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Functional quantization rate and mean pathwise regularity of processes with an application to Lévy processes

HARALD LUSCHGY*, AND GILLES PAGÈS †

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Abstract

We investigate the connections between the mean pathwise regularity of stochastic processes and their $L^r(\mathbb{P})$ -functional quantization rate as random variables taking values in some $L^p([0, T], dt)$ -spaces ($< 0p \leq r$). Our main tool is the Haar basis. We then emphasize that the derived functional quantization rate may be optimal (like for the Brownian motion) or not (like for the Poisson process). Then, we focus on the specific family of Lévy processes for which we derive a general quantization rate based on the regular variation properties of its Lévy measure at 0. The case of compound Poisson processes which appears as degenerate in the former approach, are studied specifically: one observes some rates which are in-between finite dimensional and infinite dimensional “usual” rates.

Key words: Functional quantization, Gaussian process, Haar basis, Lévy process, Poisson process

2000 Mathematics Subject Classification: 60E99, 60G51, 60G15, 60G52, 60J60.

1 Introduction

In this paper, we investigate the connection between the functional $L^r(\mathbb{P})$ -quantization rate for a process $X = (X_t)_{t \in [0, T]}$ and the $L^r(\mathbb{P})$ -mean pathwise regularity of the mapping $t \mapsto X_t$ from $[0, T] \rightarrow L^r(\mathbb{P})$ in an abstract setting with a constructive approach (we mean that all the rates are established using some explicit sequences of quantizers).

First let us briefly recall what functional quantization is and how it was introduced. Let $(E, \|\cdot\|)$ denote a finite-dimensional ($E = \mathbb{R}$ or \mathbb{R}^d) or infinite-dimensional ($E = L^p([0, T], dt), \mathcal{C}([0, T]), \dots$) separable Banach space and let $\alpha \subset E$ be a finite subset of size $\text{card}(\alpha) \leq N$, $N \geq 1$. The *Voronoi quantization* of an E -valued random vector $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow E$ with respect to the codebook α is simply the projection of X onto α following the nearest neighbour rule *i.e.*

$$\widehat{X}^\alpha = \pi_\alpha(X)$$

where

$$\pi_\alpha = \sum_{a \in \alpha} \mathbf{1}_{C_a(\alpha)},$$

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$(C_a(\alpha))_{a \in \alpha}$ Borel partition of E satisfying for every $a \in \alpha$

$$C_a(\alpha) \subset \left\{ u \in E : \|u - a\| \leq \min_{b \in \alpha \setminus \{a\}} \|u - b\| \right\}.$$

Then, the L^r -mean quantization error is defined by

$$\|X - \hat{X}^\alpha\|_{L^r_E(\mathbb{P})} = \left(\mathbb{E} \min_{a \in \alpha} \|X - a\|^r \right)^{\frac{1}{r}}$$

(this quantity is finite as soon as $X \in L^r_E(\mathbb{P})$). The set α is called N -codebook or N -quantizer. One shows that such random vectors \hat{X}^α are the best approximation of X among all α -valued random vectors. The minimal N^{th} quantization error of X is then defined by

$$e_{N,r}(X, E) := \inf \{ (\mathbb{E} \min_{a \in \alpha} \|X - a\|^r)^{1/r} : \alpha \subset E, \text{card}(\alpha) \leq N \}. \quad (1.1)$$

When $E = L^p([0, T], dt)$ (with its usual norm denoted $\|\cdot\|_{L^p_T}$ from now on) or $\mathcal{C}([0, T])$ (endowed with $\|\cdot\| = \|\cdot\|_\infty$ is an infinite dimensional Banach space of functions, an E -valued random variable X is but a (bi-measurable) stochastic process $X = (X_t)_{t \in [0, T]}$ defined on the product space $(\Omega \times [0, T], \mathcal{A} \otimes \mathcal{B}([0, T]), \mathbb{P} \otimes dt)$ whose trajectories $(X_t(\omega))_{0 \leq t \leq T}$ (almost) all belong to $L^p([0, T], dt)$. The L^r -integrability assumption then reads

$$\mathbb{E} \left(\left(\int_0^T |X_t|^p \right)^{\frac{r}{p}} \right) < +\infty.$$

It is still an open question whether L^r -optimal N -quantizers for Gaussian random vectors always exist (see [11]) in an abstract Banach setting. However in many situations of interest for process, including all the $L^p([0, T], dt)$ -spaces, $1 \leq p < +\infty$, the existence of at least one such L^r -optimal codebook has been established (provided $\mathbb{E}\|X\|^r < +\infty$). Note however that this is not the case for the space $\mathcal{C}(I)$ of continuous functions. For more details on the existence problem for optimal quantizers we refer to [11].

On the other hand, optimal L^r -quantizers always exist when $E = \mathbb{R}^d$, $d \geq 1$. In this finite dimensional setting this problem is known as *optimal vector quantization* and has been extensively investigated since the early 1950' with some applications to Signal processing and transmission (see [7] or [8]). In d -dimension the convergence rate of $e_{N,r}$ is ruled by the so-called Zador theorem

$$\lim_N N^{\frac{1}{d}} e_{N,r}(X, \mathbb{R}^d) = J_{r,d} \left(\int_{\mathbb{R}^d} g^{\frac{d}{d+r}}(\xi) d\xi \right)^{1+\frac{r}{d}}$$

where g denotes the density of the absolutely continuous part of the distribution \mathbb{P}_X of X .

Since the early 2000's much attention has been paid to the infinite dimensional case. This is the so-called *functional quantization* problem for stochastic processes: the aim is to quantize some processes viewed as random vectors taking values in their path-spaces, supposed to be an $L^p([0, T], dt)$ space, $1 \leq p < +\infty$. Many results have been obtained for several families of processes with a special attention paid to gaussian processes and (Brownian) diffusion processes by several authors. Thus, in the purely Hilbert setting ($r = 2$, $E = L^2([0, T], dt)$) the sharp rate of quantization of the Brownian motion $(W_t)_{t \in [0, T]}$ is given (see [15]) by

$$e_{N,r}(W, L^2([0, T], dt)) \sim \frac{\sqrt{2}T}{\pi(\log N)^{\frac{1}{2}}} \quad (1.2)$$

The existence of such a sharp rate for Brownian motion has been extended to $L^p([0, T], dt)$ spaces for $1 \leq p \leq \infty$ (see [6]). Similar sharp rates hold for a wide class of Gaussian processes including the fractional Brownian motions with rate

$$e_{N,r}(W^H, L^2([0, T], dt)) \sim \frac{c(H, T)}{(\log N)^H}$$

where H denotes the Hurst parameter of the fractional Brownian motion W^H , the Ornstein-Uhlenbeck process, the Brownian sheet, etc, in the purely Hilbert setting (see [15]). The exact rate has also been established in [14] for a wider class of Gaussian processes. In [14, 15], these results are based on the (sharp or exact) asymptotic behaviour of the eigenvalues of high order of the Karhunen-Loève expansion of the Gaussian process. As a by-product, this approach provides very simple explicit sequences of rate optimal asymptotic quantizers (provided the Karhunen-Loève expansion of the process itself is accessible). Their numerical implementation has lead to some unexpectedly promising numerical applications in Finance, especially for the pricing of path-dependent options like Asian options in several popular models of asset dynamics (Black-Scholes, stochastic volatility Heston and SABR models, etc). For these aspects we refer to [18] or [20].

Still for Gaussian processes an important connection with small ball probability problem has been made (see [4, 10]). Some exact or sharp rates of convergence for different classes of Brownian diffusions have also been proved recently (see [17], [5]) with a rate driven by $(\log N)^{-\frac{1}{2}}$.

The common feature shared by all these results is that there is a one-to-one correspondence between the exponent a that rules the $(L^r(\mathbb{P}), L^p(dt))$ -quantization rate of these processes in the $\log(N)^{-a}$ scale and their mean pathwise regularity *i.e.* the lowest exponent a that satisfies

$$\forall s, t \in [0, T], \quad \|X_t - X_s\|_r \leq C_r |t - s|^a. \quad (1.3)$$

Although such a correspondence is not really surprising given the connection between quantization rate and small ball probabilities in the Gaussian setting, this leads naturally to try deriving a general abstract result that connects these two features of a process. This is the aim of the Section 2 of this paper in which we show that the mean pathwise regularity always provide an upper-bound for the $(L^r(\mathbb{P}), L^p(dt))$ -quantization rate ($0 < p \leq r$). We retrieve then the rate obtained by more specific approaches for the all processes mentioned above. We also extend, to general Brownian diffusion processes and even general Itô processes the rate formerly obtained for specific classes of diffusions in [5, 16]. We also obtain some first quantization rates for some classes of Lévy processes like α -stable processes. The main tool is to expand a process on the simplest unconditional wavelet basis – the Haar basis – and to use a uniform version of the Zador theorem (coming out as a slight improvement of the Pierce Lemma, see [9])

At this point, the next question is to wonder conversely whether this always provide the true quantization rate. In this naive form, the answer to this question is clearly no because equation (1.3) only takes into account the mean-pathwise Hölder regularity of a process and one can trivially build (see [14]) some processes which smoother mean-pathwise regularity (like processes with C^k , $k \geq 1$ trajectories). We did not extend our approach in that direction for the sake of simplicity but there is no doubt that developing similar techniques as those used in Section 2 one can connect higher order mean pathwise regularity and quantization rate like in the Hölder setting. In fact we point out in section 3 devoted to general Lévy processes that the answer maybe negative – the quantization rate can be infinitely faster than the mean pathwise regularity – for different reasons in connection with the dimensionality of process: a Poisson process is in some sense an almost finite dimensional random which induces a very fast quantization rate which does not take place in the $\log(N)^{-a}$ scale although the mean-pathwise $L^r(\mathbb{P})$ -regularity of a Poisson process is Hölder (and depends on r , see *e.g.* (2.12) and (2.13)).

The main result of Section 3 is Theorem 2 which provides a functional quantization for a general Lévy process X : this rate is ruled by the behaviour of the Lévy measure ν around 0. As an example for Lévy processes which do have infinitely many small jumps, if the Lévy measure satisfies

$$\exists c > 0, \quad \mathbf{1}_{(0,c]}(x)\nu(dx) \leq \frac{C}{|x|^{\underline{\theta}+1}}\mathbf{1}_{(0,c]}(x)dx$$

for some $\underline{\theta} > 0$, then, for every p, r such that $0 < p \leq r$ and $X_1 \in L^r(\mathbb{P})$

$$\forall r, p \in (0, r^*), \quad \inf_{\text{card}\beta \leq N} \| |X - \widehat{X}|_{L_T^p} \|_r = O((\log N)^{-\frac{1}{\underline{\theta}}}).$$

A more general result is established that connects the quantization rate with the regular variation index of the zero tail function $x \mapsto \nu([-1, 1] \setminus [-x, x]^c)$ as x goes to 0. This result is obtained by applying the result of Section 2 to the dominating term for quantization which turns out to be a non-homogeneous martingale processes with independent increments which “excludes” the jumps of X with a rate $t^{\frac{1}{2}}$ as t goes to 0 (when looking at the process in a backward way). We also establish some quantization rates for compound Poisson processes and show they are infinitely faster than the above ones. To this end, we design an explicit sequence of quantizers which can clearly be implemented for numerical purposes. In fact the whole proof is constructive provided the Lévy measure is “tractable” enough

The paper is organized as follows: Section 2 is devoted to the abstract connection between mean-pathwise regularity and quantization rate of processes, with some applications to various families of processes. As far as we know, some of this rates are new. Its main result is Theorem 1. Section 3 provides an upper-bound for the quantization rate of general Lévy process in connection with the behaviour of the Lévy measure around 0. The main results are Theorem 2 and Proposition 4.

NOTATIONS: • $L_T^p := L^p([0, T], dt)$ and $|f|_{L_T^p} = (\int_0^T |f(t)|^p dt)^{\frac{1}{p}}$.

- Let $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ be two sequences of positive real numbers. The symbol $a_n \sim b_n$ means $a_n = b_n + o(b_n)$.
- $[x]$ denotes the integral part of the real number x and $x_+ = \max(x, 0)$ its positive part.

2 Mean pathwise regularity and quantization error rate: an upper bound

In this section, we derive in full generality an upper-bound for the $(L^r(\mathbb{P}), L_T^p)$ -quantization error $e_{N,r}(X, L_T^p)$ based on the path regularity of the mapping $t \mapsto X_t$ from $[0, T]$ to $L^p(\mathbb{P})$. The main result of this section is Theorem 1 below. Then we will illustrate on several examples that this rate may be optimal or not.

One key of the proof is the following extension of Pierce Lemma (see [9], p. 82) which is the main step of Zador’s Theorem for unbounded random variables. Its proof relies on random quantization.

