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# On Mukai flops for Scorza varieties

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## Abstract

I give three descriptions of the Mukai flop of type  $E_{6,I}$ , one in terms of Jordan algebras, one in terms of projective geometry over the octonions, and one in terms of  $\mathbb{O}$ -blow-ups. Each description shows that it is very similar to certain flops of type  $A$ . The Mukai flop of type  $E_{6,II}$  is also described.

## Introduction

In this article, I study a class of birational transformations called ‘‘Mukai flops’’. Let  $G/P$  be a flag variety. Recall [Ric 74] that the natural map  $T^*G/P \rightarrow \mathfrak{g}^*$ , where  $\mathfrak{g}$  is the Lie algebra of  $G$ , has image the closure of a single nilpotent orbit.

Sometimes, it happens that for two parabolic subgroups  $P, Q \subset G$ , the images in  $\mathfrak{g}^*$  of  $T^*G/P$  and  $T^*G/Q$  are equal to the same orbit closure  $\overline{\mathcal{O}}$ , and that moreover, the above maps are birational isomorphisms. We therefore get a birational map

$$\begin{array}{ccc} T^*G/P & \dashrightarrow & T^*G/Q, \\ & \searrow & \swarrow \\ & \overline{\mathcal{O}} & \end{array}$$

called a Mukai flop.

Since  $T^*G/P$  is a symplectic variety, nilpotent orbit closures provide a wide class of examples of symplectic singularities and were studied also for this reason. If  $\overline{\mathcal{O}}$  is a nilpotent orbit closure, then B. Fu showed that any symplectic resolution of  $\overline{\mathcal{O}}$  is given by a map  $T^*G/P \rightarrow \overline{\mathcal{O}}$  [Fu 03]. On the other hand, in [Nam 04], it is proved that any Mukai flop can be described using fundamental ones, when  $P$  (and  $Q$ ) is a maximal parabolic subgroup :  $G$  is then of type  $A, D_{2n+1}$  or  $E_6$ . In some sense, this provides a complete understanding of the different symplectic resolutions of  $\overline{\mathcal{O}}$  and the relations between them.

In fact, the classical fundamental flops, when  $G$  is of type  $A_n$  or  $D_{2n+1}$ , are easy to describe. The only items which are not very well understood in this matter are the fundamental Mukai flops of type  $E_6$ , and the purpose of this article is to fill this gap.

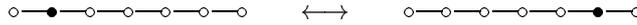
Along with this motivation in birational geometry, these flops are key ingredients for the definition of generalized dual varieties [Cha 06] for a subvariety of the homogeneous space  $G/P$ , when  $G$  is of type  $E_6$  and  $P$  is the parabolic subgroup corresponding to the root  $\alpha_1$  or  $\alpha_3$ , with Bourbaki’s notations [Bou 68].

For example, an easy consequence of theorem 3.3 is theorem 2.1 in [Ch 06], which generalizes the fact that the dual variety of the smooth quadric in  $\mathbb{P}V$  defined by an invertible symmetric map  $f : V \rightarrow V^*$  is the quadric in  $\mathbb{P}V^*$  defined by  $f^{-1}$ , when the usual projective space  $\mathbb{P}V$  is replaced by any Scorza variety (see subsection 3.1 for the definition of Scorza varieties; for example, a grassmannian of 2-dimensional subspaces of an even-dimensional fixed space, and  $E_6/P_1, P_1$  the parabolic subgroup of the adjoint group of type  $E_6$  corresponding to the root  $\alpha_1$ , are Scorza varieties).

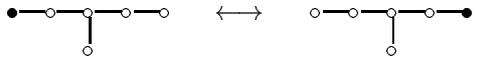
Finally, a third motivation is the study of the geometry of exceptional homogeneous spaces. For example, subsection 3.5 starts a study of the geometric properties of  $E_6/P_3$ , with a rather detailed description of its tangent bundle.

There are two flops of type  $E_6$ , denoted  $E_{6,I}$  (then  $P$  corresponds to the root  $\alpha_1$  and  $Q$  to  $\alpha_6$ ) and  $E_{6,II}$  ( $P = P_2, Q = P_5$ ). I give three descriptions of the flop  $E_{6,I}$ : one via the geometry of the corresponding flag variety, one using Jordan algebras, and one using a new class of birational transformations that I call  $\mathbb{O}$ -blow-ups.

In fact, these three constructions work uniformly for  $G/P$  any Scorza variety. This gives for example a common description of the flop



and the flop



This allows to understand better the latter.

I now describe more precisely the contents of this article. Let  $k$  be a field and let  $x \in \mathbb{P}_k^n$ . Then a non-vanishing tangent vector  $t \in T_x X$  defines a unique line  $l$  with the following properties :

- $x \in l$ ,
- $t \in T_x l$ .

Moreover, the rational map  $t \mapsto l$  is clearly the quotient map  $T_x X \simeq \mathbb{A}_k^n \dashrightarrow \mathbb{P}_k^{n-1}$ , where  $\mathbb{P}_k^{n-1}$  denotes the variety of lines in  $\mathbb{P}_k^n$  through  $x$ . Dually, we have a similar rational map  $T_x^* X \dashrightarrow (\mathbb{P}_k^{n-1})^\vee$ .

Section 3 is devoted to proving the same kind of results when the variety  $\mathbb{P}_k^n$  is replaced by a Scorza variety (see subsection 3.1), which after [Cha 05] is considered as a projective space  $\mathbb{P}_{\mathcal{A}}^n$  over a composition algebra  $\mathcal{A}$ , so that when  $\mathcal{A} = k$ , we recover  $\mathbb{P}_k^n$ . So, in this section, I show theorems 3.2 and 3.3, which have the following interpretation in terms of projective geometry over  $\mathcal{A}$ : given a generalized projective space  $\mathbb{P}_{\mathcal{A}}^n$  and a point  $x \in \mathbb{P}_{\mathcal{A}}^n$ , there is a rational quadratic map  $\overline{\nu}_x^+ : T_x \mathbb{P}_{\mathcal{A}}^n \dashrightarrow \mathbb{P}_{\mathcal{A}}^{n-1}$ , which maps a tangent vector to the unique  $\mathcal{A}$ -line through it. Dually, there is a similar map  $\overline{\nu}_x^- : T_x^* \mathbb{P}_{\mathcal{A}}^n \dashrightarrow (\mathbb{P}_{\mathcal{A}}^{n-1})^\vee$ .

Propositions 3.4 and 3.5 show that polarizing  $\overline{\nu}_x^+$  (resp.  $\overline{\nu}_x^-$ ), one gets an isomorphism between the variety of lines in  $\mathbb{P}_{\mathcal{A}}^n$  through  $x$  and the Fano variety

of maximal linear subspaces included in  $\mathbb{P}_{\mathcal{A}}^{n-1}$  (resp.  $(\mathbb{P}_{\mathcal{A}}^{n-1})^{\vee}$ ). These two results don't have analogs when  $\mathcal{A} = k$ .

Note that this last  $(\mathbb{P}_{\mathcal{A}}^{n-1})^{\vee}$  is the projective space of hyperplanes containing  $x$ ; it is therefore included in  $(\mathbb{P}_{\mathcal{A}}^n)^{\vee}$ . The connection with Mukai flops is as follows : assume that  $G/P$  is the Scorza variety  $\mathbb{P}_{\mathcal{A}}^n$ . Let  $x \in G/P$ . We will see that there is a Mukai flop  $T^*\mathbb{P}_{\mathcal{A}}^n \dashrightarrow T^*(\mathbb{P}_{\mathcal{A}}^n)^{\vee}$ . The structure map  $T^*G/Q \rightarrow G/Q$  and this Mukai flop yield a composition  $T_x^*\mathbb{P}_{\mathcal{A}}^n = T_x^*G/P \dashrightarrow T^*G/Q \rightarrow G/Q = (\mathbb{P}_{\mathcal{A}}^n)^{\vee}$ . Theorem 3.3 shows that this composition is the map  $\nu_x^- : T_x^*\mathbb{P}_{\mathcal{A}}^n \dashrightarrow (\mathbb{P}_{\mathcal{A}}^{n-1})^{\vee} \subset (\mathbb{P}_{\mathcal{A}}^n)^{\vee}$ .

Then, I show a general canonical isomorphism of quotients of tangent spaces to homogeneous spaces (theorem 4.1). As a particular case, this theorem gives a way of computing a Mukai flop  $T^*G/P \dashrightarrow T^*G/Q$  once we know the composition  $T^*G/P \dashrightarrow G/Q$ . I deduce a description of the flop of type  $E_{6,I}$  (proposition 4.1), in terms of Jordan algebras.

In subsection 4.3, I give a maybe more geometric description of the flop  $E_{6,I}$ . Recall that the minimal resolution of the simplest Mukai flop  $T^*\mathbb{P}^n \dashrightarrow T^*(\mathbb{P}^n)^{\vee}$  is the blow-up of  $T^*\mathbb{P}^n$  along the zero section. I show that the same result holds for the  $E_{6,I}$ -flop  $T^*\mathbb{P}_{\mathbb{O}}^2 \dashrightarrow T^*(\mathbb{P}_{\mathbb{O}}^2)^{\vee}$ , if one replaces the usual notion of blow-up with an octonionic version of it (theorem 4.2).

Finally, concerning the Mukai flop of type  $E_{6,II}$ , I use the fact that the homogeneous space  $E_6/P_3$  can be realized as the space of lines included in  $E_6/P_1$ . Theorem 4.3 uses this model and the study of the tangent bundle  $T(E_6/P_3)$  performed in subsection 3.5 to give also a description of the Mukai flop of type  $E_{6,II}$ .

Sections 1 and 2 study the restriction of the flops to a cotangent space in the two cases when  $G$  is of type  $E_6$ . They are of course  $L$ -equivariant rational maps, if  $L$  is a Levi factor of  $P$ , and happen to be quite subtil. In each case, I show that they are the only  $P$ -equivariant rational map  $T_x^*G/P \dashrightarrow (G/Q)_x$  (propositions 1.5 and 2.1), if  $(G/Q)_x$  denotes the variety of  $y$ 's in  $G/Q$  with stabilizer  $Q_y$  such that  $P_x \cap Q_y$  is parabolic.

In the case of a flop of type  $E_{6,I}$  for instance, we get a  $Spin_{10}$ -equivariant rational map;  $G/P$  is often called the Moufang plane  $\mathbb{P}_{\mathbb{O}}^2$  (it is some kind of octonionic projective plane). As an example of the above discussion, the restriction of the flop to a cotangent space should interpret as the "quotient map"  $\mathbb{A}_{\mathbb{O}}^2 \dashrightarrow \mathbb{P}_{\mathbb{O}}^1$ . In the first section, I show that this map has some properties of such a quotient; for example, its fibers carry a natural structure of algebra isomorphic with the octonions (corollary 1.12). I also study the projective geometry of the corresponding spinor variety.

Similarly, section 2 gives a model for the restriction of the Mukai flop of type  $E_{6,II}$  to a cotangent space. In this case, a Levi factor contains  $SL_2 \times SL_5$  and the relevant factor of  $T^*E_6/P_3$  is  $Hom(\mathbb{C}^2, (\wedge^2 \mathbb{C}^5)^*)$ . The given classification of  $(GL_2 \times GL_5)$ -orbits in  $Hom(\mathbb{C}^2, (\wedge^2 \mathbb{C}^5)^*)$  allows to understand the  $E_6$ -orbits in  $T^*E_6/P_3$ . Finally, corollary 4.3 states that the Mukai flop is defined only on the open orbit of  $T^*E_6/P_3$ , and describes the image of all orbits in  $T^*E_6/P_3$  as nilpotent orbits in  $\mathfrak{e}_6$ .

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## 1 The group $Spin_{10}$ and $\mathbb{P}_{\mathbb{O}}^1$ .

### 1.1 Geometric definition of composition algebras

For more details on composition algebras, the reader may consult [Cha 05]. I recall that if  $R$  is a ring, then  $\mathbb{R}_R, \mathbb{C}_R, \mathbb{H}_R$  and  $\mathbb{O}_R$  denote the four usual split composition algebras over  $R$ . Therefore,  $\mathbb{R}_R = R, \mathbb{C}_R = R \oplus R, \mathbb{H}_R$  is the algebra of  $2 \times 2$ -matrices with coefficients in  $R$ , and  $\mathbb{O}_R$  is obtained from  $\mathbb{H}_R$  by Cayley-Dickson's process.

Their norms will be denoted  $N$ . If  $\mathcal{A}$  is one of those and  $z \in \mathcal{A}$ , then  $L_z$  and  $R_z$  denote the endomorphisms of  $\mathcal{A}$  of left and right multiplication by  $z$ , and  $L(z), R(z) \subset \mathcal{A}$  their images.

In the following, we will have to define a composition algebra structure on a vector space by geometric means. This subsection explains how it is possible. In this section,  $k$  is an algebraically closed field of characteristic different from 2.

**Proposition 1.1.** *Let  $V$  be a  $k$ -vector space of dimension  $a$ , with  $a \in \{1, 2, 4\}$ . Let  $x_0 \in V - \{0\}$  and  $\mathcal{N} \subset \mathbb{P}V$  be a smooth quadric such that the class of  $x_0$  in  $\mathbb{P}V$  does not belong to  $\mathcal{N}$ .*

*Then if  $a \in \{1, 2\}$ , there exists a unique composition algebra structure on  $V$  with unit  $x_0$  and such that  $\mathcal{N}$  is the quadric of elements with vanishing norm. If  $a = 4$ , there are two such composition algebras.*

Therefore, giving a composition algebra structure on  $V$  is equivalent with giving a smooth quadric in  $\mathbb{P}V$  and a point out of its affine cone (if  $a = 4$ , we must moreover choose a component of the variety of maximal isotropic subspaces of the quadric).

**Proof:** The existence of the algebra is an immediate consequence of the fact that  $\text{Aut}(\mathcal{N})$  acts transitively on  $\mathbb{P}V - \mathcal{N}$ .

The unicity in the cases when  $a \in \{1, 2\}$  is easy. Assume  $a = 4$  and let  $(x, y) \mapsto xy$  be a composition product satisfying the conditions of the proposition. Let  $\mathcal{L}, \mathcal{R} \simeq \mathbb{P}^1$  be the two families of isotropic lines. For  $x \in \mathcal{Q}$ , denote  $l(x)$  (resp.  $r(x)$ ) the isotropic line in  $\mathcal{L}$  (resp.  $\mathcal{R}$ ) containing  $x$ . Up to changing the algebra structure  $(x, y) \mapsto xy$  into  $(x, y) \mapsto yx$ , we may assume that  $\forall x \in \mathcal{N}, L(x) \in \mathcal{L}$ . Therefore,  $L(x) = l(x)$  and  $R(x) = r(x)$ .

If  $z \in V$ , let  $[z]$  denote its class in  $\mathbb{P}V$ . Then, for generic  $x, y \in \mathcal{N}$ ,  $[xy] = L(x) \cap R(y) = l(x) \cap r(y)$ . Therefore, the product of two elements of  $\mathcal{N}$  is fixed up to a scale once  $\mathcal{N}$  is. One checks also that  $xy = 0$  if and only if  $r(x) \cap l(y)$  is orthogonal to the unit, with respect to the scalar product defined by  $\mathcal{N}$ . In view of lemma 1.1 applied to left multiplication by  $x \in \mathcal{N}$ , the proposition is proved.  $\square$

**Lemma 1.1.** *Let  $V$  and  $W$  be vector spaces and  $f, g : V \rightarrow W$  linear maps. Let  $X \subset \mathbb{P}V$  be an irreducible variety included in no hyperplane of  $\mathbb{P}V$ . Assume that the induced rational maps  $[f]_{|X}, [g]_{|X} : X \dashrightarrow \mathbb{P}W$  are equal and that  $\ker f = \ker g$ . Then there exists  $\lambda \in k - \{0\} : f = \lambda g$ .*

**Proof:**  $f$  and  $g$  have the same image, spanned by  $f(X) = g(X)$ . They also have the same kernel by hypothesis. Therefore, there is a linear automorphism  $h$  of this common image, such that  $g = hf$ . Since  $[f]_{|X} = [g]_{|X}$ , any vector in  $f(X)$  is an eigenvector for  $h$ , from which the lemma follows.  $\square$

We now consider the case of the octonions.

**Proposition 1.2.** *Let  $V$  be an 8-dimensional vector space and  $\mathcal{N} \subset \mathbb{P}V$  a smooth quadric. Let  $G$  denote the grassmannian of maximal isotropic subspaces of  $\mathcal{N}$ , and let  $l$  be an isomorphism between  $\mathcal{N}$  and an irreducible component of  $G$ . Assume  $\forall x \in \mathcal{N}, x \in l(x)$  and let  $x_0 \in V - \{0\}$  such that  $[x_0] \notin \mathcal{N}$ .*

*Then there exists a unique composition algebra structure on  $V$  with unit  $x_0$  and such that for all  $x \in \mathcal{N}$ , we have  $l(x) = L(x)$ .*

**Proof:** Given an octonionic structure on  $V$ , it is known as ‘‘triality principle’’ [Che 97, chapter IV] that  $L$  is an isomorphism on its image, which is a connected component of  $G$ .

The unicity of the algebra structure follows the lines of the previous proposition. Let  $(x, y) \mapsto xy$  be an algebra structure on  $V$  with unit  $x_0$  and such that  $L = l$ . If  $x \in \mathcal{N}$  is generic, then the line  $(x, x_0)$  meets  $\mathcal{N}$  at  $x$  and  $\bar{x}$ . Therefore,  $x_0$  determines the conjugation. By hypothesis,  $L(x) = l(x)$ , therefore we get  $R(x)$  as  $\overline{l(\bar{x})}$ . Now, the class of the product  $xy$  in  $\mathbb{P}V$  is again  $L(x) \cap R(y)$ , and  $xy = 0$  if and only if  $\dim L(x) \cap R(x) = 3$ , as is well-known [Che 97, IV.4.2]. Therefore, lemma 1.1 proves the unicity of the algebra.

Let us prove its existence. Put on  $V$  an arbitrary structure of composition algebra  $(x, y) \mapsto xy$  such that the quadric of elements of vanishing norm is  $\mathcal{N}$ . This induces isomorphisms  $L, R$  between  $\mathcal{N}$  and the components of  $G$ . Set  $r(x) = \overline{l(\bar{x})}$ . We can assume that  $L$  and  $l$  have the same image. Therefore,

there exist  $f, g \in \text{Aut}(\mathcal{N})$  such that  $l(x) = L(f(x))$  and  $r(x) = R(g(x))$ . The hypothesis  $x \in l(x)$  implies  $x \in r(x)$ , and so  $f(x) \in L(x)$  and  $g(x) \in R(x)$  [Cha 05, proposition 1.1]. By the following lemma 1.2, there exist invertible  $\alpha, \beta$  such that  $f(x) = x\alpha$  and  $g(y) = \beta y$ . The composition algebra  $x * y = (x\alpha)(\beta y)$ , with unit  $\beta^{-1}\alpha^{-1}$ , satisfies the conditions of the proposition.  $\square$

**Lemma 1.2.** *Let  $m : \mathbb{O}_k \rightarrow \mathbb{O}_k$  a linear map preserving  $\mathcal{N}$  and such that  $\forall x \in \mathcal{N}, m(x) \in L(x)$ . Then there exists  $\alpha \in \mathbb{O}_k$  such that  $\forall x \in \mathbb{O}_k, m(x) = x\alpha$ .*

**Proof :** Left to the reader [Cha 03, p.48].  $\square$

## 1.2 The 8-dimensional quadric as $\mathbb{P}_{\mathbb{O}}^1$

I have just recalled the triality principle, which implies that the three 8-dimensional fundamental representations of  $Spin_8$  can be identified with the algebra of octonions. The goal of this subsection is to relate the group  $Spin_{10}$  with the octonions, see proposition 1.4. To study the representations of  $Spin_{10}$ , my strategy is to restrict them to representations of  $Spin_8$ . Before proving proposition 1.4, I need to make a computation in Clifford algebras. My notations are those of [Che 97].

Let  $V$  be a  $k$ -vector space of even dimension and equipped with a non-degenerate quadratic form  $q$ . Let  $V' \subset V$  be a codimension two subspace in  $V$  such that  $q|_{V'}$  is non-degenerate. Let  $C, C'$  denote the Clifford algebra of  $V, V'$  (the Clifford algebra of  $V$  is the tensor algebra of  $V$  mod out by the relations  $x \otimes x = q(x)$ ). Let  $\alpha$  be the ‘‘main antiautomorphism’’ of  $V$ , defined by  $\alpha(v_1 \dots v_k) = v_k \dots v_1$ .

