

A Kesten-Spitzer result for a random walk in a stationary scenery with strong decorrelation properties

Stéphane Le Borgne and Françoise Pène

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Abstract. *In this paper, we extend a result of Kesten and Spitzer [15]. Let us consider an invertible probability dynamical system (M, \mathcal{F}, ν, T) and $f : M \rightarrow \mathbb{R}$ some function with null expectation. We define the stationary sequence $(\xi_k := f \circ T^k)_{k \in \mathbb{Z}}$. Let $(S_n)_{n \geq 0}$ be a simple symmetric random walk on \mathbb{Z} independent of $(\xi_k)_{k \in \mathbb{Z}}$. We are interested in the study of the sequence of random variables of the form $(\sum_{k=1}^n \xi_{S_k})_{n \geq 1}$. We give examples of partially hyperbolic dynamical systems (M, \mathcal{F}, ν, T) and of functions f such that $(\frac{1}{n^{\frac{3}{4}}} \sum_{k=1}^n \xi_{S_k})_{n \geq 1}$ converges in distribution.*

1 Introduction

Let us consider two sequences of random variables $(\xi_\ell)_{\ell \in \mathbb{Z}}$ (with values in \mathbb{R}) and $(S_n)_{n \geq 1}$ (with values in \mathbb{Z}). We suppose that these two sequences are independent one to the other. Let us define $W_n = \sum_{k=1}^n \xi_{S_k}$. This random variable can be understood as the cumulative amount got at time n if one starts from 0 at time 0 and get the amount ξ_ℓ if he is at position $\ell = S_k$ at time k . In [15], Kesten and Spitzer consider the situation when $(\xi_\ell)_{\ell \in \mathbb{Z}}$ and $(S_n - S_{n-1})_{n \geq 1}$ are two sequences of independent identically distributed random variables, with distributions in the attraction basin of some stable distributions. They got different results depending on the limit stable distributions. In particular, $(\xi_\ell)_{\ell \in \mathbb{Z}}$ is centered and square integrable and if $(S_n)_{n \geq 1}$ is the simple symmetric random walk on \mathbb{Z} , they prove that $(\frac{1}{n^{\frac{3}{4}}} \sum_{i=1}^n \xi_{S_i})_{n \geq 1}$ converges in distribution and they identify the limit distribution.

Our idea is to prove that this result of Kesten and Spitzer remains true in some situations where $(\xi_\ell)_{\ell \in \mathbb{Z}}$ is stationary but satisfies strong conditions of decorrelation. In this paper, $(S_n)_n$ will be the simple symmetric random walk on \mathbb{Z} and $(\xi_\ell)_{\ell \in \mathbb{Z}}$ will satisfy a central limit theorem. Before presenting our result, let us mention some extensions of the results of Kesten and Spitzer we are interested in.

Extension of the Kesten-Spitzer result to higher dimension have been done by Boltausen in [1] (for 2-dimensional) and by Revesz and Shi in [20] ($d \geq 3$). In [16], approximation results are given. Extensions to transient random walk and stationary $(\xi_\ell)_{\ell \in \mathbb{Z}}$ are also given by Guillotin-Plantard and Schneider [10]. Another possible extension is to get rid off the independence condition for the variables $S_n - S_{n-1}$. This has been done for some cases in [17] in order to give examples of non-standard limit theorem for regular invertible dynamical systems. Other related works are [19], Deo 1973 + Dehling 1980.

In the present paper, $(S_n)_{n \geq 1}$ and $(\xi_\ell)_{\ell \in \mathbb{Z}}$ are still supposed to be independent one to the other. Here $(S_n)_{n \geq 1}$ is still the simple symmetric random walk on \mathbb{Z} . But we do not suppose anymore that $(\xi_\ell)_{\ell \in \mathbb{Z}}$ is a sequence of independent variables. We suppose that $(\xi_\ell)_{\ell \in \mathbb{Z}}$ is a stationary sequence of random variables satisfying some strong decorrelation properties. This applies to situations when $(\xi_k)_{k \in \mathbb{Z}}$ is a stationary sequence of random variables given by a dynamical system with some hyperbolic properties. More precisely, we study the cases when $(\xi_k = f \circ T^k)_{k \in \mathbb{Z}}$, with f a ν -centered Hölder continuous function and when (M, \mathcal{F}, ν, T) is one of the following dynamical systems :

- the transformation T is an ergodic algebraic automorphism of the torus $M = \mathbb{T}^{d_0}$ endowed with its

normalised Haar measure ν (for some $d_0 \geq 2$);

- the transformation T is a diagonal transformation on a compact quotient M of $Sl_{d_0}(\mathbb{R})$ by a discrete subgroup, M being endowed with a natural T -invariant probability measure ν ;
- the transformation T is the Sinai billiard transformation.

In these situations, we prove that $\left(\frac{1}{n^{\frac{3}{4}}} \sum_{i=1}^n \xi_{S_k}\right)_{n \geq 1}$ converges in distribution to the random variable $\sqrt{\sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_m]} \Delta_1$, where Δ_1 has the limit distribution of $\left(\frac{1}{n^{\frac{3}{4}}} \sum_{i=1}^n \hat{\xi}_{S_k}\right)_{n \geq 1}$ obtained by Kesten and Spitzer when $(\hat{\xi}_m)_m$ is a sequence of independent identically distributed random variables with null expectation and with variance 1. Let us notice that, in our cases, $\sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_m]$ is well defined and is nonnegative since it is the limit of the variance of $\frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \xi_l$ as n goes to infinity.

We also get the same result of convergence in distribution for the following sequence $(\xi_k)_{k \in \mathbb{Z}}$. Let us consider the same examples of dynamical systems (M, \mathcal{F}, ν, T) . Instead of taking $\xi_k = f \circ T^k$, we suppose that, conditionally to $\omega \in M$, $(\xi_k)_{k \in \mathbb{Z}}$ is an independent sequence of random variables with values in $\{-1; 1\}$. We suppose that, conditionally to $\omega \in M$, $\xi_k(\omega, \cdot)$ is equal to 1 with probability $h \circ T^k(\omega)$, for some nonnegative Hölder continuous function h with expectation $\frac{1}{2}$. This model is envisaged by Guillotin-Plantard and Le Ny in [9] for other questions and with other hypotheses on (M, \mathcal{F}, ν, T) and on f .

Moreover we generalize this to the case when ξ_k takes p values (conditionally to $\omega \in M$, $(\xi_k)_{k \in \mathbb{Z}}$ is an independent sequence of random variables, ξ_k being equal to θ_j with probability $f_j \circ T^k(\omega)$, with $f_1 + \dots + f_p = 1$ and with f_1, \dots, f_p are nonnegative Hölder continuous functions).

In section 2, we state a general result under technical hypotheses of decorrelation (our theorem 1). Section 5 is devoted to the proof of this result (the idea of the proof is inspired by one step of an inductive method of Jan [12, 13] used in [18]).

In section 3, we give some applications of our abstract theorem 1. We apply our theorem 1 to the examples mentionned previously (ergodic algebraic automorphisms of the torus, diagonal transformation of a compact quotient of $Sl_{d_0}(\mathbb{R})$, billiard transformation). The proofs of the results of section 3 are done in sections 3 and 4.

2 A technical result

Theorem 1. *Let $(S_n)_{n \geq 1}$ and $(\xi_k)_{k \in \mathbb{Z}}$ be two sequences of random variables defined on the same probability space $(\Omega, \mathcal{T}, \mathbb{P})$ such that :*

1. $(S_n)_{n \geq 0}$ and $(\xi_k)_{k \in \mathbb{Z}}$ are independent one of the other;
2. $(S_n)_{n \geq 0}$ is a simple symmetric random walk on \mathbb{Z} ;
3. $(\xi_k)_{k \in \mathbb{Z}}$ is a stationary sequence of centered random variables admitting moments of the fourth order;
4. we have :

$$\sum_{p \geq 0} \sqrt{1+p} |\mathbb{E}[\xi_0 \xi_p]| < +\infty$$

$$\text{and } \sup_{N \geq 1} N^{-2} \sum_{k_1, k_2, k_3, k_4=0, \dots, N-1} |\mathbb{E}[\xi_{k_1} \xi_{k_2} \xi_{k_3} \xi_{k_4}]| < +\infty.$$

5. There exists some $C > 0$, some $(\varphi_{p,s})_{p,s \in \mathbb{N}}$ and some integer $r \geq 1$ such that :

$$\forall (p, s) \in \mathbb{N}^2, \varphi_{p+1,s} \leq \varphi_{p,s} \text{ and } \lim_{s \rightarrow +\infty} \sqrt{s} \varphi_{rs,s} = 0$$

and such that, for all integers n_1, n_2, n_3, n_4 with $0 \leq n_1 \leq n_2 < n_3 \leq n_4$, for all real numbers $\alpha_{n_1}, \dots, \alpha_{n_2}$ and $\beta_{n_3}, \dots, \beta_{n_4}$, we have :

$$\left| \text{Cov} \left(e^{i \sum_{k=n_1}^{n_2} \alpha_k \xi_k}, e^{i \sum_{k=n_3}^{n_4} \beta_k \xi_k} \right) \right| \leq C \left(1 + \sum_{k=n_1}^{n_2} |\alpha_k| + \sum_{k=n_3}^{n_4} |\beta_k| \right) \varphi_{n_3 - n_2, n_4 - n_3}.$$

Then, the sequence of random variables $\left(\frac{1}{n^{\frac{3}{4}}} \sum_{i=1}^n \xi_{S_k} \right)_{n \geq 1}$ converges in distribution to $\sqrt{\sum_{p \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_p]} \Delta_1$, where $\Delta_1 := \int_{\mathbb{R}} L_1(x) dB_x$, where $(B_x)_{x \in \mathbb{R}}$ and $(b_t)_{t \geq 0}$ are two independent standard brownian motions and $(L_t(x))_{t \geq 0}$ is the local time at x of $(b_t)_{t \geq 0}$, i.e. $L_t(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{(x-\varepsilon, x+\varepsilon)}(b_s) ds$.

3 Applications

Now we'll give some examples of stationary sequences $(\xi_k)_k$ satisfying our theorem 1.

We say that (M, \mathcal{F}, ν, T) is an invertible dynamical system if (M, \mathcal{F}, ν) is a probability space endowed with an invertible bi-measurable transformation $T : M \rightarrow M$.