Proposition 1 (*Pierce Lemma*) *Let $r, \delta > 0$. There exists a real constant $C_{r,\delta}$ such that, for every random variable $X : (\Omega, \mathcal{A}) \rightarrow (\mathbb{R}, \mathbb{B}(\mathbb{R}))$,*

$$\forall N \geq 1, \quad e_{N,r}(X, \mathbb{R}^d) = \inf_{\text{card}(\alpha) \leq N} \|X - \widehat{X}^\alpha\|_r \leq C_{r,\delta} \|X\|_{r+\delta} N^{-1}.$$

Proof: It follows from the original Pierce Lemma as stated that there exists a universal real constant $C_{r,\delta}^0 > 0$ and an integer $N_{r,\delta} \geq 1$ such that, for every random variable $X : (\Omega, \mathcal{A}) \rightarrow (\mathbb{R}, \mathbb{B}(\mathbb{R}))$,

$$\forall N \geq N_{r,\delta}, \quad \inf_{\text{card}(\alpha) \leq N} \mathbb{E}|X - \widehat{X}^\alpha|^r \leq C_{r,\delta}^0 (1 + \mathbb{E}|X|^{r+\delta}) N^{-r}.$$

Using the scaling property of quantization: for every $\lambda > 0$,

$$\|X - \widehat{X}^\alpha\|_r = \frac{1}{\lambda} \|(\lambda X) - \widehat{\lambda X}^{\lambda\alpha}\|_r$$

where $\lambda\alpha = \{\lambda a, a \in \alpha\}$, one derives from Pierce Lemma by considering $X/\|X\|_{r+\delta}$ and setting $\lambda := 1/\|X\|_{r+\delta}$ that

$$\forall N \geq N_{r,\delta}, \quad \inf_{\text{card}(\alpha) \leq N} \|X - \widehat{X}^\alpha\|_r \leq 2^{\frac{1}{r}} C_{r,\delta}^0 \|X\|_{r+\delta} N^{-1}.$$

Now, for every $N \in \{1, \dots, N_{r,\delta} - 1\}$, setting $\alpha := \{0\}$ yields

$$\inf_{\text{card}(\alpha) \leq N} \|X - \widehat{X}^\alpha\|_r \leq \|X\|_r \leq N_{r,\delta} \|X\|_{r+\delta} N^{-1}.$$

Combining the last two inequalities and setting $C_{r,\delta} = \max(2^{\frac{1}{r}} C_{r,\delta}^0, N_{r,\delta})$ completes the proof. \diamond

Let $(e_n)_{n \geq 0}$ denote the Haar basis defined as the restrictions on $[0, T]$ of the following functions

$$e_0 := T^{-\frac{1}{2}} \mathbf{1}_{[0,T]}, \quad e_1 := T^{-\frac{1}{2}} (\mathbf{1}_{[0,T/2]} - \mathbf{1}_{[T/2,T]}), \quad e_{2^n+k} := 2^{\frac{n}{2}} e_1(2^n \cdot - kT), \quad n \geq 0, \quad k \in \{0, \dots, 2^n - 1\}.$$

With this normalization, it makes up an orthonormal basis of the Hilbert space $(L^2([0, T], dt), (\cdot | \cdot)_2)$ where $(f|g)_2 = \int_0^T fg(t) dt$. Furthermore, it clearly satisfies for every $p > 0$,

$$\forall n \geq 0, \quad \int_0^T \left| \sum_{k=0}^{2^n-1} (f|e_{2^n+k}) e_{2^n+k}(t) \right|^p dt = 2^{n(\frac{p}{2}-1)} T^{1-\frac{p}{2}} \sum_{k=0}^{2^n-1} |(f|e_{2^n+k})|^p. \quad (2.1)$$

The second key to establish a general connection between quantization rate and mean pathwise regularity is the following standard properties of the Haar basis: for every $f \in L^2([0, T], dt)$,

$$\begin{aligned} (f|e_{2^n+k}) &= 2^{\frac{n}{2}} T^{-\frac{1}{2}} \left(\int_{kT2^{-n}}^{(2k+1)T2^{-(n+1)}} f(u) du - \int_{(2k+1)T2^{-(n+1)}}^{(k+1)T2^{-n}} f(u) du \right) \\ &= 2^{\frac{n}{2}} T^{-\frac{1}{2}} \int_0^{T2^{-(n+1)}} \left(f(kT2^{-n} + u) - f((2k+1)T2^{-(n+1)} + u) \right) du. \end{aligned} \quad (2.2)$$

Let $(X_t)_{t \in [0, T]}$ be a càdlàg (right continuous, left limited) process defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ (or more generally a bi-measurable process with \mathbb{P} -almost all paths lying in L_T^2) such that $X_t \in L^\rho(\mathbb{P})$ for every $t \in [0, T]$ for some positive real exponent $\rho > 0$.

We make the following φ -Lipschitz assumption on the map $t \mapsto X_t$ from $[0, T]$ into $L^\rho(\mathbb{P})$: there is a non-decreasing function $\varphi : (0, +\infty) \rightarrow \mathbb{R}_+$, continuous at 0 with $\varphi(0) = 0$ such that

$$(L_{\varphi,\rho}) \equiv \begin{cases} \forall s, t \in [0, T], \quad \mathbb{E}|X_t - X_s|^\rho \leq (\varphi(|t-s|))^\rho & \text{if } \rho \geq 1, \\ \forall h \in (0, h_0], \quad \mathbb{E}(\sup_{t \leq s \leq (t+h) \wedge T} |X_s - X_t|^\rho) \leq (\varphi(h))^\rho & \text{if } 0 < \rho < 1. \end{cases} \quad (2.3)$$

Note that this assumption implies that $\mathbb{E}(|X|_{L_T^\rho}^\rho) < +\infty$ so that, in particular, that, $\mathbb{P}(d\omega)$ -a.s., $t \mapsto X_t(\omega)$ lies in L_T^ρ (note this implies that the paths are L_T^2 -valued if $\rho \geq 2$).

We make a *regularly varying assumption* on φ at 0 with index $b \geq 0$ i.e. for every $t > 0$

$$\lim_{x \rightarrow 0} \frac{\varphi(tx)}{\varphi(x)} = t^b. \quad (2.4)$$

In accordance with the literature (see [2]) this means that $x \mapsto \varphi(1/x)$ is regularly varying at infinity with index $-b$ (which is a more usual notion in that field).

Let $r, p \in (0, \rho)$. Our aim is to evaluate the $L^r(\mathbb{P})$ -quantization rate of the process X viewed as an L_T^p -valued random variable induced by the ‘‘Haar product quantizations’’ of X defined by

$$\widehat{X} = \widehat{\xi}_0^{N_0} e_0 + \sum_{n \geq 0} \sum_{k=0}^{2^n-1} \widehat{\xi}_{2^{n+k}}^{N_{2^n+k}} e_{2^n+k} \quad (2.5)$$

where $\xi_k := (X|e_k)_2 \in L^\rho(\mathbb{P})$, $k \geq 0$, and $\widehat{\xi}^N$ denotes an N -quantization ($N \geq 1$) of the (real-valued) random variable ξ i.e. a quantization of X by a codebook α^N having N elements. A quantization taking finitely many values, this implies that $N_{2^n+k} = 1$ and $\widehat{\xi}_{2^n+k}^{N_{2^n+k}} = 0$ for large enough n (which may be a non optimal 1-quantizer for $\xi_{2^n+k}^{N_{2^n+k}}$).

We will see that this local behaviour of φ at 0 induces an upper-bound for the functional quantization error rate of X (regardless of the values of r and p except for constants).

Relying on the following two well-known inequalities:

$$|f|_{L_T^p} \leq T^{1/p-1/p'} |f|_{L_T^{p'}}, \quad p \leq p'$$

for every Borel functions $f : [0, T] \rightarrow \mathbb{R}$ and

$$\|Z\|_r \leq \|Z\|_{r'}, \quad r \leq r'$$

for every random variables $Z : \Omega \rightarrow \mathbb{R}$, one may assume without loss of generality that, either

$$1 \leq p = r < \rho \quad \text{or} \quad 0 < p = r < \rho \leq 1.$$

Theorem 1 *Let be $X = (X_t)_{t \in [0, T]}$ a bi-measurable process defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that $\int_0^T \mathbb{E}|X_t|^\rho dt < +\infty$ for an exponent $\rho > 0$. Assume that X satisfies the φ -Lipschitz assumption (2.3) for this exponent ρ where φ is regularly varying (in the sense of (2.4)) with index $b \geq 0$ at 0. If $b = 0$ assume moreover $\int_0^1 \varphi(\xi) \frac{d\xi}{\xi} < +\infty$. Then*

$$\forall r, p \in (0, \rho), \quad e_{N,r}(X, L_T^p) \leq C_{r,p} \begin{cases} \varphi(1/\log N) & \text{if } b > 0, \\ \psi(1/\log N) & \text{if } b = 0, \end{cases}$$

with $\psi(x) = \int_0^x \varphi(\xi)/\xi d\xi$. In particular if $\varphi(u) = u^b$, $b > 0$, then

$$e_{N,r}(X, L_T^p) = O((\log N)^{-b}). \quad (2.6)$$

Proof. CASE $1 \leq p = r < \rho$: Let $N \geq 1$ be a fixed integer. One considers a Haar product quantization \widehat{X} of X with a (product) codebook having at most N elements i.e. such that $N_0 \times \prod_{n,k} N_{2^n+k} \leq N$. Its characteristics will be specified further on. Then,

$$\begin{aligned} |X - \widehat{X}|_{L_T^r} &\leq T^{\frac{1}{r}-\frac{1}{2}} |\xi_0 - \widehat{\xi}_0^{N_0}| + \sum_{n \geq 0} \left| \sum_{k=0}^{2^n-1} (\xi_{2^n+k} - \widehat{\xi}_{2^n+k}^{N_{2^n+k}}) e_{2^n+k} \right|_{L_T^r} \\ &= T^{\frac{1}{r}-\frac{1}{2}} |\xi_0 - \widehat{\xi}_0^{N_0}| + T^{\frac{1}{r}-\frac{1}{2}} \sum_{n \geq 0} 2^{n(\frac{1}{2}-\frac{1}{r})} \left(\sum_{k=0}^{2^n-1} |\xi_{2^n+k} - \widehat{\xi}_{2^n+k}^{N_{2^n+k}}|^r \right)^{\frac{1}{r}} \end{aligned}$$

so that, both $\|\cdot\|_r$ and $\|\cdot\|_1$ being norms,

$$\begin{aligned}
\| |X - \widehat{X}|_{L_T^p} \|_r &\leq T^{\frac{1}{r}-\frac{1}{2}} \|\xi_0 - \widehat{\xi}_0^{N_0}\|_r + T^{\frac{1}{r}-\frac{1}{2}} \sum_{n \geq 0} 2^{n(\frac{1}{2}-\frac{1}{r})} \left\| \left(\sum_{k=0}^{2^n-1} |\xi_{2^n+k} - \widehat{\xi}_{2^n+k}^{N_{2^n+k}}|^r \right)^{\frac{1}{r}} \right\|_r \\
&= T^{\frac{1}{r}-\frac{1}{2}} \|\xi_0 - \widehat{\xi}_0^{N_0}\|_r + T^{\frac{1}{r}-\frac{1}{2}} \sum_{n \geq 0} 2^{n(\frac{1}{2}-\frac{1}{r})} \left\| \sum_{k=0}^{2^n-1} |\xi_{2^n+k} - \widehat{\xi}_{2^n+k}^{N_{2^n+k}}|^r \right\|_1^{\frac{1}{r}} \\
&\leq T^{\frac{1}{r}-\frac{1}{2}} \|\xi_0 - \widehat{\xi}_0^{N_0}\|_r + T^{\frac{1}{r}-\frac{1}{2}} \sum_{n \geq 0} 2^{n(\frac{1}{2}-\frac{1}{r})} \left(2^n \max_{0 \leq k \leq 2^n-1} \left\| |\xi_{2^n+k} - \widehat{\xi}_{2^n+k}^{N_{2^n+k}}|^r \right\|_1 \right)^{\frac{1}{r}} \\
&= T^{\frac{1}{r}-\frac{1}{2}} \|\xi_0 - \widehat{\xi}_0^{N_0}\|_r + T^{\frac{1}{r}-\frac{1}{2}} \sum_{n \geq 0} 2^{\frac{n}{2}} \max_{0 \leq k \leq 2^n-1} \left\| |\xi_{2^n+k} - \widehat{\xi}_{2^n+k}^{N_{2^n+k}}|^r \right\|_1^{\frac{1}{r}} \\
&= T^{\frac{1}{r}-\frac{1}{2}} \|\xi_0 - \widehat{\xi}_0^{N_0}\|_r + T^{\frac{1}{r}-\frac{1}{2}} \sum_{n \geq 0} 2^{\frac{n}{2}} \max_{0 \leq k \leq 2^n-1} \left\| \xi_{2^n+k} - \widehat{\xi}_{2^n+k}^{N_{2^n+k}} \right\|_r. \tag{2.7}
\end{aligned}$$

Let $\delta := \rho - r$. It follows from Proposition 1 (Pierce Lemma) that, for every $N \geq N_{r,\rho}$, and every r.v. $\xi \in L^r(\mathbb{P})$,

$$\inf_{\text{card}(\alpha) \leq N} \|\xi - \widehat{\xi}^\alpha\|_r \leq C_{r,\rho} \|\xi\|_\rho N^{-1}.$$

