1} a_n$$

On the one hand, $N_m \geq 1$ reads

$$N^{-\frac{1}{pm}} \left(\prod_{k=1}^m a_k \right)^{1/m} \leq a_m.$$

On the other hand the log-concavity, the monotony of the function a over $[x_0 + 1, \infty)$ (and the fact that a' is non zero) imply that

$$\sum_{n \geq m+1} a_n \leq \left| \frac{a(x_0 + 1)}{a'(x_0 + 1)} \right| a_m = o(m a_m)$$

(this follows from a straightforward adaptation of the proof of Proposition 4.4 in [19] to which we refer for details).

$$\begin{aligned} m a_m &= m \frac{A^m}{(m!)^{\frac{1}{\mu p}}} \\ &\leq \exp \left(-\frac{1}{\mu p} m \log m + O(m) \right) \\ &\leq C \exp \left(-\frac{1}{p\mu} \sqrt{\mu \log N \log_2 N} \left(1 + O \left(\frac{\log_3 N}{\log_2 N} \right) \right) \right) \end{aligned} \quad (4.44)$$

Note that $p\sqrt{\mu} = \sqrt{p \cdot p\mu} = \sqrt{(r + \delta)p}$. Finally this yields in particular that for every $\varepsilon > 0$,

$$\| |K - \widehat{K}|_{L_T^p} \|_r = O \left(\exp \left(-\frac{1}{\sqrt{rp + \varepsilon}} \sqrt{\log N \log_2 N} \right) \right).$$

STEP 2 (*Compound case*): Starting from Equation (4.38), it is natural to quantize (X_t) by setting

$$\widehat{X}_t = \sum_{k=1}^{\widehat{K}_t} \widehat{U}_k.$$

where \widehat{K} is a $N^{(1)}$ -quantization of the standard Poisson process K as described in Step 1 and, for every $n \geq 1$, \widehat{U}_n is an L^r -optimal $N_n^{(2)}$ -quantization of U_n with $1 \leq N_1^{(2)} \times \dots \times N_n^{(2)} \dots \leq N^{(2)}$ and $N^{(1)}N^{(2)} \leq N$. Then, setting $\widehat{K}_t^U := \sum_{k=1}^{\widehat{K}_t} U_k$ and $K_t^{\widehat{U}} := \sum_{k=1}^{K_t} \widehat{U}_k$, one gets

$$\begin{aligned} |K^{\widehat{U}} - \widehat{K}^{\widehat{U}}|_{L_T^p} &\leq \sum_{n \geq 1} |\widehat{U}_n| |\mathbf{1}_{\{S_n \leq \lambda T\}} - \mathbf{1}_{\{\widehat{S}_n \leq \lambda T\}}|_{L_T^p} \\ \text{so that } \| |X - \widehat{K}^U|_{L_T^p} \|_r &\leq \frac{1}{\lambda^{\frac{1}{p}}} \sum_{n \geq 1} \|\widehat{U}_n\|_r \|S_n \wedge (\lambda T) - \widehat{S}_n \wedge (\lambda T)\|_{\frac{p}{p}}^{\frac{1}{p}} \\ &= \frac{\sup_{n \geq 1} \|\widehat{U}_n\|_r}{\lambda^{\frac{1}{p}}} \sum_{n \geq 1} \|S_n \wedge (\lambda T) - \widehat{S}_n \wedge (\lambda T)\|_{\frac{p}{p}}^{\frac{1}{p}} \end{aligned}$$

where we used the fact that the sequences (U_n) and (S_n) being independent, so are (\widehat{U}_n) and (S_n) . Using

$$\|\widehat{U}_n\|_r \leq \|U_n - \widehat{U}_n\|_r + \|U_1\|_r = \|U_1 - \widehat{U}_1^{N_n^{(2)}}\|_r + \|U_1\|_r$$

shows that $\sup_{n \geq 1} \|\widehat{U}_n\|_r < +\infty$. Hence, it follows from Step 1 that, for every $c < \frac{1}{\sqrt{pr}}$,

$$\| |X - \widehat{K}^U|_{L_T^p} \|_r = O \left(\exp \left(-c \sqrt{\log(N^{(1)}) \log_2(N^{(1)})} \right) \right).$$