Let  $V' = N' \oplus P'$  be a decomposition into isotropic subspaces. Let  $x_0, y_0 \in V$  be orthogonal to  $V'$  and such that  $q(x_0, y_0) = 1$ . Denote  $N = N' \oplus k.x_0$  and  $P = P' \oplus k.y_0$ .

Let  $C_N \subset C$  (resp.  $C'_N \subset C'$ ) be the subalgebra of  $C$  (resp.  $C'$ ) generated by  $N$  (resp.  $N'$ ). Let  $f' \in C'_N$  be the product of the elements of a basis of  $N'$  and  $f = f'y_0$ . Let  $\mathbb{S}^\pm$  and  $\mathbb{S}^{\pm'}$  be the spin representations of  $Spin(V)$  and  $Spin(V')$ . We may choose  $\mathbb{S}^+$  (resp.  $\mathbb{S}^-$ ) be the subspace of even (resp. odd) elements of  $C_N$ , and similarly for  $\mathbb{S}^{\pm'}$ .

There are isomorphisms  $\varphi^\pm$  between  $\mathbb{S}'^+ \oplus \mathbb{S}'^- = C'_N$  and  $\mathbb{S}^\pm = C_N^\pm$ , given by  $\varphi^+(u'_+ + u'_-) = u'_+ + u'_-x_0$  and  $\varphi^-(u'_+ + u'_-) = u'_+x_0 + u'_-$ . Finally, there is a quadratic map  $\beta : C_N \times C_N \rightarrow \wedge V$ , where  $\beta(u, v)$  is the image of  $u\alpha(v) \in C$  in  $\wedge V$  under the canonical vector space isomorphism  $C \simeq \wedge V$  [Che 97, p.102,103 and II 1.6]. Let  $\beta' : C'_N \times C'_N \rightarrow \wedge V'$  be the similar map for  $V'$ .

**Proposition 1.3.** *Let  $r' = \dim V'/2$ . Let  $u'_+, v'_+ \in \mathbb{S}'^+$  and let  $u'_-, v'_- \in \mathbb{S}'^-$ . We have*

$$\begin{aligned} & \beta[\varphi^+(u'_+ + u'_-), \varphi^+(v'_+ + v'_-)] \\ = & \beta'(u'_+, v'_+) \wedge y_0 - x_0 \wedge y_0 \wedge \beta'(u'_+, v'_-) + \beta'(u'_+, v'_-) \\ + & (-1)^{r'}(x_0 \wedge y_0 \wedge \beta'(u'_-, v'_+) + \beta'(u'_-, v'_+)) - x_0 \wedge \beta'(u'_-, v'_-), \end{aligned}$$

and

$$\begin{aligned} & \beta[\varphi^-(u'_+ + u'_-), \varphi^-(v'_+ + v'_-)] \\ = & x_0 \wedge \beta'(u'_+, v'_+) + (-1)^{r'}(x_0 \wedge y_0 \wedge \beta'(u'_+, v'_-) + \beta'(u'_+, v'_-)) \\ + & y_0 \wedge x_0 \wedge \beta'(u'_-, v'_+) + \beta'(u'_-, v'_+) - \beta'(u'_-, v'_-) \wedge y_0. \end{aligned}$$

**Proof:** We have, in the Clifford algebra  $C$ ,  $u'_+ f' y_0 \alpha(v'_+) = u'_+ f' \alpha(v'_+) y_0$ , so  $\beta[\varphi^+(u'_+), \varphi^+(v'_+)] = \beta'(u'_+, v'_+) \wedge y_0$ . We can compute the other terms  $\beta[\varphi^+(u'_\pm), \varphi^+(v'_\pm)]$  using the facts

$$\begin{aligned} u'_+ f' y_0 \alpha(v'_- x_0) &= y_0 x_0 u'_+ f' \alpha(v'_-), & u'_- x_0 f' y_0 \alpha(v'_+) &= (-1)^{r'} x_0 y_0 u'_- f' \alpha(v'_+), \\ \text{and } u'_- x_0 f' y_0 \alpha(v'_-) &= x_0 y_0 x_0 u'_- f' \alpha(v'_-) = x_0 u'_- f' \alpha(v'_-). \end{aligned}$$

The computation of  $\beta[\varphi^-(u'_+ + u'_-), \varphi^-(v'_+ + v'_-)]$  is similar.  $\square$

Our second task is to describe spinor representations using octonions. Let  $V = H_2(\mathbb{O}_k)$  denote the 10-dimensional  $k$ -vector space of  $2 \times 2$  hermitian matrices with entries in  $\mathbb{O}_k$ . Let  $\det$  be the quadratic form on  $H_2(\mathbb{O}_k)$  defined by  $\det \left( \begin{pmatrix} t & z \\ \bar{z} & u \end{pmatrix} \right) = tu - N(z)$  ( $t, u \in k$  and  $z \in \mathbb{O}_k$ ). Recall [Che 97, III 1.2, III 1.4] that the variety of maximal isotropic subspaces of  $V$  has two components; they will be denoted  $G_Q^+(5, V)$  and  $G_Q^-(5, V)$ . Moreover, there are natural projective embeddings  $G_Q^\pm(5, V) \subset \mathbb{P}\mathbb{S}^\pm$  in the projectivized spinor representations, the elements of  $\mathbb{S}^\pm$  which class are in  $G_Q^\pm(5, V)$  being called ‘‘pure spinors’’.

Let  $\nu_2^+ : \mathbb{O}_k \times \mathbb{O}_k \rightarrow H_2(\mathbb{O}_k)$  the quadratic map defined by  $\nu_2^+(a, b) = \begin{pmatrix} N(a) & a\bar{b} \\ b\bar{a} & N(b) \end{pmatrix}$  and  $\mu^+$  the polarization of  $\nu^+$  :  $\mu^+((a, b), (c, d)) = \nu_2^+(a + c, b + d) - \nu_2^+(a, b) - \nu_2^+(c, d)$ . Similarly, let  $\nu_2^-(a, b) = \begin{pmatrix} N(b) & a\bar{b} \\ b\bar{a} & N(a) \end{pmatrix}$  and  $\mu^-$  the polarization of  $\nu_2^-$ .

Let  $X^\pm = X^\mp \subset \mathbb{P}(\mathbb{O}_k \oplus \mathbb{O}_k)$  be defined by  $[(a, b)] \in X^\pm \iff \nu_2^\pm(a, b) = 0$ .

**Proposition 1.4.** *The variety  $X^\pm$  is isomorphic with  $G_Q^\pm(5, V)$ . An isomorphism  $X^\pm \rightarrow G_Q^\pm(5, V)$  maps  $(u, v)$  on the image of  $\mu^\pm((u, v), \cdot)$ .*

**Proof:** Let  $q = -\det$ ,  $V' = \left\{ \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix} \right\} \simeq \mathbb{O}_k$ ,  $x_0 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$  and  $y_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ . Let  $\beta_k$  denote the component in  $\wedge^k V \subset \wedge V$  of  $\beta$ . Since  $q$  restricts to the norm of octonions on  $V' \simeq \mathbb{O}_k$ , by the triality principle [Che 97, Chapter IV], with the notations of proposition 1.3, there are linear isomorphisms  $\mathbb{S}'^\pm \rightarrow \mathbb{O}_k$  such that the map  $\beta'_1 : \mathbb{S}'^+ \times \mathbb{S}'^- \rightarrow V'$  identifies with the product of octonions, and  $(\beta'_0)^+ : \mathbb{S}^+ \times \mathbb{S}^+ \rightarrow k$ ,  $(\beta'_0)^- : \mathbb{S}^- \times \mathbb{S}^- \rightarrow k$  identify with the scalar product of octonions. Composing with the automorphism  $b \mapsto \bar{b}$ , of  $\mathbb{S}'^- \simeq \mathbb{O}_k$ , we may assume that  $\beta'_1$  is in fact given by  $(a, b) \mapsto a\bar{b}$ .

By proposition 1.3,  $\mathbb{S}^+$  and  $\mathbb{S}^-$  therefore identify with  $\mathbb{O}_k \oplus \mathbb{O}_k$  in such a way that  $\beta_1^+((a, b), (a, b)) = 2N(a)y_0 - 2N(b)x_0 + 2a\bar{b}$  and  $\beta_1^-((a, b), (a, b)) = -2N(a)x_0 + 2N(b)y_0 + 2a\bar{b}$ , that is to say,  $\beta_1^\pm = \mu^\pm$ .

By proposition [Che 97, III 5.2] the spinor varieties  $G_Q^\pm(5, V) \subset \mathbb{P}\mathbb{S}^\pm$  are defined by the equations  $N(a) = N(b) = 0, a\bar{b} = 0$ , which is equivalent to  $\nu_2^\pm = 0$ . Therefore, they are isomorphic with  $X^\pm$ . Moreover, since the linear space corresponding to  $s$  is the image of  $\mu^\pm(s, \cdot)$  [Che 97, III 4.4], the proposition is proved.  $\square$

In the sequel, we will identify both  $\mathbb{S}^+$  and  $\mathbb{S}^-$  with  $\mathbb{O}_k \oplus \mathbb{O}_k$ , keeping however in mind the fact that  $\mathbb{S}^+$  and  $\mathbb{S}^-$  are non-equivalent  $Spin_{10}$ -representations. The projectivization  $\mathbb{P}\mathbb{S}^\pm$  of  $\mathbb{S}^\pm$  have two  $Spin_{10}$ -orbits, by [Igu 70, prop. 2 p.1011]. The closed orbits are  $X^+$  and  $X^-$ .

Now comes the explanation of the title of this subsection : the variety of classes of matrices  $\left[ \begin{pmatrix} t & z \\ \bar{z} & u \end{pmatrix} \right] \in \mathbb{P}V$  with  $tu - N(z) = 0$  is a  $Spin_8$ -conformal compactification of the variety of classes of matrices of the form  $\left[ \begin{pmatrix} 1 & z \\ \bar{z} & N(z) \end{pmatrix} \right]$  which is isomorphic with  $\mathbb{O}_k \simeq \mathbb{A}_0^1$ , therefore, it can be thought as  $\mathbb{P}_0^1$ . Moreover, the projectivisations  $\overline{\nu_2^\pm} : \mathbb{S}^\pm \dashrightarrow \mathbb{P}\{\det = 0\}$  of the maps  $\nu_2^\pm$  are some kind of quotient maps  $\mathbb{A}_0^2 \dashrightarrow \mathbb{P}_0^1$ . Proposition 1.8 and corollary 1.12 illustrate this viewpoint.

For the moment, we show that  $\overline{\nu_2^+}$  and  $\overline{\nu_2^-}$  are the only natural (ie  $Spin_{10}$ -equivariant) candidates for such a kind of quotient (proposition 1.5). Let  $Q \subset \mathbb{P}V$  denote the quadric defined by  $\det$ .

**Lemma 1.3.** *There is a unique 15-dimensional  $Spin_{10}$ -orbit in  $(\mathbb{P}\mathbb{S}^+ - X^+) \times Q$ .*

**Proof :** Let  $(s_1, x_1), (s_2, x_2) \in (\mathbb{P}\mathbb{S}^+ - X^+) \times Q$ . We may assume that  $s_1 = s_2 = s$ . Let  $G_0 \subset Spin_{10}$  be the stabilizer of  $s$ . From the proof of [Igu 70, prop. 2 p.1011], it follows that  $\overline{\nu_2^+}(s) \in Q$  is the only line in  $Q$  stabilized by  $G_0$ . Therefore,  $x_1 = x_2 = \nu_2^+(s)$ .  $\square$

**Proposition 1.5.**  *$\overline{\nu_2^+} : \mathbb{S}^+ \dashrightarrow Q$  is the only  $(k^* \times Spin_{10})$ -equivariant rational map  $\mathbb{S}^+ \dashrightarrow Q$ .*

**Proof :** Let  $\nu : \mathbb{S}^+ \dashrightarrow Q$  be any  $(k^* \times Spin_{10})$ -equivariant rational map  $\mathbb{S}^+ \dashrightarrow Q$ . Then  $\overline{\nu_2^+}$  and  $\nu$  induce rational maps  $\mathbb{P}\mathbb{S}^+ \dashrightarrow Q$ , which will be denoted with the same letter. Since  $\nu$  is  $Spin_{10}$ -equivariant, it is defined on  $\mathbb{P}\mathbb{S}^+ - X^+$ . Therefore, the variety of  $\{(s, \nu(s)) : s \in \mathbb{P}\mathbb{S}^+ - X^+\}$  is a 15-dimensional orbit in  $(\mathbb{P}\mathbb{S}^+ - X^+) \times Q$ . By lemma 1.3, it is equal to the orbit  $\{(s, \overline{\nu_2^+}(s)) : s \in \mathbb{P}\mathbb{S}^+ - X^+\}$ .  $\square$

### 1.3 Projective geometry of the spinor variety

We keep the notations of the previous subsection; namely,  $V = H_2(\mathbb{O}_k)$ ,  $\mathbb{S}^+ = \mathbb{S}^- = \mathbb{O}_k \oplus \mathbb{O}_k$  are the two spinor representations of  $Spin_{10}$ , and  $\nu_2^\pm : \mathbb{S}^\pm \rightarrow V$  are the quadratic  $Spin_{10}$ -equivariant maps defined above. Their polarizations are denoted  $\mu^\pm$ . We denote  $Q \subset \mathbb{P}V$  the smooth quadric defined by  $\det$ . If  $(a, b) \in \mathbb{O}_k \oplus \mathbb{O}_k$  we denote  $[a, b]$  its class in  $\mathbb{P}(\mathbb{O}_k \oplus \mathbb{O}_k)$ . Finally, if  $X \subset \mathbb{P}^n$  is a variety and  $x \in X$ , let  $T_x X$  its tangent space and let  $\widehat{X} \subset \mathbb{A}_k^{n+1}$  denote the affine cone over  $X$ .

Recall from [Che 97, III 2.3] that there is a  $Spin_{10}$ -equivariant perfect pairing  $\mathbb{S}^+ \times \mathbb{S}^- \rightarrow k$ . This allows identifying  $\mathbb{S}^-$  with the dual of  $\mathbb{S}^+$ . Recall that the dual variety of a variety  $X$  is the closure of the set of tangent hyperplanes, where a tangent hyperplane is by definition a hyperplane containing a tangent space  $T_x X$  at a smooth point  $x \in X$ .

**Proposition 1.6.** *The equivariant isomorphism  $\mathbb{P}\mathbb{S}^{+*} \simeq \mathbb{P}\mathbb{S}^-$  identifies the dual variety of  $X^+$  with  $X^-$ .*

**Proof :** The dual variety of  $X^+$  is a  $Spin_{10}$ -stable closed variety. Since in  $\mathbb{P}S^-$  there are only two orbits, by [Igu 70, prop 2 p.1011], it is either the whole projective space  $\mathbb{P}S^-$ , which is absurd, or the variety  $X^-$ .  $\square$

If  $X \subset \mathbb{P}^n$  is a subvariety of projective space, and if  $z \in \mathbb{P}^n - X$ , the entry locus of  $z$  is classically defined as the closure of the set of points  $x \in X$  such that the line joining  $x$  and  $z$  meets  $X$  at at least two distinct points.

If  $s \in \mathbb{P}S^\pm - X^\pm$ , denote  $L_s^\pm$  the variety  $(\nu_2^\pm)^{-1}(k^* \cdot \nu_2^\pm(t)) \subset \mathbb{S}^\pm$ , where  $t \in \mathbb{S}^\pm$  is such that  $[t] = s$ . Let  $L^\pm$  denote the variety  $\{(s, v) \in (\mathbb{P}S^\pm - X^\pm) \times \mathbb{S}^\pm : v \in L_s^\pm\}$ . Finally, let  $\bar{\nu}_2^\pm : \mathbb{P}S^\pm - X^\pm \rightarrow Q$  denote the map induced by  $\nu_2^\pm : \mathbb{S}^\pm \rightarrow V$ .

**Proposition 1.7.** *Let  $s \in \mathbb{P}S^+ - X^+$ . Then the entry locus  $Q_s^+$  of  $s$  in  $X^+$  is a smooth 6-dimensional quadric in the 7-dimensional projective space  $\mathbb{P}L_s^+$ . Moreover, the fibration  $L \rightarrow \mathbb{P}S^+ - X^+$  is locally trivial and is the push-back by  $\bar{\nu}_2^+$  of a vector bundle on  $Q \subset \mathbb{P}V$ .*

**Remark :** The bundle over  $Q$  of the proposition is often called the spinor bundle.

**Proof :** Since  $\mathbb{P}S^+ - X^+$  is a single  $Spin_{10}$ -orbit, it is enough to check the first claim of the proposition for  $s = [1, 0]$ . Computing  $Q_s^+$  is equivalent with solving the equation  $(1, 0) = (a, x) + (b, y)$  in  $\mathbb{O}_k \oplus \mathbb{O}_k$ , with  $(a, x)$  and  $(b, y)$  in the affine cone over  $X^+$ . Equivalently,  $(a, x)$  satisfies  $N(a) = N(x) = 0$  and  $a\bar{x} = 0$ , and similarly for  $(b, y)$ .

Now, the equality  $a + b = 1$  implies  $N(a, b) \neq 0$  ( $N(\cdot, \cdot)$  denotes the polarization of  $N$ ). This, in turn, implies  $R(a) \cap R(b) = \{0\}$  [Che 97, IV 4.4]. Since  $a\bar{x} = 0$ ,  $x \in R(a)$  [Cha 05, proposition 1.1] and similarly  $y \in R(b)$ . Since  $x = -y$ , it follows that  $x \in R(a) \cap R(b)$ , so  $x = 0$ .

We thus have proved that the entry locus  $Q_s^+$  is included in the variety of elements  $[a, 0]$  with  $N(a) = 0$ . Conversely, this smooth quadric is included in  $Q_s^+$ . Since if  $N(a) \neq 0$ , then left multiplication by  $a$  is invertible, and a direct computation shows that  $L_s^+ = \mathbb{O}_k \oplus \{0\}$ .

To show that  $L^+$  is a vector bundle, let  $s \in \mathbb{S}^+$  and  $x = [\nu_2^+(s)] \in Q$ . Let  $L_x \subset V$  denote the line corresponding to  $x$ . First recall by definition that the image of the restriction of  $\nu_2^+$  to  $L_s^+$  is the line of multiples of  $\nu_2^+(s)$ . Therefore,  $\mu^+(s, L_s^+) = k \cdot \nu_2^+(s)$ . The linear space  $\mu^+(s, \mathbb{S}^+)$  is  $\widehat{T_x Q}$ , thus the kernel of the composition  $\mathbb{S}^+ \xrightarrow{\mu^+(s, \cdot)} \widehat{T_x X} \rightarrow \widehat{T_x X}/L_x$  is exactly  $L_s^+$ . Therefore,  $L^+$  is the kernel of a morphism of vector bundles over  $\mathbb{P}S^+ - X^+$  with constant rank; so, it is locally free.

Since  $L_s^+$  is constant on a fiber  $(\bar{\nu}_2^+)^{-1}(x)$ ,  $L^+$  is the push-back of a vector bundle on  $Q$ .  $\square$

We now study the family of quadrics  $\{Q_s^+\}$ . Let  $G(8, \mathbb{S}^+)$  denote the grassmannian of 8-dimensional linear spaces in  $\mathbb{S}^+$  and consider the variety  $\mathcal{Q} \subset G(8, \mathbb{S}^+)$  of 8-dimensional linear spaces  $L$  in  $\mathbb{S}^+$  such that  $X^+ \cap \mathbb{P}L$  is a smooth 6-dimensional quadric.

**Proposition 1.8.** *The variety  $\mathcal{Q}$  is  $Spin_{10}$ -equivariantly isomorphic with  $Q$  and any element of  $\mathcal{Q}$  is of the form  $Q_s$ . Moreover, let  $s, t \in \mathbb{P}S^+ - X^+$ ; one of the following holds :*

1.  $L_s^+ = L_t^+$  and  $Q_s = Q_t$ .
2.  $Q_s^+ \cap Q_t^+ = \mathbb{P}L_s^+ \cap \mathbb{P}L_t^+ \simeq \mathbb{P}^3$ .
3.  $\mathbb{P}L_s^+ \cap \mathbb{P}L_t^+ = \emptyset$ .