Now, using the monotony of the L^p -norms with respect to the probability measure $2^{n+1} \mathbf{1}_{[0, 2^{-(n+1)T}]}(t) dt / T$, Fubini's Theorem, the φ -Lipschitz continuity assumption (2.3) and (2.2), yields

$$\begin{aligned}
\mathbb{E} |\xi_{2^n+k}|^\rho &= \mathbb{E} |(X|_{e_{2^n+k}})|^\rho \\
&\leq 2^{\frac{n}{2}\rho} T^{-\rho/2} \mathbb{E} \left(\int_0^{2^{-(n+1)T}} |X_{\frac{k}{2^n}T+u} - X_{\frac{2k+1}{2^{n+1}}T+u}| du \right)^\rho \\
&\leq 2^{-n\rho} 2^{-(n+1)\rho} T^{\rho/2} \mathbb{E} \left(\int_0^{2^{-(n+1)T}} |X_{\frac{k}{2^n}T+u} - X_{\frac{2k+1}{2^{n+1}}T+u}|^\rho 2^{n+1} du / T \right) \\
&\leq 2^{-\rho} 2^{-\frac{n}{2}\rho+n+1} T^{\rho/2-1} \int_0^{2^{-(n+1)T}} \mathbb{E} |X_{\frac{k}{2^n}T+u} - X_{\frac{2k+1}{2^{n+1}}T+u}|^\rho du \\
&\leq 2^{-\frac{n}{2}\rho+n+1-\rho} T^{\rho/2-1} \int_0^{2^{-(n+1)T}} (\varphi(T/2^{n+1}))^\rho du \\
&\leq C_{X,T,r,\rho} 2^{-\frac{n}{2}\rho} (\varphi(T/2^{n+1}))^\rho. \tag{2.8}
\end{aligned}$$

At this stage, we assume *a priori* that that the size sequence $(N_{2^n+k})_{n \geq 0, k=0, \dots, 2^n-1}$ of the marginal codebooks is nonincreasing as $2^n + k$ increases and satisfies

$$1 \leq \prod_{k \geq 0} N_k \leq N.$$

We assume that all the quantizations induced by these codebooks are L^r -optimal up to $n \leq m$ i.e.

$$\|\xi_{2^n+k} - \widehat{\xi}_{2^n+k}\|_r = \inf_{\text{card}(\alpha) \leq N_{2^n+k}} \|\xi_{2^n+k} - \widehat{\xi}_{2^n+k}^\alpha\|_r$$

and that $\widehat{\xi}_{2^n+k} = 0$ otherwise. Then combining (2.7), (2.8) and Proposition 1 yields

$$\| |X - \widehat{X}|_{L_T^r} \|_r \leq C_{X,T,r,\rho} \left(\frac{1}{N_0} + \sum_{n \geq 0} \frac{\varphi(T 2^{-(n+1)})}{N_{2^{n+1}}} \right)$$

$$\begin{aligned}
&\leq C_{X,T,r,\rho} \left(\frac{1}{N_0} + \frac{1}{T} \sum_{n \geq 0} \sum_{k=0}^{2^{n+1}-1} \frac{\Phi(2T/(2^{n+1}+k))}{N_{2^{n+1}+k}} \right) \\
&= C_{X,T,r,\rho} \left(\frac{1}{N_0} + \frac{1}{T} \sum_{k \geq 2} \frac{\Phi(2T/k)}{N_k} \right)
\end{aligned}$$

where $\Phi(x) := x \varphi(x)$, $x \in (0, +\infty)$. This function Φ is regularly varying (at 0) with index $b + 1$. This implies in particular that there is a real constant $c > 0$ such that $\Phi(T/k) \leq c \Phi(1/(k+1))$ for every $k \geq 2$. Hence, inserting for convenience the term $\Phi(1/2)/N_1$ and modifying the real constant $C_{X,T,r,\rho}$ in an appropriate way finally yields

$$\| |X - \widehat{X}|_{L_T^r} \|_r \leq C_{X,T,r,\rho} \sum_{k \geq 1} \frac{\Phi(1/k)}{N_{k-1}}.$$

Now set for convenience $\nu_k = \Phi(T/k)$, $k \geq 1$. Consequently an upper-bound for the quantization rate is given by the solution of the following optimal allocation problem

$$\begin{aligned}
\inf_{\alpha \subset L_T^r, \text{card}(\alpha) \leq N} \| |X - \widehat{X}|_{L_T^r} \|_r &\leq C_{X,T,r,\rho} \min \left\{ \sum_{k \geq 1} \frac{\nu_k}{N_{k-1}}, \prod_{k \geq 0} N_k \leq N, N_0 \geq \dots \geq N_k \geq \dots \geq 1 \right\} \\
&\leq C_{X,T,r,\rho} \min \left\{ \sum_{k=1}^m \frac{\nu_k}{N_{k-1}} + \sum_{k \geq m+1} \nu_k, \prod_{0 \leq k \leq m-1} N_k \leq N, N_0 \geq \dots \geq N_{m-1} \geq 1 \right\}. \quad (2.9)
\end{aligned}$$

The rest of the proof follows the approach developed in [14] (Section 4.1, especially Lemma 4.2, Theorem 4.6 (i)-(iii) and its proof) and [15]. However, one must be aware that we had to modify some notations.

Proposition 2 (See [14] for a proof) *Assume $\nu_k = \Phi(1/k)$, $k \geq 1$, where $\Phi(x) = x \varphi(x)$, $\varphi : (0, +\infty)$ is an increasing, regularly varying function at 0 with index $b \geq 0$ (with $\int_0^1 \varphi(\xi) \frac{d\xi}{\xi} < +\infty$ when $b = 0$). Then*

(i) $\lim_k \nu_k / \nu_{k+1} = 1$,

(ii) $\left(\prod_{k=1}^n \nu_k \right)^{\frac{1}{n}} \sim e^{b+1} \nu_n$,

(iii) $\sum_{k=n+1}^{\infty} \nu_k + n \nu_n \sim c \psi(1/n)$ where $c = 1 + 1/b$ if $b > 0$ and $c = 1$ if $b = 0$ and

$$\psi(x) = \varphi(x) \quad \text{if } b > 0 \quad \text{and} \quad \psi(x) := \int_0^x \varphi(\xi) \frac{d\xi}{\xi} \quad \text{if } b = 0.$$

Set
$$m = m^*(N) = \max \left\{ m \geq 1 : N^{\frac{1}{m}} \nu_m \left(\prod_{j=1}^m \nu_j \right)^{-\frac{1}{m}} \geq 1 \right\},$$

and
$$N_{k-1} = N_{k-1}(N) := \left\lceil N^{\frac{1}{m}} \nu_k \left(\prod_{j=1}^m \nu_j \right)^{-\frac{1}{m}} \right\rceil \geq 1, \quad k = 1, \dots, m.$$

It follows from Proposition 2 (ii) that

$$m = m^*(N) \sim \frac{\log N}{b+1} \quad \text{as} \quad N \rightarrow \infty.$$

Then,

$$\begin{aligned} \sum_{k=1}^m \frac{\nu_k}{N_{k-1}} &\leq \max_{k \geq 1} (1 + 1/N_{k-1}) m N^{-\frac{1}{m}} \left(\prod_{j=1}^m \nu_j \right)^{\frac{1}{m}} \\ &\leq 2m e^{-\frac{\log N}{m}} \left(\prod_{j=1}^m \nu_j \right)^{\frac{1}{m}} \\ &\leq 2m \nu_m. \end{aligned}$$

Consequently using this time (iii) in Proposition 2

$$\begin{aligned} \sum_{k=1}^m \frac{\nu_k}{N_{k-1}} + \sum_{k \geq m+1} \nu_k &\leq C \left(m \nu_m + \sum_{k \geq m+1} \nu_k \right) \\ &= O(\psi(1/\log N)) \end{aligned}$$

so that

$$\| |X - \widehat{X}|_{L_T^r} \|_r = O(\psi(1/\log N)).$$

CASE 2 ($\rho \leq 1$). One relies this time on the pseudo-triangular inequality

$$|f + g|_{L_T^r}^r \leq |f|_{L_T^r}^r + |g|_{L_T^r}^r$$

which follows from the elementary inequality $(u + v)^r \leq u^r + v^r$.

$$\begin{aligned} |X - \widehat{X}|_{L_T^r}^r &\leq T^{1-\frac{r}{2}} |\xi_0 - \widehat{\xi}_0^{N_0}|^r + \sum_{n \geq 0} \left| \sum_{k=0}^{2^n-1} (\xi_{2^{n+k}} - \widehat{\xi}_{2^{n+k}}^{N_{2^n+k}}) e_{2^{n+k}} \right|_{L_T^r}^r \\ &= T^{1-\frac{r}{2}} |\xi_0 - \widehat{\xi}_0^{N_0}|^r + T^{1-\frac{r}{2}} \sum_{n \geq 0} 2^{n(\frac{r}{2}-1)} \sum_{k=0}^{2^n-1} |\xi_{2^{n+k}} - \widehat{\xi}_{2^{n+k}}^{N_{2^n+k}}|^r \end{aligned}$$

so that,

$$\begin{aligned} \| |X - \widehat{X}|_{L_T^r} \|_r^r &= T^{1-\frac{r}{2}} \| |X - \widehat{X}|_{L_T^r} \|_1^r \\ &\leq T^{1-\frac{r}{2}} \| |\xi_0 - \widehat{\xi}_0^{N_0}|^r \|_1 + T^{1-\frac{r}{2}} \sum_{n \geq 0} 2^{n(\frac{r}{2}-1)} \left\| \sum_{k=0}^{2^n-1} |\xi_{2^{n+k}} - \widehat{\xi}_{2^{n+k}}^{N_{2^n+k}}|^r \right\|_1 \\ &= T^{1-\frac{r}{2}} \| |\xi_0 - \widehat{\xi}_0^{N_0}|^r \|_r + T^{1-\frac{r}{2}} \sum_{n \geq 0} 2^{n(\frac{r}{2}-1)} 2^n \max_{0 \leq k \leq 2^n-1} \| |\xi_{2^{n+k}} - \widehat{\xi}_{2^{n+k}}^{N_{2^n+k}}|^r \|_1 \\ &= T^{1-\frac{r}{2}} \| |\xi_0 - \widehat{\xi}_0^{N_0}|^r \|_r + T^{1-\frac{r}{2}} \sum_{n \geq 0} 2^{\frac{nr}{2}} \max_{0 \leq k \leq 2^n-1} \| |\xi_{2^{n+k}} - \widehat{\xi}_{2^{n+k}}^{N_{2^n+k}}|^r \|_1 \\ &= T^{1-\frac{r}{2}} \| |\xi_0 - \widehat{\xi}_0^{N_0}|^r \|_r + T^{1-\frac{r}{2}} \sum_{n \geq 0} 2^{\frac{nr}{2}} \max_{0 \leq k \leq 2^n-1} \| \xi_{2^{n+k}} - \widehat{\xi}_{2^{n+k}}^{N_{2^n+k}} \|_r^r. \quad (2.10) \end{aligned}$$

This inequality replaces (2.7). Then, one concludes like in the case $\rho > 1$. \diamond

Remarks. • When $p \geq \rho > r$, the $(L^r(\mathbb{P}), L_T^p)$ -quantization problem remains consistent. But there is a price to be paid for considering a p exponent higher than ρ . Thus, if $\varphi(u) := u^b$ with $b + \frac{1}{p} - \frac{1}{r} > 0$ the same approach yields the rate

$$e_{N,r}(X, L_T^p) \leq \frac{C_{X,r,\delta,T,p}}{(\log N)^{b + \frac{1}{p} - \frac{1}{r}}}.$$

We do not know whether it comes from our approach or if it is the best possible rate.

2.1 Applications and examples

In this section, we give some examples which illustrate that the upper-bound derived from the mean pathwise regularity may be optimal or not.

