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A limit theorem for a random walk in a stationary scenery coming from a hyperbolic dynamical system

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Abstract. *In this paper, we extend a result of Kesten and Spitzer [13]. Let us consider an invertible probability dynamical system (M, \mathcal{F}, ν, T) and $f : M \rightarrow \mathbb{R}$ some function with null expectation. We define the stationary sequence $(\xi_k := f \circ T^k)_{k \in \mathbb{Z}}$. Let $(S_n)_{n \geq 0}$ be a simple symmetric random walk on \mathbb{Z} independent of $(\xi_k)_{k \in \mathbb{Z}}$. We are interested in the study of the sequence of random variables of the form $(\sum_{k=1}^n \xi_{S_k})_{n \geq 1}$. We give examples of partially hyperbolic dynamical systems (M, \mathcal{F}, ν, T) and of functions f such that $(\frac{1}{n^{\frac{3}{4}}} \sum_{k=1}^n \xi_{S_k})_{n \geq 1}$ converges in distribution.*

1 Introduction

In [13], Kesten and Spitzer prove that, if $(\xi_n)_{n \in \mathbb{Z}}$ is a sequence of independent identically distributed satisfying a central limit theorem and if $(S_n)_{n \geq 0}$ is the simple symmetric random walk on \mathbb{Z} independent of $(\xi_k)_{k \in \mathbb{Z}}$, then $(\frac{1}{n^{\frac{3}{4}}} \sum_{i=1}^n \xi_{S_i})_{n \geq 1}$ converges in distribution. In this paper, our goal is to establish such a result when $(\xi_k)_{k \in \mathbb{Z}}$ is a stationary sequence of random variables given by a dynamical system with some hyperbolic properties. More precisely, we study the cases when $(\xi_k = f \circ T^k)_{k \in \mathbb{Z}}$, with f a ν -centered Hölder continuous function and when (M, \mathcal{F}, ν, T) is one of the following dynamical systems :

- the transformation T is an ergodic algebraic automorphism of the torus $M = \mathbb{T}^{d_0}$ endowed with its normalised Haar measure ν (for some $d_0 \geq 2$);
- the transformation T is a diagonal transformation on a compact quotient M of $Sl_{d_0}(\mathbb{R})$ by a discrete subgroup, M being endowed with a natural T -invariant probability measure ν ;
- the transformation T is the Sinai billiard transformation.

In these situations, we prove that $(\frac{1}{n^{\frac{3}{4}}} \sum_{i=1}^n \xi_{S_i})_{n \geq 1}$ converges in distribution to the random variable $\sqrt{\sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_m]} \Delta_1$, where Δ_1 has the limit distribution of $(\frac{1}{n^{\frac{3}{4}}} \sum_{i=1}^n \hat{\xi}_{S_i})_{n \geq 1}$ obtained by Kesten and Spitzer when $(\hat{\xi}_m)_m$ is a sequence of independent identically distributed random variables with null expectation and with variance 1. Let us notice that, in our cases, $\sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_m]$ is well defined and is nonnegative since it is the limit of the variance of $\frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \xi_l$ as n goes to infinity.

We also get the same result of convergence in distribution for the following sequence $(\xi_k)_{k \in \mathbb{Z}}$. Let us consider the same examples of dynamical systems (M, \mathcal{F}, ν, T) . Instead of taking $\xi_k = f \circ T^k$, we suppose

that, conditionally to $\omega \in M$, $(\xi_k)_{k \in \mathbb{Z}}$ is an independent sequence of random variables with values in $\{-1; 1\}$. We suppose that, conditionally to $\omega \in M$, $\xi_k(\omega, \cdot)$ is equal to 1 with probability $h \circ T^k(\omega)$, for some nonnegative Hölder continuous function h with expectation $\frac{1}{2}$. This model is envisaged by Guillotin-Plantard and Le Ny in [8] for other questions and with other hypotheses on (M, \mathcal{F}, ν, T) and on f .

Moreover we generalize this to the case when ξ_k takes p values (conditionally to $\omega \in M$, $(\xi_k)_{k \in \mathbb{Z}}$ is an independent sequence of random variables, ξ_k being equal to θ_j with probability $f_j \circ T^k(\omega)$, with $f_1 + \dots + f_p = 1$ and with f_1, \dots, f_p are nonnegative Hölder continuous functions).