Definition 2. We say that an invertible dynamical system (M, \mathcal{F}, ν, T) is **strongly mixing** if there exists $C_0 > 0$, there exist two real sequences $(\varphi_n)_{n \geq 0}$ and $(\kappa_m)_{m \geq 0}$ and, for any function $g : M \rightarrow \mathbb{C}$, there exist $K_g^{(1)} \in [0; +\infty]$ and $K_g^{(2)} \in [0; +\infty]$ such that, for all bounded functions $g, h : M \rightarrow \mathbb{C}$:

1. for all integer $n \geq 0$, we have : $|\text{Cov}_\nu(g, h \circ T^n)| \leq c_0 \left(\|g\|_\infty \|h\|_\infty + \|h\|_\infty K_g^{(1)} + \|g\|_\infty K_h^{(2)} \right) \varphi_n$;
2. for all integer $m \geq 0$, we have : $K_{g \circ T^{-m}}^{(1)} \leq c_0 K_g^{(1)}$;
3. for all integer $m \geq 0$, we have : $K_{h \circ T^m}^{(2)} \leq c_0 K_h^{(2)} (1 + \kappa_m)$;
4. we have : $K_{g \times h}^{(1)} \leq \|g\|_\infty K_h^{(1)} + \|h\|_\infty K_g^{(1)}$;
5. we have : $K_{g \times h}^{(2)} \leq \|g\|_\infty K_h^{(2)} + \|h\|_\infty K_g^{(2)}$;
6. the sequence $(\varphi_n)_{n \geq 0}$ is decreasing, the sequence $(\kappa_m)_{m \geq 0}$ is increasing and there exists an integer $r \geq 1$ such that : $\sup_{n \geq 1} n^6 (1 + \kappa_n) \varphi_{nr} < +\infty$.

For some hyperbolic or partially hyperbolic transformations, such properties are satisfied with $K_g^{(1)}$ some Hölder constant of g along the unstable manifolds and $K_h^{(2)}$ some Hölder constant of h along the stable-central manifolds, with $\varphi_n = \alpha^n$ for some $\alpha \in]0; 1[$ and $\kappa_m = m^\beta$ for some $\beta \geq 0$. Let us mention, for example, the ergodic algebraic automorphisms of the torus as well as the diagonal transformation on compact quotient of $Sl_{d_0}(\mathbb{R})$ (cf. [14]). Moreover, in the case of the Sinai (citeSin70 billiard transformation, these properties will come from estimations got in [7, 6].

Theorem 3. Let (M, \mathcal{F}, ν, T) be a **strongly mixing** dynamical system. Let one define the sequence (ξ_k) in one of the three following way.

- (a) Let us suppose that $f : M \rightarrow \mathbb{R}$ bounded, ν -centered, function such that $K_f^{(1)} < +\infty$ and $K_f^{(2)} < +\infty$. We suppose that there exists some real number $c_1 > 0$ such that, for any real number α , we have : $K_{\exp(i\alpha f)}^{(1)} \leq c_1 |\alpha|$ and $K_{\exp(i\alpha f)}^{(2)} \leq c_1 |\alpha|$.
- (b) Let us suppose that f takes its values in $[0; 1]$ and that $\int_M f d\nu = \frac{1}{2}$. Moreover let us suppose that there exists some $c_1 > 0$ such that, for any $a, b \in \mathbb{C}$, we have $K_{af+b}^{(1)} \leq c_1 |a|$ and $K_{af+b}^{(2)} \leq c_1 |a|$.

Let $(\Omega_1 :=]0; 1]^{\mathbb{Z}}, \mathcal{F}_1 := (\mathcal{B}(]0; 1])^{\times \mathbb{Z}}, \nu_1 := \lambda^{\otimes \mathbb{Z}})$ where λ is the Lebesgue measure on $]0; 1[$. We define $(\xi_k)_{k \in \mathbb{Z}}$ on the product $(\Omega_2 := M \times \Omega_1, \mathcal{F}_2 := \mathcal{F} \otimes \mathcal{F}_1, \nu_2 := \nu \otimes \nu_1)$ as follows :

$$\xi_k(\omega, (z_m)_{m \in \mathbb{Z}}) := 2 \cdot \mathbf{1}_{\{z_k \leq f \circ T^k(\omega)\}} - 1.$$

(c) Let us fix an integer $p \geq 2$. Let us fix p real numbers $\theta_1, \dots, \theta_p$ (and $\theta_0 := 0$) and p non-negative functions $f_1, \dots, f_p : M \rightarrow [0; 1]$ such that $\int_M (\theta_1 f_1 + \dots + \theta_p f_p) d\nu = 0$ and $f_1 + \dots + f_p = 1$ and such that there exists $c_2 > 0$ such that, for all complex numbers a_1, \dots, a_{p-1}, b , we have

$$\max(K_{a_1 f_1 + \dots + a_{p-1} f_{p-1} + b}^{(1)}, K_{a_1 f_1 + \dots + a_{p-1} f_{p-1} + b}^{(2)}) \leq c_2(|a_1| + \dots + |a_{p-1}|).$$

Let $(\Omega_1 :=]0; 1[^\mathbb{Z}, \mathcal{F}_1 := (\mathcal{B}(]0; 1[))^\otimes \mathbb{Z}, \nu_1 := \lambda^{\otimes \mathbb{Z}})$ where λ is the Lebesgue measure on $]0; 1[$. We define $(\xi_k)_{k \in \mathbb{Z}}$ on the product $(\Omega_2 := M \times \Omega_1, \mathcal{F}_2 := \mathcal{F} \otimes \mathcal{F}_1, \nu_2 := \nu \otimes \nu_1)$ as follows :

$$\xi_k(\omega, (z_m)_{m \in \mathbb{Z}}) = \sum_{l=1}^p (\theta_l - \theta_{l-1}) \mathbf{1}_{\{z_k \leq \sum_{j=1}^l f_j(T^k(\omega))\}}.$$

Then, if S_n is a simple symmetric random walk on \mathbb{Z} independent of (ξ_k) , the sequence of random variables $\left(\frac{1}{n^{\frac{3}{4}}} \sum_{i=1}^n \xi_{S_i}\right)_{n \geq 1}$ converges in distribution to $\sqrt{\sum_{p \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_p]} \Delta_1$, where $\Delta_1 := \int_{\mathbb{R}} L_1(x) dB_x$, where $(B_x)_{x \in \mathbb{R}}$ and $(b_t)_{t \geq 0}$ are two independent standard brownian motions and $(L_t(x))_{t \geq 0}$ is the local time at x of $(b_t)_{t \geq 0}$, i.e. $L_t(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{(x-\varepsilon, x+\varepsilon)}(b_s) ds$.

Let us make some comments on the point (b). Conditionally to $\omega \in M$, $(\xi_k(\omega, \cdot))_{k \in \mathbb{Z}}$ is a sequence of independent random variables with values in $\{-1; 1\}$ and $\xi_k(\omega, \cdot)$ is equal to 1 with probability $f \circ T^k(\omega)$. This model is envisaged by Guillin-Plantard and Le Ny in [9].

The case (c) is a generalization of the case (b) to the case when ξ_k takes p values (conditionally to $\omega \in M$, $\xi_k(\omega, \cdot)$ is equal to θ_j with probability $f_j \circ T^k(\omega)$).

Now we give some examples of strongly mixing dynamical systems.

Proposition 4. *Let us consider an integer $d_0 \geq 2$. The following two dynamical systems (M, \mathcal{F}, ν, T) are strongly mixing :*

- (i) M is the d_0 -dimensional torus $\mathbb{T}^{d_0} = \mathbb{R}^{d_0} / \mathbb{Z}^{d_0}$ endowed with its Borel σ -algebra \mathcal{F} and with the normalised Haar measure ν on \mathbb{T}^{d_0} and T is an algebraic automorphism of \mathbb{T}^{d_0} given by a matrix $S \in Sl_{d_0}(\mathbb{Z})$ the eigenvalues of which are not root of the unity. We endow \mathbb{T}^{d_0} with the metric d induced by the natural metric on \mathbb{R}^{d_0} .
- (ii) M is a compact quotient of $Sl_{d_0}(\mathbb{R})$ by a discrete subgroup Γ of $Sl_{d_0}(\mathbb{R}) : M := \{x\Gamma; x \in Sl_{d_0}(\mathbb{R})\}$; endowed with the normalised measure ν induced by the Haar measure on $Sl_{d_0}(\mathbb{R})$. The transformation T corresponds to the multiplication on the left by a diagonal matrix $S = \text{diag}(T_1, \dots, T_{d_0}) \in Sl_{d_0}(\mathbb{R})$ not equal to the identity and such that, for all $i = 1, \dots, d_0 - 1$, $T_i \geq T_{i+1} > 0$. We endow M with the metric d induced by a right-translations invariant riemanian metric on $SL_{d_0}(\mathbb{R})$.

Proof. Let $\eta > 0$. Let us denote by $\Gamma^{(s,e)}$ the set of stable-central manifolds and by Γ^u the set of unstable manifolds. In [14], each $\gamma^u \in \Gamma^u$ is endowed with some metric d^u and each $\gamma^{(s,e)} \in \Gamma^{(s,e)}$ is endowed with some metric $d^{(s,e)}$ such that there exist $\tilde{c}_0 > 0$, $\delta_0 \in]0; 1[$ and $\beta > 0$ such that, for any integer $n \geq 0$, for any $\gamma^u \in \Gamma^u$ and any $\gamma^{(s,e)} \in \Gamma^{(s,e)}$, we have :

- For any $y, z \in \gamma^u$, $d^u(y, z) \geq d(y, z)$ and for any $y', z' \in \gamma^{(s,e)}$, $d^{(s,e)}(y', z') \geq d(y', z')$.
- For any $y, z \in \gamma^u$, there exists $\gamma_{(n)}^u$ such that $T^{-n}(y)$ and $T^{-n}(z)$ belong to $\gamma_{(n)}^u$ and we have : $d^u(T^{-n}(y), T^{-n}(z)) \leq \tilde{c}_0(\delta_0)^n d^u(y, z)$.
- For any $y, z \in \gamma^{(s,e)}$, there exists $\gamma_{(n)}^{(s,e)}$ such that $T^n(y)$ and $T^n(z)$ belong to $\gamma_{(n)}^{(s,e)}$ and we have : $d^{(s,e)}(T^n(y), T^n(z)) \leq \tilde{c}_0(1 + n^\beta) d^{(s,e)}(y, z)$.