2.1.1 Application to Itô processes and d -dimensional diffusion processes

• Let W denote an \mathbb{R}^d -valued standard Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let \mathcal{F}_t^W denote its natural filtration (completed with all the \mathbb{P} -negligible sets). Let X be a 1-dimensional Itô process defined by

$$dX_t = H_t dt + K_t \cdot dW_t, \quad X_0 = x_0 \in \mathbb{R}$$

where $(H_t)_{t \in [0, T]}$ is a real-valued process and $(K_t)_{t \in [0, T]}$ is an \mathbb{R}^d -valued process, both assumed \mathcal{F}_t^W -progressively measurable. Assume there exists a real number $\rho \geq 2$ such that

$$\sup_{t \in [0, T]} \mathbb{E}|H_t|^\rho + \sup_{t \in [0, T]} \mathbb{E}|K_t|^\rho < +\infty. \quad (2.11)$$

Then, it is classical background, see *e.g.* [3], that Assumption (2.3) is satisfied with $\varphi(u) = u^{\frac{1}{2}}$. It follows from Theorem 1

$$\forall r, p \in (0, \rho), \quad \| |X|_{L_T^p} \|_r = O\left((\log N)^{-1/2}\right).$$

• Let $X = (X^1, \dots, X^d)$ be an \mathbb{R}^d -valued diffusion process defined by

$$dX_t = b(t, X_t) dt + \sigma(t, X_t) dW_t, \quad X_0 = x_0 \in \mathbb{R}^d$$

where $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathcal{M}(d \times q, \mathbb{R})$ are Borel functions satisfying

$$\forall t \in [0, T], \forall x \in \mathbb{R}^d, \quad |b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|)$$

and W is an \mathbb{R}^q -valued standard Brownian motion¹. Then, every component X^i is an Itô process (with $H_t = b^i(t, X_t)$ and $K_t := \sigma^{i \cdot}(t, X_t)$) for which Assumption (2.11) is satisfied for every $\rho > 0$ (see *e.g.* [3]). On the other hand, if (u^1, \dots, u^d) denotes the canonical basis of \mathbb{R}^d and $|\cdot|$ denotes any norm on \mathbb{R}^d , then for every $f := \sum_{1 \leq i \leq d} f^i u^i : [0, T] \rightarrow \mathbb{R}^d$,

$$|f|_{L_{\mathbb{R}^d}^p([0, T], dt)} \leq \sum_{i=1}^d |f^i|_{L_T^p} |u^i|.$$

¹The above assumption does not imply that such a diffusion process X exists. The existence holds as soon as b and σ are Lipschitz in x uniformly with respect to $t \in [0, T]$

Now, one can quantize each Itô process (X_t^i) , $i = 1, \dots, d$ using an (L^r, L_T^p) -optimal quantizer $\alpha^{(i)}$ of size $[\sqrt[d]{N}]$. It is clear that the resulting product quantizer $\prod_{i=1}^d \alpha^{(i)}$ of size $[\sqrt[d]{N}]^d \leq N$ induces an $(L^r, L_{\mathbb{R}^d}^p([0, T], dt))$ -quantization error $O((\log N)^{-1/2})$ (see *e.g.* [17]). Combining these obvious remarks finally yields

$$\forall r, p > 0, \quad e_{n,r}(X, L_{\mathbb{R}^d}^p([0, T], dt)) = O((\log N)^{-1/2}).$$

As far as quantization rate is concerned, this extends to general d -dimensional diffusions a first result obtained in [17] by stochastic calculus techniques for a more restricted class of Brownian diffusions (which included 1-dimensional ones). This also extends (the upper bound part of the) the result obtained in [5] for another class of (possibly multi-dimensional) Brownian diffusions. For the class investigated in [17] it is shown that under a mild ellipticity assumption on σ , this rate is optimal. In [5], still with an ellipticity assumption, the rate is sharp. This leads us to conjecture that this rate is optimal for non too degenerate Brownian diffusions.

2.1.2 Application to fractional Brownian motion

The Fractional Brownian Motion B^H with Hurst constant $H \in (0, 1]$ is a Gaussian process satisfying for every $\rho > 0$,

$$\mathbb{E} |W_t^H - W_s^H|^\rho = C_{H,p} |t - s|^{\rho H} \quad \text{and} \quad (W_s^H)_{0 \leq s \leq t} \stackrel{\mathcal{L}}{\sim} t^H (W_{s/t}^H)_{0 \leq s \leq t}.$$

So, using Theorem 1, we obtain $e_{N,r}(W^H, L_T^p) = O((\log N)^{-H})$ as an $(L^r(\mathbb{P}), |\cdot|_{L_T^p})$ -quantization rate. This rate is known to be optimal: a sharp rate is established (see [15] when $p = r = 2$ or [6]).

2.1.3 Stationary processes

Let X be a centered weakly (square integrable) stationary process. Then

$$\mathbb{E} |X_t - X_s|^2 = \mathbb{E} |X_{t-s} - X_0|^2 = 2 \text{Var}(X_0)(1 - c(|t - s|))$$

where $c(t)$ denotes the correlation between X_t and X_0 . Hence if

$$c(u) = 1 - \kappa u^{2a}$$

then the $L^r(\mathbb{P})$ -rate for L_T^p -quantization, $0 < p, r < 2$, will be $O((\log(N))^{-a})$. If furthermore, X is a Gaussian process, then this $O((\log(N))^{-a})$ rate holds for any $r, p > 0$ since for every $\rho \in \mathbb{N}^*$,

$$\mathbb{E} |X_t - X_s|^\rho = \mathbb{E} |X_{t-s} - X_0|^\rho = C_\rho (\text{Var}(X_0)(1 - c(|t - s|)))^{\rho/2}.$$

2.1.4 Lévy processes: a first approach

A Lévy process – or Process with Stationary Independent Increments – is characterized by its so-called local characteristics appearing in the Lévy-Khintchine formula. These characteristics depend on the way the “big” jumps are truncated. We will adopt in the following the convention that the truncation occurs at size 1. So that

$$\mathbb{E}(e^{iuX_t}) = e^{-t\psi(u)} \quad \text{where} \quad \psi(u) = -iua + \frac{1}{2}\sigma^2 u^2 - \int_{\mathbb{R} \setminus \{0\}} (e^{iux} - 1 - iux \mathbf{1}_{\{|x| \leq 1\}}) \nu(dx)$$

where $a, \sigma \in \mathbb{R}$, ν is a non-negative measure on $\mathbb{R} \setminus \{0\}$ such that $\nu(x^2 \wedge 1) < +\infty$. The measure ν is called the *Lévy measure* of the process. One shows that a Lévy process is a compound Poisson

process if and only if ν is a finite measure and has finite variation if and only if $\int_{\{|x|\leq 1\}} |x|\nu(dx) < +\infty$. Furthermore

$$X_t \in L^\rho(\mathbb{P}) \quad \text{if and only if} \quad \int_{\{|x|\geq 1\}} |x|^\rho \nu(dx) < +\infty.$$

As concerns Assumption (2.3) note that the very definition of a Lévy process implies that

$$\mathbb{E} |X_t - X_s|^\rho = \mathbb{E} |X_{t-s}|^\rho \quad \text{and} \quad \mathbb{E} \sup_{s \in [t, t+h]} |X_t - X_s|^\rho = \mathbb{E} \sup_{s \in [0, h]} |X_s|^\rho$$

so that we may focus on the distribution of X_t and $X_t^* := \sup_{s \in [0, t]} |X_s|$. Finally, note that it follows from the usual symmetry principle that for any Lévy process, $\mathbb{P}(X_t^* > u + v) \leq \mathbb{P}(|X_t| > u) / \mathbb{P}(X_t^* \leq v/2)$ so that $\mathbb{E}|X_t|^r$ and $\mathbb{E}|X_t^*|^r$ are simultaneously finite or infinite when $0 < r \leq 2$.

• **The α -stable processes** The α -stable processes are families of Lévy processes indexed by $\alpha \in (0, 2)$ satisfying a self-similarity property, namely

$$\forall t \in \mathbb{R}_+, \quad X_t \stackrel{\mathcal{L}}{\sim} t^{1/\alpha} X_1 \quad \text{and} \quad \sup_{0 \leq s \leq t} |X_s| \stackrel{\mathcal{L}}{\sim} t^{1/\alpha} \sup_{0 \leq s \leq 1} |X_s|.$$

Furthermore,

$$\sup\{r \text{ such that } \mathbb{E}(\sup_{0 \leq s \leq 1} |X_s|^r) < +\infty\} = \alpha \quad \text{and} \quad \mathbb{E}|X_1|^\alpha = +\infty.$$

Consequently it follows from Theorem 1 that

$$\forall p, r \in (0, \alpha), \quad e_{N,r}(X, L_T^p) = O\left(\frac{1}{(\log(N))^{\frac{1}{\alpha}}}\right).$$

Although we have no evidence for it in this paper, it is clear that this rate is optimal.

• **The Γ -processes** These are Lévy processes whose distribution \mathbb{P}_{X_t} at time t is a $\gamma(\alpha, t)$ -distribution

$$\mathbb{P}_{X_t}(dx) = \frac{\alpha^t}{\Gamma(t)} \mathbf{1}_{(0, \infty)}(x) x^{t-1} e^{-\alpha x} dx.$$

So, easy computations show that for every $\rho \geq 1$,

$$\mathbb{E}|X_t|^\rho = \frac{\Gamma(t + \rho)}{\alpha^\rho \Gamma(t + 1)} t \sim \frac{\Gamma(\rho)}{\alpha^\rho \Gamma(1)} t \quad \text{as} \quad t \rightarrow 0.$$

Consequently it follows from Theorem 1 that

$$\forall r \in [1, +\infty), \forall p \in [0, r], \quad e_{N,r}(X, L_T^p) = O\left(\frac{1}{(\log(N))^{\frac{1}{r} - \varepsilon}}\right).$$

• **Compound Poisson processes from the pathwise regularity viewpoint** One considers a compound Poisson process

$$N_t^U = \sum_{k=0}^{N_t} U_k,$$

where $N = (N_t)_{t \in [0, T]}$ denotes a standard Poisson process with intensity $\lambda = 1$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $(U_k)_{k \geq 1}$ an i.i.d. sequence of random variables defined on the same probability space, with $U_1 \in L^\rho(\mathbb{P})$. Then, standard computations show that,

$$\mathbb{E} \sup_{0 < s \leq t} \left| \sum_{k=1}^{N_s} U_k \right|^\rho \leq \mathbb{E} \sum_{k=1}^{N_t} |U_k|^\rho = t \|U_1\|_\rho^\rho \quad \text{if } 0 < \rho \leq 1, \quad (2.12)$$

$$\mathbb{E} \left| \sum_{k=1}^{N_t} U_k \right|^\rho \leq t \|U_1\|_\rho^\rho \times \left[e^{-t} \sum_{k \geq 1} \frac{t^{k-1} k^\rho}{k!} \right] \quad \text{if } \rho > 1, \quad (2.13)$$

$$\begin{aligned} \mathbb{E} \left| \sum_{k=1}^{N_t} U_k - t \mathbb{E} U_1 \right|^\rho &\leq \mathbb{E} \left| \sum_{k=1}^{N_t} U_k \right|^\rho + 2^{\rho-1} \left(\mathbb{E} \left| \sum_{k=1}^{N_t} U_k \right|^{\rho-1} t |\mathbb{E} U_1| + t^\rho |\mathbb{E} U_1|^\rho \right) \\ &= C_\rho t + O(t^{2 \wedge \rho}) \quad \text{if } \rho > 1. \end{aligned}$$

The inequality for the compensated process follows from the second one and from the elementary inequality $(u + v)^\rho \leq u^\rho + 2^{\rho-1}(u^{\rho-1}v + v^\rho)$ ($\rho \geq 1$). Besides, it is clear that $(N_t^U)_{t \geq 0}$ and its compensated counterpart $N_t^U - t \mathbb{E}(U_1)$ have the same quantization rate since $t \mapsto t \mathbb{E} U_1$ lies in any L_T^ρ (when $\rho \geq 1$). Consequently, Assumption (2.3) is fulfilled with $\varphi(u) = u^b$, $b = 1/(\rho \vee 1)$. Theorem 2 then yields

$$\forall r, p \in (0, \rho), \quad e_{N,r}(N^U, L_T^p) = O((\log N)^{1/(\rho \vee 1)}).$$

These rates are very far from optimality as we will see further on (in Section 3 some sharper rates are established by a completely different approach based on the almost finite-dimensional feature of the paths of such elementary jump processes). This means that the mean pathwise regularity of $t \mapsto X_t$ is not the only parameter that rules the quantization rate.

3 A general quantization rate for Lévy processes

One aim of this section is to prove the following theorem for Lévy processes having no Brownian component.

Theorem 2 *Let $X = (X_t)_{t \in [0, T]}$ be a Lévy process with Lévy measure ν without Brownian component. Set*

$$\underline{\theta} := \inf \left\{ \theta : \int_{\{|x| \leq 1\}} |x|^\theta \nu(dx) < +\infty \right\} \in [0, 2], \quad (3.14)$$

$$r^* := \sup \left\{ \theta : \int_{\{|x| > 1\}} |x|^\theta \nu(dx) < +\infty \right\} \leq +\infty \quad (3.15)$$

and assume $r^* > 0$

(a) *If the Lévy measure satisfies*

$$\exists c > 0, \quad \mathbf{1}_{(0, c]}(x) \nu(dx) \leq \frac{C}{|x|^{\underline{\theta} + 1}} \mathbf{1}_{(0, c]}(x) dx \quad (3.16)$$

then

$$\forall r, p \in (0, r^*), \quad e_{N,r}(X, L_T^p) = O((\log N)^{-\frac{1}{\underline{\theta}}}). \quad (3.17)$$

When $\underline{\theta} = 0$, the right hand side of (3.17) reads $O((\log N)^{-a})$, $a > 0$.

(b) If the tail function of the Lévy measure $\nu: u \mapsto \underline{\nu}^c[-c, c] \setminus [-u, u]$ defined on $\mathbb{R}_+ \setminus \{0\}$ has regular variation with index $-b$ (as $u \rightarrow 0$), then $b = \underline{\theta}$. If furthermore

$$\underline{\theta} \in (0, 2) \quad \text{and} \quad \liminf_{t \rightarrow 0} t^{\underline{\theta}} \underline{\nu}(t) > 0, \quad (3.18)$$

then,

$$\forall r, p \in (0, +\infty), \quad e_{N,r}(X, L_T^p) = O\left((\log N)^{-\frac{1}{\underline{\theta}}} \sqrt{\underline{\ell}((\log N)^{-1})}\right) \quad (3.19)$$

where $\underline{\ell}(t) := t \underline{\nu}(t^{\frac{1}{\underline{\theta}}})$ is a slowly varying function.