On the other hand, with obvious notations, and using that $(\widehat{U}_n - U_n)$ and (S_n) are independent

$$\begin{aligned} \| |X - K^{\widehat{U}}|_{L_T^p} \|_r &= \| |K^{U - \widehat{U}}|_{L_T^p} \|_r \\ &\leq \sum_{n \geq 1} \|U_n - \widehat{U}_n\|_r \| \mathbf{1}_{\{S_n \leq \lambda \cdot\}} \|_{L_T^p} \\ &= \frac{1}{\lambda^{\frac{1}{p}}} \sum_{n \geq 1} \|U_n - \widehat{U}_n\|_r \|(\lambda T - S_n)_+\|_{r'}^{\frac{1}{p}} \\ &\leq \frac{1}{\lambda^{\frac{1}{p}}} \sum_{n \geq 1} \|U_n - \widehat{U}_n\|_r \frac{(\lambda T)^{\frac{1}{p} + \frac{n}{r}}}{(n!)^{\frac{1}{r}}} \\ &\leq C \sum_{n \geq 1} \|U_n - \widehat{U}_n\|_r \frac{(\lambda T)^{\frac{n}{r}}}{(n!)^{\frac{1}{r}}}. \end{aligned}$$

Now, it follows from the (extended) Pierce Lemma that

$$\begin{aligned} \| |K^U - K^{\widehat{U}}|_{L_T^p} \|_r &\leq C_{U_1, r} \sum_{n \geq 1} \frac{(\lambda T)^{\frac{n}{r}}}{(n!)^{\frac{1}{r}} N_n^{(2)}} \\ &= O \left(\exp \left(-\frac{1}{\sqrt{r}} \sqrt{\log(N^{(2)}) \log_2(N^{(2)})} \right) \right). \end{aligned}$$

The rate follows from the resolution of the optimal bit allocation problem (4.41) obtained by setting formally $\mu p = r$ and $p = 1$. Then, note that, on the one hand

$$\|X - \widehat{X}\|_{L_T^p} \leq \|X - K^{\widehat{U}}\|_{L_T^p} + \|K^{\widehat{U}} - \widehat{K}^{\widehat{U}}\|_{L_T^p},$$

and on the other hand

$$\widehat{K}_t^{\widehat{U}} = \sum_{n \geq 1} \widehat{U}_n^{N_n^{(2)}} \mathbf{1}_{\{\widehat{S}_n^{N_n^{(1)}} \leq \lambda t\}}$$

can take at most

$$\prod_{n \geq 1} N_n^{(1)} N_n^{(2)} \leq N^{(1)} \times N^{(2)} \leq N$$

values. Let $c < \frac{1}{\sqrt{pr}}$. Setting $N^{(1)} = \lceil N^{\frac{rc^2}{1+rc^2}} \rceil$, $N^{(2)} = \lceil N^{\frac{1}{1+rc^2}} \rceil$ yields a rate

$$\|X - \widehat{X}\|_{L_T^p} = O\left(\exp\left(-\frac{1}{\sqrt{1/c^2 + r}} \sqrt{\log(N) \log_2(N)}\right)\right)$$

i.e.

$$\forall \varepsilon > 0, \quad \|X - \widehat{X}\|_{L_T^p} = O\left(\exp\left(-\frac{1}{\sqrt{r(p+1+\varepsilon)}} \sqrt{\log(N) \log_2(N)}\right)\right). \quad \diamond$$

4.3 Proof of Theorem 2

Any Lévy process X can be decomposed as the sum $X = X^{(1)} + X^{(2)}$ of two (independent) Lévy processes, one having bounded jumps and one being a compound Poisson process – according to the decomposition of its Lévy measure

$$\nu(dx) = \nu^{(1)}(dx) + \nu^{(2)}(dx) \quad \text{with } \nu^{(1)}(dx) := \mathbf{1}_{\{|x| \leq 1\}} \nu(dx) \text{ and } \nu^{(2)}(dx) := \mathbf{1}_{\{|x| > 1\}} \nu(dx). \quad (4.45)$$