Let us define :

$$K_f^{(1)} := \sup_{\gamma^u \in \Gamma^u} \sup_{y, z \in \gamma^u : y \neq z} \frac{|f(y) - f(z)|}{(d^u(y, z))^\eta} \quad \text{and} \quad K_f^{(2)} := \sup_{\gamma^{(s,e)} \in \Gamma^{(s,e)}} \sup_{y, z \in \gamma^{(s,e)} : y \neq z} \frac{|f(y) - f(z)|}{(d^{(s,e)}(y, z))^\eta}.$$

Hence, points 2, 3, 4 and 5 of the definition of strongly mixing dynamical systems are satisfied with $\kappa_n = n^\beta$. Moreover, these two quantities are less than the Hölder constant of order η of f . In [14], the point 1 is proved in the particular case (ii). The same proof can be used in the case (i). We get a sequence $(\varphi_n)_n$ decreasing exponentially fast (cf. lemme 1.3.1 in [14]). \square

For the billiard transformation, we prove the following (see appendix for details and for the proof) :

Proposition 5. *Let (M, \mathcal{F}, ν, T) be the time-discrete dynamical system given by the discrete Sinai billiard (corresponding to the reflection times on a scatterer). We suppose that the billiard domain is $\mathcal{D} := \mathbb{T}^2 \setminus \left(\bigcup_{i=1}^I O_i\right)$, where the scatterers O_i are open convex subsets of \mathbb{T}^2 , the closures of which are pairwise disjoint and the boundaries of which are C^3 smooth with non-null curvature. Then the system (M, \mathcal{F}, ν, T) is strongly mixing.*

Proposition 6. *If $(\xi_k)_{k \in \mathbb{Z}}$ is a stationary sequence of bounded random variables satisfying the following α -mixing condition*

$$\sup_{n \geq 1} n^6 \alpha_n < +\infty, \quad \text{with} \quad \alpha_n := \sup_{p \geq 0; m \geq 0} \sup_{A \in \sigma(\xi_{-p}, \dots, \xi_0)} \sup_{B \in \sigma(\xi_n, \dots, \xi_{n+m})} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|,$$

then points 3, 4 and 5 of theorem 1 hold.

Proof. According to [11], lemma 1.2, the definition of a strongly dynamical system is satisfied with the following choice of $K^{(1)}$ and of $K^{(2)}$. If g is $\sigma(\xi_k, k \leq 0)$ -measurable, we have $K_g^{(1)} := 1$; otherwise we have $K_g^{(1)} := \infty$. If h is $\sigma(\xi_k, k \geq 0)$ -measurable, we have $K_h^{(2)} := 1$; otherwise we have $K_h^{(2)} := \infty$. We conclude with theorem 3, *qed*.

Let us observe that, in the case of proposition 5, we can take the function f constant on each atom of \mathcal{C}_m for some integer $m \geq 0$. For example $f = \mathbf{1}_{\{\varphi > \frac{\pi}{2} - \frac{1}{k_0^2}\}} - \mathbf{1}_{\{\varphi < -\frac{\pi}{2} + \frac{1}{k_0^2}\}}$ satisfies the case (a) of theorem ?? for the Sinai billiard (with the notation k_0 of [6] page 5). In the case (c) of theorem ??, we can take $p = 3$, $\theta_1 = 1$, $\theta_2 = -1$, $\theta_3 = 0$, $f_1 = \mathbf{1}_{\{\varphi > \frac{\pi}{2} - \frac{1}{k_0^2}\}}$, $f_2 = \mathbf{1}_{\{\varphi < -\frac{\pi}{2} + \frac{1}{k_0^2}\}}$ and $f_3 = \mathbf{1} - f_1 - f_2$ in the case of the Sinai billiard (with again the notations of [6] page 5).

4 Proof of theorem 3

In the cases (a), (b) and (c), it is easy to see that $(\xi_k)_k$ is a stationary sequence of bounded random variables

4.1 Proof of (a)

We have :

$$\begin{aligned} \sum_{p \geq 0} \sqrt{1+p} |\mathbb{E}[\xi_0 \xi_p]| &= \sum_{p \geq 0} \sqrt{1+p} |\mathbb{E}_\nu[f \circ T^p]| \\ &\leq c_0 \|f\|_\infty \left(\|f\|_\infty + K_f^{(1)} + K_f^{(2)} \right) \sum_{p \geq 0} \sqrt{1+p} \varphi_p < +\infty. \end{aligned}$$

Let us consider an integer $N \geq 1$. We have :

$$\frac{1}{N^2} \sum_{k_1, k_2, k_3, k_4 = 0, \dots, N-1} |\mathbb{E}[\xi_{k_1} \xi_{k_2} \xi_{k_3} \xi_{k_4}]| \leq \frac{24}{N^2} \sum_{0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq N-1} |\mathbb{E}[\xi_{k_1} \xi_{k_2} \xi_{k_3} \xi_{k_4}]|.$$

Let us consider the set $E_N^{(1)}$ of (k_1, k_2, k_3, k_4) such that $0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq N-1$ and $k_4 - k_3 \geq N^{\frac{1}{3}}$. We have :

$$\begin{aligned} \sum_{(k_1, k_2, k_3, k_4) \in E_N^{(1)}} |\mathbb{E}[\xi_{k_1} \xi_{k_2} \xi_{k_3} \xi_{k_4}]| &= \sum_{(k_1, k_2, k_3, k_4) \in E_N^{(1)}} |Cov_\nu(f \circ T^{k_1 - k_3} f \circ T^{k_2 - k_3} f, f \circ T^{k_4 - k_3})| \\ &\leq c_0 N^4 \left(\|f\|_\infty^4 + \|f\|_\infty^3 (K_f^{(2)} + 3c_0 K_f^{(1)}) \right) \varphi_{\lfloor N^{\frac{1}{3}} \rfloor} \\ &\leq c_0 N^2 \left(\|f\|_\infty^4 + \|f\|_\infty^3 (K_f^{(2)} + 3c_0 K_f^{(1)}) \right) \sup_{n \geq 1} n^6 \varphi_n. \end{aligned}$$

Let us consider the set $E_N^{(2)}$ of (k_1, k_2, k_3, k_4) such that $0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq N-1$ and $k_4 - k_3 < N^{\frac{1}{3}}$ and $k_3 - k_2 \geq rN^{\frac{1}{3}}$. We have :

$$\begin{aligned} \sum_{(k_1, k_2, k_3, k_4) \in E_N^{(2)}} |Cov(\xi_{k_1} \xi_{k_2}, \xi_{k_3} \xi_{k_4})| &= \sum_{(k_1, k_2, k_3, k_4) \in E_N^{(2)}} |Cov_\nu(f \circ T^{k_1 - k_2} f, (f \circ T^{k_4 - k_3}) \circ T^{k_3 - k_2})| \\ &\leq c_0 N^4 \left(\|f\|_\infty^4 + 2c_0 \|f\|_\infty^3 (K_f^{(2)} + K_f^{(1)}) \right) (1 + \kappa_{\lfloor N^{\frac{1}{3}} \rfloor}) \varphi_{\lfloor N^{\frac{1}{3}} \rfloor} \\ &\leq 2^6 c_0 N^2 \left(\|f\|_\infty^4 + 2c_0 \|f\|_\infty^3 (K_f^{(2)} + K_f^{(1)}) \right) \sup_{n \geq 1} n^6 (1 + \kappa_n) \varphi_{rn}. \end{aligned}$$

Moreover, we have :

$$\begin{aligned} \sum_{(k_1, k_2, k_3, k_4) \in E_N^{(2)}} |\mathbb{E}[\xi_{k_1} \xi_{k_2}] \mathbb{E}[\xi_{k_3} \xi_{k_4}]| &\leq \left(\sum_{0 \leq k_1 \leq k_2 \leq N-1} |\mathbb{E}[\xi_{k_1} \xi_{k_2}]| \right)^2 \\ &\leq \left(N \sum_{k \geq 0} |\mathbb{E}_\nu[f \circ T^k]| \right)^2 \\ &\leq N^2 \left(c_0 \left(\|f\|_\infty^2 + \|f\|_\infty (K_f^{(1)} + K_f^{(2)}) \right) \sum_{k \geq 0} \varphi_k \right)^2. \end{aligned}$$

Let us consider the set $E_N^{(3)}$ of (k_1, k_2, k_3, k_4) such that $0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq N-1$ and $k_4 - k_3 < N^{\frac{1}{3}}$ and $k_3 - k_2 < rN^{\frac{1}{3}}$ and $k_2 - k_1 \geq r(1+r)N^{\frac{1}{3}}$. By the same method, we get :

$$\sum_{(k_1, k_2, k_3, k_4) \in E_N^{(3)}} |\mathbb{E}[\xi_{k_1} \xi_{k_2} \xi_{k_3} \xi_{k_4}]| \leq N^2 \frac{c_0 2^6}{(1+r)^6} \left(\|f\|_\infty^4 + 3c_0 \|f\|_\infty^3 (K_f^{(2)} + K_f^{(1)}) \right) \sup_{n \geq 1} n^6 (1 + \kappa_n) \varphi_{rn}.$$

Since the number of (k_1, k_2, k_3, k_4) such that $0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq N-1$ and that do not belong to $E_N^{(1)} \cup E_N^{(2)} \cup E_N^{(3)}$ is bounded by $N^2 2(r+1)^3$, we get :

$$\sup_{N \geq 1} \frac{1}{N^2} \sum_{k_1, k_2, k_3, k_4 = 0, \dots, N-1} |\mathbb{E}[\xi_{k_1} \xi_{k_2} \xi_{k_3} \xi_{k_4}]| < +\infty.$$

Now, let us prove the point 5. Let n_1, n_2, n_3 and n_4 be four integers such that $0 \leq n_1 \leq n_2 \leq n_3 \leq n_4$. Let us consider any real numbers $\alpha_{n_1}, \dots, \alpha_{n_2}$ and $\beta_{n_3}, \dots, \beta_{n_4}$. We have :

$$\begin{aligned} |Cov(e^{i \sum_{k=n_1}^{n_2} \alpha_k \xi_k}, e^{i \sum_{k=n_3}^{n_4} \beta_k \xi_k})| &= |Cov_\nu(e^{i \sum_{k=n_1}^{n_2} \alpha_k f \circ T^{-(n_2-k)}}, (e^{i \sum_{k=n_3}^{n_4} \beta_k f \circ T^{k-n_3}}) \circ T^{n_3-n_2})| \\ &\leq c_0 \left(1 + K_{\exp(i \sum_{k=n_1}^{n_2} \alpha_k f \circ T^{-(n_2-k)})}^{(1)} + K_{\exp(i \sum_{k=n_3}^{n_4} \beta_k f \circ T^{k-n_3})}^{(2)} \right) \varphi_{n_3-n_2} \\ &\leq c_0 \left(1 + \sum_{k=n_1}^{n_2} K_{\exp(i \alpha_k f \circ T^{-(n_2-k)})}^{(1)} + \sum_{k=n_3}^{n_4} K_{\exp(i \beta_k f \circ T^{k-n_3})}^{(2)} \right) \varphi_{n_3-n_2} \\ &\leq c_0 \left(1 + \sum_{k=n_1}^{n_2} c_0 c_1 |\alpha_k| + \sum_{k=n_3}^{n_4} c_0 c_1 |\beta_k| (1 + \kappa_{n_4-n_3}) \right) \varphi_{n_3-n_2}. \end{aligned}$$

We conclude by taking $\varphi_{p,s} := (1 + \kappa_s)\varphi_p$.