Remark: When $\underline{\theta} = 2$, the right hand part of Assumption (3.18) cannot be satisfied: if it were then

$$\int x^2 \nu(dx) = \int_0^{c^2} \underline{\nu}(\sqrt{s}) ds \geq C \int_0^{c^2} \frac{ds}{s} = +\infty.$$

The complete proof of this theorem is deferred to Section 3.2. The reason is that it relies on a specific decomposition of X and on several quantization rate results of the processes involved in this decomposition.

Note that, in case of equality in (3.16), claim (a) is a sub-case of claim (b) corresponding to a (non-zero) constant function ℓ . In general claim (a) yields a result when Assumption (3.18) fails e.g. when $\nu(dx) = \kappa x^{-\underline{\theta}-1} (-\ln x)^{-a} \mathbf{1}_{(0,1]}(x) dx$, $a > 0$.

Note that rate (3.19) coincides for stable processes with that obtained in Section 2.1.4 and is most likely optimal.

On the other hand when $\underline{\theta} = 0$, which includes the case of compound Poisson processes, (3.19) only shows that the quantization rate is faster than any negative power of $\log N$. A sharper result is given in Section 3.1.1.

3.1 Lévy processes with bounded jumps

We consider in this section a Lévy process X without Brownian component ($\sigma = 0$) and whose jumps are bounded by a real constant $c > 0$ which is not a Poisson process. This means in term of the Lévy measure ν of X that

$$\nu^c([-c, c]) = 0 \quad \text{and} \quad \forall \varepsilon \in (0, c), \quad \nu([-c, c] \setminus [-\varepsilon, \varepsilon]) > 0. \quad (3.20)$$

Furthermore for every $\rho > 0$ and every $t \geq 0$, $X_t \in L^\rho(\mathbb{P})$. In Proposition 3 below we show that Theorem 2 is true in that setting.

Proposition 3 *let $(X_t)_{t \in [0, T]}$ be a Lévy process satisfying (3.20). Then claims (a) and (b) in Theorem 2 both hold true with $r^* = \infty$.*

Proof. The proof of this proposition is decomposed into several steps. First note that we may assume $p, r \geq 1$.

STEP 1 (Decomposition of (X_t) and the quantization error): We consider $\underline{\theta}$ as defined in Theorem 1. Note that in the present setting $\underline{\theta} = \inf\{\theta : \int_{\{|x| \leq c\}} |x|^\theta \nu(dx) < +\infty\}$. Assume first that $\underline{\theta} > 0$.

The starting idea is to part the “small” and the “big” jumps of X in a non homogeneous way with respect to the function $s \mapsto s^{\frac{1}{2}}$. Indeed one may decompose X as follows

$$X = \xi(t) + X_t^{(\varrho)} + M_t^{(\varrho)} \quad (3.21)$$

where $\xi(t) := t \mathbb{E}(X_t) - \int_0^t ds \int_{\{s^{\frac{1}{2}} < |x| \leq c\}} x \nu(dx)$ is a continuous deterministic function,

$$X_t^{(\varrho)} := \sum_{0 < s \leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s| > s^{\frac{1}{2}}\}},$$

$M_t^{(\varrho)}$ is a martingale with (non-homogeneous) independent increments

with a predictable conditional variance process given by

$$\langle M^{(\varrho)} \rangle_t = \int_0^t ds \int_{\{|x| \leq s^{\frac{1}{2}}\}} x^2 \nu(dx).$$

Note that $(X_t^{(\varrho)})$ is well defined (and has finite variations on $[0, T]$) since

$$\int_0^t ds \int_{\{s^{\frac{1}{2}} \leq |x| \leq c\}} |x| \nu(dx) = \int_{\{|x| \leq c\}} |x| (|x|^{\varrho} \wedge t) \nu(dx) \leq \int_{\{|x| \leq c\}} |x|^{1+\varrho} \nu(dx) < +\infty.$$

Our aim is to prove that the martingale is the dominating term as far as quantization is concerned. First note that the deterministic function ξ does not affect the quantization rate since it is invariant by translation (provided $\xi \in L_T^p$ as a continuous function).

This follows from the following general fact: for any N -quantizer, $\beta \in L_T^p$ and any $\xi \in L_T^p$

$$\|X - \widehat{X}^\beta\|_{L_T^p} = \| (X + \xi) - \widehat{(X + \xi)}^{\xi + \beta} \|_{L_T^p}, \quad \text{with } \xi + \beta = \{\xi + f, f \in \beta\}.$$

Consequently, one derives from (3.21) that, for every $N \geq 1$,

$$e_{N,r}(X, L_T^p) \leq \min \left\{ e_{N_1,r}(X^{(\varrho)}, L_T^p) + e_{N_2,r}(M^{(\varrho)}, L_T^p), N_1, N_2 \geq 1, N_1 N_2 \leq N \right\} \quad (3.22)$$

STEP 2 (*Time reversal argument*): One easily checks that for any Lévy process $X = (X_t)_{0 \leq t \leq T}$ has the same distribution – as a process – as Y defined by

$$Y_t = X_T - X_{(T-t)_-}, \quad t \in [0, T].$$

(see *e.g.* [1]). On the other hand

$$X_T^{(\varrho)} - X_{(T-t)_-}^{(\varrho)} = \sum_{0 < s \leq t} \Delta Y_s \mathbf{1}_{\{(T-s)^{\frac{1}{2}} < |\Delta Y_s| \leq c\}}, \quad t \in [0, T].$$

so that, setting $\bar{Y}_t^T := X_T^{(\varrho)} - X_{(T-t)_-}^{(\varrho)}$, yields the identity

$$X_t^{(\varrho)} = \tilde{X}_t^{(\varrho)} - \bar{Y}_{(T-t)_-}^T, \quad t \in [0, T].$$

This implies that for ever $p \geq 1$ and every $x \in \mathbb{R}$, $g \in L_T^p$ that

$$\begin{aligned} |X^{(\varrho)} - (x - g((T - \cdot)_-))|_{L_T^p} &\leq |X_T^{(\varrho)} - x| + |\bar{Y}_{(T-\cdot)_-}^T - g((T - \cdot)_-)|_{L_T^p} \\ &= |X_T^{(\varrho)} - x| + |\bar{Y}_{\cdot}^T - g(\cdot)|_{L_T^p} \\ &= |X_T^{(\varrho)} - x| + |\bar{Y}^T - g|_{L_T^p}. \end{aligned} \quad (3.23)$$

where we used that the mapping $s \mapsto T - s$ leaves the Lebesgue measure on $[0, T]$ invariant. Then, using that $X \stackrel{\mathcal{L}}{\sim} Y$, one gets for every $r \geq 1$

$$e_{N,r}(X^{(\underline{\theta})}, L_T^p) \leq \inf \left\{ e_{N_1,r}(X_T^{(\underline{\theta})}) + e_{N_2,r}(\bar{X}^T, L_T^p), N_1, N_2 \geq 1, N_1 N_2 \leq N \right\} \quad (3.24)$$

where

$$\bar{X}_t^T = \sum_{0 < s \leq t} \Delta X_s \mathbf{1}_{\{(T-s)^{\frac{1}{\underline{\theta}}} < |\Delta X_s| \leq c\}}.$$

Since the random variable $X_T^{(\underline{\theta})}$ lies in every $L^r(\mathbb{P})$ it follows from Pierce Lemma that

$$e_{N_1,r}(X_T^{(\underline{\theta})}) \leq C_{r,\underline{\theta}} N_1^{-1}. \quad (3.25)$$

STEP 3 (Quantization of the martingale term): In this step we will evaluate

Lemma 1 For every $\theta \in (0, +\infty)$, set $k_\theta := \max\{k : 2^k < \theta\}$.

$$\forall t \geq 0, \quad \mathbb{E} \left(\sup_{0 \leq s \leq t} |M_s^{(\underline{\theta})}|^\theta \right) \leq C_\theta \sum_{k=1}^{k_\theta+1} \left(\int_0^t ds \int_{\{|x| \leq s^{\frac{1}{\underline{\theta}}}\}} x^{2k} \nu(dx) \right)^{\frac{\theta}{2k}}. \quad (3.26)$$

Proof: Note that $2^{k_\theta} < \theta \leq 2^{k_\theta+1}$. It follows from the combination of the Burkholder-Davis-Gundy inequality and the more elementary inequality $(u+v)^a \leq 2^{a-1}(u^a + v^a)$, ($u, v \geq 0, a \geq 1$) that

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} |M_s^{(\underline{\theta})}|^\theta &\leq C_\theta \mathbb{E} \left(\sum_{0 \leq s \leq t} (\Delta X_s)^2 \mathbf{1}_{\{|\Delta X_s| \geq s^{\frac{1}{\underline{\theta}}}\}} \right)^{\frac{\theta}{2}} \\ &\leq C_\theta \left(\left(\int_0^t ds \int_{\{|x| \leq s^{\frac{1}{\underline{\theta}}}\}} x^2 \nu(dx) \right)^{\frac{\theta}{2}} + \mathbb{E} \left(\sum_{0 \leq s \leq t} (\Delta X_s)^2 \mathbf{1}_{\{|\Delta X_s| \leq s^{\frac{1}{\underline{\theta}}}\}} - \int_0^t ds \int_{\{|x| \leq s^{\frac{1}{\underline{\theta}}}\}} x^2 \nu(dx) \right)^{\frac{\theta}{2}} \right) \\ &\leq C_\theta \left(\left(\int_0^t ds \int_{\{|x| \leq s^{\frac{1}{\underline{\theta}}}\}} x^2 \nu(dx) \right)^{\frac{\theta}{2}} + \left(\int_0^t ds \int_{\{|x| \leq s^{\frac{1}{\underline{\theta}}}\}} x^{2^2} \nu(dx) \right)^{\frac{\theta}{2^2}} \right) \end{aligned}$$

where C_θ is a real constant depending on θ that may vary from line to line. Then introducing successively the martingales

$$N_t^k = \sum_{0 \leq s \leq t} (\Delta X_s)^{2k} \mathbf{1}_{\{|\Delta X_s| \geq s^{\frac{1}{\underline{\theta}}}\}} - \int_0^t ds \int_{\{|x| \leq s^{\frac{1}{\underline{\theta}}}\}} |x|^{2k} \nu(dx), \quad k < k_\theta,$$

and applying again the Burkholder-Davis-Gundy inequality finally yields

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} |M_s^{(\underline{\theta})}|^\theta &\leq C_\theta \sum_{k \geq 1, 2^k < \theta} \left(\int_0^t ds \int_{\{|x| \leq s^{\frac{1}{\underline{\theta}}}\}} x^{2k} \nu(dx) \right)^{\frac{\theta}{2k}} + C_\theta \mathbb{E} \left(\sum_{0 \leq s \leq t} (\Delta X_s)^{2^{k_\theta+1}} \mathbf{1}_{\{|\Delta X_s| \leq s^{\frac{1}{\underline{\theta}}}\}} \right)^{\frac{\theta}{2^{k_\theta+1}}} \\ &= C_\theta \sum_{k=1}^{k_\theta+1} \left(\int_0^t ds \int_{\{|x| \leq s^{\frac{1}{\underline{\theta}}}\}} x^{2k} \nu(dx) \right)^{\frac{\theta}{2k}}. \quad \diamond \end{aligned} \quad (3.27)$$

This lemma yields the following result for the martingale term.

Lemma 2 (a) *If the Lévy measure satisfies Assumption (3.16) then*

$$\forall r, p \in (0, +\infty), \quad e_{N,r}(M^{(\underline{\theta})}, L_T^p) = O((\log N)^{-\frac{1}{\underline{\theta}}}). \quad (3.28)$$

(b) *Assume that the function $u \mapsto \underline{\nu}(u) = \nu([-c, c] \setminus [-u, u])$ defined on $\mathbb{R}_+ \setminus \{0\}$ has regular variation with index $-b$. Then $b = \underline{\theta}$. If furthermore Assumption (3.18) holds then,*

$$\forall r, p \in (0, +\infty), \quad e_{N,r}(M^{(\underline{\theta})}, L_T^p) = O\left((\log N)^{-\frac{1}{\underline{\theta}}} \sqrt{\underline{\ell}((\log N)^{-1})}\right) \quad (3.29)$$

where $\underline{\ell}(t) := t \underline{\nu}(t^{\frac{1}{\underline{\theta}}})$ is a slowly varying function.

Proof: (a) When $\underline{\theta} > 0$, the result follows by inspecting all the integrals in (3.27). When $\underline{\theta} = 0$, one checks that the first three steps are still true by replacing $\underline{\theta}$ by a arbitrary same $\varepsilon > 0$.

(b) It follows from Theorem 1.4.1 in [2] that $\underline{\nu}(u) = u^{-b} \ell(u)$ where ℓ is a (non-negative) slowly varying function. Consequently by the Markov inequality, one has, for every $\theta > 0$ and every $u > 0$

$$u^{\theta-b} \ell(u) \leq \int_{\{|x|>u\}} |x|^\theta \nu(dx).$$

Assume $b \neq 0$. Now, the left hand side of the above inequality goes to ∞ as $u \rightarrow 0$ as soon as $\theta < b$ since ℓ has slow variations (see Proposition 1.3.6 in [2]). Consequently $\theta \leq \underline{\theta}$. Letting θ go to b implies that $b \leq \underline{\theta}$.