**Remark :** Although this does not make sense due to the lack of associativity of the octonions, the maps  $\nu_2^\pm : \mathbb{S}^\pm \dashrightarrow Q$  should be some kind of quotient maps  $\mathbb{A}_0^2 \dashrightarrow \mathbb{P}_0^1$ . The linear space  $L_s^+$  can be interpreted as the set of  $\mathbb{O}_k$ -multiples of  $s$  (in  $\mathbb{O}_k \oplus \mathbb{O}_k$ ). With this point of view, the proposition says that for two non-degenerate (out of  $X^+$ ) vectors in  $\mathbb{O}_k \oplus \mathbb{O}_k$ , there are three possibilities : either they are linked (1), either they are free (3), either they are “weakly linked” (2). This last case would not occur if we would consider non-split octonions, say for example over the field  $\mathbb{R}$  of real numbers. The same situation holds when one considers two non-degenerate vectors  $v, w \in \mathbb{H}_k \oplus \mathbb{H}_k$ . In fact, it is easy to check that

$$\dim (\{\lambda.v : \lambda \in \mathbb{H}_k\} \cap \{\lambda.w : \lambda \in \mathbb{H}_k\}) \in \{0, 2, 4\}.$$

(see the remark after lemma 2.1 in [Cha 05]).

**Proof :** Proposition 1.7 and the functorial property of grassmannians show that there is a map  $\psi : Q \rightarrow Q$ . In the other way, let  $l \in Q$ . Let  $\delta$  be a generic line in  $\mathbb{P}l$ ; this line meets the quadric  $X^+ \cap \mathbb{P}l$  in two points  $x$  and  $y$ . Since  $\nu_2^+$  vanishes on  $\widehat{X}^+$ , for any  $s$  in  $\delta$ , we have  $\overline{\nu}_2^+(s) = \overline{\mu}^+(x, y)$ . Therefore,  $\overline{\nu}_2^+$  is constant on the generic lines in  $\mathbb{P}l$ , so it is constant on  $\mathbb{P}l$ . This proves that there is a map  $\varphi : Q \rightarrow Q$ , induced by  $\nu_2^+$ .

It is obvious, by construction, that  $\varphi$  and  $\psi$  are inverse maps, so the first point of the proposition is proved.

The rest of the proposition follows. In fact, set  $s = (1, 0)$ , so that  $L_s^+ = \mathbb{O}_k \oplus 0$ . If  $t = s$ , then  $L_s^+ = L_t^+$ . If  $t = (1, b)$ , with  $N(b) = 0$ , then an easy computation shows that  $\mathbb{P}L_s^+ \cap \mathbb{P}L_t^+ = Q_s^+ \cap Q_t^+ = \{(c, 0) : c \in R(b)\}$ . If  $t = (0, 1)$ , then  $L_t^+ = 0 \oplus \mathbb{O}_k$  and so  $\mathbb{P}L_s^+ \cap \mathbb{P}L_t^+ = \emptyset$ .

Since there are three  $Spin_{10}$ -orbits in  $Q \times Q$ , these three examples exhaust all the possibilities for a couple  $(L_s^+, L_t^+) \in Q \times Q$ .  $\square$

Let  $s \in \mathbb{P}\mathbb{S}^+ - X^+$ . Define  $Q_s^-$  as the intersection of  $X^-$  with the orthogonal of  $L_s^+$  (in other words,  $Q_s^- \subset (X^+)^*$  is the variety of tangent hyperplanes which contain  $L_s^+$ ).

**Proposition 1.9.** *With notations above,  $Q_s^-$  is a 6-dimensional smooth quadric in  $X^-$ . Moreover, its linear span in  $\mathbb{S}^-$  is the closure of  $(\nu_2^-)^{-1}(k^*. \nu_2^+(s)) =: L_s^-$ .*

**Proof :** Arguing as in the proof of proposition 1.4 one can show that the equivariant duality between  $\mathbb{S}^+$  and  $\mathbb{S}^-$  is  $\langle (a, b), (c, d) \rangle = N(a, c) + N(b, d)$ . Therefore, if  $s = [1, 0]$ , then  $Q_s^-$  is the variety  $\{[0, b] : N(b) = 0\}$ . Its linear span is  $0 \oplus \mathbb{O}_k$ , which is sent by  $\nu_2^-$  on  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \nu_2^+(s)$ .  $\square$

Let  $\varphi^\pm$  denote the isomorphisms between  $X^\pm$  and the components of the grassmannian of maximal isotropic subspaces in  $Q$ . We have another characterisation of the quadrics  $Q_s^+$  and  $Q_s^-$  :

**Proposition 1.10.** *Let  $x \in X^\pm$  and  $s \in \mathbb{S}^+$ . Then  $x \in Q_s^\pm$  if and only if  $\nu_2^\pm(s) \in \varphi^\pm(x)$ .*

**Proof:** By proposition 1.7, there exists a quadratic form  $q_s$  on  $L_s^+$ , which zero locus is  $Q_s^+$ , and such that  $\forall u \in L_s^+, \nu_2^+(u) = q_s(u)\nu_2^+(s)$ . Let  $x \in \widehat{Q}_s^+ \subset L_s^+$ , then for  $u \in L_s^+$ , we have  $\mu^+(x, u) = q_s(x, u)\nu_2^+(s)$ . Therefore,  $\nu_2^+(s) \in \{\mu^+(x, u) : u \in \mathbb{S}^+\} = \varphi^+(x)$ .

The converse implication  $\nu_2^+(s) \in \varphi^+(x) \Rightarrow x \in Q_s^+$  follows by a dimension count argument.

In view of proposition 1.9, the proof of the same result for  $Q_s^-$  is similar.  $\square$

#### 1.4 Equivariant octonionic structure on the fibers on $\nu_2^\pm$

In a honest projective space  $\mathbb{P}_k^n$ , over a field  $k$ , the choice of an element  $v \in \mathbb{A}_k^{n+1}$  identifies the closure of the fiber of the quotient map  $\mathbb{A}_k^{n+1} \dashrightarrow \mathbb{P}_k^n$  with  $k$ , since any element in this fiber can uniquely be written as  $\lambda.v$ , with  $\lambda \in k$ . Therefore, this fiber carries the structure of a field, isomorphic with  $k$ .

We will see (corollary 1.12) in this subsection something analogous for  $\nu_2^\pm$ , which is interpreted as a quotient map. However, let  $s \in \mathbb{S}^+$ ; the image of the stabilizer of  $s$  in  $GL(L_s^+)$  contains  $Spin_7$  by [Igu 70, prop 2 p.1011]. Therefore, there is no hope to give  $L_s^+$  an equivariant octonionic structure.

I will show that given two generic spinors  $s, t \in \mathbb{S}^+$ , there are equivariant octonionic structures on  $L_s^+$  and  $L_t^+$  (and indeed the stabilizer of two elements has a quotient isomorphic with  $G_2$ ). I don't know how to interpret the necessity of two spinors to define such a structure in terms of octonionic projective geometry.

Let  $s, t \in \mathbb{P}\mathbb{S}^+ - X^+$  such that  $\langle \nu_2^+(s), \nu_2^+(t) \rangle \neq 0$ . The idea of the geometric definition of an octonionic structure on  $L_s^+$  is as follows : we have the two quadrics  $Q_s^+$  and  $Q_s^-$ . Let  $Q_s$  denote variety of lines in  $Q$  containing  $[\nu_2^+(s)]$ . Then  $Q_s$  is isomorphic with a 6-dimensional quadric. By proposition 1.10,  $Q_s^+$  and  $Q_s^-$  parametrize the maximal isotropic linear spaces of  $Q_s$ . The point is to show that  $s$  and  $t$  yield an isomorphism  $Q_s^+ \xrightarrow{\sim} Q_s^-$ . Then, proposition 1.2 gives the octonionic structure.

The next proposition yields the isomorphism  $\psi : Q_s^+ \xrightarrow{\sim} Q_s^-$ . Let  $x \in Q_s^+$  be such that the line through  $x$  and  $s$  is not a tangent line to  $Q_s^+$ . Call  $\bar{x}$  the other point of intersection of this line with  $Q_s$ . Moreover, set  $r(x) = \langle T_x X^+, L_s^+ \rangle^\perp \subset \mathbb{P}\mathbb{S}^{+*} = \mathbb{P}\mathbb{S}^-$ .

**Proposition 1.11.** *For all  $x \in Q_s^+$ ,  $r(x)$  is a maximal isotropic subspace of  $Q_s^-$ . Moreover, if  $(x, s)$  is not a tangent line to  $Q_s$ , then  $r(x)$  and  $r(\bar{x})$  are supplementary subspaces of  $L_s^-$ . Call  $\psi(x)$  the image of  $t$  by the projection on  $r(x)$  with center  $r(\bar{x})$ . Then  $\psi : Q_s^+ \rightarrow Q_s^-$  is an isomorphism.*

**Proof:** Assume  $s = [1, 0]$  and  $t = [0, 1]$ . Let  $x = [a, 0] \in Q_s^+$  (therefore  $N(a) = 0$ ). Since  $X^+$  is the variety of pairs  $[a, b]$  with  $N(a) = N(b) = 0$  and  $a\bar{b} = 0$ ,  $T_x X^+ = \{[c, d] : N(a, c) = 0 \text{ and } a\bar{d} = 0\}$ . Therefore, its orthogonal is the set of  $[c, d]$  with  $c$  colinear with  $a$  and  $d \in R(c)$ . So  $r(x) = \{[0, d] : d \in R(a)\}$ . This is indeed a maximal isotropic subspace of  $Q_s^-$ .

Moreover, we have  $\bar{x} = [\bar{a}, 0]$ , and so  $r(\bar{x}) = \{[0, d] : d \in R(\bar{a})\}$ . Therefore,  $r(x)$  and  $r(\bar{x})$  are supplementary.

Finally, since  $t = [0, 1] = [0, (a + \bar{a})/2]$ , we deduce that  $\psi(x) = [0, a]$ . We have proved that  $\psi([a, 0]) = [0, a]$ , so  $\psi$  is an isomorphism.  $\square$

**Corollary 1.12.** *Let  $s, t \in \mathbb{P}\mathbb{S}^+ - X^+$  such that  $\langle \nu_2^+(s), \nu_2^+(t) \rangle \neq 0$ . Then  $L_s^+$  has a natural structure of algebra, isomorphic with  $\mathbb{O}_k$ .*

*Moreover, when  $(s, t)$  vary, this octonionic structure on the vector bundle with fiber  $L_s^+$  varies algebraically.*

**Proof :** We have isomorphisms  $\psi^\pm$  between  $Q_s^\pm$  and the components of the variety of maximal isotropic subspaces in  $Q_s$ , as explained at the beginning of this paragraph.

If  $s = (1, 0)$  and  $t = (0, 1)$ , it follows from the proof of proposition 1.11 that  $\forall x \in Q_s^+$ ,  $\dim(\psi^+(x) \cap \psi^-(\psi(x))) = 3$ . This is analogous to the condition  $x \in l(x)$  of proposition 1.2, and therefore  $s$  and the isomorphisms  $\psi, \psi^+, \psi^-$  define a unique octonionic structure on  $L_s^+$ .

It follows by general arguments that this octonionic structure varies algebraically. Alternatively, one can give another construction of this octonionic structure, where the algebraicity is clear.

Let  $x = \nu_2^+(s)$  and  $M = \widehat{T_x Q}/k.x$ . Then, as one checks one the example  $s = (1, 0), t = (0, 1)$ ,  $\mu^+(s, \cdot)$  restricts to an isomorphism  $\nu_t$  between  $L_t^+$  and  $M$  and  $\mu^+(t, \cdot)$  to an isomorphism  $\nu_s$  between  $L_s^+$  and  $M$ . We can therefore give an octonionic structure to  $L_s^+$  by setting

$$\forall u, v \in L_s^+, uv = \nu_s^{-1}[\mu^+(u, \nu_t^{-1}(\nu_s(v)))].$$

A direct computation shows that this octonionic structure is the same as the previous one.  $\square$

## 2 Geometry associated with two skew-forms in $k^5$

In this section, we consider a model for the restriction of the Mukai flop of the second kind to a tangent space. In the first subsection, we prove lemmas which will suffice defining this restriction, in section 4. The second subsection will be used when classifying the orbits in  $T^*G/P$ , for  $G$  of type  $E_6$  and  $P$  corresponding to  $\alpha_3$ . The third subsection shows that the involved rational map is the unique equivariant rational map.

Let  $k$  denote an arbitrary field.

### 2.1 A rational map $Hom(k^2, (\wedge^2 k^5)^*) \dashrightarrow G(3, k^5)$

Let  $r$  be an integer and let  $F$  be a vector space of dimension  $2r + 1$ . An element  $\omega$  in  $\wedge^2 F^*$  yields a skew-symmetric map  $F \rightarrow F^*$  which will be denoted  $L_\omega$ . The rank, image, and kernel of  $\omega$  will be those of  $L_\omega$ . If  $f_1, f_2 \in F$ ,  $\omega(f_1, f_2)$  will denote the number  $L_\omega(f_1)(f_2)$ .

**Lemma 2.1.** *Let  $\omega \in \wedge^2 F^*$  of rank  $2t$  and  $U \subset F$  a linear subspace of dimension  $2r + 1 - t$  and such that  $\omega \perp \wedge^2 U$ . Then*

- *If  $u \in U$ , then  $\omega(u, U) \equiv 0$ .*
- *$\ker \omega \subset U$ .*

**Proof :** Taking a basis of  $F$  containing a basis of  $U$  and decomposing  $\omega$  along this basis, one checks that the condition  $\omega \perp \wedge^2 U$  is equivalent to  $\forall u, v \in U, \omega(u, v) = 0$ , proving the first point.

Therefore, we have  $L_\omega(U) \subset U^\perp$ , and since  $2t = \mathbf{rg}(L_\omega) \leq \mathbf{rg}(L_{\omega|U}) + t$ , we have  $\mathbf{rg}(L_{\omega|U}) = t$  and so  $L_\omega(U) = U^\perp$ . Since moreover  $L_\omega$  is skew-symmetric, it follows that  $\ker L_\omega = (\mathbf{Im} L_\omega)^\perp \subset (U^\perp)^\perp = U$ .  $\square$

**Notation 2.1.** Let  $\omega_1, \omega_2 \in \Lambda^2 F^*$ . We denote

$$l(\omega_1, \omega_2) := \{f \in F : \forall u \in \ker \omega_1, \omega_2(u, f) = 0\}.$$

**Lemma 2.2.** Assume  $2(2r+1) = 5t$ . Let  $\omega_1, \omega_2 \in \Lambda^2 F^*$  with rank  $2t$  be such that

1.  $\ker \omega_1 \cap \ker \omega_2 = \{0\}$ , and
2.  $L_{\omega_2}(\ker \omega_1) \cap L_{\omega_1}(\ker \omega_2) = \{0\}$ .

If a linear subspace  $U \subset F$  of dimension  $2r+1-t$  is such that  $\Lambda^2 U \perp \omega_i, i = 1, 2$ , then  $U = l(\omega_1, \omega_2) \cap l(\omega_2, \omega_1)$ .

We will see in lemma 2.3 that for the minimal possible values of  $r, t$ , which are those of interest to describe Mukai's flop,  $U = l(\omega_1, \omega_2) \cap l(\omega_2, \omega_1)$  satisfies indeed  $\Lambda^2 U \perp \omega_i, i = 1, 2$ ; this is not the case in general.

**Proof:** Let  $u \in \ker \omega_1$ . By the previous lemma, we have  $\ker \omega_1 \subset U$ , so  $u \in U$ . If  $f \in U$ , it follows that  $\omega_2(u, f) = 0$ , so  $f \in l(\omega_1, \omega_2)$  and  $U \subset l(\omega_1, \omega_2)$ . By symmetry, we have also  $U \subset l(\omega_2, \omega_1)$ . By condition (1),

$$\dim \ker \omega_1 = \dim L_{\omega_2}(\ker \omega_1) = \dim L_{\omega_1}(\ker \omega_2) = 2r+1-2t,$$

and by condition (2),

$$\dim(L_{\omega_2}(\ker \omega_1) + \dim L_{\omega_1}(\ker \omega_2)) = 2(2r+1-2t).$$

Since we know that  $U$  is orthogonal to this space, and since by hypothesis  $2r+1-2(2r+1-2t) = 4t - (2r+1) = 2r+1-t = \dim U$ ,  $U$  is exactly the orthogonal of this space, proving the lemma.  $\square$

**Notation 2.2.** Denote  $U(\omega_1, \omega_2) := l(\omega_1, \omega_2) \cap l(\omega_2, \omega_1)$ .

**Lemma 2.3.** Assume  $r = t = 2$  and let  $\omega_1, \omega_2 \in \Lambda^2 F^*$  be arbitrary. Then there exists  $U \subset F$  of dimension 3 such that  $\Lambda^2 U \perp \omega_1, \omega_2$ . Therefore, if the two conditions of lemma 2.2 are satisfied, then  $\Lambda^2 U(\omega_1, \omega_2) \perp \omega_1, \omega_2$ . If moreover  $\omega'_1, \omega'_2$  are linear combinations of  $\omega_1, \omega_2$  which also satisfy the two conditions, then  $U(\omega'_1, \omega'_2) = U(\omega_1, \omega_2)$ .

**Proof:** The second claim is a consequence of the first and the lemma 2.2. The third claim follows from the second since  $\Lambda^2 U(\omega_1, \omega_2) \perp \omega'_1, \omega'_2$ . It is therefore enough to prove the first claim.

Let  $G = \Lambda^2 F^* \oplus \Lambda^2 F^* \simeq \text{Hom}(k^2, \Lambda^2 F^*)$ . There is a natural  $GL_2 \times GL(F)$  action on  $G$ . Let  $G(3, F)$  denote the grassmannian of 3-spaces in  $F$  and consider the incidence variety  $I \subset G(3, F) \times \mathbb{P}G$  defined by  $(U, [\omega_1, \omega_2]) \in I$  if and only if  $\Lambda^2 U \perp \omega_1, \omega_2$ . It is a closed projective  $GL_2 \times GL(F)$ -stable variety. Therefore, its projection  $p_2(I) \subset \mathbb{P}G$  also.

Now, let  $f_1, \dots, f_5$  be a basis of  $F$  and  $f_1^*, \dots, f_5^*$  be the dual basis of  $F^*$ . Set  $\omega_1 = f_4^* \wedge f_1^* + f_5^* \wedge f_2^*$  and  $\omega_2 = f_4^* \wedge f_2^* + f_5^* \wedge f_3^*$ . It is clear that if  $U = \mathbf{Vect}(f_1, f_2, f_3)$ , then  $\Lambda^2 U \perp \omega_1, \omega_2$ ; therefore  $[\omega_1, \omega_2] \in p_2(I)$ . It is proved in [KS 77, proof of proposition 13 p.94] that the  $GL_2 \times GL(F)$ -orbit through  $[\omega_1, \omega_2]$  is dense (it also follows from lemma 2.5); therefore  $p_2(I) = G$  and the existence claim of the lemma is proved.  $\square$

## 2.2 $GL_2 \times GL_5$ -orbits

Let as above  $F$  a 5-dimensional vector space over  $k$ . In this subsection, I describe the  $GL_2 \times GL(F)$ -orbits in  $\text{Hom}(k^2, \wedge^2 F^*)$ , and prove where the previous rational map  $U : \mathbb{P}\text{Hom}(k^2, \wedge^2 F^*) \dashrightarrow G(3, F)$ , defined on the open orbit by notation 2.2, extends.

We start with a result of co-diagonalisation of 2-forms of maximal rank :

**Lemma 2.4.** *Let  $\omega_1, \omega_2 \in \wedge^2 F^*$  be forms such that  $\forall (\alpha_1, \alpha_2) \in k^2 - \{(0, 0)\}$ ,  $\alpha_1 \omega_1 + \alpha_2 \omega_2$  has rank 4. Then there exists a basis  $f_1^*, \dots, f_5^*$  of  $F^*$  such that*

$$\begin{aligned}\omega_1 &= f_2^* \wedge f_4^* + f_3^* \wedge f_5^* \\ \omega_2 &= f_1^* \wedge f_5^* + f_3^* \wedge f_4^*.\end{aligned}$$

**Proof :** For  $i \in \{1, 2\}$  and  $u, v \in F$ , we denote  $\langle u, v \rangle_i := L_{\omega_i}(u)(v)$ .

Assume that  $\ker \omega_1 = \ker \omega_2$ . Denote  $K$  this 1-dimensional vector space. Then  $\omega_1, \omega_2$  belong to  $\wedge^2(F/K)^*$ . The variety of degenerate 2-forms in  $(F/K)^*$  is a hypersurface, so there exists  $(\alpha_1, \alpha_2) \in k^2 - \{(0, 0)\}$  such that  $\alpha_1 \omega_1 + \alpha_2 \omega_2$  is degenerate, contradicting the hypothesis of the lemma.

We consider  $0 \neq f_1 \in \ker \omega_1$  and  $0 \neq f_2 \in \ker \omega_2$ ;  $f_1$  and  $f_2$  are therefore not colinear.