4.2 Proof of (b) and of (c)

Let us consider (c) which is an extension of the case (b) (by taking $p = 2$, $\theta_1 = 1$, $\theta_2 = -1$, $f_1 = f$ and $f_2 = 1 - f$). Let us define the function $g := \sum_{j=1}^p \theta_j f_j$ (in the case (b), we have : $g = 2f - 1$). This function is ν -centered. More generally, for any integer $m \geq 1$, let us define : $g_m := \sum_{j=1}^p \theta_j^m f_j$. These functions satisfy $K_{g_m}^{(1)} + K_{g_m}^{(2)} < +\infty$. We observe that, conditionally to $\omega \in M$, the expectation of $(\xi_k(\omega, \cdot))^m$ is equal to $g_m \circ T^k(\omega)$. Using the Fubini theorem and starting by integrating over Ω_1 , we observe that, for any integers $p \geq 1$, we have : $\mathbb{E}[\xi_0 \xi_p] = \mathbb{E}_\nu[g \circ T^p]$ and that, for any integers k_1, k_2, k_3, k_4 , we have : $\mathbb{E}[\xi_{k_1}^{n_1} \xi_{k_2}^{n_2} \xi_{k_3}^{n_3} \xi_{k_4}^{n_4}] = \mathbb{E}_\nu \left[\prod_{j=1}^4 g_{n_j} \circ T^{k_j} \right]$. Hence, we can prove the point 4 of theorem 1 as we proved it for (a).

Now, let us prove the point 5 of theorem 1. We observe that, conditionally to $\omega \in M$, the ξ_k are independent and that the expectation of $\exp(iu\xi_k(\omega, \cdot))$ is $h_u \circ T^k$ with $(h_u := \sum_{l=1}^p e^{i\theta_l u} f_l)$. This function can be rewritten : $h_u = e^{i\theta_p u} + \sum_{l=1}^{p-1} (e^{i\theta_l u} - e^{i\theta_p u}) f_l$. The modulus of this function is bounded by 1 and we have :

$$\max \left(K_{h_u}^{(1)}, K_{h_u}^{(2)} \right) \leq c_2 2p \max_{j=0, \dots, p} |\theta_j| |u|.$$

Let n_1, n_2, n_3 and n_4 be four integers such that $0 \leq n_1 \leq n_2 < n_3 \leq n_4$. Let us consider any real numbers $\alpha_{n_1}, \dots, \alpha_{n_2}$ and $\beta_{n_3}, \dots, \beta_{n_4}$. We have :

$$\begin{aligned} & \left| Cov \left(e^{i \sum_{k=n_1}^{n_2} \alpha_k \xi_k}, e^{i \sum_{k=n_3}^{n_4} \beta_k \xi_k} \right) \right| = \\ & = \left| Cov_\nu \left(\prod_{k=n_1}^{n_2} h_{\alpha_k} \circ T^k, \prod_{k=n_3}^{n_4} h_{\beta_k} \circ T^k \right) \right| \\ & \leq c_0 \left(1 + c_0 c_2 2p \max_{j=0, \dots, p} |\theta_j| \left(\sum_{k=n_1}^{n_2} |\alpha_k| + \sum_{k=n_3}^{n_4} |\beta_k| \right) \right) (1 + \kappa_{n_4 - n_3}) \varphi_{n_3 - n_2}. \end{aligned}$$

5 Proof of theorem 1

To prove our result of convergence in distribution, we use characteristic functions. Let us fix some real number t . We will show that :

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{k=1}^n \xi_{S_k} \right) \right] = \mathbb{E} \left[\exp \left(it \sqrt{\sum_{p \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_p]} \Delta_1 \right) \right].$$

Let us notice that we have (cf [15] lemma 5, for example) :

$$\mathbb{E} [\exp(iu\Delta_1)] = \mathbb{E} \left[\exp \left(-\frac{u^2}{2} \int_{\mathbb{R}} (L_1(x))^2 dx \right) \right].$$

Hence, it is enough to prove that :

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{k=1}^n \xi_{S_k} \right) \right] = \mathbb{E} \left[\exp \left(-\frac{t^2}{2} \sum_{p \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_p] \int_{\mathbb{R}} (L_1(x))^2 dx \right) \right].$$

In the following, for any integer $m \geq 1$ and any integer k , we define :

$$N_m(k) := Card\{j = 1, \dots, m : S_j = k\}.$$

We notice that, for any integer $n \geq 1$, we have :

$$\sum_{j=1}^n \xi_{S_j} = \sum_{k \in \mathbb{Z}} \xi_k N_n(k).$$

In the step 1 of our proof, we will use the following facts :

$$C_0 := \sup_{n \geq 1} \sup_{K > 0} K^2 n^{-1} \mathbb{P} \left(\max_{m=1, \dots, n} |S_m| \geq K \right) < +\infty,$$

$$C_1 := \sup_{n \geq 1} \sup_{k \in \mathbb{Z}} n^{-\frac{1}{2}} \|N_n(k)\|_6 < +\infty,$$

$$C_2 := \sup_{n \geq 1} \sup_{k, \ell \in \mathbb{Z}} \frac{\|N_n(\ell) - N_n(k)\|_2}{\sqrt{1 + |\ell - k|} n^{\frac{1}{4}}} < +\infty.$$

The first fact comes from the Kolmogorov inequality. The second fact comes from [15] lemma 1. The third fact comes from [15] lemmas 2 and 3.

5.1 Step 1 : Technical part

This is the big part of our proof. In this part, we prove that the following quantity goes to zero as n goes to $+\infty$:

$$\left| \mathbb{E} \left[\exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell \in \mathbb{Z}} \xi_\ell N_n(\ell) \right) \right] - \mathbb{E} \left[\exp \left(-\frac{t^2}{2n^{\frac{3}{2}}} \sum_{\ell, k \in \mathbb{Z}} \mathbb{E}[\xi_\ell \xi_k] N_n(\ell)^2 \right) \right] \right|.$$

Let us fix $\varepsilon > 0$. We will prove that, if n is large enough, this quantity is less than ε .

Our proof is inspired by a method used by Jan to establish central limit theorem with rate of convergence (cf. [13], [12], method also used in [18]). More precisely, we adapt the idea of the first step of the inductive method of Jan.

- For any $K \geq 1$ and any integer $n \geq 1$, we have :

$$\mathbb{P} \left(\max_{m=1, \dots, n} |S_m| \geq K\sqrt{n} \right) \leq \frac{C_0 n}{K^2 n} = \frac{C_0}{K^2}.$$

Let us fix $K \geq 1$ such that $2\frac{C_0}{K^2} < \frac{\varepsilon}{10}$. Then, we have

$$\left| \mathbb{E} \left[\exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell \in \mathbb{Z}} \xi_\ell N_n(\ell) \right) \right] - \mathbb{E} \left[\exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell = -\lceil K\sqrt{n} \rceil}^{\lceil K\sqrt{n} \rceil} \xi_\ell N_n(\ell) \right) \right] \right| \leq 2\frac{C_0}{K^2} < \frac{\varepsilon}{10} \quad (1)$$

and :

$$\left| \mathbb{E} \left[\exp \left(-\frac{t^2}{2n^{\frac{3}{2}}} \sum_{\ell, k \in \mathbb{Z}} \mathbb{E}[\xi_\ell \xi_k] N_n(\ell)^2 \right) \right] - \mathbb{E} \left[\exp \left(-\frac{t^2}{2n^{\frac{3}{2}}} \sum_{\ell = -\lceil K\sqrt{n} \rceil}^{\lceil K\sqrt{n} \rceil} \sum_{k \in \mathbb{Z}} \mathbb{E}[\xi_\ell \xi_k] N_n(\ell)^2 \right) \right] \right| < \frac{\varepsilon}{10}. \quad (2)$$

Hence we have to estimate :

$$A_n := \left| \mathbb{E} \left[\exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell = -\lceil K\sqrt{n} \rceil}^{\lceil K\sqrt{n} \rceil} \xi_\ell N_n(\ell) \right) \right] - \mathbb{E} \left[\exp \left(-\frac{t^2}{2n^{\frac{3}{2}}} \sum_{\ell = -\lceil K\sqrt{n} \rceil}^{\lceil K\sqrt{n} \rceil} \sum_{k \in \mathbb{Z}} \mathbb{E}[\xi_\ell \xi_k] N_n(\ell)^2 \right) \right] \right|. \quad (3)$$

- In the following, L will be some real number bigger than 8 and large enough and n any integer bigger than 1 and large enough so that : $\frac{2K\sqrt{n}}{L} \geq L$. We will have : $\frac{K\sqrt{n}}{L} \leq \left\lfloor \frac{2\lceil K\sqrt{n} \rceil + 1}{L} \right\rfloor \leq \frac{5K\sqrt{n}}{L}$.