The rest of the proof makes use of the following easy equality which follows from the very definition of $\underline{\nu}$: for every non-negative Borel function f ,

$$\int_{\mathbb{R}} f(|x|) \nu(dx) = - \int_{\mathbb{R}_+} f(x) d\underline{\nu}(x). \quad (3.30)$$

In particular, for every $x \in (0, c]$ and every $a > 0$,

$$\int_{\{|u|\geq x\}} |u|^a \nu(du) = - \int_x^c u^a d\underline{\nu}(u)$$

Assume $b < \underline{\theta}$. Then, it follows from Theorem 1.6.4 in [2] that for every $a \in (b, \underline{\theta})$,

$$\int_x^c u^a d\underline{\nu}(u) \sim \frac{b}{b-a} x^a \underline{\nu}(x) = \frac{b}{b-a} x^{a-b} \ell(x) \rightarrow 0 \quad \text{as } x \rightarrow 0,$$

since ℓ is slowly varying. This contradicts $\int |u|^a \nu(du) = +\infty$. Consequently $b = \underline{\theta}$.

Now, Theorem 1.6.5 in [2] implies that for any $a > \underline{\theta}$

$$\int_{\{|u|\leq x\}} |u|^a \nu(du) = - \int_{(0,x]} u^a d\underline{\nu}(u) \sim \frac{\underline{\theta}}{a-\underline{\theta}} x^a \underline{\nu}(x) \quad \text{as } x \rightarrow 0.$$

Since $\underline{\theta} \neq 2$, one may apply that with $a = 2^k$, $k = 1, \dots, k_\theta + 1$. This yields yields

$$\int_{\{|x|\leq s^{\frac{1}{\underline{\theta}}}\}} x^{2^k} \nu(dx) \sim \frac{\underline{\theta}}{\underline{\theta}-2^k} s^{\frac{2^k}{\underline{\theta}}} \underline{\nu}(s^{\frac{1}{\underline{\theta}}}) \quad \text{as } s \rightarrow 0$$

which in turn implies that

$$\int_0^t ds \int_{\{|x|\leq s^{\frac{1}{\underline{\theta}}}\}} x^{2^k} \nu(dx) \sim \frac{\underline{\theta}}{\underline{\theta}-2^k} \int_0^t s^{\frac{2^k}{\underline{\theta}}} \underline{\nu}(s^{\frac{1}{\underline{\theta}}}) ds \quad \text{as } t \rightarrow 0.$$

The function $s \mapsto \underline{\nu}(s^{\frac{1}{\underline{\theta}}})$ has regular variation with index -1 , hence Theorem 1.6.1 in [2] implies that for every $k = 1, \dots, k_{\underline{\theta}} + 1$,

$$\int_0^t ds \int_{\{|x| \leq s^{\frac{1}{\underline{\theta}}}\}} x^{2k} \nu(dx) \sim C_{\underline{\theta}, \underline{\theta}, k} t^{\frac{2k}{\underline{\theta}} + 1} \underline{\nu}(t^{\frac{1}{\underline{\theta}}}) \quad \text{as } t \rightarrow 0.$$

Finally, still for every $k = 1, \dots, k_{\underline{\theta}} + 1$,

$$\left(\int_0^t ds \int_{\{|x| \leq s^{\frac{1}{\underline{\theta}}}\}} u^{2k} \nu(du) \right)^{\frac{\underline{\theta}}{2k}} \sim \bar{C}_{\underline{\theta}, \underline{\theta}, k} \left(t^{\frac{1}{\underline{\theta}}} (\underline{\ell}(t))^{\frac{1}{2k}} \right)^{\underline{\theta}} \quad \text{as } t \rightarrow 0. \quad (3.31)$$

If $\liminf_{t \rightarrow 0} t \underline{\nu}(t^{\frac{1}{\underline{\theta}}}) > 0$, then it follows from (3.31) that the dominating term in the right hand side of (3.26) is obtained with $k = 1$. Consequently for every $t \in [0, T]$,

$$\mathbb{E} \left(\sup_{0 \leq s \leq t} |M_s^{(\underline{\theta})}|^{\underline{\theta}} \right) \leq C_{T, \underline{\theta}, \underline{\theta}} \left(t^{\frac{1}{\underline{\theta}}} \sqrt{\underline{\ell}(t)} \right)^{\underline{\theta}}.$$

One concludes by applying Theorem 1 with the function with regular variation $\varphi(t) = t^{\frac{1}{\underline{\theta}}} \sqrt{\underline{\ell}(t)}$. \diamond

STEP 4 (*Quantization of \bar{X}^T*): First we introduce

$$\Psi(t) = \int_{\{|x| \leq c\}} (|x|^{\underline{\theta}} - t)_+ \nu(dx), \quad t > 0.$$

The function Ψ is non-negative, non-increasing and convex, $\Psi(0) = \int |x|^{\underline{\theta}} \nu(dx) \leq +\infty$, $\Psi(0) > 0$, $\Psi(t) < +\infty$ if $t > 0$ since $\Psi(t) \leq t^{-\frac{2}{\underline{\theta}} - 1} \int x^{2\nu}(dx) < +\infty$. It follows from (3.20) that $\Psi(t) = 0$ if $t \in [c^{\underline{\theta}}, +\infty)$ and $\Psi(t) > 0$ on $[0, c^{\underline{\theta}})$. Consequently Ψ is a decreasing homeomorphism from $[0, c^{\underline{\theta}}]$ onto $[0, \Psi(0)]$ (with the convention $\Psi^{-1}(0) = +\infty$ if $\Psi(0) = +\infty$).

An alternate expression for Ψ using identity (3.30) and an integration by parts can be derived:

$$\Psi(t) = \int_t^{c^{\underline{\theta}}} \underline{\nu}(s^{\frac{1}{\underline{\theta}}}) ds, \quad t \in [0, c^{\underline{\theta}}]$$

which shows that Ψ is a slowly varying function as $t \rightarrow 0$ since it is the antiderivative of a regularly varying function with index -1 .

Now, the compensate formula shows that, for every $t \in [0, T]$,

$$\mathbb{E} \left(\sum_{0 < s \leq t} \mathbf{1}_{\{(T-s)^{\frac{1}{\underline{\theta}}} < |\Delta X_s| \leq c\}} \right) = \int_{\{|x| \leq c\}} (|x|^{\underline{\theta}} - T + t)_+ \nu(dx) = \Psi(T - t) < +\infty.$$

Consequently, the jump times of the process $(\bar{X}_t^T)_t$ make up an increasing sequence converging toward T if $\Psi(0) = +\infty$ and \bar{X}^T admits the following representation

$$\bar{X}^T = \sum_{n \geq 1} U_n \mathbf{1}_{\{S_n \leq \Psi(T-t)\}}, \quad t \in [0, T]$$

where $(U_n)_{n \geq 1}$ is a sequences of independent random variables bounded by c and $S_n = Z_1 + \dots + Z_n$, $(Z_k)_{k \geq 1}$, i.i.d., $Z_1 \sim \text{Exp}(1)$. We will start from this representation to quantize \bar{X}^T by considering some appropriate quantizations of the random variables U_n and R_n . As concerns U_n we consider for every $n \geq 1$, a rate optimal $L^\infty(\mathbb{P})$ -quantization U_n of size N_n^U . As concerns S_n we first set for

every $n \geq 1$, $R_n := T \wedge \psi^{-1}(S_n \wedge \Psi(0))$. These random variables are $[0, T_c]$ -valued where we set $T_c := T \wedge c^\ell$ for notational convenience. Then, we set

$$\tilde{S}_n := \psi(\hat{R}_n), \quad n \geq 1,$$

where \hat{R}_n is a rate optimal $L^\infty(\mathbb{P})$ -quantization of R_n of size N_n^R , taking its values in $(0, T_c)$ so that \tilde{S}_n is a positive finite random variable. Finally, we define the (non Voronoi) quantization process \hat{X}^T of \bar{X}^T by

$$\hat{X}_t^T := \sum_{n \geq 1} \hat{U}_n \mathbf{1}_{\{\tilde{S}_n \leq \Psi(T-t)\}}, \quad t \in [0, T] \quad (3.32)$$

At this stage, note that Pierce lemma in the $L^\infty(\mathbb{P})$ -setting is trivial in the sense that for any random variable $Z : \Omega \rightarrow [m, M]$

$$e_{N, \infty}(Z, \mathbb{R}^d) = \inf_{\alpha \subset [m, M], \text{card}(\alpha) \leq N} \mathbb{P}\text{-esssup}_{\omega \in \Omega} |Z(\omega) - \hat{Z}^\alpha(\omega)| \leq \inf_{\alpha \subset [m, M], \text{card}(\alpha) \leq N} \sup_{|\zeta| \leq c} \min_{a \in \alpha} |\zeta - a| \leq \frac{M - m}{2N}. \quad (3.33)$$

This is obtained by considering $\alpha := \{m + (M - m)(2i - 1)/(2N), i = 1, \dots, N\}$ which produces a rate optimal (m, M) -valued quantization Z^α of Z .

It follows from (3.33) that, $\mathbb{P}(d\omega)$ -a.s. (in fact for every $\omega \in \Omega$),

$$|U_n(\omega) - \hat{U}_n(\omega)| \leq \frac{c}{N_n^U} \quad \text{and} \quad |R_n(\omega) - \hat{R}_n(\omega)| \leq \frac{T_c}{2N_n^R}.$$

It follows from Minkowski's inequality applied with the L_T^p norm that

$$|\hat{X}^T - \bar{X}^T|_{L_T^p} \leq \sum_{n \geq 1} |U_n - \hat{U}_n|_{L_T^p} \mathbf{1}_{\{S_n \leq \Psi(T-\cdot)\}} + \sum_{n \geq 1} |\hat{U}_n|_{L_T^p} |\mathbf{1}_{\{S_n \leq \Psi(T-\cdot)\}} - \mathbf{1}_{\{\tilde{S}_n \leq \Psi(T-\cdot)\}}|_{L_T^p}$$

Now, using the invariance of the Lebesgue measure by $s \mapsto T - s$, that $\Psi(t) < \Psi(0)$ for every $t \in (0, T]$, $\Psi(t) = 0$ on $[c^\ell, T]$ and $\Psi^{-1} \circ \Psi = Id_{[0, c^\ell]}$, one has on the event $\{S_n > 0\}$,

$$\begin{aligned} |\mathbf{1}_{\{S_n \leq \Psi(T-\cdot)\}}|_{L_T^p}^p &= |\mathbf{1}_{\{S_n \wedge \Psi(0) \leq \Psi(T-\cdot)\}}|_{L_T^p}^p = \int_0^T \mathbf{1}_{\{S_n \wedge \Psi(0) \leq \Psi(t)\}} dt \\ &= \int_0^{T_c} \mathbf{1}_{\{S_n \wedge \Psi(0) \leq \Psi(t)\}} dt \\ &= \int_0^{T_c} \mathbf{1}_{\{\Psi^{-1}(S_n \wedge \Psi(0)) \geq t\}} dt \\ &= T \wedge \Psi^{-1}(S_n \wedge \Psi(0)) \\ &= R_n. \end{aligned}$$

Similar arguments show that on the event $\{S_n > 0\}$,

$$\begin{aligned} |\mathbf{1}_{\{S_n \leq \Psi(T-\cdot)\}} - \mathbf{1}_{\{\tilde{S}_n \leq \Psi(T-\cdot)\}}|_{L_T^p}^p &= |T \wedge \Psi^{-1}(S_n \wedge \Psi(0)) - (T \wedge \Psi^{-1}(\tilde{S}_n \wedge \Psi(0)))| \\ &= |R_n - \tilde{R}_n|. \end{aligned}$$

This finally yields

$$|\hat{X}^T - \bar{X}^T|_{L_T^p} \leq c \sum_{n \geq 1} \frac{|R_n|^{\frac{1}{p}}}{N_n^U} + \sum_{n \geq 1} |R_n - \hat{R}_n|^{\frac{1}{p}} \quad \mathbb{P}\text{-a.s.}$$

Consequently, for every $r \geq 1$ (setting $r' = r/p$)

$$\| |\widehat{X}^T - \bar{X}^T |_{L_T^p} \|_r \leq c \sum_{n \geq 1} \frac{\|R_n\|_{r'}^{\frac{1}{p}}}{N_n^U} + \sum_{n \geq 1} \|R_n - \widehat{R}_n\|_{r'}^{\frac{1}{p}}. \quad (3.34)$$

Now, using that $\{\Psi^{-1}(S_n \wedge \Psi(0)) \geq s\} = \{S_n \leq \Psi(T-s)\}$ and the expression for the p.d.f. of the $\gamma(n)$ distribution, one easily checks that

$$\begin{aligned} \mathbb{E} R_n^{r'} &= r' \int_0^{T_c} \mathbb{P}(R_n \geq s) s^{r'-1} ds \\ &= r' \int_0^{T_c} \mathbb{P}(\Psi^{-1}(S_n \wedge \Psi(0)) \geq s) s^{r'-1} ds \\ &\leq r' \int_0^T \mathbb{P}(S_n \wedge \Psi(0) \leq \Psi(s)) s^{r'-1} ds \\ &= r' \int_0^T \mathbb{P}(S_n \leq \Psi(s)) s^{r'-1} ds \\ &= \frac{r'}{(n-1)!} \int_0^T s^{r'-1} \int_0^{\Psi(s)} u^{n-1} e^{-u} du ds. \end{aligned}$$