Assume now that  $L_{\omega_1}(f_2)$  and  $L_{\omega_2}(f_1)$  are colinear. Denote  $I$  this common image. The map  $F^* \rightarrow F^*/I$  induces a map  $\wedge^2 F^* \rightarrow \wedge^2(F^*/I)$ . Let  $\bar{\omega}_i \in \wedge^2(I^\perp)^*$  denote the image of  $\omega_i$  under this map. Both  $\bar{\omega}_1$  and  $\bar{\omega}_2$  vanish on  $f_1, f_2$ . Therefore, they are proportional 2-forms : let  $\alpha_1 \bar{\omega}_1 + \alpha_2 \bar{\omega}_2$  be a non-trivial relation. Since  $I^\perp$  is an isotropic subspace for  $\alpha_1 \omega_1 + \alpha_2 \omega_2$ , this form does not have rank 4, contradicting the hypothesis.

We set  $f_5^* = L_{\omega_2}(f_1)$  and  $f_4^* = L_{\omega_1}(f_2)$ ;  $f_4^*$  and  $f_5^*$  are therefore not colinear. Note that  $\langle f_4^*, f_1 \rangle = \langle f_2, f_1 \rangle_1 = 0$  because  $L_{\omega_1}(f_1) = 0$ , and that  $\langle f_4^*, f_2 \rangle = \langle f_2, f_2 \rangle_1 = 0$ ; therefore,  $f_4^* \in \langle f_1, f_2 \rangle^\perp$ , and similarly  $f_5^* \in \langle f_1, f_2 \rangle^\perp$ .

We now let  $[\omega_i]$  be the composition  $F \xrightarrow{L_{\omega_i}} F^* \rightarrow F^*/\langle f_4^*, f_5^* \rangle$ . I claim that  $\ker[\omega_i] = \langle f_4^*, f_5^* \rangle^\perp$ .

Note that  $\text{Im} L_{\omega_i} = f_i^\perp \supset \langle f_4^*, f_5^* \rangle$ . I will prove the claim when  $i = 1$ . Both spaces are 3-dimensional and contain  $\langle f_1, f_2 \rangle$ . So let  $f$  such that  $L_{\omega_1}(f) = f_5^*$ , and let us see that  $f \in \langle f_4^*, f_5^* \rangle^\perp$ . Since  $L_{\omega_1}(f) = f_5^*$  by assumption,  $\langle f_5^*, f \rangle = 0$ . Similarly,  $\langle f_4^*, f \rangle = \langle f_2, f \rangle_1 = -\langle f_5^*, f_2 \rangle = -\langle f_1, f_2 \rangle_2 = 0$ , since  $L_{\omega_2}(f_2) = 0$ . So the claim is proved.

Looking at the surjective maps  $\langle f_4^*, f_5^* \rangle^\perp \xrightarrow{L_{\omega_1}, L_{\omega_2}} \langle f_4^*, f_5^* \rangle$ , one proves that there exists  $f_3 \in \langle f_4^*, f_5^* \rangle^\perp$  such that  $L_{\omega_1}(f_3) \in \langle f_5^* \rangle - \{0\}$  and  $L_{\omega_2}(f_3) \in \langle f_4^* \rangle - \{0\}$ . Up to scaling  $f_1$  (and so  $f_5^* = L_{\omega_1}(f_1)$ ) and  $f_2$  (and so  $f_4^*$ ), we may assume that  $L_{\omega_1}(f_3) = f_5^*$  and  $L_{\omega_2}(f_3) = f_4^*$ .

Up to now, the vectors  $f_1, f_2, f_3, f_4^*, f_5^*$  were determined, up to a scale, by  $\omega_1$  and  $\omega_2$ . We now make a more significant choice for  $f_4$  : let  $f_4 \in (f_5^*)^\perp$  such that  $\langle f_4^*, f_4 \rangle = 1$ . Note that this implies  $\langle f_3, f_4 \rangle_2 = \langle f_2, f_4 \rangle_1 = 1$ , by definition of  $f_3$  and  $f_4^*$ . We moreover choose  $f_5 \in (f_4^*)^\perp$  such that  $\langle f_5^*, f_5 \rangle = 1$  and  $\langle f_4, f_5 \rangle_1 = \langle f_4, f_5 \rangle_2 = 0$ . Note that this implies  $\langle f_5, f_3 \rangle_1 = \langle f_5, f_1 \rangle_2 = -1$ .

It is then easy to check that for  $i \in \{1, \dots, 5\}$ , we have  $f_5^*(f_i) = \delta_{i,5}$  and  $f_4^*(f_i) = \delta_{i,4}$ . So it will not conflict notations to consider the dual basis  $(f_1^*, \dots, f_5^*)$  of the basis  $(f_1, \dots, f_5)$  of  $F$ . In this dual basis,  $\omega_1$  and  $\omega_2$  are as in the proposition.  $\square$

Let  $(f_1^*, \dots, f_5^*)$  be a basis of  $F^*$ . Let  $\omega_1, \omega_2$  denote the forms  $f_2^* \wedge f_4^* + f_3^* \wedge f_5^*, f_1^* \wedge f_5^* + f_3^* \wedge f_4^*$ . We now classify the  $GL_2 \times GL(F)$ -orbits in  $Hom(k^2, \wedge^2 F^*)$ .

**Lemma 2.5.** *There are eight  $GL_2 \times GL_5$ -orbits in  $Hom(k^2, \wedge^2 F^*)$ . The following array gives elements in each orbit, its dimension and a label.*

| label        | $f((1, 0))$          | $f((0, 1))$          | dim |
|--------------|----------------------|----------------------|-----|
| $A_2 + 2A_1$ | $\omega_1$           | $\omega_2$           | 20  |
| $A_2 + A_1$  | $\omega_1$           | $f_1^* \wedge f_2^*$ | 18  |
| $A_2$        | $\omega_1$           | $f_2^* \wedge f_4^*$ | 16  |
| $3A_{1,a}$   | $\omega_1$           | $f_2^* \wedge f_3^*$ | 15  |
| $3A_{1,b}$   | $f_1^* \wedge f_2^*$ | $f_1^* \wedge f_3^*$ | 12  |
| $3A_{1,c}$   | $\omega_1$           | $\omega_1$           | 11  |
| $2A_1$       | $f_1^* \wedge f_2^*$ | $f_1^* \wedge f_2^*$ | 8   |
| $A_1$        | 0                    | 0                    | 0   |

Finally, the closure of an orbit  $\mathcal{O}$  contains the orbit  $\mathcal{O}'$  if and only if  $\mathcal{O}$  lies above  $\mathcal{O}'$  in this array, except that the closure of the orbit labelled  $3A_{1,b}$  does not contain the orbit labelled  $3A_{1,c}$ .

**Proof:** Granting the classification of the orbits, I leave it to the reader to check the dimensions of the orbits and the decomposition of their closures.

So let again  $F = k^5$  and  $f \in Hom(k^2, \wedge^2 F^*)$ . If the rank of  $f$ , as a morphism of vector spaces, is one, then there are three cases (labelled  $A_1, 2A_1, 3A_{1,c}$ ), according to the rank (as an element of  $\wedge^2 F^*$ ) of a generic element of its image.

Assume  $f$  has rank two. If all non-vanishing elements of the image of  $f$  have rank 4, then, by lemma 2.4, we are in case  $A_2 + 2A_1$ . If all these elements are degenerate, then it is well-known that we are in case  $3A_{1,b}$ .

Otherwise, we may assume that  $f((1, 0))$  has rank 4 and  $\omega := f((0, 1))$  has rank 2. There is a basis  $f_1, \dots, f_5$  of  $F$  such that in terms of the dual basis  $f_1^*, \dots, f_5^*$ ,  $f((1, 0)) = \omega_1$ . The kernel of  $L_{\omega_1}$  is generated by  $f_1$ . Consider the 4-dimensional subspace  $F' := \langle f_2^*, f_3^*, f_4^*, f_5^* \rangle \subset F^*$  and the 5-dimensional projective space  $\mathbb{P} \wedge^2 F'$  containing the 4-dimensional quadric  $G(2, F')$  of classes of elements of rank 2. The generic element  $\omega_1 \in \wedge^2 F'$  defines a polar hyperplane (with respect to the quadric) in  $\mathbb{P} \wedge^2 F'$ , which will be denoted  $H$ . Note that  $H \cap G(2, F')$  is a smooth 3-dimensional quadric.

- Assume first that  $L_{\omega}$  does not vanish on  $f_1$  and let  $g_1^*$  be an element in  $\text{Im } L_{\omega}$  not vanishing on  $f_1$ . Let  $g_2^* \neq 0$  be an element in  $\text{Im } L_{\omega} \cap \text{Im } L_{\omega_1}$ . The variety of classes of elements of the form  $[g_2^* \wedge g^*], g^* \in F'$  is a  $\mathbb{P}^2$  in the quadric  $G(2, F')$ . Therefore, it can't be included in  $H$ . Let  $g_4^* \in F'$  such that  $[g_2^* \wedge g_4^*] \notin H$ ; the projective line through  $[g_2^* \wedge g_4^*]$  and  $[\omega_1]$  is therefore a secant line: let  $[g_3^* \wedge g_5^*]$  be an element in the intersection of this line and  $G(2, F')$ . We can assume  $\omega_1 = g_2^* \wedge g_4^* + g_3^* \wedge g_5^*$  and  $\omega = g_1^* \wedge g_2^*$ ; therefore, we are in the case labelled  $A_2 + A_1$ .

- Assume now that  $L_\omega(f_1) = 0$ . In this case, both  $\omega$  and  $\omega_1$  are in  $\Lambda^2 F'$ . There are two  $GL(F')$ -orbits for the projective line through  $\omega$  and  $\omega_1$ : either it is a secant line to the quadric  $G(2, F')$ , either it is a tangent line; this corresponds to the cases  $A_2$  and  $3A_{1,a}$ .  $\square$

Recall the rational map  $U$  of notation 2.2. It is a model for the restriction of the Mukai flop of the second kind to a tangent space, so it is interesting to know where it is defined.

**Lemma 2.6.** *The open orbit is the locus where  $U$  is defined.*

**Proof:** Let as before  $\omega_1 = f_2^* \wedge f_4^* + f_3^* \wedge f_5^*$ , and let  $\omega_2(t) = f_1^* \wedge f_5^* + t \cdot f_2^* \wedge f_3^*$ . Let  $f : k^2 \rightarrow \Lambda^2 F^*$  be defined by  $f((1, 0)) = \omega_1$  and  $f((0, 1)) = \omega_2(t)$ ; we have  $U(f) = \langle f_5^*, f_2^* \rangle$ .

The same construction with  $\omega'_2(t) = f_1^* \wedge f_5^* + t \cdot f_3^* \wedge f_4^*$  yields  $U(f') = \langle f_5^*, f_4^* \rangle$ . Now, since  $\omega_2(t)$  and  $\omega'_2(t)$  converge to  $f_1^* \wedge f_5^*$ , this proves that  $U$  is not defined at the point  $f_0$  defined by  $f_0((1, 0)) = f_2^* \wedge f_4^* + f_3^* \wedge f_5^*$  and  $f_0((0, 1)) = f_1^* \wedge f_5^*$ .

Therefore, the indeterminacy locus of  $U$  contains the orbit labelled  $A_2 + A_1$ ; since it is closed, it contains all the orbits but the open one.  $\square$

### 2.3 Unicity of the equivariant rational map

Recall that  $F$  is a 5-dimensional vector space over  $k$ . In this subsection, I show that the rational map  $U : Hom(k^2, \Lambda^2 F^*) \dashrightarrow G(3, F)$  of notation 2.2 is the unique  $(GL_2 \times GL(F))$ -equivariant rational map  $Hom(k^2, \Lambda^2 F^*) \dashrightarrow G(3, F)$ . This is a result analogous to proposition 1.5, and the strategy of proof is the same: we characterize its graph as an orbit of minimal dimension.

Let  $O$  denote the open  $(GL_2 \times GL(F))$ -orbit in  $Hom(k^2, \Lambda^2 F^*)$ .

**Lemma 2.7.** *In  $O \times G(3, F)$ , there is a unique 20-dimensional orbit.*

**Proof:** The set of  $(f, \alpha)$  with  $\alpha = U(f)$  is such an orbit. Let  $(f, \alpha)$  be in an orbit of dimension 20. Since  $O$  is homogeneous, we can assume that  $f((1, 0)) = \omega_1 = f_2^* \wedge f_4^* + f_3^* \wedge f_5^*$  and  $f((0, 1)) = \omega_2 = f_1^* \wedge f_5^* + f_3^* \wedge f_4^*$ . Let  $G_0$  denote the stabilizer of  $f$  in  $GL_2 \times GL(F)$ ; we have  $G_0 = \{1\} \times G_1$ , with  $G_1 =$

$$\left\{ \left( \begin{pmatrix} t & 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 \\ a & c & d & t^{-1} & 0 \\ b & d & a & 0 & t^{-1} \end{pmatrix} : t \in k^*; a, b, c, d \in k \right) \right\} \text{ (these matrices express the}$$

action of  $G_1$  on  $F^*$  in the dual basis  $f_1^*, \dots, f_5^*$ ). This fact can be proved by a direct computation; we will only use the obvious fact that  $G_1$  stabilizes  $f$ .

Let  $F_\alpha \subset F$  denote the 3-dimensional subspace corresponding to  $\alpha$ , and let  $F_\alpha^\perp \subset F^*$  denote its orthogonal. Since  $(f, \alpha)$  belongs to an orbit of dimension 20,  $G_1$  must stabilize  $\alpha$ . Assume there exists  $f^* \in F_\alpha^\perp$ , with  $f^* = \sum x_i f_i^*$  and  $x_1 \neq 0$ . The action of  $G_1$  implies that  $f_5^*$  belongs to  $F_\alpha^\perp$ , and so also  $f_4^*$ . Therefore we have a contradiction. The same contradiction arises if  $F_\alpha^\perp$  contains a form with non-vanishing coefficient along  $f_2^*$ . From this it follows easily that  $F_\alpha^\perp$  is generated by  $f_4^*$  and  $f_5^*$ , so  $\alpha = U(f)$ .  $\square$

**Proposition 2.1.** *There is a unique  $(GL_2 \times GL(F))$ -equivariant rational map  $Hom(k^2, \Lambda^2 F^*) \dashrightarrow G(3, F)$ .*

**Proof:**  $U$  is such a rational map. Let  $u$  denote any  $(GL_2 \times GL(F))$ -equivariant rational map  $Hom(k^2, \wedge^2 F^*) \dashrightarrow G(3, F)$ .

Recall that  $O \subset Hom(k^2, \wedge^2 F^*)$  denotes the open orbit; since  $u$  is equivariant, it is defined on  $O$  and surjective. The variety  $\{(f, u(f)) : f \in O\}$  is a 20-dimensional orbit in  $O \times G(3, F)$ ; therefore, by lemma 2.7, it is equal to the variety  $\{(f, U(f)) : f \in O\}$ .  $\square$

### 3 Tangency in Scorza varieties

In a projective space, given a point  $x$  and a non-vanishing tangent vector  $t \in T_x X$ , there is a unique line  $l$  through  $x$  and such that  $t \in T_x l$ . Similarly, given a non-vanishing cotangent form  $f \in T_x^* X$ , there is a unique hyperplane  $h$  such that  $f$  vanishes on  $T_x h$ . Therefore, a tangent vector defines a line and a cotangent form a hyperplane. This will be extended to a projective space over a composition algebra in this section. Both of these maps will be also defined using Jordan algebras.

#### 3.1 Notations for Scorza varieties

Let  $\mathcal{A}$  be a composition algebra over  $\mathbb{C}$ , of dimension  $a$ . If  $n$  is an integer, let  $H_n(\mathcal{A})$  denote the space of  $(n \times n)$  hermitian matrices with coefficients in  $\mathcal{A}$ . Let

$$\begin{aligned} \nu_2 & : \quad \mathcal{A}^n & \rightarrow & H_n(\mathcal{A}) \\ & (z_1, \dots, z_n) & \mapsto & (z_i \bar{z}_j)_{1 \leq i, j \leq n} \end{aligned}$$

be the map generalizing that of section 1. Recall from [Cha 05] that in  $H_n(\mathcal{A})$  there is a notion of rank. The variety of rank  $n - 1$ -elements is a hypersurface; let  $\det$  denote a reduced equation of this hypersurface. The variety of rank one matrices may be described, by [Cha 05, theorem 3.1 (4) and proposition 4.2], as the closure  $X = \{[\nu_2(1, z_2, \dots, z_n)] : z_i \in \mathcal{A}\}$ .

Scorza varieties were defined and classified by F. Zak as varieties having some extremal properties with respect to their secant varieties [Zak 93, Cha 03]. For our purpose, the following theorem will serve as a definition :

**Theorem 3.1 (Zak).** *Let  $a \in \{1, 2, 4, 8\}$  and  $n$  be integers. A Scorza variety of type  $(n, a)$  is a pair  $(V, X)$ , where  $V$  is a  $\mathbb{C}$ -vector space, and  $X \subset \mathbb{P}V$  is a projective variety projectively isomorphic to the variety of classes of rank one matrices in the projectivisation of the space  $H_n(\mathcal{A})$  (with  $\dim \mathcal{A} = a$ ).*

$X$  is a kind of projective space; moreover, one can define a dual ‘‘projective space’’  $X^\vee \subset \mathbb{P}V^*$ , non-canonically isomorphic with  $X$ , and an incidence relation for  $(x, h) \in X \times X^\vee$  denoted  $x \vdash h$ . In fact,  $X^\vee$  is the variety of hyperplanes containing  $n - 1$  general tangent spaces to  $X$  and  $x \vdash h$  if and only if  $T_x X \subset h$ . For  $h \in X^\vee$ , the Schubert cell of  $x$ ’s incident to  $h$  will be denoted  $C_h$ . The quadratic representation corresponding to the Scorza variety  $(V^*, X^\vee)$  will be denoted  $U^\vee$ ; therefore,  $U^\vee$  is a quadratic map  $V^* \rightarrow Hom(V, V^*)$ .

For the convenience of the reader, I recall, given  $a$  and  $n$ , the corresponding Scorza varieties and their automorphism group  $(G(n_1, n_2))$  denotes the grassmannian of  $n_1$ -dimensional subspaces in  $\mathbb{C}^{n_2}$ .

| $a$ | $\mathcal{A}$                  | $V$                                 | $X$  | $X^\vee$   | $Aut(X)$             |
|-----|--------------------------------|-------------------------------------|--|--|----------------------|
| 1   | $\mathbb{C}$                   | $S^2\mathbb{C}^n$                   | $\mathbb{P}^{n-1}$                         | $(\mathbb{P}^{n-1})^\vee$                                | $PGL_n$              |
| 2   | $\mathbb{C} \oplus \mathbb{C}$ | $\mathbb{C}^n \otimes \mathbb{C}^n$ | $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ | $(\mathbb{P}^{n-1})^\vee \times (\mathbb{P}^{n-1})^\vee$ | $PGL_n \times PGL_n$ |
| 4   | $M_2(\mathbb{C})$              | $\Lambda^2\mathbb{C}^{2n}$          | $G(2, 2n)$                                 | $G(2n-2, 2n)$  | $PGL_{2n}$           |
| 8   | $\mathbb{O}_{\mathbb{C}}$      | $\dim 27$                           | $E_6/P_1$                                  | $E_6/P_6$  | $E_6$                |

In each case, it is well-known that there is a Mukai flop  $T^*X \dashrightarrow T^*X^\vee$ . One aim of the rest of the article is to describe this flop.

Let  $(X, V)$  be a Scorza variety of type  $(n, a)$ . Recall from [Ch 06, section 1] the ‘‘quadratic representation’’ : it is a quadratic map  $V \rightarrow Hom(V^*, V)$ , canonically defined using only  $X$ . If  $A \in V$ , we will denote  $U_A \in Hom(V^*, V)$  the image of  $A$  under the quadratic representation.

In concrete terms, when we will have to compute a quadratic representation in  $V$ , we will always do the following. First, we will identify  $V$  with  $H_n(\mathcal{A})$ . Second, we will choose the scalar product  $(A, B) = \text{tr}(AB)$ , which identifies  $V$  and  $V^*$ . These two choices will not affect the final result. Then, to compute  $U_A(B)$ , for  $A \in V$  and  $B \in V^* \simeq V$ , we will always manage to be in the situation when all the coefficients of  $A$  and  $B$  belong to an associative subalgebra of  $\mathcal{A}$  (this holds, for example, if  $\mathcal{A}$  itself is associative). Then we use the fact that  $U_A(B)$  is  $ABA$ , where juxtaposition stands for the usual product of matrices [Ch 06].