- We split our sums $\sum_{\ell=-\lceil K\sqrt{n} \rceil}^{\lceil K\sqrt{n} \rceil}$ in L sums over $\left\lfloor \frac{2\lceil K\sqrt{n} \rceil + 1}{L} \right\rfloor$ terms and one sum over less than L terms and so over less than $\left\lfloor \frac{2\lceil K\sqrt{n} \rceil + 1}{L} \right\rfloor$ terms. Let us introduce the following notation :

$$\alpha^{(k)} = \alpha_{(k,n,L)} = \begin{cases} -\lceil K\sqrt{n} \rceil + k \left\lfloor \frac{2\lceil K\sqrt{n} \rceil + 1}{L} \right\rfloor & \text{if } k = 0, \dots, L \\ \lceil K\sqrt{n} \rceil + 1 & \text{if } k = L + 1 \end{cases}.$$

For all $k = 0, \dots, L$, we define :

$$a_{k,n,L} = \exp \left(-\frac{t^2}{2n^{\frac{3}{2}}} \sum_{\ell=\alpha^{(k)}}^{\alpha^{(k+1)}-1} \sum_{k \in \mathbb{Z}} \mathbb{E}[\xi_\ell \xi_k] N_n(\ell)^2 \right)$$

and

$$b_{k,n,L} = \exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell=\alpha^{(k)}}^{\alpha^{(k+1)}-1} \xi_\ell N_n(\ell) \right).$$

Let us notice that, for any $k = 0, \dots, L$, we have : $|a_{k,n,L}| \leq 1$ and $|b_{k,n,L}| \leq 1$. We have :

$$\begin{aligned} |A_n| &= \left| \mathbb{E} \left[\prod_{k=0}^L b_{k,n,L} - \prod_{k=0}^L a_{k,n,L} \right] \right| \\ &= \left| \sum_{k=0}^L \mathbb{E} \left[\left(\prod_{m=0}^{k-1} b_{m,n,L} \right) (b_{k,n,L} - a_{k,n,L}) \prod_{m'=k+1}^L a_{m',n,L} \right] \right|. \end{aligned} \quad (4)$$

- Now we explain how we can restrict our study to the sum over the k such that $(r+1)^3 \leq k \leq L-1$. Indeed, the number of k that do not satisfy this is equal to $(r+1)^3 + 1$. Let us consider any $k = 0, \dots, L$. We have :

$$\mathbb{E} [|b_{k,n,L} - 1|] \leq \frac{|t|}{n^{\frac{3}{4}}} \mathbb{E} \left[\left| \sum_{\ell=\alpha^{(k)}}^{\alpha^{(k+1)}-1} \xi_\ell N_n(\ell) \right| \right]$$

and :

$$\mathbb{E} [|a_{k,n,L} - 1|] \leq \frac{t^2}{2n^{\frac{3}{2}}} \mathbb{E} \left[\left| \sum_{\ell=\alpha^{(k)}}^{\alpha^{(k+1)}-1} \sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_\ell \xi_m] N_n(\ell)^2 \right| \right].$$

But, for any integers α and β with $\beta \geq 1$, we have :

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{\ell=\alpha+1}^{\alpha+\beta} \xi_\ell N_n(\ell) \right)^2 \right] &\leq \sum_{\ell=\alpha+1}^{\alpha+\beta} \sum_{m=\alpha+1}^{\alpha+\beta} |\mathbb{E}[\xi_\ell \xi_m]| |\mathbb{E}[N_n(\ell) N_n(m)]| \\ &\leq \sum_{\ell=\alpha+1}^{\alpha+\beta} \sum_{m=\alpha+1}^{\alpha+\beta} |\mathbb{E}[\xi_\ell \xi_m]| \|N_n(\ell)\|_2 \|N_n(m)\|_2 \\ &\leq (C_1)^2 n \beta \sum_{m \in \mathbb{Z}} |\mathbb{E}[\xi_0 \xi_m]|. \end{aligned}$$

From which, we get :

$$\begin{aligned} \mathbb{E} [|b_{k,n,L} - 1|] &\leq \frac{|t|}{n^{\frac{3}{4}}} \sqrt{(C_1)^2 n \frac{5K\sqrt{n}}{L} \sum_{m \in \mathbb{Z}} |\mathbb{E}[\xi_0 \xi_m]|} \\ &\leq \frac{|t|}{\sqrt{L}} C_1 \sqrt{5K \sum_{m \in \mathbb{Z}} |\mathbb{E}[\xi_0 \xi_m]|}. \end{aligned}$$

Moreover we have :

$$\begin{aligned}
\mathbb{E} [|a_{k,n,L} - 1|] &\leq \frac{t^2}{2n^{\frac{3}{2}}} \mathbb{E} \left[\sum_{\ell=\alpha(k)}^{\alpha(k+1)-1} \sum_{m \in \mathbb{Z}} \mathbb{E} [\xi_\ell \xi_m] N_n(\ell)^2 \right] \\
&\leq \frac{t^2}{2n^{\frac{3}{2}}} \frac{5K\sqrt{n}}{L} \sum_{m \in \mathbb{Z}} \mathbb{E} [\xi_0 \xi_m] (C_1)^2 n \\
&\leq \frac{5t^2}{2} \frac{K(C_1)^2}{L} \sum_{m \in \mathbb{Z}} \mathbb{E} [\xi_0 \xi_m].
\end{aligned}$$

Let $L_1 \geq 8$ be such that, for all $L \geq L_1$, we have :

$$((r+1)^3 + 1) \frac{|t|}{\sqrt{L}} C_1 \sqrt{5K \sum_{m \in \mathbb{Z}} |\mathbb{E} [\xi_0 \xi_m]|} < \frac{\varepsilon}{20}$$

and

$$((r+1)^3 + 1) \frac{5t^2}{2} \frac{K(C_1)^2}{L} \sum_{m \in \mathbb{Z}} \mathbb{E} [\xi_0 \xi_m] < \frac{\varepsilon}{20}.$$

Then, if we have $L \geq L_1$ and $n \geq 1$ such that $\frac{2K\sqrt{n}}{L} \geq L$, we have :

$$\mathbb{E} [|b_{L,n,L} - a_{L,n,L}|] + \sum_{k=0}^{(r+1)^3-1} \mathbb{E} [|b_{k,n,L} - a_{k,n,L}|] < \frac{\varepsilon}{10}. \quad (5)$$

It remains to estimate :

$$\sum_{k=(r+1)^3}^{L-1} \left| \mathbb{E} \left[\left(\prod_{m=0}^{k-1} b_{m,n,L} \right) (b_{k,n,L} - a_{k,n,L}) \prod_{m'=k+1}^L a_{m',n,L} \right] \right|. \quad (6)$$

• We estimate :

$$\begin{aligned}
B_{n,L} &:= \sum_{k=(r+1)^3}^{L-1} \left| \mathbb{E} \left[\left(\prod_{m=0}^{k-(r+1)^3} b_{m,n,L} \right) \left(\left(\prod_{m=k-(r+1)^3+1}^{k-(r+1)^2} b_{m,n,L} \right) - 1 \right) \times \right. \right. \\
&\times \left. \left. \left(\left(\prod_{m=k-(r+1)^2+1}^{k-r-1} b_{m,n,L} \right) - 1 \right) \left(\prod_{m'=k-r}^{k-1} b_{m',n,L} \right) (b_{k,n,L} - a_{k,n,L}) \prod_{m'=k+1}^L a_{m',n,L} \right] \right|.
\end{aligned}$$

We have :

$$B_{n,L} \leq \sum_{k=(r+1)^3}^{L-1} \left\| \left(\prod_{m=k-(r+1)^3+1}^{k-(r+1)^2} b_{m,n,L} \right) - 1 \right\|_3 \left\| \left(\prod_{m=k-(r+1)^2+1}^{k-r-1} b_{m,n,L} \right) - 1 \right\|_3 \|b_{k,n,L} - a_{k,n,L}\|_3.$$

– We have :

$$\|b_{k,n,L} - 1\|_3 \leq \frac{|t|}{n^{\frac{3}{4}}} \left\| \sum_{\ell=\alpha(k)}^{\alpha(k+1)-1} \xi_\ell N_n(\ell) \right\|_3.$$

For any integers α and β with $\beta \geq 1$, we have :

$$\begin{aligned}
\mathbb{E} \left[\left(\sum_{\ell=\alpha+1}^{\alpha+\beta} \xi_\ell N_n(\ell) \right)^4 \right] &\leq \sum_{\ell_1, \ell_2, \ell_3, \ell_4=\alpha+1}^{\alpha+\beta} |\mathbb{E} [\xi_{\ell_1} \xi_{\ell_2} \xi_{\ell_3} \xi_{\ell_4}]| (C_1)^4 n^2 \\
&\leq (C_1)^4 n^2 C_2' \beta^2.
\end{aligned} \quad (7)$$

with $C'_2 := \sup_{N \geq 1} N^{-2} \sum_{k_1, k_2, k_3, k_4=0, \dots, N-1} |\mathbb{E}[\xi_{k_1} \xi_{k_2} \xi_{k_3} \xi_{k_4}]|$. Hence, we have :

$$\begin{aligned} \|b_{k,n,L} - 1\|_3 &\leq \frac{|t|}{n^{\frac{3}{4}}} \left((C_1)^4 n^2 C'_2 \left(\frac{5K\sqrt{n}}{L} \right)^2 \right)^{\frac{1}{4}} \\ &\leq |t| C_1 (C'_2)^{\frac{1}{4}} \sqrt{\frac{5K}{L}}. \end{aligned}$$

– We have :

$$\begin{aligned} \|a_{k,n,L} - 1\|_3 &\leq \frac{t^2}{2n^{\frac{3}{2}}} \sum_{k \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_k] \left\| \sum_{\ell=\dots}^{\dots} N_n(\ell)^2 \right\|_3 \\ &\leq \frac{t^2}{2n^{\frac{3}{2}}} \sum_{k \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_k] \sum_{\ell=\dots}^{\dots} \|N_n(\ell)\|_6^2 \\ &\leq \frac{t^2}{2n^{\frac{3}{2}}} \sum_{k \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_k] \frac{5K\sqrt{n}}{L} (C_1)^2 n \\ &\leq \frac{5t^2}{2} \sum_{k \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_k] \frac{K}{L} (C_1)^2. \end{aligned}$$

– Using formula (7), we get :

$$\begin{aligned} \left\| \left(\prod_{m=k-(r+1)^3+1}^{k-(r+1)^2} b_{m,n,L} \right) - 1 \right\|_3 &\leq \frac{|t|}{n^{\frac{3}{4}}} \left\| \sum_{\ell=\alpha_{(k-(r+1)^3+1)}^{\alpha_{(k-(r+1)^2+1)-1}} \xi_\ell N_n(\ell) \right\|_3 \\ &\leq \frac{|t|}{n^{\frac{3}{4}}} \left((C_1)^4 n^2 C'_2 (r(r+1)^2)^2 \left(\frac{5K\sqrt{n}}{L} \right)^2 \right)^{\frac{1}{4}} \\ &\leq |t| C_1 (C'_2)^{\frac{1}{4}} \sqrt{r(r+1)} \sqrt{\frac{5K}{L}}. \end{aligned}$$

– Analogously, we get :

$$\left\| \left(\prod_{m=k-(r+1)^2+1}^{k-r-1} b_{m,n,L} \right) - 1 \right\|_3 \leq |t| C_1 (C'_2)^{\frac{1}{4}} \sqrt{r(r+1)} \sqrt{\frac{5K}{L}}.$$

Hence, we have :

$$\begin{aligned} B_{n,L} &\leq L \left(|t| C_1 (C'_2)^{\frac{1}{4}} \sqrt{r(r+1)} \sqrt{\frac{5K}{L}} \right)^2 \left(|t| C_1 (C'_2)^{\frac{1}{4}} \sqrt{\frac{5K}{L}} + \frac{5t^2}{2} \sum_{k \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_k] \frac{K}{L} (C_1)^2 \right) \\ &\leq |t|^2 (C_1)^2 (C'_2)^{\frac{1}{2}} r(r+1)^2 5K \left(|t| C_1 (C'_2)^{\frac{1}{4}} \sqrt{\frac{5K}{L}} + \frac{5t^2}{2} \sum_{k \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_k] \frac{K}{L} (C_1)^2 \right). \end{aligned}$$

Let $L'_1 \geq L_1$ be such that, for all $L \geq L'_1$, the right term of this last inequality is less than $\frac{\varepsilon}{10}$.