Let $\varepsilon > 0$. Since the function Ψ is slowly varying, there is a real constant $C_\varepsilon > 0$ such that

$$\Psi(t) \leq \kappa_\varepsilon t^{-\varepsilon}, \quad t \in (0, T].$$

As a consequence, as soon as $n \geq \frac{r'}{\varepsilon}$,

$$\begin{aligned} \mathbb{E} R_n^{r'} &\leq \frac{r'}{(n-1)!} \int_0^T s^{r'-1} \int_0^{\kappa_\varepsilon t^{-\varepsilon}} u^{n-1} e^{-u} du ds \\ &= \frac{r'}{(n-1)!} \int_0^{+\infty} u^{n-1} e^{-u} \int_0^{(\frac{\kappa_\varepsilon}{u})^{\frac{1}{\varepsilon}} \wedge T} s^{r'-1} ds du \\ &\leq \frac{r'}{(n-1)!} \kappa_\varepsilon^{\frac{r'}{\varepsilon}} \int_0^{+\infty} u^{n-1-\frac{r'}{\varepsilon}} e^{-u} du \\ &\leq \kappa_\varepsilon^{\frac{r'}{\varepsilon}} \frac{\Gamma(n - \frac{r'}{\varepsilon})}{\Gamma(n)} \\ &\leq \tilde{\kappa}_{\varepsilon, r'} n^{-\frac{r'}{\varepsilon}}. \end{aligned} \quad (3.35)$$

One can change the real constant $\tilde{\kappa}_{\varepsilon, r'}$ so that (3.35) holds true for every $n \geq 1$. On the other hand, $|R_n| \leq T_c \leq T$, consequently the L^∞ version of the Pierce Lemma implies the existence of a positive real constant $K_{\varepsilon, r', T}$ such that

$$\|R_n\|_{r'} \leq K_{\varepsilon, r', T} n^{-\frac{1}{\varepsilon}}, \quad n \geq 1.$$

As concerns $\|R_n - \widehat{R}_n\|_{r'}$, it follows again from the above computations, the Pierce Lemma and (3.35) applied with $r' + \delta$ that

$$\|R_n - \widehat{R}_n\|_{r'} \leq C \|R_n\|_{r'+\delta} \frac{1}{N_n^S} \leq K_{\varepsilon, r'+\delta, T} \frac{1}{N_n^U R_n^{\frac{1}{\varepsilon}}}, \quad n \geq 1.$$

STEP 3 Optimization: Using (3.34) and (3.25), the N -level quantization problem (3.24) for the process $X^{(\ell)}$ amounts to considering the following two problems:

$$(U) \equiv \min \left\{ \sum_{n=1}^{m_U} \frac{1}{N_n^U n^{\frac{1}{\varepsilon p}}} + \sum_{n \geq m_U+1} \frac{1}{n^{\frac{1}{\varepsilon p}}}, m'_1 \geq 1, N_1^U \times \dots \times N_{m_U}^U \leq N'_1 \right\}$$

and

$$(R) \equiv \min \left\{ \sum_{n=1}^{m_R} \frac{1}{(N_n^R n^{\frac{1}{\varepsilon}})^{\frac{1}{p}}} + \sum_{n \geq m_R+1} \sum_{n \geq m_U+1} \frac{1}{n^{\varepsilon p}}, m_1'' \geq 1, N_1^R \times \dots \times N_{m_R}^R \leq N_1'' \right\}.$$

with the constraint $N_1' \times N_1'' \times N_2 \leq N$.

The usual optimal allocation procedure (see [14, 15] or Section 2) leads to the rates

$$C_{U,\varepsilon,r',p}(\log(N_1'))^{1-\frac{1}{\varepsilon p}} \quad \text{and} \quad C_{R,\varepsilon,r'+\delta,p}(\log(N_1''))^{1-\frac{1}{\varepsilon p}}$$

for problems (U) and (R) respectively. Then setting $N_1' = N_1'' = N_2 = \lceil N^{1/3} \rceil$ yields, for every $N \geq 1$,

$$e_{N,r}(\tilde{X}^{(\underline{\ell})}, L_T^p) \leq C_{R,\varepsilon,r,\delta,p}((\log(N))^{1-\frac{1}{\varepsilon p}} + N^{-\frac{1}{3}}).$$

Since this holds true for arbitrary $\varepsilon > 0$, it follows that, for every $\eta > 0$,

$$e_{N,r}(\tilde{X}^{(\underline{\ell})}, L_T^p) = o((\log N)^{-\frac{1}{\eta}}) \quad \text{as } N \rightarrow \infty. \quad (3.36)$$

STEP 5 (*Quantization of X*): Combining the result obtained for the martingale $M^{(\underline{\ell})}$ in Lemma 2 and the above rate (3.36) for $X^{(\underline{\ell})}$ finally yields

$$e_{N,r}(\tilde{X}^{(\underline{\ell})}, L_T^p) = o\left((\log N)^{-\frac{1}{2}} \sqrt{\underline{\ell}(\log N)^{-1}}\right) \quad \text{as } N \rightarrow \infty. \quad \diamond$$

Remark. In fact we made absolutely no use if the independence of (U_n) et (S_n) .

3.1.1 Compound Poisson process

In this section we consider a compound Poisson process $(X_t)_t$ defined by

$$X_t := \sum_{n \geq 1} U_n \mathbf{1}_{\{S_n \leq \lambda T\}}, \quad t \geq 0.$$

where $S_n = Z_1 + \dots + Z_n$, $(Z_n)_{n \geq 1}$ is an i.i.d. sequence of $\text{Exp}(1)$ distributed random variables, $(U_n)_{n \geq 1}$ is an i.i.d. sequence of random variables, independent of $(Z_n)_{n \geq 1}$ with $U_1 \in L^\rho$, $\rho > 0$ and $\lambda > 0$ is the the jump intensity. For convenience we also introduce the underlying standard Poisson process $(N_t)_{t \geq 0}$ defined by

$$N_t := \sum_{n \geq 1} \mathbf{1}_{\{S_n \leq \lambda T\}}, \quad t \geq 0.$$

so that (with the convention that $\sum_\emptyset = 0$)

$$X_t = \sum_{k=1}^{N_t} U_k. \quad (3.37)$$

Proposition 4 *Let X be a compound Poisson process. Then for every $p, r \in (0, r^*)$, $p \leq r$,*

$$\forall \varepsilon > 0, \quad e_{N,r}(X, L_T^p) = O\left(\exp\left(-\frac{1}{\sqrt{r(p+1+\varepsilon)}} \sqrt{\log(N) \log_2(N)}\right)\right). \quad (3.38)$$

Furthermore when X is a standard Poisson process, then, one van replace $p+1+\varepsilon$ by $p+\varepsilon$ in (3.38).

Remark. In fact the rate obtained in the above proposition holds as soon as X has the form (3.37) where (Z_n) is as above and (U_n) is $L^r(\mathbb{P})$ -bounded for every $r < r^*$, independent of $(U_n)_{n \geq 1}$.

Proof. We divide the proof in two steps, one is devoted to the standard Poisson process, the second one to the general case. We will assume that $r^* > 1$ throughout the proof so that, so that as it was already emphasized in the proof of Theorem 1 we may assume without loss of generality that $r, p \in (0, r^*) \cap [1, +\infty)$. The case $r^* \leq 1$ is left to the reader but can be treated by replacing the “triangular” Minkowski inequality by the pseudo-triangular inequalities $|f + g|_{L_T^p}^p \leq |f|_{L_T^p}^p + |g|_{L_T^p}^p$ and $\|X + Y\|_r^r \leq \|X\|_r^r + \|Y\|_r^r$.

STEP 1: One quantizes the standard Poisson N in a very natural way by setting

$$\widehat{N}_t := \sum_{n \geq 1} \mathbf{1}_{\{\widehat{S}_n \leq \lambda t\}}, \quad t \geq 0,$$

with

$$\widehat{S}_n := \widehat{S}_n^{\alpha_n},$$

where $\alpha_n = \alpha'_n \cup \{\lambda T\}$, α'_n is an $L^{r'}$ -optimal $(N_n - 1)$ -quantization of $S_n^{tr} := S_n \mathbf{1}_{\{S_n \leq \lambda T\}}$ and $r' = \frac{r}{p}$. Furthermore, one assume that the sequence (N_n) is non-increasing and satisfies $\prod_n N_n \leq N$ (so that $N_n = 1$ for large enough n). Then, for every $p \geq 1$, it follows from the (extended) Minkowski inequality

$$|N - \widehat{N}|_{L_T^p} \leq \sum_{n \geq 1} |\mathbf{1}_{\{S_n \leq \lambda\}} - \mathbf{1}_{\{\widehat{S}_n \leq \lambda\}}|_{L_T^p}.$$

Now

$$\begin{aligned} |\mathbf{1}_{\{S_n \leq \lambda t\}} - \mathbf{1}_{\{\widehat{S}_n \leq \lambda t\}}|_{L_T^p}^p &= \int_0^T |\mathbf{1}_{\{S_n \leq \lambda t\}} - \mathbf{1}_{\{\widehat{S}_n \leq \lambda t\}}|^p dt \\ &= \frac{1}{\lambda} |S_n \wedge (\lambda T) - \widehat{S}_n \wedge (\lambda T)| = \frac{1}{\lambda} |S_n \wedge (\lambda T) - \widehat{S}_n|. \end{aligned}$$

Now $\{S_n > \lambda T\} \subset \{\widehat{S}_n = \lambda T\}$ since $\max \alpha_n = \lambda T$ so that and $\widehat{S}_n = \lambda T$ on $\{S_n > \lambda T\}$. On the other hand $S_n = S_n^{tr}$ on $\{S_n \leq \lambda T\}$ so that

$$|S_n \wedge (\lambda T) - \widehat{S}_n| = |S_n \wedge (\lambda T) - \widehat{S}_n \mathbf{1}_{\{S_n \leq \lambda T\}}| = |S_n^{tr} - \widehat{S}_n^{tr} \mathbf{1}_{\{S_n \leq \lambda T\}}| \leq |S_n^{tr} - \widehat{S}_n^{tr}|.$$

Also note that when $N_n = 1$, $\widehat{S}_n = \lambda T$ so that $|S_n \wedge (\lambda T) - \widehat{S}_n| = (\lambda T - S_n)_+$. Consequently, for every $r \geq 1$,

$$\begin{aligned} \| |N - \widehat{N}|_{L_T^p} \|_r &\leq \sum_{n \geq 1} \| |\mathbf{1}_{\{S_n \leq \lambda\}} - \mathbf{1}_{\{\widehat{S}_n \leq \lambda\}}|_{L_T^p} \|_r \\ &\leq \frac{1}{\lambda^{\frac{1}{p}}} \sum_{n \geq 1} \| |S_n \wedge (\lambda T) - \widehat{S}_n|_{r'}^{\frac{1}{p}} \\ &= \frac{1}{\lambda^{\frac{1}{p}}} \left(\sum_{n, N_n \geq 2} \| |S_n^{tr} - \widehat{S}_n^{tr \alpha_n}|_{r'}^{\frac{1}{p}} + \sum_{n, N_n = 1} \| |(\lambda T - S_n)_+|_{r'}^{\frac{1}{p}} \right) \\ &\leq \frac{1}{\lambda^{\frac{1}{p}}} \left(\sum_{n, N_n \geq 2} \| |S_n^{tr} - \widehat{S}_n^{tr \alpha'_n}|_{r'}^{\frac{1}{p}} + \sum_{n, N_n = 1} \| |(\lambda T - S_n)_+|_{r'}^{\frac{1}{p}} \right). \end{aligned}$$

The extended Pierce Lemma 1 yields that, for every $n \geq 1$ such that $N_n \geq 2$,

$$\begin{aligned} \| |S_n^{tr} - \widehat{S}_n^{tr \alpha'_n}|_{r'} \|_r &\leq \| |S_n^{tr}|_{r'+\delta/p} \|_{r,p,\delta} C_{r,p,\delta} |N_n - 1|^{-1} \\ &\leq 2 \| |S_n \mathbf{1}_{\{S_n \leq \lambda T\}} \|_{r+\delta/p} C_{r,p,\delta} N_n^{-1}. \end{aligned}$$

Set $\mu := r' + \delta/p = \frac{r+\delta}{p}$ so that $\mu p = r + \delta$.

$$\begin{aligned} \| |N - \widehat{N}|_{L_T^p} \|_r &\leq C_{p,r,\delta} \frac{1}{\lambda^{\frac{1}{p}}} \left(\sum_{n, N_n \geq 2} \|S_n \mathbf{1}_{\{S_n \leq \lambda T\}}\|_{\mu}^{\frac{1}{p}} \frac{1}{N_n^{\frac{1}{p}}} + \sum_{n, N_n = 1} \|(\lambda T - S_n)_+\|_{\mu}^{\frac{1}{p}} \right) \\ &\leq C_{p,r,\delta} T^{\frac{1}{p}} \left(\sum_n (\mathbb{P}(S_n \leq \lambda T))^{\frac{1}{\mu p}} \frac{1}{N_n^{\frac{1}{p}}} \right). \end{aligned}$$

Now, standard computations show that

$$\mathbb{P}(\{S_n \leq \lambda T\}) = \frac{(\lambda T)^n}{(n-1)!} \int_0^1 u^{n-1} e^{-\lambda T u} du \leq \frac{(\lambda T)^n}{n!}$$

so that

$$(\mathbb{P}(S_n \leq \lambda T))^{\frac{1}{\mu p}} \leq \frac{(\lambda T)^{\frac{n}{\mu p}}}{(n!)^{\frac{1}{\mu p}}} \leq \frac{A^n}{(n!)^{\frac{1}{\mu p}}}$$

where $A = (\lambda T)^{\frac{1}{\mu p}} \vee 1$. Finally, this leads to the following optimal integral ‘‘bit allocation’’ problem

$$\min \left\{ \sum_{n \geq 1} \frac{A^n}{(n!)^{\frac{1}{\mu p}} N_n^{\frac{1}{p}}}, N_n \geq 1, \prod_n N_n \leq N \right\}. \quad (3.39)$$

Temporarily set $a_n = \frac{A^n}{(n!)^{\frac{1}{\mu p}}}$. This sequence is non-increasing for $n \geq n_0$ and converges to 0.