Assume $r^* > 1$. Then, it is clear that, for every $r, p \in (0, r^*)$

$$e_{N,r}(X, L_T^p) \leq C_{r,p,T} e_{\lfloor \sqrt{N} \rfloor, r'}(X, L_T^{r'}) \leq C_{r,p,T} (e_{\lfloor \sqrt{N} \rfloor, r'}(X^{(1)}, L_T^{r'}) + e_{\lfloor \sqrt{N} \rfloor, r'}(X^{(2)}, L_T^{r'})) \quad (4.46)$$

where $r' = r \vee p \vee 1$. Now, it follows from Proposition 3 that $e_{N,r'}(X^{(2)}, L_T^{r'}) = o(e_{N,r'}(X^{(1)}, L_T^{r'}))$ so that

$$e_{N,r}(X, L_T^p) \leq C'_{r,p,T} e_{\lfloor \sqrt{N} \rfloor, r'}(X^{(1)}, L_T^{r'}).$$

Now, using that $\underline{\ell}$ has slow variations at 0, one derives that

$$e_{\lfloor \sqrt{N} \rfloor, r'}(X^{(1)}, L_T^{r'}) = O(e_{N,r'}(X^{(1)}, L_T^{r'})).$$

Proposition 2 completes the proof of Theorem 2. When $r^* \leq 1$, one uses

$$e_{N,r}(X, L_T^p)^r \leq C'_{r,p,T} e_{\lfloor \sqrt{N} \rfloor, r'}(X, L_T^{r'}) \leq C'_{r,p,T} (e_{\lfloor \sqrt{N} \rfloor, r'}(X^{(1)}, L_T^{r'})^{r'} + e_{\lfloor \sqrt{N} \rfloor, r'}(X^{(2)}, L_T^{r'})^{r'}).$$

with $r' = r \vee p < 1$ (based on the pseudo-triangular inequality satisfied by L^s -pseudo-norms when $s < 1$). \diamond

5 Further results for Lévy processes

5.1 An exact rate for Lévy processes with a Brownian component

In that case the quantization rate of the Brownian motion rules the global rate of convergence.

Proposition 4 *Let X be a Lévy process with a non-vanishing Brownian component. Let $r^* = r^*(X)$ defined by (4.22). Then*

$$\forall r, p \in (0, r^* \wedge 2), \quad e_{N,r}(X, L_T^p) = O((\log N)^{-\frac{1}{2}}).$$

and

$$\forall r, p \in (0, +\infty), \quad e_{N,r}(X, L_T^p) = \Omega(e_{N,r}(W, L_T^p)).$$

In particular, $\forall r \in (0, +\infty), \forall p \in [1, +\infty), e_{N,r}(X, L_T^p) = \Omega((\log N)^{-\frac{1}{2}})$.

Proof. One can decompose X as

$$X = cW + X^{(1)} + X^{(2)}$$

where $X^{(i)}, i = 1, 2$, have $\nu^{(i)}$ as Lévy measure as defined in (4.45) in the above proof of Theorem 2. Then, if $r^* > 1$ and $r, p \in (0, r^* \wedge 2)$, one easily checks that, for every $N \geq 1$,

$$\begin{aligned} e_{N,r'}(X, L_T^p) &\leq e_{[\frac{3}{\sqrt{N}}]^{3,r'}}(X, L_T^{r'}) \\ &\leq e_{[\frac{3}{\sqrt{N}}]^{3,r'}}(cW, L_T^{r'}) + e_{[\frac{3}{\sqrt{N}}]^{3,r'}}(X^{(1)}, L_T^{r'}) + e_{[\frac{3}{\sqrt{N}}]^{3,r'}}(X^{(2)}, L_T^{r'}). \end{aligned}$$

where $r' = r \vee p \vee 1$. It follows from Proposition 3 (see remark right below) that $e_{N,r'}(X^{(2)}, L_T^{r'}) = o(e_{N,r'}(W, L_T^{r'}))$. Now, $\int_{\mathbb{R} \setminus \{0\}} x^2 \nu^{(1)}(dx) < +\infty$ hence, by Millar's Lemma,

$$\mathbb{E} \sup_{s \in [0,t]} |X_s^{(1)}|^2 \leq Ct.$$

Then one easily derives from (3.16) (or directly from Theorem 1) that $e_{N,r'}(X^{(1)}, L_T^{r'}) = O((\log N)^{-\frac{1}{2}})$. This yields the announced upper bound since $e_{N,r'}(W, L_T^{r'}) = O((\log N)^{-\frac{1}{2}})$. If $r^* \leq 1$, one proceeds as above using the pseudo-triangular inequality for L^s -pseudo-norms (with $r' = r \vee p < 1$).