Then, for any $L \geq L'_1$ and any $n \geq 1$ such that $\frac{2K\sqrt{n}}{L} \geq L$, we have : $B_{n,L} \leq \frac{\varepsilon}{10}$.

- In the following, we suppose $L \geq L'_1$ and $\frac{2K\sqrt{n}}{L} \geq L$. It remains to estimate :

$$\sum_{k=(r+1)^3+1}^{L-1} C_{n,k,L,1,3} + C_{n,k,L,1,2} + C_{n,k,L,2,3}$$

where C_{n,k,L,j_0,j_1} is the following quantity :

$$\left| \mathbb{E} \left[\left(\prod_{m=0}^{k-(r+1)^{j_1}} b_{m,n,L} \right) \left(\prod_{m=k-(r+1)^{j_0+1}}^{k-1} b_{m,n,L} \right) (b_{k,n,L} - a_{k,n,L}) \prod_{m'=k+1}^L a_{m',n,L} \right] \right|$$

- Let j_0, j_1 be fixed. We estimate C_{n,k,L,j_0,j_1} . We have :

$$C_{n,k,L,j_0,j_1} \leq D_{n,k,L,j_0,j_1} + E_{n,k,L,j_0,j_1},$$

where :

$$D_{n,k,L,j_0,j_1} := \left| \mathbb{E} \left[\text{Cov}_{|(S_p)_{p \geq 0}} (\Delta_{n,k,L,j_1}, \Gamma_{n,k,L,j_0}) \prod_{m'=k+1}^L a_{m',n,L} \right] \right|$$

and

$$E_{n,k,L,j_0,j_1} := \left| \mathbb{E} \left[\mathbb{E} [\Delta_{n,k,L,j_1} | (S_p)_{p \geq 0}] \mathbb{E} [\Gamma_{n,k,L,j_0} | (S_p)_{p \geq 0}] \prod_{m'=k+1}^L a_{m',n,L} \right] \right|.$$

with $\Delta_{n,k,L,j_1} := \prod_{m=0}^{k-(r+1)^{j_1}} b_{m,n,L}$ and $\Gamma_{n,k,L,j_0} := \left(\prod_{m=k-(r+1)^{j_0+1}}^{k-1} b_{m,n,L} \right) (b_{k,n,L} - a_{k,n,L})$.

- Control of the terms with the product of the expectations.

Let j_0, j_1 be fixed. Let $k = (r+1)^3, \dots, L-1$. We can notice that E_{n,k,L,j_0,j_1} is bounded from above by the following quantity :

$$F_{n,k,L,j_0,j_1} := \mathbb{E} \left[\left| \mathbb{E} \left[\prod_{m=k-(r+1)^{j_0+1}}^k b_{m,n,L} - \left(\prod_{m=k-(r+1)^{j_0+1}}^{k-1} b_{m,n,L} \right) a_{k,n,L} \middle| (S_p)_{p \geq 0} \right] \right| \right].$$

We use the Taylor expansions of the exponential function.

- Let us show that, in F_{n,k,L,j_0,j_1} , we can replace

$$\prod_{m=k-(r+1)^{j_0+1}}^k b_{m,n,L} = \exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell=\alpha_{(k-(r+1)^{j_0+1})}}^{\alpha_{(k+1)}-1} \xi_\ell N_n(\ell) \right)$$

by the formula given by the Taylor expansion of the exponential function at the second order :

$$1 + \frac{it}{n^{\frac{3}{4}}} \sum_{\ell=\alpha_{(k-(r+1)^{j_0+1})}}^{\alpha_{(k+1)}-1} \xi_\ell N_n(\ell) - \frac{t^2}{2n^{\frac{3}{2}}} \left(\sum_{\ell=\alpha_{(k-(r+1)^{j_0+1})}}^{\alpha_{(k+1)}-1} \xi_\ell N_n(\ell) \right)^2. \quad (8)$$

Indeed the L^1 -norm of the error between these two quantities is less than :

$$\frac{|t|^3}{6n^{\frac{9}{4}}} \mathbb{E} \left[\left| \sum_{\ell=\alpha_{(k-(r+1)^{j_0+1})}}^{\alpha_{(k+1)}-1} \xi_\ell N_n(\ell) \right|^3 \right]$$

which, according to formula (7), is less than :

$$\frac{|t|^3}{6n^{\frac{9}{4}}} \left((C_1)^4 n^2 C_2' \left(((r+1)^{j_0}) \frac{5K\sqrt{n}}{L} \right)^2 \right)^{\frac{3}{4}} = \frac{|t|^3}{6} (C_1)^3 (C_2')^{\frac{3}{4}} \left((r+1)^{j_0} \frac{5K}{L} \right)^{\frac{3}{2}}.$$

Hence, the sum over $k = (r+1)^3, \dots, L-1$ of the L^1 -norm of these errors is less than :

$$\frac{1}{\sqrt{L}} \frac{|t|^3}{6} (C_2')^{\frac{3}{4}} (C_1)^3 ((r+1)^{j_0} 5K)^{\frac{3}{2}}.$$

Let us consider $L_2 \geq L_1'$ such that, for all $L \geq L_2$, this last quantity is less than $\frac{\varepsilon}{10}$.

- Let us introduce $Y_k := \sum_{\ell=\alpha_{(k-(r+1)j_0+1)}^{\alpha(k)-1}} \xi_\ell N_n(\ell)$ and $Z_k := \sum_{\ell=\alpha_{(k)}^{\alpha(k+1)-1}} \sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_\ell \xi_m] N_n(\ell)^2$. We show that, in F_{n,k,L,j_0,j_1} , we can replace

$$\left(\prod_{m=k-(r+1)j_0+1}^{k-1} b_{m,n,L} \right) a_{k,n,L} = \exp \left(\frac{it}{n^{\frac{3}{4}}} Y_k - \frac{t^2}{2n^{\frac{3}{2}}} Z_k \right)$$

by the formula given by the Taylor expansion of the exponential function at the second order :

$$1 + \frac{it}{n^{\frac{3}{4}}} Y_k - \frac{t^2}{2n^{\frac{3}{2}}} Z_k + \frac{1}{2} \left(\frac{it}{n^{\frac{3}{4}}} Y_k - \frac{t^2}{2n^{\frac{3}{2}}} Z_k \right)^2, \quad (9)$$

Indeed, the L^1 -norm of the error between these two quantities is less than :

$$\frac{1}{6} \mathbb{E} \left[\left| \frac{it}{n^{\frac{3}{4}}} Y_k - \frac{t^2}{2n^{\frac{3}{2}}} Z_k \right|^3 \right] \leq \frac{4}{3} \mathbb{E} \left[\left| \frac{it}{n^{\frac{3}{4}}} Y_k \right|^3 + \left| \frac{t^2}{2n^{\frac{3}{2}}} Z_k \right|^3 \right].$$

According to formula (7), we have :

$$\frac{4}{3} \mathbb{E} \left[\left| \frac{it}{n^{\frac{3}{4}}} Y_k \right|^3 \right] \leq \frac{4|t|^3}{3} (C_1)^3 (C_2')^{\frac{3}{4}} \left((r+1)^{j_0} \frac{5K}{L} \right)^{\frac{3}{2}}.$$

Moreover, we have :

$$\begin{aligned} \frac{4}{3} \mathbb{E} \left[\left| \frac{t^2}{2n^{\frac{3}{2}}} Z_k \right|^3 \right] &= \frac{t^6}{6n^{\frac{9}{2}}} \left(\sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_m] \right)^3 \mathbb{E} \left[\sum_{\ell_1 \ell_2, \ell_3 = \alpha_{(k)}^{\alpha(k+1)-1}} N_n(\ell_1)^2 N_n(\ell_2)^2 N_n(\ell_3)^2 \right] \\ &\leq \frac{t^6}{6n^{\frac{9}{2}}} \left(\sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_m] \right)^3 \left(\frac{5K\sqrt{n}}{L} \right)^3 (C_1)^6 n^3. \end{aligned}$$

The sum over $k = (r+1)^3, \dots, L-1$ of the L^1 -norm of these errors is less than :

$$\frac{1}{\sqrt{L}} \frac{4|t|^3}{3} (C_2')^{\frac{3}{4}} (C_1)^3 \left((r+1)^{j_0} 5K \right)^{\frac{3}{2}} + \frac{1}{L^2} \frac{t^6}{6} \left(\sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_m] \right)^3 (5K)^3 (C_1)^6.$$

Let us consider $L'_2 \geq L_2$ such that, for all $L \geq L'_2$, this last quantity is less than $\frac{\varepsilon}{10}$.

- Now we show that, in formula (9), we can omit the term with $(Z_k)^2$. Indeed, we have :

$$\frac{1}{2} \mathbb{E} \left[\left(\frac{t^2}{2n^{\frac{3}{2}}} Z_k \right)^2 \right] \leq \frac{t^4}{8n^3} \left(\sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_m] \right)^2 \left(\frac{5K\sqrt{n}}{L} \right)^2 (C_1)^4 n^2.$$

The sum over $k = (r+1)^3, \dots, L-1$ of the L^1 -norm of these errors is less than :

$$\frac{1}{L} \frac{t^4}{8} \left(\sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_m] \right)^2 (5K)^2 (C_1)^4.$$

Let us consider $L''_2 \geq L'_2$ such that, for all $L \geq L''_2$, this last quantity is less than $\frac{\varepsilon}{10}$.