Recall also that for any integer  $r < n$  there is a well-defined variety  $G_{\mathcal{A}}(r, X)$  parametrizing Scorza subvarieties of type  $(r, a)$  in  $X$ . To an element  $A \in \mathbb{P}V$  of rank  $r$  is associated a subvariety  $X_A \in G_{\mathcal{A}}(r, X)$  and its linear span in  $\mathbb{P}V$  is denoted  $\Sigma_A$  [Ch 06, proposition 1.3].

As explained in [Cha 05] and [Ch 06], the Scorza varieties admit a model over  $\mathbb{Z}$ , and the quadratic representation is defined over  $\mathbb{Z}$ . Therefore, all the following constructions are valid on this base, and we get a description of Mukai flops over  $\mathbb{Z}$ . For the clarity of redaction, I will work over  $\mathbb{C}$ , since it is the usual context of Mukai flops.

In the following,  $(V, X)$  will be a Scorza variety of type  $(n, a)$ , and  $G$  denotes the automorphism group of  $X$ .

### 3.2 A generic tangent vector defines a line

Let  $x \in X$  and let  $L_x \subset V$  be the line it represents. We have  $T_x X = Hom(L_x, \widehat{T_x X}/L_x)$ . Let  $t \in T_x X$ ; in the next proposition, I say that  $T \in \widehat{T_x X}$  represents  $t$  if the morphism  $t \in Hom(L_x, \widehat{T_x X}/L_x)$  has image the line generated by the class modulo  $L_x$  of  $T$ .

By [Ch 06, proposition 1.5], the  $\mathcal{A}$ -lines through a point  $x \in X$  are naturally parametrized by a subvariety of  $\mathbb{P}(V/\widehat{T_x X})$ . I say that a representative of an  $\mathcal{A}$ -line through  $x$  is  $l$  (with  $L \in V/\widehat{T_x X}$ ) if the class of  $L$  in  $\mathbb{P}(V/\widehat{T_x X})$  corresponds to  $l$ .

**Theorem 3.2.** *Let  $x \in X$  and  $t \in T_x X$  generic. There exists a unique  $\mathcal{A}$ -line  $l \in G_{\mathcal{A}}(2, X)$  such that  $x \in l$  and  $t \in T_x l$ . A representative for  $l$  in  $V/\widehat{T_x X}$  is  $L = [U_T(A)]$ , if  $T \in \widehat{T_x X}$  represents  $t$  and  $A$  is a generic element in  $V^*$ .*

**Notation 3.1.** Let  $\nu_x^+$  denote the quadratic map  $T_x X \rightarrow V/\widehat{T_x X}$ ,  $T \mapsto U_T(A)$  of this theorem.

**Proof:** Let  $x \in X$  and  $t \in T_x X$  be generic. Let  $T$  represent  $t$ . Then  $T$  has rank two, so by [Ch 06, proposition 1.4],  $T$  defines the  $\mathcal{A}$ -line  $X_T$ . We will prove that  $X_T$  is the unique  $\mathcal{A}$ -line with the properties of the proposition. To this end, we assume that  $n = 3$  to simplify notations, since larger values of  $n$  would not change the argument.

We assume that  $V = H_n(\mathcal{A})$  and  $X$  is the variety of rank one matrices. By [Ch 06, proposition 1.3], we can assume  $x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  and  $T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ . Then  $X_T$  is the set of rank one matrices of the form  $\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix}$ . We therefore

check that  $x \in X_T$  and  $T \in \widehat{T_x X_T} = \begin{pmatrix} * & * & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ .

Conversely, let  $l \in G_{\mathcal{A}}(2, X)$  such that  $x \in l$  and  $t \in T_x l$ . Let  $B \in \mathbb{P}V$  of rank 2 such that  $l = X_B$ . Then since  $T$  represents  $t$  and  $t \in T_x l$ , we have  $T \in \widehat{T_x l} \subset \Sigma_B$ . By [Ch 06, proposition 1.4],  $\Sigma_B = \Sigma_T$  and  $l = X_T$ .

The fact that  $L = [U_T(A)] \in V/\widehat{T_x X}$  is a representative for  $l$  follows from the fact that by [Ch 06, proposition 1.4] again,  $\Sigma_T$  is the image of  $U_T$ , and the fact that the isomorphism of [Ch 06, proposition 1.5] maps the  $\mathcal{A}$ -line  $X_T$  on the line  $\mathbf{Im} U_T/\widehat{T_x X} \subset V/\widehat{T_x X}$ .  $\square$

Let  $d$  be an integer and  $\mathcal{A}$  a composition algebra; recall the map  $\nu_2 : \mathcal{A}^d \rightarrow H_d(\mathcal{A})$  defined in subsection 3.1. Its projectivisation  $\bar{\nu}_2 : \mathcal{A}^d \dashrightarrow \mathbb{P}H_d(\mathcal{A})$  may be considered as a kind of quotient map  $\mathcal{A}^d \dashrightarrow \mathbb{P}_{\mathcal{A}}^{d-1}$  [Cha 05, subsection 3.4].

**Corollary 3.1.** *There are identifications of  $T_x X$  with  $\mathcal{A}^{n-1}$  and  $V/\widehat{T_x X}$  with  $H_{n-1}(\mathcal{A})$  such that  $\nu_x^+$  identifies with  $\nu_2 : \mathcal{A}^{n-1} \rightarrow H_{n-1}(\mathcal{A})$ .*

**Proof:** With the notations of the previous proof, to see that  $\nu_x^+$  identifies with  $\nu_2$ , we choose the scalar product  $(A, B) \mapsto \text{tr}(AB)$  on  $V = H_n(\mathcal{A})$ , which identifies  $V$  and  $V^*$ , and moreover we choose  $A \in V^*$  to be the linear form corresponding to the identity matrix in  $V = H_n(\mathcal{A})$ . Then, by subsection 3.1, if  $T = \begin{pmatrix} t & \bar{z}_1 & \bar{z}_2 \\ z_1 & 0 & 0 \\ z_2 & 0 & 0 \end{pmatrix}$ , then  $U_T(A) = T^2 = \begin{pmatrix} * & * & * \\ * & N(z_1) & z_1 \bar{z}_2 \\ * & z_2 \bar{z}_1 & N(z_2) \end{pmatrix}$ . Therefore,  $\nu_x^+$  identifies with  $\nu_2$ .  $\square$

### 3.3 A generic cotangent form defines a hyperplane

A cotangent form  $f \in T_x X^*$  is an element  $f \in \text{Hom}(L_x^*, (\widehat{T_x X}/L_x)^*)$ . I say that  $\tilde{f} \in V^*$  represents  $f$  if  $\tilde{f}|_{\widehat{T_x X}}$  generates the image of  $f$ . Recall (subsection 3.1) that for  $h \in X^\vee$ ,  $C_h$  denotes the Schubert cell in  $X$  defined by  $h$ . Let  $\mu : T^* X \dashrightarrow T^* X^\vee$  denote the Mukai flop and  $\pi : T^* X^\vee \rightarrow X^\vee$  the projection.

**Theorem 3.3.** *Let  $x \in X$ ,  $x_0 \in L_x - \{0\}$ , and  $f \in T_x^* X$  generic. There exists a unique  $h \in X^\vee$  such that  $f$  vanishes on  $T_x C_h$ . If  $\tilde{f} \in V^*$  represents  $f$ , then a representative of  $h$  is  $U_{\tilde{f}}^\vee(x_0) \in (V/\widehat{T_x X})^*$ . Finally,  $\pi \circ \mu(x, f) = h$ .*

**Notation 3.2.** Let  $\nu_x^-$  denote the quadratic map  $T_x^*X \rightarrow (V/\widehat{T_x X})^*$ ,  $\tilde{f} \mapsto U_{\tilde{f}}^\vee(x_0)$  of this theorem.

**Proof:** The last claim follows from the first and [Cha 06], where it is proved that  $\pi \circ \mu(x, f)$  is the only  $h \in X^\vee$  such that  $f$  vanishes on  $T_x C_h$ .

To simplify notations, we assume in the proof that  $V = H_3(\mathcal{A})$  and we identify  $V$  and  $V^*$  via the scalar product  $(A, B) = \text{tr}(AB)$ . Assume as before

that  $x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  and that  $f \left( \begin{pmatrix} t & \bar{z}_1 & \bar{z}_2 \\ z_1 & 0 & 0 \\ z_2 & 0 & 0 \end{pmatrix} \right) = \text{Re}(z_2)$ .

Let  $h_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ . Let  $y = \begin{pmatrix} t & z & 0 \\ \bar{z} & u & 0 \\ 0 & 0 & 0 \end{pmatrix}$ , with  $t, u \in \mathbb{C}$  and  $z \in \mathcal{A}$

and  $tu - N(a) = 0$ , be an element of  $\widehat{X}$ . By [Cha 05, theorem 3.1 (4) and proposition 4.2],  $\widehat{X} = \{[\nu_2(1, z_1, z_2)] : z_i \in \mathcal{A}\}$ . We deduce that if  $(m_{i,j}) \in \widehat{X}$ , then the minors  $m_{i,i}m_{j,j} - N(m_{i,j})$  vanish. It follows that if  $t \neq 0$ ,  $T_y \widehat{Y}$  is orthogonal to  $h_0$ .

Therefore, by continuity, the intersection of  $\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix}$  with  $X$  lies in the Schubert cell  $C_{h_0}$ , and for dimension reasons we have equality. This shows that  $f$  vanishes on  $T_x C_{h_0}$ . Therefore  $h = h_0$ .

Finally, let  $\tilde{f} = \begin{pmatrix} 0 & \bar{z}_1 & \bar{z}_2 \\ z_1 & t & \bar{z} \\ z_2 & z & u \end{pmatrix}$  be a linear form ( $t, u \in \mathbb{C}$  and  $z, z_1, z_2 \in \mathcal{A}$

are arbitrary); then  $U^\vee(\tilde{f}) \cdot x = \tilde{f} \cdot x \cdot \tilde{f} = \tilde{f} \begin{pmatrix} 0 & \bar{z}_1 & \bar{z}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & N(z_1) & z_1 \bar{z}_2 \\ 0 & z_2 \bar{z}_1 & N(z_2) \end{pmatrix}$ , so

that  $\nu_x^-$  identifies with  $\nu_2$ . Moreover, if  $z_1 = 0$  and  $z_2 = 1$ , then  $\tilde{f}$  represents  $f$  and we have  $[\nu_x(\tilde{f})] = h_0$ , as claimed.  $\square$

In the proof of the theorem, we showed :

**Corollary 3.2.** *There are identifications of  $T_x X^*$  and  $(V/\widehat{T_x X})^*$  with  $\mathcal{A}^{n-1}$  and  $H_{n-1}(\mathcal{A})$  such that  $\nu_x^-$  identifies with  $\nu_2 : \mathcal{A}^{n-1} \rightarrow H_{n-1}(\mathcal{A})$ .*

### 3.4 The variety of lines through a point in $\mathbb{P}_{\mathcal{A}}^n$ as Fano variety of maximal linear subspaces of $\mathbb{P}_{\mathcal{A}}^{n-1}$

The goals of this subsection are propositions 3.4 and 3.5.

The normal bundle to  $X$  in  $\mathbb{P}V$  twisted by  $(-1)$  will be denoted  $N$  and let  $\pi : N \rightarrow X$  (resp.  $\bar{\pi} : \mathbb{P}N \rightarrow X$ ) be the structure map of this vector bundle (resp. its projectivisation). Similarly, let  $\psi$  and  $\bar{\psi}$  denote the natural maps  $TX(-1) \rightarrow X$  and  $\mathbb{P}TX(-1) \rightarrow X$ . Let  $x \in X$ ; the quotient map  $V \rightarrow V/\widehat{T_x X} = N_x$  will be denoted  $\pi_x$ . The normal bundle  $N$  admits an interesting subvariety : the image of  $\widehat{X}$ . This variety will be denoted  $N(X)$  : by definition, the fiber  $N(X)_x := \pi^{-1}(x) \cap N(X)$  is  $\pi_x(\widehat{X})$ . Recall [Ch 06] that  $(N_x, \mathbb{P}N(X)_x)$  is a Scorza variety of type  $(n-1, a)$ .

Assume  $a > 1$ . Let  $F(0, 1, X)$  denote the variety of couples  $(x, l)$  where  $x \in \mathbb{P}V$ ,  $l \subset \mathbb{P}V$  is a projective line, and  $x \in l \subset X$ . The map which sends a

pair  $(x, l) \in F(0, 1, X)$  to  $(x, t) \in \mathbb{P}T_x X$ , where  $t \in \mathbb{P}T_x X$  is the projectivisation of the tangent vector of  $l$  at  $x$  shows that  $F(0, 1, X)$  can be considered as a subvariety of  $\mathbb{P}T_x X$ . By [Ch 06, lemma 1.2 and proposition 1.3],  $F(0, 1, X)$  is homogeneous.

The first interesting point is that  $(\overline{\psi}^* N)|_{F(0,1,X)}$  admits a subbundle included in  $\overline{\psi}^{-1}(N(X))$ . For  $(x, l) \in F(0, 1, X)$  and  $x \neq y \in l$ , define  $T_y := \pi_x(\widehat{T}_y X)$ .

**Proposition 3.3.**  *$T_y$  does not depend on  $y \in l$  and  $T_y \rightarrow F(0, 1, X)$  defines a rank  $(ra/2 + 1)$ -subbundle of  $(\overline{\psi}^* N)|_{F(0,1,X)}$ , entirely included in  $\overline{\psi}^{-1}(N(X))$ .*

**Proof:** Assume for the simplicity of notations that  $n = 3$ . I use the fact that if  $z_1, z_2, z_3$  generate an associative subalgebra of  $\mathcal{A}$ , then  $\nu_2(z_1, z_2, z_3) \in \widehat{X}$  [Cha 05, proposition 4.2]. The condition on  $z_1, z_2, z_3$  holds for example if  $\mathcal{A}$  itself is associative or if  $z_1 = 1$ , since in  $\mathbb{O}_{\mathbb{C}}$ , the subalgebra generated by two elements is always associative.

Let  $x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  and  $y = \begin{pmatrix} 1 & \bar{z} & 0 \\ z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  in  $H_3(\mathcal{A})$ , with  $z \in \mathcal{A}$  and  $N(z) = 0$ . Then the line through  $[x]$  and  $[y]$  in  $\mathbb{P}H_3(\mathcal{A})$  lies in  $X$ , because  $x + ty = \nu_2(1, tz, 0)$ . Moreover, differentiating  $\nu_2$ , we have  $\widehat{T}_x X = \left\{ \begin{pmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix} \right\}$

and  $\widehat{T}_y X = \left\{ \begin{pmatrix} \operatorname{Re}(u) & u\bar{z} + \bar{v} & \bar{w} \\ z\bar{u} + v & \operatorname{Re}(zv) & z\bar{w} \\ w & w\bar{z} & 0 \end{pmatrix} : u, v, w \in \mathcal{A} \right\}$ . It follows that

$$\widehat{T}_y X / \widehat{T}_x X \simeq \left\{ \begin{pmatrix} * & z\bar{w} \\ w\bar{z} & 0 \end{pmatrix} : w \in \mathcal{A} \right\}.$$

Therefore, this space does not change if  $y$  is replaced by a point of the line through  $x$  and  $y$ . Since  $F(0, 1, X)$  is homogeneous, this holds for any of its elements. Therefore,  $T_{(x,l)}$  is always a  $(a/2 + 1)$ -linear subspace of  $V/\widehat{T}_x X$ . It follows that it is a subbundle of  $(\overline{\psi}^* N)|_{F(0,1,X)}$ , as it is locally the image of the bundle  $\widehat{T}_y X$  ( $y$  a local section of  $l$  different from  $x$ ) under a constant rank vector bundle map.

Moreover, since  $\begin{pmatrix} 0 & 0 & 0 \\ 0 & * & z\bar{w} \\ 0 & w\bar{z} & 0 \end{pmatrix}$  belongs to  $\widehat{X}$ ,  $\begin{pmatrix} * & z\bar{w} \\ w\bar{z} & 0 \end{pmatrix}$  belongs to  $N(X)_x$ ,

and so the subbundle  $T$  is included in  $\overline{\psi}^{-1}(N(X)_x)$ .  $\square$

Let  $x \in X$  and denote  $F(x, 1, X)$  the variety of lines through  $x$  and included in  $X$ . Theorem 3.2 yields a quadratic map  $\nu_x^+ : \widehat{T}_x X / L_x \rightarrow N_x$ , well-defined up to a scale. Let  $\mu_x^+(\cdot, \cdot)$  denote its polarization.

**Proposition 3.4.** *The map  $l \mapsto T_{(x,l)}$  defines a map between  $F(x, 1, X)$  and some components of the Fano variety of maximal linear subspaces in  $\mathbb{P}N(X)_x$ . This map is an isomorphism when  $a \neq 4$ , and is surjective with fibers isomorphic to  $\mathbb{P}^1$  when  $a = 4$ . Moreover, let  $0 \neq t \in T_x l$ ;  $T_{(x,l)}$  is the image of  $\mu_x^+(t, \cdot)$ .*

In particular, in the case  $\mathcal{A} = \mathbb{O}_{\mathbb{C}}$  this proposition proves that the variety of lines in  $X$  through a fixed point  $x \in X$  is isomorphic with a 10-dimensional

spinor variety; this fact is proved in [LM 03, prop 3.4 p.77], but I give here a direct proof which makes it clear which isotropic spaces this spinor variety parametrizes.

**Proof :** First of all, by [Ch 06, proposition 1.4], there are two  $P$ -orbits in  $\mathbb{P}T_x X$ , if  $P$  denotes the stabilizer of  $x$  in  $G$ . Therefore,  $F(x, 1, X) = \mathbb{P}\{\nu_x = 0\} \subset \mathbb{P}T_x X$ . We know that  $\nu_x$  identifies with  $\nu_2$  and in [Cha 05, section 3.4], I described the locus where  $\nu_2$  vanishes; therefore, we get the following array ( $G_Q^+(5, 10)$  denotes a 10-dimensional spinor variety and  $Q$  an 8-dimensional projective quadric;  $G(2, 2n - 2)$  is the grassmannian of 2-dimensional spaces in  $\mathbb{C}^{2n-2}$ ) :

| $V$                            | $F(x, 1, X)$                               | $\mathbb{P}N(X)_x$                         | $\frac{a}{2}(n-2) + 1$ |
|--------------------------------|--|--|------------------------|
| $H_n(\mathbb{C}_{\mathbb{C}})$ | $\mathbb{P}^{n-2} \amalg \mathbb{P}^{n-2}$ | $\mathbb{P}^{n-2} \times \mathbb{P}^{n-2}$ | $n - 1$                |
| $H_n(\mathbb{H}_{\mathbb{C}})$ | $\mathbb{P}^1 \times \mathbb{P}^{2n-3}$    | $G(2, 2n - 2)$                             | $2n - 3$               |
| $H_3(\mathbb{O}_{\mathbb{C}})$ | $S^+$                                      | $Q$  | $5$                    |

Let now  $z \in \mathcal{A}$  such that  $N(z) = 0$ ; the image of  $\mu_x^+((z, 0, \dots, 0), \cdot)$  is

$$\left\{ \left( \begin{array}{c} * \quad z\bar{u}_1, \dots, z\bar{u}_{n-2} \\ u_1\bar{z} \\ \vdots \\ u_{n-2}\bar{z} \end{array} \quad \begin{array}{c} \\ \\ 0 \\ \end{array} \right) : u_i \in \mathcal{A} \right\} ;$$

it is of dimension  $1 + \frac{a}{2}(n-2)$ , so it is a maximal linear subspace of  $N(X)_x$ . If  $l \in F(x, 1, X)$ , the fact that the image of  $\mu_x^+(l, \cdot)$  is  $T_{(x,l)}$  is a consequence of the formula for  $\nu_x^+$  and the computation of  $T_{(x,l)}$  made in the proof of proposition 3.3.

In the case when  $\mathcal{A} = \mathbb{O}_{\mathbb{C}}$ , proposition 1.4 shows that the map of the proposition is an isomorphism. I leave it to the reader to check that in case  $\mathcal{A} = \mathbb{C}_{\mathbb{C}}$ , it is an isomorphism, and in case  $\mathcal{A} = \mathbb{H}_{\mathbb{C}}$ , it has fibers isomorphic with  $\mathbb{P}^1$ .  $\square$

Let  $\nu_x^- : (\widehat{T_x X}/L_x)^* \rightarrow N_x X^* \subset V^*$  be the quadratic map of theorem 3.3 and  $\mu_x^-$  its polarization. We know that  $(N_x, \mathbb{P}N(X)_x)$  is a Scora variety of type  $(n-1, a)$ ; let  $\mathbb{P}N(X)_x^{\vee} \subset \mathbb{P}N_x^*$  denote its dual Scora variety.