As concerns the lower bound, note that if Y and Z are L_T^r -valued independent random vectors, for every $r, p > 0$

$$\begin{aligned} (e_{N,r}(Y + Z, L_T^p))^r &= \inf_{\alpha \subset L_T^p, \text{card}(\alpha) \leq N} \int \mathbb{E} \min_{a \in \alpha} |Y - z - a|_{L_T^p}^r \mathbb{P}_Z(dz) \\ &\geq \int_{L_T^p} \inf_{\alpha \subset L_T^p, \text{card}(\alpha) \leq N} \mathbb{E} \min_{a \in \alpha} |Y - z - a|_{L_T^p}^r \mathbb{P}_Z(dz) \\ &= (e_{N,r}(Y, L_T^p))^r \end{aligned}$$

so that

$$e_{N,r}(Y + Z, L_T^p) \geq \max(e_{N,r}(Y, L_T^p), e_{N,r}(Z, L_T^p)).$$

This holds true by induction to any finite sum of independent random variables. In particular

$$e_{N,r}(X, L_T^p) \geq e_{N,r}(cW, L_T^p) = c e_{N,r}(W, L_T^p).$$

This completes the proof. \diamond

5.2 Subordinated Lévy processes

We consider now subordination of the Brownian motion *i.e.* Lévy processes of the form

$$X_t = W_{A_t}, \quad t \geq 0,$$

where W denotes a standard Brownian motion and A a subordinator independent of W . A subordinator is a non-decreasing Lévy process (hence non negative). What follows is borrowed from [1]. Its Lévy-Khintchine characteristics (a, σ^2, ν_A) satisfy $\sigma^2 = 0$ and $\nu_A((-\infty, 0)) = 0$ and $\int_0^1 x \nu_A(dx) < +\infty$ and $\gamma := a - \int_0^1 x \nu_A(dx) \geq 0$ (so that $\underline{\theta}(A) \leq 1$). Consequently a subordinator reads

$$A_t = \gamma t + \sum_{s \leq t} \Delta A_s, \quad t \geq 0.$$

Its Laplace transform is given by $\mathbb{E} e^{-uA_t} = e^{-t\Phi(u)}$ with, for every $u \geq 0$,

$$\Phi(u) = \gamma u + \int_0^{+\infty} (1 - e^{-ux}) \nu_A(dx) = \gamma u + u \int_0^{+\infty} e^{-ux} \underline{\nu}_A(x) dx \quad (5.47)$$

where $\underline{\nu}_A(x) = \nu_A((x, +\infty))$ denotes the tail of the Lévy measure ν_A and $\lim_{u \rightarrow +\infty} \frac{\Phi(u)}{u} = \gamma$. Furthermore, for every $t \geq 0$ and $u \in \mathbb{R}$,

$$\mathbb{E}(e^{iuX_t}) = \mathbb{E}(e^{-\frac{u^2}{2}A_t}) = \exp\left(-\frac{u^2}{2}\gamma - \int_0^{+\infty} (1 - e^{-\frac{u^2}{2}x}) \nu_A(dx)\right)$$

so that one easily derives that

$$\nu_x(f) = \int_{(0, \infty)} \mathbb{E}(f(\sqrt{x}Z)) \nu_A(dx)$$

(with $Z \sim \mathcal{N}(0; 1)$) and that X has a Brownian component if and only if $\gamma > 0$ (see also [25], p. 198).

The small deviation of subordinator has been deeply investigated in [18]: it is established that if $\liminf_{u \rightarrow +\infty} \frac{\Phi(u)}{\log u} > 0$, then

$$\forall p \in [1, +\infty), \quad -\log(\mathbb{P}(|X|_{L_T^p} \leq \varepsilon)) \approx \Phi(\varepsilon^{-2}) \quad \text{as } \varepsilon \rightarrow 0. \quad (5.48)$$

These processes preserve a Gaussian feature which will be the key to estimate their quantization rate: they satisfy the Anderson inequality as briefly recalled in the lemma below.