Consequently one may assume furthermore that for every $n \geq n_0$, $a_n = \min_{1 \leq k \leq n} a_k$. Then let $m \geq n_0$ be an integer. We set

$$N_n = \left\lceil \frac{a_n^p N^{\frac{1}{m}}}{(\prod_{1 \leq k \leq m} a_k)^{\frac{p}{m}}} \right\rceil, \quad 1 \leq n \leq m, \quad N_n = 1, \quad n \geq m+1.$$

The sequence $N_n, 1 \leq n \leq m$, is nonincreasing too for $n \geq n_0$. We wish to choose $m \geq n_0$ as a function of N so that

$$a_m N^{\frac{1}{pm}} \geq \left(\prod_{1 \leq k \leq m} a_k \right)^{\frac{1}{m}}. \quad (3.40)$$

This will ensure that

$$N_n \geq N_m \geq 1, \quad 1 \leq n \leq m.$$

Inequality (3.40) reads, using that $\prod_{1 \leq k \leq m} A^k = A^{m(m+1)/2}$ and taking log, to

$$\frac{m-1}{2} \log A + \frac{1}{\mu m} \log N \geq \frac{1}{\mu p} \left(\log(m!) - \frac{1}{m} \sum_{k=1}^m \log(k!) \right).$$

We will make use of the following classical expansions

$$\log(\Gamma(t+1)) = m(\log t - 1) + \frac{1}{2} \log t + \log(\sqrt{2\pi}) + \frac{1}{12t} + O(t^{-2}) \quad \text{as } t \rightarrow \infty \quad (3.41)$$

$$\text{and } \frac{1}{m} \sum_{k=1}^m \log(k!) = m \left(\frac{1}{2} \log m - \frac{3}{4} \right) + \log m + \log(\sqrt{2\pi}) - 1 + O(\log(m)/m) \quad \text{as } m \rightarrow \infty \quad (3.42)$$

Then, the above inequality reads

$$\frac{m-1}{2} \log A + \frac{1}{\mu m} \log N \geq \frac{1}{\mu p} \left(\frac{m}{2} \log m - \frac{m}{4} - \frac{1}{2} \log(m) + 1 + O(\log(m)/m) \right).$$

If one sets (this is probably not optimal)

$$m = \left\lceil 2\sqrt{\mu \frac{\log N}{\log_2 N}} \right\rceil$$

then the above inequality is satisfied, at least for large enough N . With N_n and m settled as above and using that $\frac{x}{\lceil x \rceil} \leq 2$ for every $x \geq 1$, one gets

$$\sum_{n \geq 1} \frac{A^n}{(n!)^{\frac{1}{\mu p}} N_n^{\frac{1}{p}}} \leq 2^{\frac{1}{p}} \frac{m \left(\prod_{1 \leq k \leq m} a_k \right)^{\frac{1}{m}}}{N^{\frac{1}{pm}}} + \sum_{n \geq m+1} \frac{A^n}{(n!)^{\frac{1}{\mu p}}}$$

It follows from (3.41) that for every $h \geq 1$ and every $t \geq 1$,

$$\begin{aligned} \Gamma(t+1) &= C_h h^{\frac{t}{h}} \Gamma\left(\frac{t}{h} + 1\right) (1 + o(1)) \\ &\geq C_h h^{\frac{t}{h}} \Gamma\left(\frac{t}{h} + 1\right) \end{aligned}$$

where C_h is a real constant only depending on h . One derives that if $B = A \times (\mu p)^{-\frac{1}{\mu p}}$ there is a real constant C depending on $\lambda, T, p, r, \delta, \dots$, (that may vary from line to line) such that

$$\begin{aligned} \sum_{n \geq m+1} \frac{A^n}{(n!)^{\frac{1}{\mu p}}} &\leq C \sum_{n \geq m+1} \frac{B^n}{\Gamma\left(\frac{n}{\mu p} + 1\right)} \\ &\leq C \frac{B^m}{\Gamma\left(\frac{m}{\mu p} + 1\right)} \\ &\leq C \exp\left(-\frac{m}{\mu p}(\log m - \log B_0) - \frac{1}{2} \log m\right) \end{aligned}$$

so that finally

$$\sum_{n \geq m+1} \frac{A^n}{(n!)^{\frac{1}{\mu p}}} \leq C \exp\left(-\frac{1}{p\mu} \sqrt{\mu \log N \log_2 N} \left(1 + O\left(\frac{\log_3 N}{\log_2 N}\right)\right)\right) \quad (3.43)$$

Let us pass now to the first term: using the second expansion in (3.41) yields

$$\begin{aligned} \frac{m \left(\prod_{1 \leq k \leq m} a_k \right)^{\frac{1}{m}}}{N^{\frac{1}{pm}}} &\leq C \frac{mA^{\frac{m+1}{2}} \exp\left(-\frac{1}{\mu p} \frac{1}{m} \sum_{1 \leq k \leq m} \log k\right)}{N^{\frac{1}{mp}}} \\ &\leq C \exp\left(-\frac{1}{2\mu p} m(\log m - \frac{3}{2}) + \frac{m}{2} \log A + \left(1 - \frac{1}{\mu p}\right) \log m - \frac{1}{pm} \log N\right) \\ &\leq C \exp\left(-\frac{1}{p\mu} \sqrt{\mu \log N \log_2 N} \left(1 + O\left(\frac{\log_3 N}{\log_2 N}\right)\right)\right) \end{aligned} \quad (3.44)$$

Note that $p\sqrt{\mu} = \sqrt{p \cdot p\mu} = \sqrt{p(r + \delta)}$. Finally this yields in particular that for every $\varepsilon > 0$,

$$\| |N - \hat{N}|_{L_T^p} \|_r = O\left(\exp\left(-\frac{1}{\sqrt{pr + \varepsilon}} \sqrt{\log N \log_2 N}\right)\right).$$

STEP 2 (*General case*): Starting from Equation (3.37), it is natural to quantize X by setting

$$\hat{X}_t = \sum_{k=1}^{\hat{N}_t} \hat{U}_k.$$

where \widehat{N} is a $N^{(1)}$ -quantization of the standard Poisson process N as described in Step 1 and, for every $n \geq 1$, \widehat{U}_n is an L^r -optimal $N_n^{(2)}$ -quantization of U_n with $1 \leq N^{(2)} \times \dots \times N_n^{(2)} \dots \leq N^{(2)}$ and $N^{(1)}N^{(2)} \leq N$. Then, setting $\widehat{N}_t^U := \sum_{k=1}^{\widehat{N}_t} U_k$ and $N_t^{\widehat{U}} := \sum_{k=1}^{N_t} \widehat{U}_k$, one gets

$$\begin{aligned} |N_t^{\widehat{U}} - \widehat{N}_t^U|_{L_T^p} &\leq \sum_{n \geq 1} |\widehat{U}_n| |\mathbf{1}_{\{S_n \leq \lambda T\}} - \mathbf{1}_{\{\widehat{S}_n \leq \lambda T\}}|_{L_T^p} \\ \text{so that } \|X - \widehat{N}^U\|_{L_T^p} &\leq \frac{1}{\lambda^{\frac{1}{p}}} \sum_{n \geq 1} \|\widehat{U}_n\|_r \|S_n \wedge (\lambda T) - \widehat{S}_n \wedge (\lambda T)\|_{\frac{p}{r}} \\ &= \frac{\sup_{n \geq 1} \|\widehat{U}_n\|_r}{\lambda^{\frac{1}{p}}} \sum_{n \geq 1} \|S_n \wedge (\lambda T) - \widehat{S}_n \wedge (\lambda T)\|_{\frac{p}{r}} \end{aligned}$$

where we used that the sequences (U_n) and (S_n) being independent, so are (\widehat{U}_n) and (S_n) . Using

$$\|\widehat{U}_n\|_r \leq \|U_n - \widehat{U}_n\|_r + \|U_1\|_r$$

shows that $\sup_{n \geq 1} \|\widehat{U}_n\|_r < +\infty$. Hence, it follows from Step 1 that, for every $c < \frac{1}{\sqrt{pr}}$,

$$\|X - \widehat{N}^U\|_{L_T^p} = O\left(\exp(-c\sqrt{\log(N^{(1)}) \log_2(N^{(1)})})\right).$$

On the other hand, with obvious notations, and using that $(\widehat{U}_n - U_n)$ and (S_n) are independent

$$\begin{aligned} \|X - N^{\widehat{U}}\|_{L_T^p} &= \|N^{U - \widehat{U}}\|_{L_T^p} \\ &\leq \sum_{n \geq 1} \|U_n - \widehat{U}_n\|_r \|\mathbf{1}_{\{S_n \leq \lambda\}}\|_{L_T^p} \\ &= \frac{1}{\lambda^{\frac{1}{p}}} \sum_{n \geq 1} \|U_n - \widehat{U}_n\|_r \|(\lambda T - S_n)_+\|_{\frac{p}{r}} \\ &\leq \frac{1}{\lambda^{\frac{1}{p}}} \sum_{n \geq 1} \|U_n - \widehat{U}_n\|_r \frac{(\lambda T)^{\frac{1}{p} + \frac{n}{r}}}{(n!)^{\frac{1}{r}}} \\ &\leq C \sum_{n \geq 1} \|U_n - \widehat{U}_n\|_r \frac{(\lambda T)^{\frac{n}{r}}}{(n!)^{\frac{1}{r}}}. \end{aligned}$$

Now, it follows from the (extended) Pierce Theorem that

$$\begin{aligned} \|N^U - N^{\widehat{U}}\|_{L_T^p} &\leq C_{U_1, r} \sum_{n \geq 1} \frac{(\lambda T)^{\frac{n}{r}}}{(n!)^{\frac{1}{r}} N_n^{(2)}} \\ &= O\left(\exp\left(-\frac{1}{\sqrt{r}} \sqrt{\log(N^{(2)}) \log_2(N^{(2)})}\right)\right). \end{aligned}$$

The rate follows from the resolution of the optimal bit allocation problem (3.39) obtained by setting formally $\mu p = r$ and $p = 1$. Then, note that, on the one hand

$$\|X - \widehat{X}\|_{L_T^p} \leq \|X - N^{\widehat{U}}\|_{L_T^p} + \|N^{\widehat{U}} - \widehat{N}^U\|_{L_T^p}$$

and on the other hand

$$\widehat{N}_t^U = \sum_{n \geq 1} \widehat{U}_n^{N_n^{(2)}} \mathbf{1}_{\{\widehat{S}_n^{N_n^{(1)}} \leq \lambda t\}}$$

can take at most

$$\prod_{n \geq 1} N_n^{(1)} N_n^{(2)} \leq N^{(1)} \times N^{(2)} \leq N$$

values. Let $c < \frac{1}{\sqrt{pr}}$. Setting $N^{(1)} = \lfloor N^{\frac{rc^2}{1+rc^2}} \rfloor$, $N^{(2)} = \lfloor N^{\frac{1}{1+rc^2}} \rfloor$ yields a rate

$$\| |X - \widehat{X}|_{L_T^p} \|_r = O \left(\exp \left(-\frac{1}{\sqrt{1/c^2 + r}} \sqrt{\log(N) \log_2(N)} \right) \right)$$

i.e.

$$\forall \varepsilon > 0, \quad \| |X - \widehat{X}|_{L_T^p} \|_r = O \left(\exp \left(-\frac{1}{\sqrt{r(p+1+\varepsilon)}} \sqrt{\log(N) \log_2(N)} \right) \right). \quad \diamond$$

3.2 Proof of Theorem 2

Any Lévy process X can be decomposed as the sum of two (independent) Lévy processes, one having bounded jumps and one being a compound Poisson process – according to the decomposition of its Lévy measure

$$\nu(dx) = \mathbf{1}_{\{|x| \leq 1\}} \nu(dx) + \mathbf{1}_{\{|x| > 1\}} \nu(dx).$$

The rate $O((\log N)^{-\frac{1}{2}})$ is given by the “small jump” component whereas the maximal integrability level r^* is given by the moment of the “big jump” (Poisson) component of the Lévy measure. This completes the proof of Theorem 2. \diamond

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