We have a similar result for the cotangent space :

**Proposition 3.5.** *The map  $l \mapsto \text{Im } \mu_x^-(l, \cdot)$ , where  $0 \neq l \in T_x l$  defines a morphism between  $F(x, 1, X)$  and some components of the Fano variety of maximal linear subspaces in  $\mathbb{P}N(X)_x^{\vee}$ . It is an isomorphism if  $a \neq 4$ , and is surjective with fibers isomorphic to  $\mathbb{P}^1$  if  $a = 4$ .*

From the array in the proof of proposition 3.4, we see that in case  $n = 3$ ,  $\mathbb{P}N(X)_x$  is a smooth quadric of dimension  $a$ . So there are two families of maximal linear subspaces in  $N(X)_x$ . In case  $\mathcal{A} = \mathbb{C}_{\mathbb{C}}$ , the two families are described by proposition 3.4. But in case  $a \geq 4$ , we only get one family. The other family comes with proposition 3.5, because we can use the canonical isomorphism  $\mathbb{P}N(X)_x^{\vee} = \mathbb{P}N(X)_x$  which holds since  $\mathbb{P}N(X)_x$  is a smooth quadric. One can check that we indeed find two different families with the two dual constructions of propositions 3.4 and 3.5.

### 3.5 The tangent bundle to the variety of lines in a Severi variety

In this subsection, we prepare the description of the Mukai flop of the second kind. Let  $X \subset \mathbb{P}V$  be a Scorza scheme of type  $(3, a)$  (these schemes are called Severi varieties in [Zak 93]) and assume  $a \geq 2$ . Note [Cha 05] that if  $a = 2$ , then  $X$  is isomorphic with  $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$  and that if  $a = 4$ , then  $X$  is isomorphic with the grassmannian  $G(2, 6) \subset \mathbb{P}^{14}$  of 2-dimensional subspaces of  $\mathbb{C}^6$ .

Let  $Y$  denote an irreducible component of the variety of projective lines in  $\mathbb{P}V$  which are included in  $X$ . If  $a = 2$ , then  $Y \simeq (\mathbb{P}^2)^\vee \times (\mathbb{P}^2)$  and if  $a = 4$ , then  $Y$  is isomorphic with the flag variety  $F(1, 3, 6)$  of 1-dimensional subspaces included in a 3-dimensional subspace included in a fixed  $\mathbb{C}^6$ . If  $a = 8$ , then it follows from [LM 03, theorem 4.3 p.82] that  $Y$  is the quotient  $G/P_3$ , where  $G$  is a simply-connected group of type  $E_6$  and  $P_3$  is the parabolic subgroup corresponding to the simple root  $\alpha_3$ . Therefore, the Mukai flop of the second kind is a rational map  $T^*Y \dashrightarrow T^*Y^\vee$ , where  $Y^\vee = G/P_5$ .

The aim of this subsection is to describe the tangent bundle  $TY$ . As before, this will be done in a unified way for all Severi varieties with  $a \geq 2$  (if  $a = 1$ , the variety  $Y$  is empty).

Let us start with an easy lemma. Let  $\det(., ., .)$  be the polarization of the degree 3 polynomial  $\det$  (that is, the unique trilinear symmetric form such that  $\forall v \in V, \det(v, v, v) = 6 \det(v)$ ).

**Lemma 3.1.** *Let  $X$  be a Severi variety and  $x \in X$ . Then we have*

$$\widehat{T_x X} = \{v : \forall w \in V, \det(x, v, w) = 0\}.$$

**Proof :** By [Cha 05, propositions 3.5 and 4.2], the ideal of  $X$  is generated by the quadratic equations  $\det(x, x, .) = 0$ . Therefore, by differentiation, we get the given equations for the tangent space at  $x$ .  $\square$

Now, let  $\alpha \in Y$ . The 2-dimensional linear space it represents will be denoted  $L_\alpha$ . We set

$$\begin{aligned} S_\alpha &:= \langle T_x \widehat{X} \rangle_{x \in L_\alpha - \{0\}} \\ I_\alpha &:= \bigcap_{x \in L_\alpha - \{0\}} T_x \widehat{X}. \end{aligned}$$

It is clear that  $S$  and  $I$  are  $G$ -homogeneous subbundles of the trivial bundle  $V \otimes \mathcal{O}_Y$  over  $Y$ . We moreover consider the quotient bundles defined by  $A_\alpha := I_\alpha/L_\alpha, B_\alpha := S_\alpha/I_\alpha, C_\alpha := V/S_\alpha$ .

**Proposition 3.6.** *The ranks of the bundles  $A, B, C$  are, respectively,  $3a/2 - 2, a + 2, a/2 + 1$ . There is a  $G$ -equivariant short exact sequence of bundles*

$$0 \rightarrow \text{Hom}(L, A) \rightarrow TY \rightarrow \wedge^2 L^* \otimes C^* \rightarrow 0.$$

**Remark :** The image of  $\text{Hom}(L, A)_\alpha$  in  $T_\alpha Y$  may be described geometrically, by [LM 03, theorem 4.3 p.82], as the linear subspace generated by the tangent vectors to lines through  $\alpha$  included in  $Y$ .

**Proof :** Let  $u \in \mathcal{A}$  such that  $N(u) = 0$ . Let  $x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  and  $y = \begin{pmatrix} 1 & 0 & \bar{u} \\ 0 & 0 & 0 \\ u & 0 & 0 \end{pmatrix}$

be vectors in  $\widehat{X}$ . The tangent spaces  $T_x\widehat{X}$  and  $T_y\widehat{X}$  were computed during the proof of proposition 3.3; it follows from this computation that  $T_x\widehat{X} \cap T_y\widehat{X} =$

$$\begin{pmatrix} * & \cdot & \cdot \\ R(u) & 0 & 0 \\ u^\perp & 0 & 0 \end{pmatrix} \text{ (the dots replace coefficients above the diagonal which are}$$

conjugates of elements under it) and that  $\langle T_x\widehat{X}, T_y\widehat{X} \rangle = \begin{pmatrix} * & * & * \\ * & 0 & * \\ * & L(u) & * \end{pmatrix}$ .

Note that these spaces do not change when  $x$  is replaced by  $\lambda.x$ , and  $y$  by  $\nu.y$ ,  $\lambda, \nu \in \mathbb{C}$ . Therefore, if  $\alpha$  represents the subspace generated by  $x$  and  $y$ , we have  $I_\alpha = T_x\widehat{X} \cap T_y\widehat{X}$  and  $S_\alpha = \langle T_x\widehat{X}, T_y\widehat{X} \rangle$ . We see that  $\dim I_\alpha = 3a/2$  and  $\dim S_\alpha = 5a/2 + 2$ . The first result on the ranks of the vector bundles therefore follows.

Let  $\alpha \in Y$ ; I now define a map  $T_\alpha Y \rightarrow \Lambda^2 L_\alpha^* \otimes C_\alpha^*$ . Let  $G(2, V)$  denote the grassmannian of 2-dimensional linear subspaces of  $V$ . We use the fact that  $T_\alpha Y$ , as a subspace of  $T_\alpha G(2, V)$ , may be described as the set of  $\varphi : L_\alpha \rightarrow V/L_\alpha$  such that  $\forall x \in L_\alpha, \varphi(x) \in T_x\widehat{X}/L_\alpha$ . So an element  $\varphi \in T_\alpha Y \subset \text{Hom}(L_\alpha, V/L_\alpha)$  defines a linear map  $\varphi_0 : L_\alpha \otimes L_\alpha \rightarrow V^*$

$$x \otimes y \mapsto (w \mapsto \det(x, \varphi(y), w)).$$

Now, if  $y = \lambda.x$ , with  $\lambda \in \mathbb{C}$ , then  $\varphi(y) \in T_y\widehat{X} = T_x\widehat{X}$ , and so by lemma 3.1,  $\det(x, \varphi(y), w) = 0$  for all  $w \in V$ . Therefore,  $\varphi_0$  induces a linear map  $\varphi_1 : \Lambda^2 L_\alpha \rightarrow V^*$ .

Moreover, assume there exists  $x \in L_\alpha - \{0\}$  such that  $w \in T_x\widehat{X}$ . Then we have  $\det(x, \varphi(y), w) = \det(x, w, \varphi(y)) = 0$  because  $w \in T_x\widehat{X}$ . Choosing  $y \in L_\alpha$  not colinear with  $x$ , this proves that  $\varphi_1(\Lambda^2 L_\alpha) \subset S_\alpha^\perp$ . Since  $S_\alpha^\perp = C_\alpha^*$ , we therefore get an element  $\varphi_2 \in \Lambda^2 L_\alpha^* \otimes C_\alpha^*$ . The map  $\varphi \mapsto \varphi_2$  is the map  $T_\alpha Y \rightarrow \Lambda^2 L_\alpha^* \otimes C_\alpha^*$  of the proposition.

From the realization of  $T_\alpha Y$  as a subspace of  $\text{Hom}(L_\alpha, V/L_\alpha)$ , it is moreover clear that  $\text{Hom}(L_\alpha, A_\alpha)$  is a subspace of  $T_\alpha Y$ . Assume now that  $\varphi_2 = 0$ . This implies that if  $x, y \in L_\alpha$  and  $w \in V$ , then  $\det(x, \varphi(y), w) = 0$ . By lemma 3.1 again, this implies that  $\varphi(y) \in T_x\widehat{X}$ . It follows that  $\text{Im } \varphi \subset A_\alpha$  and  $\varphi \in \text{Hom}(L_\alpha, A_\alpha)$ . Since  $\dim Y = 25$ , the above map  $\varphi \mapsto \varphi_2$  is surjective and the sequence of the proposition is exact.  $\square$

We will see (proposition 3.8) that the projectivised bundle  $\mathbb{P}A$  contains a subvariety which is isomorphic to the relative grassmannian  $G(2, C)$  of 2-dimensional subspaces in  $C$ . Here is a first result in this direction.

**Proposition 3.7.** *There is a  $G$ -equivariant injective map of bundles  $\psi : \Lambda^2 C \otimes \Lambda^2 L \rightarrow A$ . The cokernel bundle is trivial except when  $a = 4$ , in which case it is a line bundle.*

**Proof:** Assume first that  $a = 4$ . Let  $E$  be a 6-dimensional vector space; we have already seen that  $Y = F(1, 3, E)$ . So a point  $\alpha$  in  $Y$  defines a 1-dimensional subspace  $E_1$  of  $E$  and a 3-dimensional subspace  $E_3$  of  $E$ ; moreover,  $E_1 \subset E_3$ . Consider now  $E_1, E_3$  as bundles over  $Y$ .

We have  $V \otimes \mathcal{O}_Y = \Lambda^2 E$ ,  $L = E_1 \wedge E_3 = E_1 \otimes (E_3/E_1)$ ,  $A = E_1 \otimes (E/E_3) \oplus \Lambda^2(E_3/E_1)$  and  $C = \Lambda^2(E/E_3)$ . Set  $A' = E_1 \otimes (E/E_3)$ . Recall that if  $Z$  is a 3-dimensional vector space, then  $\Lambda^2(\Lambda^2 Z)$  is canonically isomorphic with  $Z \otimes \Lambda^3 Z$ .

Therefore,

$$\begin{aligned}
\wedge^2 C \otimes \wedge^2 L &= (E/E_3) \otimes \wedge^3(E/E_3) \otimes E_1 \otimes E_1 \otimes \wedge^2(E_3/E_1) \\
&= (E/E_3) \otimes \wedge^3(E/E_3) \otimes E_1 \otimes \wedge^3 E_3 \\
&= E_1 \otimes (E/E_3) \otimes \wedge^6 E \\
&= A'.
\end{aligned}$$

The last equality follows from the fact that  $\wedge^6 E$  is the trivial line bundle on  $Y$ . We therefore get the map  $\wedge^2 C \otimes \wedge^2 L \rightarrow A$ , which is injective and has 1-dimensional cokernel.

The case when  $a = 2$  is similar.

Assume now that  $a = 8$ . In this case, I don't know any better proof than checking the weights. Recall from [Bou 68] the following : the highest weight of  $V$  is  $\lambda = \frac{1}{3} \begin{bmatrix} 4 & 5 & 6 & 4 & 2 \\ & & & 3 & \end{bmatrix}$  and the lowest is  $\frac{1}{3} \begin{bmatrix} -2 & -4 & -6 & -5 & -4 \\ & & & -3 & \end{bmatrix}$ . Let  $x$  be a vector of weight  $\lambda$  and  $y$  a vector of weight  $s_{\alpha_1}(\lambda) = \frac{1}{3} \begin{bmatrix} 1 & 5 & 6 & 4 & 2 \\ & & & 3 & \end{bmatrix}$ . We may assume that  $L_\alpha$  is the space generated by  $x$  and  $y$ . I claim that the weights of  $C_\alpha$  are  $\frac{1}{3} \begin{bmatrix} -2 & -4 & -6 & -5 & -4 \\ & & & -3 & \end{bmatrix}$ ,  $\frac{1}{3} \begin{bmatrix} -2 & -4 & -6 & -5 & -1 \\ & & & -3 & \end{bmatrix}$ ,  $\frac{1}{3} \begin{bmatrix} -2 & -4 & -6 & -2 & -1 \\ & & & -3 & \end{bmatrix}$ ,  $\frac{1}{3} \begin{bmatrix} -2 & -4 & -3 & -2 & -1 \\ & & & -3 & \end{bmatrix}$  and  $\frac{1}{3} \begin{bmatrix} -2 & -4 & -3 & -2 & -1 \\ & & & 0 & \end{bmatrix}$ . In fact, first, we see that these weights are obtained from the lowest adding successively  $\alpha_6, \alpha_5, \alpha_4, \alpha_2$  (this proves by the way that if  $L \simeq SL_2 \times SL_5$  is included in a Levi factor of  $P_3$ , then  $C_\alpha$  is an irreducible  $SL_5$ -module). Second, the corresponding weight lines are not in  $T_x \widehat{X}$  (resp. neither in  $T_y \widehat{X}$ ) since the weights of this linear subspace are the sum of  $\lambda$  (resp.  $s_{\alpha_1}(\lambda)$ ) and a root. Since no root has a coefficient  $-3$  in  $\alpha_4$ , the claim follows.

Adding the two highest weights of  $C_\alpha$  and the two weights of  $L_\alpha$ , one gets  $\frac{1}{3} \begin{bmatrix} 1 & 2 & 6 & 4 & 2 \\ & & & 3 & \end{bmatrix}$ . This is exactly the highest weight of  $A$ . Therefore, there is an  $L$ -equivariant map  $\wedge^2 C_\alpha \otimes \wedge^2 L_\alpha \rightarrow A_\alpha$ . Since this is a map between irreducible  $L$ -representations, it is also a  $P_3$ -equivariant map, proving the proposition.  $\square$

**Lemma 3.2.** *Let  $\alpha \in Y$  and  $x, y \in \mathbb{P}I_\alpha - \mathbb{P}L_\alpha$  such that  $x \equiv y \pmod{L_\alpha}$ . Then  $x \in X$  if and only if  $y \in X$ .*

**Proof :** Let  $z_1 \neq z_2 \in \mathbb{P}L_\alpha$  and  $i \in \{1, 2\}$ . By definition of  $I_\alpha$ ,  $x \in T_{z_i} X$ . If  $x \in X$ , then the projective line  $(xz_i)$  through  $x$  and  $z_i$  meets  $X$  at the points  $z_i$  and  $x$ , and with multiplicity at least two at  $z_i$ . Since  $X$  is defined by quadratic equations,  $(xz_i) \subset X$ . Therefore, the plane  $(xz_1 z_2)$  meets  $X$  along the three lines  $(z_1 z_2), (xz_1), (xz_2)$ ; so this plane is included in  $X$ . Therefore,  $y \in X$ .  $\square$

**Notation 3.3.** *Let  $A' \subset A$  denote the image of  $\wedge^2 C \otimes \wedge^2 L$  under the map of proposition 3.7. Let  $X(\alpha) \subset \mathbb{P}A'_\alpha$  denote the intersection of the image of  $X$  under the rational projection  $\mathbb{P}I_\alpha \dashrightarrow \mathbb{P}A_\alpha$  and  $\mathbb{P}A'_\alpha$ .*

**Proposition 3.8.** *Assume  $a \geq 4$ . Let  $\alpha \in Y$  and  $x \in X(\alpha)$ . The projectivisation of the inverse of the isomorphism  $\psi_\alpha : \wedge^2 C_\alpha \otimes \wedge^2 L_\alpha \rightarrow A'_\alpha$  maps  $x$  on the element in  $G(2, C_\alpha)$  representing the 2-dimensional space  $T_y \widehat{X}/S_\alpha \subset V/S_\alpha$ , if  $y \in I_\alpha$  is any vector with class  $x$  in  $\mathbb{P}A_\alpha$ .*

**Proof:** Assume first that  $a = 8$ . Let  $\alpha \in Y$ . Since  $A_\alpha$  is an irreducible  $SL_5$ -representation isomorphic with  $\wedge^2 \mathbb{C}^5$ , there is a unique non-trivial invariant subvariety in  $\mathbb{P}A_\alpha$ , and therefore it is  $X(\alpha)$ . If  $a = 4$ , then obviously we also have  $X(\alpha) = \mathbb{P}A'_\alpha$ .

Let  $a \in \{4, 8\}$  and assume as in the proof of proposition 3.6 that  $x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $y = \begin{pmatrix} 1 & 0 & \bar{u} \\ 0 & 0 & 0 \\ u & 0 & 0 \end{pmatrix}$  and  $L_\alpha$  is spanned by  $x$  and  $y$ . If  $z = \begin{pmatrix} 1 & 0 & \bar{v} \\ 0 & 0 & 0 \\ v & 0 & 0 \end{pmatrix}$ , with  $\langle u, v \rangle = 0$ , then the 3-dimensional space generated by  $x, y, z$  lies in  $X$  and  $T_z \widehat{X}/S_\alpha \subset C_\alpha$  is 2-dimensional and does not change if  $z$  is replaced by a linear combination of  $x, y$  and  $z$ . By homogeneity of  $X(\alpha)$ , this fact holds for any  $[z] \in X(\alpha)$  and so we have a well-defined map  $X(\alpha) \rightarrow G(2, C_\alpha)$ . Since there is only one such  $P_3$ -equivariant map, this map also coincides with the restriction of the projectivisation of  $\psi^{-1}$ .  $\square$

We now assume  $a = 8$ , and conclude this subsection classifying the  $E_6$ -orbits in  $T^*Y$ . By propositions 3.6 and 3.7, there is a vector bundle map  $T^*Y \rightarrow \text{Hom}(L^* \otimes \wedge^2 L, \wedge^2 C^*) = \text{Hom}(L, \wedge^2 C^*)$ ; I denote it  $h$ .

**Proposition 3.9.** *Let  $\alpha \in Y$  and  $f, g \in T^*Y$ , and assume  $f$  and  $g$  both don't vanish. Then  $f, g$  lay in the same  $E_6$ -orbit if and only if the two elements  $h(f), h(g) \in \text{Hom}(L_\alpha, \wedge^2 C_\alpha^*)$  lay in the same  $(GL(L_\alpha) \times GL(C_\alpha))$ -orbit.*

In view of lemma 2.5, this gives a complete understanding of the  $E_6$ -orbits in  $T^*Y$ .

**Proof:** Let  $P \subset E_6$  be the stabilizer of  $\alpha$  and  $L(P)$  a Levi factor of  $P$ . We know that the image of  $L(P)$  in  $\text{End}(\text{Hom}(L_\alpha, \wedge^2 C_\alpha^*))$  is the same as that of  $GL(L_\alpha) \times GL(C_\alpha)$ . If  $h(f) = h(g) = 0$ , then, by proposition 3.6,  $f$  and  $g$  are elements in  $(\wedge^2 L_\alpha \otimes C_\alpha) - \{0\}$ , which is obviously homogeneous under  $L(P)$ , and so lay in the same  $P$ -orbit.

Assume  $h(f) \neq 0$  and  $h(g) \neq 0$ . Since by hypothesis  $h(f)$  and  $h(g)$  lay in the same  $L(P)$ -orbit, we may assume that  $h(f) = h(g)$ . Let  $R_u(P)$  denote the unipotent radical of  $P$ ;  $R_u(P)$  acts trivially on the irreducible  $P$ -representation  $\text{Hom}(L_\alpha, \wedge^2 C_\alpha^*)$ . Therefore, it is enough to prove that the  $R_u(P)$ -orbit of  $f$  is dense in  $h^{-1}(h(f))$ . Equivalently, we will prove that the image of the action of the Lie algebra of  $R_u(P)$  on  $f$  contains  $\wedge^2 L_\alpha \otimes C_\alpha$ .