Lemma 6 *A subordinated Lévy process is unimodal for every L_T^p -norm for every $p \in [1, +\infty)$. The result still holds if one replaces W by any e.g. pathwise continuous centered Gaussian process (like e.g. the fractional Brownian motion, etc).*

Proof. Using that A and W are independent it suffices to show that for every non decreasing function $a : [0, T] \rightarrow [0, \alpha(T)]$, $a(0) = 0$, and every $x \in L_T^p$,

$$\mathbb{P}\left(\int_0^T |W_{a(s)} - x(s)|^p ds \leq \varepsilon^p\right) \leq \mathbb{P}\left(\int_0^T |W_{a(s)}|^p ds \leq \varepsilon^p\right), \quad \varepsilon > 0.$$

It is clear that $(W_{a(t)})_{t \in [0, T]}$ is a centered (bi-measurable) Gaussian process and has sample paths in L_T^p *a.s.*. Hence $(W_{a(t)})_{t \in [0, T]}$ can be seen as an L_T^p -valued centered Gaussian random vector and the assertion follows from Anderson Inequality. \diamond

Now, we make the connection between Blumenthal-Gettoor indexes of X and A (and between the finiteness of moments).

Lemma 7 $\underline{\theta}(X) = 2\underline{\theta}(A)$ and $r^*(X) = 2r^*(A)$.

Proof. As a consequence of the expression for ν_x , one checks that for every $\theta \in (0, 2]$,

$$\int_{\{|x| \leq 1\}} |x|^\theta \nu_x(dx) = \int u^{\frac{\theta}{2}} \int_{\{|y| \leq 1/\sqrt{u}\}} |y|^\theta e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} = \int_{\{u > 0\}} u^{\frac{\theta}{2}} \nu_A \ell_\theta(u) (du).$$

where ℓ_θ satisfies: $\ell_\theta(u) > 0$ when $u > 0$ and $\lim_{u \rightarrow 0} \ell_\theta(u) = C_\theta \in (0, +\infty)$. Hence the first equality. As concerns the second one, $r^*(X)$ coincides with the (absolute) moments of X , it is obvious that

$$\mathbb{E}(|X_t|^r) = \mathbb{E}(|W_{A_t}|^r) = \mathbb{E}(A_t^{2r}).$$

Consequently $\mathbb{E}(|X_t|^r) < +\infty$ iff $\mathbb{E}(A_t^{2r}) < +\infty$ so that $r^*(X) = 2r^*(A)$. \diamond

As concerns upper bounds, we cannot apply Theorem 2 since a subordinated Lévy process may have a Brownian component. So we need to go back to Theorem 1.

Proposition 5 (a) If $\gamma > 0$,

$$\forall r, p \in (0, r^*(X) \wedge 2), \quad e_{N,r}(X, L_T^p) = O((\log N)^{-\frac{1}{2}}).$$

(b) If $\underline{\theta}(A) \in (0, 1)$, $\gamma = 0$ and $\nu_A(dx) \mathbf{1}_{\{0 < x \leq \eta\}} \leq c \mathbf{1}_{\{0 < x \leq \eta\}} \frac{dx}{x^{1+\underline{\theta}(A)}}$ for some real constants $c, \eta > 0$, then

$$\forall r, p \in (0, \underline{\theta}(X) \wedge r^*(X)), \quad e_{N,r}(X, L_T^p) = O((\log N)^{-\frac{1}{\underline{\theta}(X)}}).$$

Proof. (a) follows from Proposition 4 since X has a Brownian component.

(b) Let $\rho < 2(\underline{\theta}(A) \wedge r^*(A))$. First note that $\mathbb{E}(|X_t|^\rho) = \mathbb{E}A_t^{\frac{\rho}{2}} \leq Ct^{\frac{\rho}{2\underline{\theta}(A)}}$, $\rho \leq 2(\underline{\theta}(A) \wedge r^*(A)) = \underline{\theta}(X) \wedge r^*(X)$ (by Lemma 4 applied to A). Then, the result follows from Theorem 1. \diamond

The following lower bounds follows from Lemma 6 and Inequality (2.12) (see Remark right after Theorem 1). The main point to be noticed is that *the obtained upper and lower bounds match, providing an exact quantization rate for subordinated Lévy processes.*

Proposition 6 (a) If $\gamma > 0$,

$$\forall r \in (0, +\infty), \forall p \in [1, +\infty), \quad e_{N,r}(X, L_T^p) = \Omega((\log N)^{-\frac{1}{2}}).$$

(b) If $\gamma = 0$, $\underline{\theta}(A) > 0$ and $\mathbf{1}_{\{0 < x \leq \eta\}} \nu_A(dx) \geq c \mathbf{1}_{\{0 < x \leq \eta\}} \frac{dx}{x^{1+\underline{\theta}(A)}}$ for some real constants $c, \eta > 0$, then

$$\forall r \in (0, +\infty), \forall p \in [1, +\infty), \quad e_{N,r}(X, L_T^p) = \Omega((\log N)^{-\frac{1}{\underline{\theta}(X)}}).$$

Proof. (a) follows from Proposition 4 since X has a Brownian component.