- From now, we fix $L := L''_2$ and we consider an integer $n \geq \frac{L^4}{4K^2}$.
- Hence, it remains to estimate the following quantity called G_{n,k,L,j_0,j_1} :

$$\begin{aligned} \mathbb{E} \left[\left| \mathbb{E} \left[\frac{it}{n^{\frac{3}{4}}} (Y_k + W_k) - \frac{t^2}{2n^{\frac{3}{2}}} (Y_k + W_k)^2 - \frac{it}{n^{\frac{3}{4}}} Y_k + \frac{t^2}{2n^{\frac{3}{2}}} Z_k + \right. \right. \right. \\ \left. \left. \left. + \frac{t^2}{2n^{\frac{3}{2}}} (Y_k)^2 + \frac{it}{n^{\frac{3}{4}}} Y_k \frac{t^2}{2n^{\frac{3}{2}}} Z_k \right| (S_p)_p \right] \right| \end{aligned}$$

with $W_k := \sum_{\ell=\alpha(k)}^{\alpha(k+1)-1} \xi_\ell N_n(\ell)$. Using the fact that the ξ_k are centered and independent of $(S_p)_p$, we get :

$$\begin{aligned} G_{n,k,L,j_0,j_1} &= \mathbb{E} \left[\mathbb{E} \left[-\frac{t^2}{2n^{\frac{3}{2}}} (Y_k + W_k)^2 + \frac{t^2}{2n^{\frac{3}{2}}} Z_k + \frac{t^2}{2n^{\frac{3}{2}}} (Y_k)^2 \middle| (S_p)_p \right] \right] \\ &= \frac{t^2}{2n^{\frac{3}{2}}} \mathbb{E} \left[\mathbb{E} \left[(W_k)^2 + 2W_k Y_k - Z_k \middle| (S_p)_p \right] \right]. \end{aligned}$$

Let us notice that we have :

$$Z_k := \sum_{\ell=\alpha(k)}^{\alpha(k+1)-1} \left(\mathbb{E}[(\xi_\ell)^2] N_n(\ell)^2 + 2 \sum_{m \leq \ell-1} \mathbb{E}[\xi_\ell \xi_m] N_n(\ell)^2 \right).$$

– Let us show that, in the last expression of G_{n,k,L,j_0,j_1} , we can replace Z_k by :

$$\tilde{Z}_k := \sum_{\ell=\alpha(k)}^{\alpha(k+1)-1} \left(\mathbb{E}[(\xi_\ell)^2] N_n(\ell)^2 + 2 \sum_{m \leq \ell-1} \mathbb{E}[\xi_\ell \xi_m] N_n(\ell) N_n(m) \right).$$

Indeed, we have :

$$\begin{aligned} \frac{t^2}{2n^{\frac{3}{2}}} \mathbb{E} \left[\left| Z_k - \tilde{Z}_k \right| \right] &\leq \frac{t^2}{n^{\frac{3}{2}}} \sum_{\ell=\alpha(k)}^{\alpha(k+1)-1} \sum_{m \leq \ell-1} |\mathbb{E}[\xi_\ell \xi_m]| \|N_n(\ell)\|_2 \|N_n(m) - N_n(\ell)\|_2 \\ &\leq \frac{t^2}{n^{\frac{3}{2}}} \frac{5K\sqrt{n}}{L} \sum_{p \geq 1} |\mathbb{E}[\xi_0 \xi_p]| C_1 \sqrt{n} C_2 n^{\frac{1}{4}} \sqrt{1+p}. \end{aligned}$$

The sum over $k = (r+1)^3, \dots, L-1$ of these quantities is less than :

$$\frac{t^2}{n^{\frac{1}{4}}} 5K C_1 C_2 \sum_{p \geq 1} \sqrt{1+p} |\mathbb{E}[\xi_0 \xi_p]|,$$

which goes to zero when n goes to infinity. Hence, there exists some $n_0 \geq \frac{L^4}{4K^2}$ such that, for any integer $n \geq n_0$, this sum is less than $\frac{\varepsilon}{10}$.

– Hence we have to estimate :

$$\tilde{G}_{n,k,L,j_0,j_1} = \frac{t^2}{2n^{\frac{3}{2}}} \mathbb{E} \left[\mathbb{E} \left[(W_k)^2 + 2W_k Y_k \middle| (S_p)_p \right] - \tilde{Z}_k \right].$$

We have :

$$\mathbb{E} \left[(W_k)^2 \middle| (S_p)_p \right] = \sum_{\ell=\alpha(k)}^{\alpha(k+1)-1} \left(\mathbb{E}[(\xi_\ell)^2] (N_n(\ell))^2 + 2 \sum_{m=\alpha(k)}^{\ell-1} \mathbb{E}[\xi_\ell \xi_m] N_n(\ell) N_n(m) \right).$$

Hence we have :

$$\mathbb{E} \left[(W_k)^2 + 2W_k Y_k \middle| (S_p)_p \right] = \sum_{\ell=\alpha(k)}^{\alpha(k+1)-1} \left(\mathbb{E}[(\xi_\ell)^2] (N_n(\ell))^2 + 2 \sum_{m=\alpha_{(k-(r+1)j_0+1)}}^{\ell-1} \mathbb{E}[\xi_\ell \xi_m] N_n(\ell) N_n(m) \right).$$

We get :

$$\begin{aligned} \tilde{G}_{n,k,L,j_0,j_1} &= \frac{t^2}{n^{\frac{3}{2}}} \mathbb{E} \left[\left| \sum_{\ell=\alpha(k)}^{\alpha(k+1)-1} \sum_{m \leq \alpha_{(k-(r+1)j_0+1)}-1} \mathbb{E}[\xi_\ell \xi_m] N_n(\ell) N_n(m) \right| \right] \\ &\leq \frac{t^2}{n^{\frac{3}{2}}} \frac{5K\sqrt{n}}{L} \sum_{m \geq r \frac{K\sqrt{n}}{L}} |\mathbb{E}[\xi_0 \xi_m]| (C_1)^2 n. \end{aligned}$$

The sum over $k = (r+1)^3, \dots, L-1$ of these quantities is less than :

$$t^2 5K \sum_{m \geq r \frac{K\sqrt{n}}{L}} |\mathbb{E}[\xi_0 \xi_m]| (C_1)^2,$$

which goes to zero when n goes to infinity. Hence, there exists some $n'_0 \geq n_0$ such that, for any integer $n \geq n_0$, this sum is less than $\frac{\varepsilon}{10}$.

- Control of the covariance terms.

Let j_0, j_1 be fixed. Let $k = (r+1)^3, \dots, L-1$. We have :

$$D_{n,k,L,j_0,j_1} \leq \left| \mathbb{E} \left[\text{Cov}_{|(S_p)_p} \left(\prod_{m=0}^{k-(r+1)^{j_1}} b_{m,n,L}, \prod_{m=k-(r+1)^{j_0+1}}^k b_{m,n,L} \right) \prod_{m'=k+1}^L a_{m',n,L} \right] \right| + \left| \mathbb{E} \left[\text{Cov}_{|(S_p)_p} \left(\prod_{m=0}^{k-(r+1)^{j_1}} b_{m,n,L}, \prod_{m=k-(r+1)^{j_0+1}}^{k-1} b_{m,n,L} \right) \prod_{m'=k}^L a_{m',n,L} \right] \right|.$$

But we have :

$$\prod_{m=\alpha}^{\alpha+\beta} b_{m,n,L} = \exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell = -\lceil K\sqrt{n} \rceil + \alpha \lfloor \frac{2\lceil K\sqrt{n} \rceil + 1}{L} \rfloor}^{-\lceil K\sqrt{n} \rceil + (\alpha+\beta+1) \lfloor \frac{2\lceil K\sqrt{n} \rceil + 1}{L} \rfloor - 1} \xi_\ell N_n(\ell) \right).$$

Therefore, according to point 4 of the hypothesis of our theorem, we have :

$$D_{n,k,L,j_0,j_1} \leq 2\mathbb{E} \left[C \left(1 + \frac{|t|}{n^{\frac{3}{4}}} \sum_{\ell \in \mathbb{Z}} N_n(\ell) \right) \left(\frac{rK\sqrt{n}}{2L} \right)^{-\frac{1}{2}} \sup_{s \geq r \frac{K\sqrt{n}}{2L}} \sqrt{s} \varphi_{rs,s} \right].$$

Hence, we have :

$$\sum_{k=(r+1)^3}^{L-1} D_{n,k,L,j_0,j_1} \leq 2CL\sqrt{L}C(1 + |t|n^{\frac{1}{4}}) \frac{n^{-\frac{1}{4}}\sqrt{2}}{\sqrt{rK}} \sup_{s \geq r \frac{K\sqrt{n}}{2L}} \sqrt{s} \varphi_{rs,s}.$$

which goes to zero as n goes to infinity. Hence, there exists some $N_0 \geq n'_0$ such that, for any integer $n \geq n_0$, this sum is less than $\frac{\varepsilon}{10}$.

Therefore, there exists N_0 (depending on t and on ε) such that, for any integer $n \geq N_0$, we have :

$$\left| \mathbb{E} \left[\exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell \in \mathbb{Z}} \xi_\ell N_n(\ell) \right) \right] - \mathbb{E} \left[\exp \left(-\frac{t^2}{2n^{\frac{3}{2}}} \sum_{\ell,k \in \mathbb{Z}} \mathbb{E}[\xi_\ell \xi_k] N_n(\ell)^2 \right) \right] \right| < \varepsilon.$$

This ends the step 1 of our proof.