It is enough to prove this when  $h(f)$  is in the minimal non-zero orbit of  $GL(L_\alpha) \times GL(C_\alpha)$  in  $\text{Hom}(L_\alpha, \wedge^2 C_\alpha^*)$ . This, in turn, can be verified at the level of weights. In fact, we assume that  $h(f)$  is a highest weight vector of  $\text{Hom}(L_\alpha, \wedge^2 C_\alpha^*)$ . Therefore,  $h(f)$  has weight  $\begin{bmatrix} 1 & 1 & 2 & 2 & 1 \\ & & & & 2 \end{bmatrix}$ . In fact, as we saw in the

proof of proposition 3.7, the highest weight of  $L_\alpha^*$  is  $\frac{1}{3} \begin{bmatrix} -1 & -5 & -6 & -4 & -2 \\ & & & & -3 \end{bmatrix}$ ,

and the two highest weights of  $C_\alpha^*$  are  $\frac{1}{3} \begin{bmatrix} 2 & 4 & 6 & 5 & 4 \\ & & & & 3 \end{bmatrix}$  and  $\frac{1}{3} \begin{bmatrix} 2 & 4 & 6 & 5 & 1 \\ & & & & 3 \end{bmatrix}$ .

Since the weight of  $\wedge^2 L$  is  $\frac{1}{3} \begin{bmatrix} 5 & 10 & 12 & 8 & 4 \\ & & & & 6 \end{bmatrix}$  and the heighest weight of  $C$  is  $\frac{1}{3} \begin{bmatrix} -2 & -4 & -3 & -2 & -1 \\ & & & & 0 \end{bmatrix}$ , the highest weight of  $\wedge^2 L_\alpha \otimes C$  is  $\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ & & & & 1 \end{bmatrix} = \omega_2$ . Since this is the highest weight of  $\text{Hom}(L_\alpha, \wedge^2 C_\alpha^*)$  plus  $\alpha_3 + \alpha_4$ , which is a root

of  $R_u(P)$ , we are done.  $\square$

## 4 Mukai flops of type $E_6$

Let  $(V, X)$  be a Scorza variety and  $(V^*, X^\vee)$  the dual Scorza variety. An element in  $T^*X$  will be denoted  $(x, \alpha)$ , where  $x \in X$  and  $\alpha$  is a linear form on  $T_x X$ . The flop is a map  $T^*X \dashrightarrow X^\vee$ ,  $(x, \alpha) \mapsto (h, \kappa) = (h(x, \alpha), \kappa(x, \alpha))$ .

For flops of type  $E_{6,I}$ , the element  $h(x, \alpha)$  was described in the preceding section. The complete description of Mukai flops should also include a formula for  $\kappa(x, \alpha)$ . However, it is not easy to follow the identification of  $T_x^*E_6/P_1$ , seen more or less as a subspace of  $V^*$ , with a subspace of the Lie algebra of  $G$ . Instead, given  $h(x, \alpha)$ , I explain in the next subsection a general geometric way to put our hands on  $\kappa(x, \alpha)$ .

### 4.1 Canonical isomorphism of quotients of tangent spaces to flag varieties

Let  $G$  be a reductive algebraic group and let  $\mathcal{P}, \mathcal{Q}$  denote two flag varieties parametrizing two classes of parabolic subgroups of  $G$ . Let  $\mathcal{R}$  denote the flag variety of parabolic subgroups which are intersections of a parabolic subgroup in  $\mathcal{P}$  and a parabolic subgroup in  $\mathcal{Q}$ . Since a parabolic subgroup in  $\mathcal{R}$  is contained in exactly one subgroup in  $\mathcal{P}$  (resp.  $\mathcal{Q}$ ),  $\mathcal{R}$  is canonically isomorphic with a subvariety of  $\mathcal{P} \times \mathcal{Q}$ ; an element in  $\mathcal{R}$  will therefore be denoted  $(x, y)$ , with  $x \in \mathcal{P}$  and  $y \in \mathcal{Q}$ .

If  $x \in \mathcal{P}$ , let  $\mathcal{Q}_x$  denote the variety of parabolic subgroups  $y$  such that  $(x, y) \in \mathcal{R}$ , and define similarly  $\mathcal{P}_y$ . The following quite general theorem allows, as a special case, describing Mukai flops  $T^*X \dashrightarrow T^*X^\vee$  of type  $E_{6,I}$  as soon as we know the composition  $T^*X \dashrightarrow T^*X^\vee \rightarrow X^\vee$ . Since it is the case by theorem 3.3, it will be easy to deduce a Jordan-theoretic formula for this Mukai flop, in proposition 4.1.

In the next theorem,  $(C, 0, t)$  will denote a pointed curve  $(C, 0)$  which is smooth at 0, together with a tangent vector  $t$  at the point 0. Moreover, if  $Y$  is an algebraic variety and  $f : C \rightarrow Y$  is a map, then  $f'(0) \in T_{f(0)}Y$  will denote the derivative  $df_0(t)$ . I say that there is a Mukai flop  $T^*\mathcal{P} \dashrightarrow T^*\mathcal{Q}$  if the natural maps  $T^*\mathcal{P} \rightarrow \mathfrak{g}, T^*\mathcal{Q} \rightarrow \mathfrak{g}$  are birational and have the same image.

**Theorem 4.1.** *Let  $(x, y) \in \mathcal{R}$ . Then there is a canonical isomorphism  $\mu(x, y) : \frac{T_x \mathcal{P}}{T_x \mathcal{P}_y} \rightarrow \frac{T_y \mathcal{Q}}{T_y \mathcal{Q}_x}$ . If  $(C, 0, t)$  is as above, and if  $\gamma : (C, 0) \rightarrow (\mathcal{P}, x)$  is any map, such an isomorphism maps the class of  $\gamma'(0)$  on the class of  $\delta'(0)$ , if  $\delta : (C, 0) \rightarrow (\mathcal{Q}, y)$  is any map such that  $(\gamma, \delta)(C) \subset \mathcal{R}$ .*

*If, moreover, there is a Mukai flop  $T^*\mathcal{P} \dashrightarrow T^*\mathcal{Q}$ , then this flop maps a generic form  $f \in (T_x \mathcal{P}/T_x \mathcal{P}_y)^*$  to  $(y, {}^t\mu(x, y)^{-1}(f))$ .*

**Proof :** Let  $\pi_{\mathcal{P}} : \mathcal{R} \rightarrow \mathcal{P}$  and  $\pi_{\mathcal{Q}} : \mathcal{R} \rightarrow \mathcal{Q}$  denote the natural projections. Consider the diagram

$$\begin{array}{ccccc} \frac{T_x \mathcal{P}}{T_x \mathcal{P}_y} & \xleftarrow{\varphi_{\mathcal{P}}} & \frac{T_{(x,y)\mathcal{R}}}{(T_{(x,y)\pi_{\mathcal{P}}^{-1}(x)}, T_{(x,y)\pi_{\mathcal{Q}}^{-1}(y)})} & \xrightarrow{\varphi_{\mathcal{Q}}} & \frac{T_y \mathcal{Q}}{T_y \mathcal{Q}_x} \\ \parallel & & \parallel & & \parallel \\ \frac{\mathfrak{g}/\mathfrak{p}}{\mathfrak{q}/\mathfrak{p}} & \simeq & \frac{\mathfrak{g}/(\mathfrak{p} \cap \mathfrak{q})}{(\mathfrak{p}, \mathfrak{q})/(\mathfrak{p} \cap \mathfrak{q})} & \simeq & \frac{\mathfrak{g}/\mathfrak{q}}{\mathfrak{p}/\mathfrak{q}} \end{array},$$

where  $\varphi_P$  (resp.  $\varphi_Q$ ) is induced by the differential  $d_{(x,y)}\pi_P$  (resp.  $d_{(x,y)}\pi_Q$ ). All the terms on the second line are canonically isomorphic with  $\mathfrak{g}/\langle \mathfrak{p}, \mathfrak{q} \rangle$ . Obviously, the diagram commutes, so  $\varphi_P$  and  $\varphi_Q$  are isomorphisms. Let  $\mu(x, y) : \frac{T_x\mathcal{P}}{T_x\mathcal{P}_y} \rightarrow \frac{T_y\mathcal{Q}}{T_y\mathcal{Q}_x}$  denote the canonical isomorphism  $\varphi_Q \circ \varphi_P^{-1}$ .

Let  $f \in (T_x\mathcal{P}/T_x\mathcal{P}_y)^*$  be generic. Let  $(y', f') \in T^*\mathcal{Q}$  denote the image of  $(x, f)$  by the flop  $T^*\mathcal{P} \dashrightarrow T^*\mathcal{Q}$ . By [Cha 06],  $y'$  is the only element in  $\mathcal{Q}$  such that  $f$  vanishes on  $T_x\mathcal{P}_y$ ; since by assumption,  $f$  vanishes on  $T_x\mathcal{P}_y$ ,  $y' = y$ . Moreover, we have canonical isomorphisms  $(T_x\mathcal{P}/T_x\mathcal{P}_y)^* \simeq \mathfrak{u}(\mathfrak{p}) \cap \mathfrak{u}(\mathfrak{q}) \simeq (T_y\mathcal{Q}/T_y\mathcal{Q}_x)^*$ , and under this isomorphism,  $f$  is mapped to  $f'$  by definition of the Mukai flop. It is clear that this isomorphism is the transpose of  $\mu(x, y)^{-1}$ , so the last claim of the proposition is proved.

If  $(\gamma, \delta)$  are as in the proposition, then  $(\gamma'(0), \delta'(0)) \in T_{(x,y)}\mathcal{R}$ ; denote by  $[\gamma'(0), \delta'(0)]$  its class in  $\frac{T_{(x,y)}\mathcal{R}}{(T_{(x,y)}\pi_P^{-1}(x), T_{(x,y)}\pi_Q^{-1}(y))}$ . By definition of  $\varphi_P$  and  $\varphi_Q$ , we have  $\varphi_P([\gamma'(0), \delta'(0)]) = [\gamma'(0)]$  and  $\varphi_Q([\gamma'(0), \delta'(0)]) = [\delta'(0)]$ . We therefore have, as expected,  $\varphi_Q \circ \varphi_P^{-1}([\gamma'(0)]) = [\delta'(0)]$ .  $\square$

## 4.2 Mukai flop of type $E_{6,I}$ in terms of Jordan algebras

Let  $\mu(x, y)$  denote the isomorphism of proposition 4.1. In this subsection, I give an expression of  $\mu(x, y)$  in the case of Scorza varieties, in terms of Jordan algebras. Therefore, this gives also a formula for the Mukai flop.

More precisely, let  $(V, X)$  be a Scorza variety of type  $(n, a)$  and let  $(V^*, X^\vee)$  be the dual Scorza variety. Let  $(x, h) \in X \times X^\vee$  such that  $x \vdash h$ . Let us choose  $(\tilde{x}, \tilde{h}) \in V \times V^*$  such that  $[\tilde{x}] = x$  and  $[\tilde{h}] = h$ . This identifies  $T_x X$  (resp.  $T_h X^\vee$ ) with  $T_{\tilde{x}}\widehat{X}/\mathbb{C}\tilde{x}$  (resp.  $T_{\tilde{h}}\widehat{X}^\vee/\mathbb{C}\tilde{h}$ ). The previous isomorphism  $\mu(x, h) : T_x X/T_x C_h \simeq T_h X^\vee/T_h C_x$  induces an isomorphism  $T_{\tilde{x}}\widehat{X}/T_{\tilde{x}}\widehat{C}_h \simeq T_{\tilde{h}}\widehat{X}^\vee/T_{\tilde{h}}\widehat{C}_x$ .

The goal of this section is to give a formula for this isomorphism in Jordan terms.

For  $A, B \in V$ , let  $\sigma_A(B) \in V^*$  denote the linear form  $U \mapsto D_A^2 \det(B, U)$ . Note that this is equal, modulo  $D_A \det$ , to  $S_A(B)$  [Cha 06]. For  $h \in X^\vee$ , let  $V(h) := (T_h X^\vee)^\perp \subset V$ .

**Lemma 4.1.** *Let  $A \in V(h)$ . Then  $\sigma_A(\tilde{x})$  is proportional to  $\tilde{h}$ .*

**Proof:** We can assume that  $V = H_3(A)$  and  $X \subset \mathbb{P}V$  is the variety of rank one elements. Identify  $V$  and  $V^*$  as usually. Since  $X^\vee$  is homogeneous under  $G$ ,

we may assume that  $h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ . Then  $V(h) = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$ . It is enough

to prove the lemma for generic  $A$  in  $V(h)$ , so we may assume that  $A$  has rank 2. Moreover, let  $G_h$  denote the stabilizer in  $\text{Aut}(X)$  of  $h$ . It is clear that  $G_h$  acts transitively on the set of rank 2 elements of  $V(h)$ , and on the set of its rank 1

elements. So we may assume  $\tilde{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ . Moreover, for the stabilizer of  $x$  in

$G_h$ , the set of  $A$ 's of rank 2 is made of two orbits and  $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  is in the open orbit, as it is easily checked case by case.

Therefore, it is enough to compute  $\sigma_A(\tilde{x})$  for these choices of  $x$  and  $A$ . Let  $m = (m_{i,j}) \in V$ . Then one computes

$$D_A^2 \det(m, m) = \sum_{i < n} m_{i,i} m_{n,n} - \sum_{i < n} N(m_{n,i}). \quad (1)$$

The lemma immediately follows.  $\square$

Let as before  $x \in X, h \in C_x, A \in V(h)$ , and let  $\tilde{x} \in V, \tilde{h} \in V^*$  represent  $x$  and  $h$ .

**Proposition 4.1.** *Let  $v \in T_{\tilde{x}}\widehat{X}$ , and let  $[v]$  denote its class in  $T_{\tilde{x}}\widehat{X}/T_{\tilde{x}}\widehat{C}_h \simeq T_x X/T_x C_h$ . If  $\sigma_A(\tilde{x}) = \tilde{h}$ , then the vector*

$$[\sigma_A(v)] \in T_{\tilde{h}}\widehat{X^\vee}/T_{\tilde{h}}\widehat{C}_x \simeq T_h X^\vee/T_h C_x$$

*identifies with  $\mu(x, h)([v])$ .*

The isomorphism  $T_{\tilde{x}}\widehat{X}/T_{\tilde{x}}\widehat{C}_h \simeq T_x X/T_x C_h$  depends on the choice of  $\tilde{x}$ , and the isomorphism  $T_{\tilde{h}}\widehat{C}_x \simeq T_h X^\vee/T_h C_x$  depends on the choice of  $\tilde{h}$ . However, the proposition says that the corresponding map  $T_x X/T_x C_h \rightarrow T_h X^\vee/T_h C_x$  does not depend on these choices, neither on the choice of  $A$ , as long as  $\sigma_A(\tilde{x}) = \tilde{h}$ .

**Proof :** As in the previous lemma, we assume that  $V = H_n(\mathcal{A})$ . Let  $X_r$  denote the variety of rank  $r$  matrices. If  $B \in X_{n-1}$ , then  $D_B \det$  belongs to  $X^\vee$ . Since  $D_A \det = h$  and  $\widehat{T_x X} \subset \widehat{T_A X_{n-1}}$ , we have the implication  $u \in \widehat{T_x X} \implies \sigma_A(u) \in \widehat{T_h X^\vee}$ .

Now, let  $v \in \widehat{T_x X}$  and let  $u$  be the class of  $v$  in  $\widehat{T_x X}/\mathbb{C}\tilde{x}$ . Let  $\varphi(\tilde{x}, \tilde{h}, A)(u)$  denote the element of  $T_h X^\vee$  corresponding to the class of  $\sigma_A(v)$  in  $\widehat{T_h X^\vee}/\mathbb{C}\tilde{h}$  (by lemma 4.1, this class depends only on  $u$ ).

We first show that if  $\tilde{x}, \tilde{h}, A$  are multiplied by a scalar, then  $\varphi(\tilde{x}, \tilde{h}, A)$  does not vary. So let  $\lambda, \mu, \nu \in \mathbb{C}$ , and assume  $\sigma_{\nu A}(\lambda \tilde{x}) = \mu \tilde{h}$ . Since by assumption  $\sigma_A(\tilde{x}) = \tilde{h}$ , this means that  $\nu^{n-2} \lambda = \mu$ . Now,  $\lambda \tilde{x}$  will identify  $u$  with  $\lambda v$ . Then,  $\sigma_{\nu A}(\lambda v) = \nu^{n-2} \lambda \sigma_A(v) = \mu \sigma_A(v)$ , and the class of this vector will identify with  $\varphi(\tilde{x}, \tilde{h}, A)(u) \in T_h X^\vee$  with the choice  $\mu \tilde{h}$  instead of  $\tilde{h}$ .

Therefore, the claim is proved, and one can choose the same elements  $\tilde{h}, \tilde{x}, A$  as in the proof of the lemma. By formula (1), if  $v = \begin{pmatrix} t & \bar{w} & \bar{z} \\ w & 0 & 0 \\ z & 0 & 0 \end{pmatrix} \in \widehat{T_x X}$ , then

$$\sigma_A(v) = \begin{pmatrix} 0 & 0 & -\bar{z} \\ 0 & 0 & 0 \\ -z & 0 & t \end{pmatrix}, \text{ if one identifies } V \text{ and } V^* \text{ via the usual scalar product.}$$

Note that  $T_{\tilde{h}}\widehat{C}_x = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & * & * \end{pmatrix} \right\}$ , so that the class of  $\sigma_A(v)$  does not depend on  $w$ , neither on  $t$ .

Let us now compute  $\mu(x, h)([v])$  using theorem 4.1, and assuming  $w = 0$ . So let  $z \in \mathcal{A}$  and  $t \in \mathbb{C}$ . Recall that generic elements of  $\widehat{X}$  can be written as  $\nu_2(\alpha, z_1, z_2)$ , with  $\alpha \in \mathbb{C}$ ,  $z_1, z_2 \in \mathcal{A}$ , and  $\nu_2(\alpha, z_1, z_2) = \begin{pmatrix} \alpha^2 & \alpha\bar{z}_1 & \alpha\bar{z}_2 \\ \alpha z_1 & N(z_1) & z_1\bar{z}_2 \\ \alpha z_2 & z_2\bar{z}_1 & N(z_2) \end{pmatrix}$ .

Denote  $x(t) = \nu_2(1, 0, tz) = \begin{pmatrix} 1 & 0 & \bar{z}t \\ 0 & 0 & 0 \\ tz & 0 & N(z)t^2 \end{pmatrix}$ ; we have  $x(t) \in \widehat{X}$  and  $x'(0) =$

$$\begin{pmatrix} 0 & 0 & \bar{z} \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix}.$$

Differentiating  $\nu_2$  we get

$$\widehat{T_{x(t)}X} = \left\{ \begin{pmatrix} 2\alpha & \bar{z}_1 & \bar{z}_2 + \alpha t\bar{z} \\ z_1 & 0 & tz_1\bar{z} \\ \alpha tz + z_2 & tz\bar{z}_1 & t(z\bar{z}_2 + z_2\bar{z}) \end{pmatrix} : \alpha \in \mathbb{C}, z_1, z_2 \in \mathcal{A} \right\}$$

Recall that the incidence relation between  $X$  and  $X^\vee$  is :  $x \vdash h$  if  $h \supset \widehat{T_x X}$ ; therefore, if we set  $h(t) = \begin{pmatrix} t^2 N(z)/2 & 0 & -t\bar{z} \\ 0 & 0 & 0 \\ -tz & 0 & 1 \end{pmatrix}$ , we have  $x(t) \vdash h(t)$ , and since

$h'(0) = -\begin{pmatrix} 0 & 0 & \bar{z} \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix}$ , the proposition follows.  $\square$

### 4.3 Mukai flops for Scorza varieties in terms of $\mathcal{A}$ -blow-up

The simplest Mukai flop  $T^*\mathbb{P}^n \dashrightarrow T^*(\mathbb{P}^n)^\vee$  can be resolved blowing-up the zero section. Let's recall this construction. Let  $Z \subset T^*\mathbb{P}^n$  be the zero section, and let  $B$  be the blow-up of  $T^*\mathbb{P}^n$  along  $Z$ . It is known that there is a map  $B \rightarrow T^*(\mathbb{P}^n)^\vee$  such that the following triangle commutes :

$$\begin{array}{ccc} & B & \\ & \swarrow & \searrow \\ T^*\mathbb{P}^n & \dashrightarrow & T^*(\mathbb{P}^n)^\vee. \end{array} \quad (2)$$

Moreover, this is the minimal resolution, in the sense that for any other  $B'$  with the same property, there is a map  $B' \rightarrow B$  and an obvious commutative diagram.