(b) It follows from the assumption made on ν_A that $\underline{\nu}_A(x) \geq c \int_x^\eta \xi^{-\underline{\theta}-1} d\xi \geq \kappa x^{-\underline{\theta}}$ for $x \in (0, \eta/2]$. Hence it follows from (5.47) that

$$\Phi(u) \geq cu \int_0^{\eta/2} e^{-ux} \underline{\nu}_A(x) dx = cu^{\underline{\theta}} \int_0^{u\eta/2} e^{-y} y^{-\underline{\theta}} dy \geq c'u^{\underline{\theta}}$$

for large enough u (with an appropriate real constant $c' > 0$). One concludes by combining (2.12) and (5.48) since X is strongly unimodal. \diamond

EXAMPLES. • If A is a tempered α -stable process with Lévy measure

$$\nu_A(dx) = \frac{2^\alpha \alpha}{\Gamma(1-\alpha)} x^{-(\alpha+1)} \exp\left(-\frac{1}{2} \delta \frac{1}{x}\right) \mathbf{1}_{(0, \infty)}(x) dx,$$

with $\alpha \in (0, 1)$, $\delta > 0$, $\gamma = 0$, so that $\underline{\theta}(A) = \alpha$ and $r^*(A) + \infty$. Then, one gets

$$\forall r \in (0, 2\alpha), \forall p \in [1, 2\alpha), \quad e_{N,r}(X, L_T^p) \approx (\log N)^{-\frac{1}{2\alpha}}.$$

• Assume $\underline{\theta}(A) \in (0, 1)$ and that the function Φ is regularly varying at ∞ with index $\alpha \in (0, 1)$ such that

$$\Phi(x) \sim c x^\alpha (\log(x))^c \quad x \rightarrow \infty$$

for some real constant $c > 0$. Since $\alpha < 1$, we have $\gamma = 0$. Then

$$\Gamma(1 - \alpha)\underline{\nu}(x) \sim \Phi(1/x) \quad \text{as } x \rightarrow 0.$$

(see [1]) so that $\underline{\nu}$ is regularly varying at zero with index $-\alpha$. By Theorem 2, $\underline{\theta}(A) = \alpha$. Set

$$\Psi(x) = x^{\frac{1}{2\underline{\theta}(A)}} (\log x)^{-\frac{c}{2\underline{\theta}(A)}}$$

for large enough $x > 0$. Then $\Psi \circ \Phi(x) \sim c \sqrt{x}$ as $x \rightarrow \infty$ so that $\Psi \circ \Phi(1/\varepsilon^2) \sim c \varepsilon^{-1}$ as $\varepsilon \rightarrow 0$. Thus

$$\forall r > 0, \forall p \in [1, +\infty), \quad e_{N,r}(X, L_T^p) = \Omega \left((\log N)^{-\frac{1}{2\underline{\theta}(A)}} (\log \log N)^{-\frac{c}{2\underline{\theta}(A)}} \right)$$

On the other hand, by Lemma 4 and Remark below Theorem 1, in case $c > 0$,

$$\mathbb{E} A_t^{\frac{\rho}{2}} \leq C t^{\frac{\rho}{2\underline{\theta}(A)}} (-\log t)^c, \quad \rho/2 < \underline{\theta}(A) \wedge r^*(A)$$

so that

$$\forall r, p \in (0, \underline{\theta}(X) \wedge r^*(X)), \quad e_{N,r}(X, L_T^p) = O \left((\log N)^{-\frac{1}{2\underline{\theta}(X)}} (\log \log N)^{\frac{\rho}{2}} \right), \quad \rho < \underline{\theta}(X) \wedge r^*(X).$$

In case $r^*(X) \geq \underline{\theta}(X)$, this matches the lower bound up a $O(\log \log N)^\varepsilon$ term, $\varepsilon > 0$.

ACKNOWLEDGEMENT: We thank Sylvain Delattre, Gérard Kerkycharian and Dominique Picard for helpful discussions.

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