In section 2, we state a general result under technical hypotheses of decorrelation (our theorem 1). Section 5 is devoted to the proof of this result (the idea of the proof is inspired by one step of an inductive method of Jan [9, 11] used in [14]).

In section 3, we give some applications of our abstract theorem 1. We apply our theorem 1 to the examples mentioned previously (ergodic algebraic automorphisms of the torus, diagonal transformation of a compact quotient of $Sl_{d_0}(\mathbb{R})$, billiard transformation). The proofs of the results of section 3 are done in sections 3 and 4.

2 A technical result

Theorem 1 *Let $(S_n)_{n \geq 1}$ and $(\xi_k)_{k \in \mathbb{Z}}$ be two sequences of random variables defined on the same probability space $(\Omega, \mathcal{T}, \mathbb{P})$ such that :*

1. $(S_n)_{n \geq 0}$ and $(\xi_k)_{k \in \mathbb{Z}}$ are independent one of the other;
2. $(S_n)_{n \geq 0}$ is a simple symmetric random walk on \mathbb{Z} ;
3. $(\xi_k)_{k \in \mathbb{Z}}$ is a stationary sequence of centered random variables admitting moments of the fourth order;
4. we have :

$$\sum_{p \geq 0} \sqrt{1+p} |\mathbb{E}[\xi_0 \xi_p]| < +\infty$$

$$\text{and } \sup_{N \geq 1} N^{-2} \sum_{k_1, k_2, k_3, k_4=0, \dots, N-1} |\mathbb{E}[\xi_{k_1} \xi_{k_2} \xi_{k_3} \xi_{k_4}]| < +\infty.$$

5. There exists some $C > 0$, some $(\varphi_{p,s})_{p,s \in \mathbb{N}}$ and some integer $r \geq 1$ such that :

$$\forall (p, s) \in \mathbb{N}^2, \quad \varphi_{p+1,s} \leq \varphi_{p,s} \quad \text{and} \quad \lim_{s \rightarrow +\infty} \sqrt{s} \varphi_{r,s} = 0$$

and such that, for all integers n_1, n_2, n_3, n_4 with $0 \leq n_1 \leq n_2 \leq n_3 \leq n_4$, for all real numbers $\alpha_{n_1}, \dots, \alpha_{n_2}$ and $\beta_{n_3}, \dots, \beta_{n_4}$, we have :

$$\left| \text{Cov} \left(e^{i \sum_{k=n_1}^{n_2} \alpha_k \xi_k}, e^{i \sum_{k=n_3}^{n_4} \beta_k \xi_k} \right) \right| \leq C \left(1 + \sum_{k=n_1}^{n_2} |\alpha_k| + \sum_{k=n_3}^{n_4} |\beta_k| \right) \varphi_{n_3-n_2, n_4-n_3}.$$

Then, the sequence of random variables $\left(\frac{1}{n^{\frac{3}{4}}} \sum_{i=1}^n \xi_{S_i} \right)_{n \geq 1}$ converges in distribution to $\sqrt{\sum_{p \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_p]} \Delta_1$, where $\Delta_1 := \int_{\mathbb{R}} L_1(x) dB_x$, where $(B_x)_{x \in \mathbb{R}}$ and $(b_t)_{t \geq 0}$ are two independent standard brownian motions and $(L_t(x))_{t \geq 0}$ is the local time at x of $(b_t)_{t \geq 0}$, i.e. $L_t(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{(x-\varepsilon, x+\varepsilon)}(b_s) ds$.

Let us notice that the point 5 of our theorem 1 is true if $(\xi_k)_{k \in \mathbb{Z}}$ is a stationary sequence of random variables satisfying the following α -mixing condition (cf. for example [10], lemma 1.2) :

$$\lim_{n \rightarrow +\infty} \sqrt{n} \alpha_n = 0, \quad \text{with } \alpha_n := \sup_{p \geq 0; m \geq 0} \sup_{A \in \sigma(\xi_{-p}, \dots, \xi_0)} \sup_{B \in \sigma(\xi_n, \dots, \xi_{n+m})} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|.$$

3 Applications

Now let us give some examples of stationary sequences $(\xi_k)_k$ satisfying the points 3, 4 and 5 of our theorem 1. We say that (M, \mathcal{F}, ν, T) is an invertible dynamical system if (M, \mathcal{F}, ν) is a probability space endowed with an invertible bi-measurable transformation $T : M \rightarrow M$.