In this subsection, I give a similar resolution of the rational map  $T^*X \dashrightarrow T^*X^\vee$ , if  $X$  is a Scorza variety and  $X^\vee$  the corresponding dual Scorza variety. In fact, the main idea is that since  $X$  behaves like a projective space  $\mathbb{P}_{\mathcal{A}}^n$  over  $\mathcal{A}$ , one should consider an " $\mathcal{A}$ -blow-up".

Let me make a heuristic comment. Given a composition algebra  $\mathcal{A}$ , I believe in the existence of a category  $\mathcal{A}\text{-Var}$  of  $\mathcal{A}$ -varieties, containing projective spaces and grassmanians over  $\mathcal{A}$ . Moreover, if  $Y \subset X$  is a closed immersion in this category, then there should be an object  $Bl_X(Y)$  over  $X$  defined by a universal property analogous to that defining usual blow-ups, but in the category  $\mathcal{A}\text{-Var}$ . Since for the moment I don't know how to define  $\mathcal{A}\text{-Var}$ , I will not give this construction here. In the following we will only have very simple  $\mathcal{A}$ -blow-ups to do, and in these simple cases we can guess what the blow-up should be.

So let  $\mathcal{A}$  be a composition algebra over  $\mathbb{C}$  of dimension  $a$  and  $n \geq 2$  an integer, with  $n = 2$  if  $\mathcal{A} = \mathbb{O}_{\mathbb{C}}$ . Let the affine space  $\mathbb{A}_{\mathcal{A}}^n$  be just  $\mathcal{A}^n$ , the affine  $(an)$ -dimensional space over  $\mathbb{C}$ .

Recall that in subsection 1.2, I introduced a map  $\bar{\nu}_2 : \mathbb{A}_{\mathbb{O}}^2 \dashrightarrow \mathbb{P}_{\mathbb{O}}^1$ , where by definition  $\mathbb{P}_{\mathbb{O}}^1$  is an 8-dimensional smooth quadric. Recall also the rank 8 vector bundle  $L$  over  $\mathbb{P}_{\mathbb{O}}^1$  of proposition 1.7. By definition,  $L$  is a subbundle of the trivial bundle  $\mathbb{A}_{\mathbb{O}}^2 \otimes \mathcal{O}_{\mathbb{P}_{\mathbb{O}}^1}$  of rank 16 over  $\mathbb{P}_{\mathbb{O}}^1$ . Therefore, if  $\mathcal{L}$  denotes the total space of the vector bundle  $L$ , there is an inclusion  $\mathcal{L} \subset \mathbb{A}_{\mathbb{O}}^2 \times \mathbb{P}_{\mathbb{O}}^1$ . Therefore we have a map  $\mathcal{L} \rightarrow \mathbb{A}_{\mathbb{O}}^2$ .

The case of associative algebras is simpler and was studied in [Cha 05] : recall that there is a rational map  $\bar{\nu}_2 : \mathbb{A}_{\mathcal{A}}^n \dashrightarrow \mathbb{P}_{\mathcal{A}}^{n-1}$  and a rank  $a$  subbundle  $\mathcal{L}$  of the trivial bundle  $\mathbb{A}_{\mathcal{A}}^n \otimes \mathcal{O}_{\mathbb{P}_{\mathcal{A}}^{n-1}}$  with fiber  $\mathbb{A}_{\mathcal{A}}^n$  over  $\mathbb{P}_{\mathcal{A}}^{n-1}$ . This subbundle is also defined by  $\mathcal{L}_x = \overline{\{v \in \mathbb{A}_{\mathcal{A}}^n : \bar{\nu}_2(v) \text{ is defined and } \bar{\nu}_2(v) = x\}}$ . Let  $\mathcal{L}$  denote its total space.

**Definition 4.1.** *The  $\mathcal{A}$ -blow-up  $Bl_{\mathbb{A}_{\mathcal{A}}^n}(0)$  of the affine space  $\mathbb{A}_{\mathcal{A}}^n$  at the origin is the map  $\mathcal{L} \rightarrow \mathbb{A}_{\mathcal{A}}^n$ .*

Recall that in  $\mathbb{A}_{\mathbb{O}}^2$  there are three  $Spin_{10}$ -orbits : the open orbit, the point 0, and the affine cone  $\widehat{\mathbb{S}}$  over a spinor variety  $\mathbb{S} \subset \mathbb{P}_{\mathbb{O}}^2$ . The map  $Bl_{\mathbb{A}_{\mathbb{O}}^2}(0) \rightarrow \mathbb{A}_{\mathbb{O}}^2$  is an isomorphism above the open orbit, and the fiber over 0 is isomorphic with  $\mathbb{P}_{\mathbb{O}}^1$ . Except for the existence of the intermediate orbit  $\widehat{\mathbb{S}} - \{0\}$  in  $\mathbb{A}_{\mathbb{O}}^2$ , the situation is therefore very similar to that of the usual blow-up of the origin in  $\mathbb{A}_{\mathbb{C}}^2$ . A similar statement holds in general for the blow-up of  $\mathbb{A}_{\mathcal{A}}^n$ . The following result gives another analogy with usual blow-ups :

**Proposition 4.2.** *This  $\mathcal{A}$ -blow-up is the minimal resolution of the rational map  $\bar{\nu}_2 : \mathbb{A}_{\mathcal{A}}^n \dashrightarrow \mathbb{P}_{\mathcal{A}}^{n-1}$ .*

**Proof :** Let  $\pi : \mathcal{L} \rightarrow \mathbb{A}_{\mathcal{A}}^n$  denote this  $\mathcal{A}$ -blow-up. The restriction of  $\pi$  to the regular locus of  $\bar{\nu}_2$  is an isomorphism. By definition, there are maps  $Bl_{\mathbb{A}_{\mathcal{A}}^n}(0) \rightarrow \mathbb{A}_{\mathcal{A}}^n$  and  $Bl_{\mathbb{A}_{\mathcal{A}}^n}(0) \rightarrow \mathbb{P}_{\mathcal{A}}^{n-1}$  such that the diagram

$$\begin{array}{ccc} & Bl_{\mathbb{A}_{\mathcal{A}}^n}(0) & \\ \swarrow & & \searrow \\ \bar{\nu}_2 : \mathbb{A}_{\mathcal{A}}^n & \dashrightarrow & \mathbb{P}_{\mathcal{A}}^{n-1} \end{array}$$

commutes.

Let

$$\begin{array}{ccc} & B' & \\ \swarrow & & \searrow \\ \bar{\nu}_2 : \mathbb{A}_{\mathcal{A}}^n & \dashrightarrow & \mathbb{P}_{\mathcal{A}}^{n-1} \end{array}$$

be another resolution. Then we have a map  $B' \rightarrow \mathbb{A}_{\mathcal{A}}^n \times \mathbb{P}_{\mathcal{A}}^{n-1}$ . Since the above diagram is commutative, the image of this map is  $Bl_{\mathbb{A}_{\mathcal{A}}^n}(0)$ , and we get the desired map  $B' \rightarrow Bl_{\mathbb{A}_{\mathcal{A}}^n}(0)$ .  $\square$

Let  $(V, X)$  be a Scorza variety of type  $(n, a)$  with  $n \geq 3$ . The above construction of the blow-up of a point in the fixed vector-space  $\mathbb{A}_{\mathcal{A}}^{n-1}$  extends readily to a blow-up of the zero section in the vector bundle  $T^*X$ . In fact, let  $x \in X$ ; recall (theorem 3.3) that we have a quadratic map  $\overline{\nu}_x^- : T_x^*X \rightarrow N_x^*X$ , where  $N_xX = V/\widehat{T_xX}$ , and so a rational map  $\overline{\nu}_x^- : T_x^*X \dashrightarrow \mathbb{P}N_x^*X$ . This map is isomorphic with our model map  $\overline{\nu}_2^- : \mathcal{A}^{n-1} \dashrightarrow \mathbb{P}_{\mathcal{A}}^{n-2}$ . Letting  $x$  vary, we get an algebraic map  $\nu^- : T^*X \rightarrow N^*X$  over  $X$ , and so a rational map  $\overline{\nu}^- : T^*X \dashrightarrow \mathbb{P}N^*X$ .

In subsection 3.4, the projectivisation of the image of  $\nu^-$  was denoted  $\mathbb{P}N(X)^\vee$ ;  $\mathbb{P}N(X)^\vee$  is a locally trivial fibration over  $X$  with fibers Scorza varieties of type  $(n-1, a)$ . Let  $p_X : \mathbb{P}N(X)^\vee \rightarrow X$  denote the natural projection.

Consider the bundle  $p_X^*T^*X$  above  $\mathbb{P}N(X)^\vee$ . An element of this bundle will be denoted  $(x, h, f)$ , with  $x \in X, h \in \mathbb{P}N(X)_x^\vee$  and  $f \in T_x^*X$ . Globalizing the above construction, let  $\mathcal{L} \subseteq p_X^*T^*X$  be defined as the closure of the set of  $(x, h, f) \in p_X^*T^*X$  such that  $\overline{\nu}_x^-(f)$  is defined and equals  $h$ .

**Lemma 4.2.**  $\mathcal{L} \subseteq p_X^*T^*X$  is a subbundle.

**Proof:** Assume first that  $a = 8$ . Then it is simply a global version of proposition 1.7. By theorem 3.3, We know that  $\nu^-$  is a global algebraic map, which on each fiber  $T^*X$  is isomorphic with the map  $\nu_2^- : \mathbb{O}_k \oplus \mathbb{O}_k \dashrightarrow \mathbb{P}_0^1$  defined in subsection 1.2. Therefore, the argument of proposition 1.7 works in this situation. The case of associative composition algebras  $\mathcal{A}$  is similar and left to the reader.  $\square$

Let  $Z \subseteq T^*X$  denote the zero section.

**Definition 4.2.** The  $\mathcal{A}$ -blow-up  $Bl_{T^*X}(Z)$  of  $T^*X$  along  $Z$  is the map  $\mathcal{L} \rightarrow T^*X$ .

**Theorem 4.2.** This  $\mathcal{A}$ -blow-up is the minimal resolution of the Mukai flop  $\mu : T^*X \dashrightarrow T^*X^\vee$ .

**Proof:** Globalizing the proof of proposition 4.2, we see that  $Bl_{T^*X}(Z)$  is the minimal resolution of the rational map  $\overline{\nu}^- : T^*X \dashrightarrow \mathbb{P}N(X)^\vee$ . In view of theorem 3.3, it is also the minimal resolution of the composition  $T^*X \xrightarrow{\mu} T^*X^\vee \rightarrow X^\vee$ . Now, by theorem 4.1, resolving the Mukai flop  $T^*X \dashrightarrow T^*X^\vee$  is equivalent with resolving its projection to  $X^\vee$ , so the theorem follows.  $\square$

#### 4.4 Mukai flop of type $E_{6,II}$

Let  $Y = E_6/P_3$  be the homogeneous space considered in subsection 3.5,  $Y^\vee = E_6/P_5$  the “dual” homogeneous space and  $A, B, C$  the homogeneous vector bundles over  $Y$  defined there. Let also  $X = E_6/P_1$  and  $X^\vee = E_6/P_6$ .

We already used the fact that  $Y$  is isomorphic with the Fano variety of projective lines included in  $X$ . Similarly,  $Y^\vee$  is the variety of lines included in  $X^\vee$ . Denote as before  $\mathbb{P}V$  the ambient space of  $X$ . As we have already seen,  $X^\vee$  identifies with the set of hyperplanes in  $V$  which contain two tangent spaces to  $X$ .

Therefore, given a point  $\alpha \in Y$ , which represents a projective line  $l_\alpha$  contained in  $X$ , and given two points  $x \neq y \in X$ , any hyperplane  $h \subseteq \mathbb{P}V$  which contains the span of  $T_xX$  and  $T_yX$  can be considered as an element of  $X^\vee$ . A codimension two subspace  $V_\beta \subseteq V$  containing this span defines a pencil of hyperplanes belonging to  $X^\vee$ , or a point in  $Y^\vee$ .

Let  $\alpha \in Y$ . Recall that the linear space  $V/\langle \widehat{T_x X}, \widehat{T_y X} \rangle$  (where  $x$  and  $y$  are different points of the line  $l_\alpha$ ) was denoted  $C_\alpha$  in subsection 3.5. Let  $\mathbb{P}C^*$  denote the projective bundle over  $Y$  and  $G(3, C)$  the relative grassmannian of 3-spaces in  $C$ . The preceding remarks show that there are natural maps  $\mathbb{P}C^* \rightarrow X^\vee$  and  $G(3, C) \rightarrow Y^\vee$ . Let  $f : G(3, C) \rightarrow Y^\vee$  be this map.

For any  $\alpha \in Y$ , let  $g_\alpha : \text{Hom}(L_\alpha, \wedge^2 C_\alpha^*) \dashrightarrow G(3, C_\alpha)$  be the map defined by lemmas 2.2 and 2.3 using  $F = C_\alpha$  (namely,  $g_\alpha(\varphi) = U(\varphi(l_1), \varphi(l_2))$  for  $\varphi \in \text{Hom}(L_\alpha, \wedge^2 C_\alpha^*)$  and any non-colinear  $l_1, l_2 \in L_\alpha$ ). By propositions 3.6 and 3.7, there is a natural vector bundle map  $T^*Y \rightarrow \text{Hom}(L^* \otimes \wedge^2 L, \wedge^2 C^*) = \text{Hom}(L, \wedge^2 C^*)$ , which I denote  $h$ .

Let finally  $\mu : T^*Y \dashrightarrow T^*Y^\vee$  be the Mukai flop and  $\pi : T^*Y^\vee \rightarrow Y^\vee$  the structure map.

**Theorem 4.3.** *The composition*

$$T^*Y \xrightarrow{h} \text{Hom}(L, \wedge^2 C^*) \xrightarrow{g} G(3, C) \xrightarrow{f} Y^\vee$$

*equals the composition*

$$T^*Y \xrightarrow{\mu} T^*Y^\vee \xrightarrow{\pi} Y^\vee.$$

**Remark :** This describes the rational map  $\pi \circ \mu$ . The rational map  $\mu$  itself is then described using proposition 4.1.

**Proof :** Let  $\alpha \in Y$  and generic  $\eta \in T_\alpha^*Y$ . We know that  $\pi \circ \mu(\eta)$  is the unique  $\beta \in f(G(3, C_\alpha))$  such that  $\eta$  vanishes on the tangent space  $T_\alpha SC_\beta$  at  $\alpha$  of the Schubert cell  $SC_\beta \subset Y$  defined by  $\beta$ . So first, we compute  $T_\alpha SC_\beta$ .

If  $\beta = f(\beta_0)$ , with  $\beta_0 \in G(3, C_\alpha)$ , let  $c_\beta \subset C_\alpha$  be the 3-dimensional subspace corresponding to  $\beta_0$ . Let  $a_\beta$  denote the image of  $\wedge^2 c_\beta \otimes \wedge^2 L_\alpha \subset \wedge^2 C_\alpha \otimes \wedge^2 L_\alpha$  in  $A_\alpha$  under the isomorphism of proposition 3.7. By proposition 3.8, if the class modulo  $L_\alpha$  of  $x \in X$  is in  $\mathbb{P}a_\beta$ , then  $\widehat{T_x X}/S_\alpha \subset c_\beta$ .

Let  $v_\beta \subset V$  denote the inverse image of  $a_\beta$  under the projection  $I_\alpha \rightarrow A_\alpha = I_\alpha/L_\alpha$ . Since  $SC_\beta$  is the variety of lines  $l \subset X$  such that  $\forall x \in l, \widehat{T_x X}/S_\alpha \subset c_\beta$ , we deduce  $G(2, v_\beta) \subset SC_\beta$ .

Now, given  $\alpha \in Y$ , the cell  $SC_\alpha \in Y^\vee$  identifies with  $G(3, C_\alpha)$ . By symmetry,  $SC_\beta$  is also isomorphic with a six-dimensional grassmannian, and so  $G(2, v_\beta) = SC_\beta$ . Therefore,  $T_\alpha SC_\beta = \text{Hom}(L_\alpha, a_\beta)$ .

Now, we complete the proof. We already saw that a cotangent form  $\eta \in T_\alpha^*Y$  defines an element  $h(\eta) \in \text{Hom}(L_\alpha, \wedge^2 C_\alpha^*)$ . Given the previous computation of  $T_\alpha SC_\beta$ ,  $\eta$  vanishes on  $T_\alpha SC_\beta$  if and only if  $\wedge^2 c_\beta \perp \text{Im } h(\eta)$ . Therefore, we can conclude thanks to lemmas 2.2 and 2.3.  $\square$

Recall that if  $V_2$  and  $V_5$  are vector spaces of respective dimensions 2 and 5, there are 8  $(GL(V_2) \times GL(V_5))$ -orbits in  $\text{Hom}(V_2, \wedge^2 V_5)$ , which were given a label in lemma 2.5. We also use the standart labels for nilpotent orbits, as in [McG 02, p.202].

**Corollary 4.3.** *Let  $\alpha \in Y$  and  $0 \neq f \in T_\alpha^*Y$ . Under the natural map  $T^*Y \rightarrow \mathfrak{e}_6$ ,  $f$  is mapped to the nilpotent orbit with the same label as that of the  $(GL(L) \times GL(C))$ -orbit  $h(f) \in Hom(L, \wedge^2 C^*)$  belongs. The Mukai flop is defined exactly on the open orbit of  $T^*Y$ .*

In this corollary, I mean that if  $h(f)$  is the orbit labelled  $3A_{1,a}, 3A_{1,b}$  or  $3A_{1,c}$ , then it is mapped on the nilpotent orbit labelled  $3A_1$ .

**Proof :** This corollary follows from dimension arguments, which are not so illuminating on the geometry of the resolution. The map  $T^*Y \rightarrow \mathfrak{e}_6$  being birational and proper, it has 50-dimensional closed image; so it is the closure of the unique 50-dimensional orbit in  $\mathfrak{e}_6$ , labelled  $A_2 + 2A_1$ . By the given graph of orbit closures [McG 02, p.212], the image of  $T^*Y$  is the union of the orbits labelled  $0, A_1, 2A_1, 3A_1, A_2, A_2 + A_1, A_2 + 2A_1$ .

Let  $\alpha \in Y$  and  $f, g \in T_\alpha^*Y$ , with  $f \neq 0$  and  $g \neq 0$ . Then  $f$  and  $g$  lay in the same  $E_6$ -orbit if and only if  $h(f), h(g) \in Hom(L_\alpha, \wedge^2 C_\alpha^*)$  lay in the same  $(GL(L_\alpha) \times GL(C_\alpha))$ -orbit, by proposition 3.9. It is clear that the zero section in  $T^*Y$  is mapped to the 0-orbit in  $\mathfrak{e}_6$ . Let us label the other  $E_6$ -orbits in  $T^*Y$  by the labels of their images in  $Hom(L, \wedge^2 C^*)$ .

We first begin with a trivial remark : the image of an orbit in  $T^*Y$  is an orbit in  $\mathfrak{e}_6$  of non-greater dimension. From this it follows that the orbits in  $T^*Y$  labelled  $A_2 + 2A_1, A_2 + A_1, A_1$  map to the orbits in  $\mathfrak{e}_6$  with the same label.

Suppose the orbit in  $T^*Y$  labelled  $3A_{1,a}$  maps to the orbit in  $\mathfrak{e}_6$  labelled  $A_2$ . Then the fibers above the nilpotent orbit  $A_2$  would have dimension 3, and the preimage of the nilpotent orbit labelled  $3A_1$  would be included in the orbits labelled  $3A_{1,b}$  and  $3A_{1,c}$ . So the fibers above this orbit would have dimension 1 or 2, contradicting the semi-continuity of the dimensions of the fibers of a morphism. Therefore,  $3A_{1,a}$  maps to  $3A_1$ .

We know that the resolution  $T^*Y \rightarrow \mathfrak{e}_6$  is semi-small, so the 38-dimensional orbit in  $T^*Y$  labelled  $2A_1$  cannot contract to the 22-dimensional orbit labelled  $A_1$ ; therefore, it maps to  $2A_1$ .

We deduce that the fibers above the  $A_1$ -orbit are 8-dimensional; by semi-continuity again, the orbits labelled  $3A_{1,b}, 3A_{1,c}$  map to the orbit  $3A_1$ .

Since by lemma 2.5 the map  $U$  of notation 2.2 is defined only on the open orbit of  $Hom(L_\alpha, \wedge^2 C_\alpha^*)$ , the Mukai flop is also defined only on the open orbit of  $T^*Y$ .  $\square$

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