5.2 Step 2 : Conclusion

In the previous section we proved that :

$$\lim_{n \rightarrow +\infty} \left| \mathbb{E} \left[\exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell \in \mathbb{Z}} \xi_\ell N_n(\ell) \right) \right] - \mathbb{E} \left[\exp \left(-\frac{t^2}{2n^{\frac{3}{2}}} \sum_{\ell,k \in \mathbb{Z}} \mathbb{E}[\xi_\ell \xi_k] N_n(\ell)^2 \right) \right] \right| = 0.$$

According to [15] lemma 6, we know that : $\left(\frac{1}{n^{\frac{3}{2}}} \sum_{\ell \in \mathbb{Z}} N_n(\ell)^2 \right)_{n \geq 1}$ converges in distribution to $Z_1 := \int_{\mathbb{R}} (L_1(x))^2 dx$. Hence, we get :

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell \in \mathbb{Z}} \xi_\ell N_n(\ell) \right) \right] = \mathbb{E} \left[\exp \left(-\frac{t^2}{2} \sum_{k \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_k] \int_{\mathbb{R}} (L_1(x))^2 dx \right) \right].$$

A Sinai billiard

The billiard domain is $\mathcal{D} := \mathbb{T}^2 \setminus \left(\bigcup_{i=1}^I O_i \right)$, where the scatterers O_i are open convex subsets of \mathbb{T}^2 , the closures of which are pairwise disjoint and the boundaries of which are C^3 smooth with non-null curvature. In this domain, we consider a point-particle moving with unit speed, with elastic reflections off $\partial\mathcal{D}$. Now, as in [21], let us define the billiard system (M, \mathcal{F}, ν, T) corresponding at reflection times :

- The set of reflected vectors at a reflection time will be identified with its parametrization given by :

$$M := \bigcup_{i=1}^I M^{(i)}, \quad \text{with} \quad M^{(i)} := \left(\{i\} \times \frac{\mathbb{R}}{\text{length}(\partial O_i)\mathbb{Z}} \times]-\pi/2; \pi/2[\right).$$

The configuration of a particle at a reflection time is parametrised by $(i, r, \varphi) \in M$ if the particle is on the obstacle i , at the point q with curvilinear absciss r and if the reflected vector makes an angle φ with the unit vector $\vec{n}(q)$ normal to $\partial\mathcal{D}$ at q directed to the inside of \mathcal{D} .

- We endow M with a metric such that :

$$d((i, r, \varphi), (i, r', \varphi')) = |r - r'| + |\varphi - \varphi'|.$$

- We endow M with its Borel σ -algebra \mathcal{F} .
- The transformation $T : M \rightarrow M$ maps a configuration x to the configuration $T(x)$ corresponding to the next reflection time.
- This transformation preserves the probability measure ν on M proportional to $\cos(\varphi) dr d\varphi$.

We use the parametrisation by (r, φ) introduced by Sinai in [21] and we denote by d the natural corresponding metric. We can define $g \mapsto K_g^{(1)}$ and $g \mapsto K_g^{(2)}$ such that hypothesis ?? is true and such that, for any bounded $g : M \rightarrow \mathbb{C}$, $K_g^{(1)}$ and $K_g^{(2)}$ are dominated by the following quantity :

$$C_g^{(\eta, m)} := \sup_{C \in \mathcal{C}_m} \sup_{x, y \in C, x \neq y} \frac{|g(x) - g(y)|}{\max(d(T^k(x), T^k(y)); k = -m, \dots, m)^\eta},$$

for some integer $m \geq 0$, with $\mathcal{C}_m = \{A \cap B; A \in \xi_m^u, B \in \xi_m^s\}$ with ξ_m^u and ξ_m^s as in [6] (page 7). (We recall that, for any $k = -m, \dots, m$, the map T^k is C^1 on each atom of \mathcal{C}_m).

Let us mention that, since the earliest work of Sinai [21], this billiard system has been studied by many authors ([2, 3, 4, 5, 8] and others).

The study of this system is complicated by the discontinuities of the transformation T . But it is known that T is C^1 -regular on $M \setminus (R_0 \cup T^{-1}(R_0))$, where the set $R_0 := \{(i, r, \varphi) \in M : |\varphi| = \pi/2\}$ corresponds to tangent vectors.

To prove proposition 5. We will need some results of [7, 6].

Proof. We will use the notations of Chernov in [6] We take k_0 large enough (as in [6]). Let us define $\mathbb{S} := \bigcup_{k \geq k_0} \{(i, r, \varphi) \in M : |\varphi| = \pi/2 - \frac{1}{k^2}\}$. Let us recall the definition of homogeneous stable curves and homogeneous unstable curves :

- A homogeneous stable curve is a C^1 curve γ of M contained in $M \setminus \left(\bigcup_{m \geq 0} T^{-m}(R_0 \cup \mathbb{S}) \right)$.
- A homogeneous unstable curve is a C^1 curve γ of M contained in $M \setminus \left(\bigcup_{m \geq 0} T^m(R_0 \cup \mathbb{S}) \right)$.

Let us consider an integer $m \geq 0$. We denote by ξ_m^s the partition of $M \setminus \left(\bigcup_{p=0}^m T^{-p}(R_0 \cup \mathbb{S}) \right)$ in connected components. Analogously, we denote by ξ_m^u the partition of $M \setminus \left(\bigcup_{p=0}^m T^p(R_0 \cup \mathbb{S}) \right)$ in connected components.

Let us consider the set Γ^s of homogeneous stable curves and the set Γ^u of homogeneous unstable curves (see [6] page 7 for the definition of these curves). We recall that there exist two constants $c_1 > 0$ and $\delta_1 \in]0; 1[$ such that :

- let y and z belonging to the same homogeneous unstable curve. Then, for any integer $n \geq 0$, $T^{-n}(y)$ and $T^{-n}(z)$ belong to a same homogeneous unstable curve and we have : $d(T^{-n}(y), T^{-n}(z)) \leq c_1 \delta_1^n$. Moreover, for any integer $p \geq 0$, y and z belong to the same atom of ξ_p^u . Moreover, if y and z belong to the same atom of ξ_m^s , then $T^m(y)$ and $T^m(z)$ belong to a same homogeneous unstable curve.
- let y and z belonging to the same homogeneous stable curve. Then, for any integer $n \geq 0$, $T^n(y)$ and $T^n(z)$ belong to a same homogeneous stable curve and we have : $d(T^n(y), T^n(z)) \leq c_1 \delta_1^n$. Moreover, for any integer $p \geq 0$, y and z belong to the same atom of ξ_p^s . Moreover, if y and z belong to the same atom of ξ_m^u , then $T^{-m}(y)$ and $T^{-m}(z)$ belong to a same homogeneous stable curve.

In [6], for any y, z , Chernov defines : $s_+(x, y) := \min\{n \geq 0 : y \notin \xi_n^s(x)\}$ and $s_-(x, y) := \min\{n \geq 0 : y \notin \xi_n^u(x)\}$, where $\xi_n^s(x)$ (resp. $\xi_n^u(x)$) is the atom of ξ_n^s (resp. ξ_n^u) containing the point x .

Following Chernov in [6] (page 15), let us introduce the following quantities :

$$\tilde{K}_f^{(1)} := \sup_{\gamma^u \in \Gamma^u} \sup_{y, z \in \gamma^u, y \neq z} \frac{|f(y) - f(z)|}{(\delta_1)^{\eta s_+(y, z)}}$$

and

$$\tilde{K}_f^{(2)} := \sup_{\gamma^s \in \Gamma^s} \sup_{y, z \in \gamma^s, y \neq z} \frac{|f(y) - f(z)|}{(\delta_1)^{\eta s_-(y, z)}}.$$

In the definition of [6], the suprema are taken over all unstable and stable curves instead of homogeneous unstable and stable curves. However, in the proofs of theorems 4.1, 4.2 and 4.3 of [6], Chernov only uses Hölder continuity on homogeneous stable and unstable curves. We observe that we have : $\tilde{K}_f^{(i)} \leq 2\|f\|_\infty \delta_1^{-\eta m} + K_f^{(i)}$, with :

$$K_f^{(1)} := \sup_{\gamma^u \in \Gamma^u} \sup_{y, z \in \gamma^u; y \neq z; s_+(y, z) \geq m+1} \frac{|f(y) - f(z)|}{(\delta_1)^{\eta s_+(y, z)}}$$

and

$$K_f^{(2)} := \sup_{\gamma^s \in \Gamma^s} \sup_{y, z \in \gamma^s; y \neq z; s_-(y, z) \geq m+1} \frac{|f(y) - f(z)|}{(\delta_1)^{\eta s_-(y, z)}}.$$

With these definitions, we have :

$$K_f^{(1)} \leq (\delta_1)^{-\eta(m+1)} (c_1)^\eta C_f^{(\eta, m)} \quad \text{and} \quad K_f^{(2)} \leq (\delta_1)^{-\eta(m+1)} (c_1)^\eta C_f^{(\eta, m)}.$$

Let us prove the first inequality. Let two points y and z belonging to the same homogeneous unstable curve such that $s^+(y, z) \geq m+1$. Then $y' := T^{s^+(y, z)-1}(y)$ and $z' := T^{s^+(y, z)-1}(z)$ belong to the same homogeneous unstable curve. Therefore, for any $k = -m, \dots, m$, we have :

$$\begin{aligned} d(T^k(y), T^k(z)) &= d(T^{-(s^+(y, z)-1-k)}(y'), T^{-(s^+(y, z)-1-k)}(z')) \\ &\leq c_1 \delta_1^{s^+(y, z)-1-k} \\ &\leq c_1 \delta_1^{s^+(y, z)-(m+1)}. \end{aligned}$$

Hence, since y and z belong to the same atom of \mathcal{C}_m , we have :

$$|f(y) - f(z)| \leq C_f^{(\eta, m)} (c_1)^\eta \delta_1^{\eta s^+(y, z)} \delta_1^{-\eta(m+1)}.$$

The proof of the second inequality is analogous.

Let two points y and z . If y and z belong to the same homogeneous unstable curve, then, for any integer $n \geq 0$, we have $s_+(T^{-n}(y), T^{-n}(z)) = s_+(y, z) + n$. In the same way, if y and z belong to the same homogeneous stable curve, then for any integer $n \geq 0$, we have $s_-(T^n(y), T^n(z)) = s_-(y, z) + n$.

Hence, we get points 2, 3, 4 and 5 of hypothesis ?? with $\kappa_n = 1$.

Moreover, Chernov establishes the existence of $c_3 > 0$ and of $\alpha_3 \in]0; 1[$ such that, for any integer $n \geq 0$, for any bounded \mathbb{C} -valued functions f and g , we have :

$$|Cov(f, g \circ T^n)| \leq c_3 \left(\|f\|_\infty \|g\|_\infty + \|f\|_\infty K_g^{(2)} + \|g\|_\infty K_f^{(1)} \right) (\alpha_3)^n$$

(cf. theorem 4.3 in [6] and the remark after theorem 4.3 in [6]). This gives points 1 and 5 of our hypothesis ??, \square

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