Hypothesis 2 *Let us consider an invertible dynamical system (M, \mathcal{F}, ν, T) such that there exists $C_0 > 0$, there exist two real sequences $(\varphi_n)_{n \geq 0}$ and $(\kappa_m)_{m \geq 0}$ and, for any function $g : M \rightarrow \mathbb{C}$, there exist $K_g^{(1)} \in [0; +\infty]$ and $K_g^{(2)} \in [0; +\infty]$ such that, for all bounded functions $g, \tilde{g}, h, \tilde{h} : M \rightarrow \mathbb{C}$:*

1. for all integer $n \geq 0$, we have : $|Cov_\nu(g, h \circ T^n)| \leq c_0 \left(\|g\|_\infty \|h\|_\infty + \|h\|_\infty K_g^{(1)} + \|g\|_\infty K_h^{(2)} \right) \varphi_n$;
2. for all integer $m \geq 0$, we have : $K_{g \circ T^{-m}}^{(1)} \leq c_0 K_g^{(1)}$;
3. for all integer $m \geq 0$, and all $k = 0, \dots, m$, we have : $K_{h \circ T^k}^{(2)} \leq c_0 K_h^{(2)} (1 + \kappa_m)$;
4. we have : $K_{g \times \tilde{g}}^{(1)} \leq \|g\|_\infty K_{\tilde{g}}^{(1)} + \|\tilde{g}\|_\infty K_g^{(1)}$;
5. we have : $K_{h \times \tilde{h}}^{(2)} \leq \|h\|_\infty K_{\tilde{h}}^{(2)} + \|\tilde{h}\|_\infty K_h^{(2)}$;
6. the sequence $(\varphi_n)_{n \geq 0}$ is decreasing and there exists an integer $r \geq 1$ such that : $\sup_{n \geq 1} n^6 (1 + \kappa_n) \varphi_{nr} < +\infty$.

For some hyperbolic or partially hyperbolic transformations, such properties are satisfied with $K_g^{(1)}$ some Hlder constant of g along the unstable manifolds and $K_h^{(2)}$ some Hlder constant of h along the stable-central manifolds, with $\varphi_n = \alpha^n$ for some $\alpha \in]0; 1[$ and $\kappa_m = m^\beta$ for some $\beta \geq 0$. Let us mention, for example, the ergodic algebraic automorphisms of the torus as well as the diagonal transformation on compact quotient of $Sl_{d_0}(\mathbb{R})$ (cf. [12]). Moreover, in the case of the Sinai billiard transformation, these properties come from [6, 5]. Since the earliest work of Sinai [15], these billiard systems have been studied by many authors (let us mention [1, 2, 3, 4, 7]). More precisely, we state :

Proposition 3 *Let us consider an integer $d_0 \geq 2$. Let (M, \mathcal{F}, ν, T) be one of the following dynamical systems :*

- (i) M is the d_0 -dimensional torus $\mathbb{T}^{d_0} = \mathbb{R}^{d_0} / \mathbb{Z}^{d_0}$ endowed with its Borel σ -algebra \mathcal{F} and with the normalised Haar measure ν on \mathbb{T}^{d_0} and T is an algebraic automorphism of \mathbb{T}^{d_0} given by a matrix $S \in Sl_{d_0}(\mathbb{Z})$ the eigenvalues of which are not root of the unity. We endow \mathbb{T}^{d_0} with the metric d induced by the natural metric on \mathbb{R}^{d_0} .
- (ii) M is a compact quotient of $Sl_{d_0}(\mathbb{R})$ by a discrete subgroup Γ of $Sl_{d_0}(\mathbb{R})$: $M := \{x\Gamma; x \in Sl_{d_0}(\mathbb{R})\}$; endowed with the normalised measure ν induced by the Haar measure on $Sl_{d_0}(\mathbb{R})$. The transformation T corresponds to the multiplication on the left by a diagonal matrix $S = \text{diag}(T_1, \dots, T_{d_0}) \in Sl_{d_0}(\mathbb{R})$ not equal to the identity and such that, for all $i = 1, \dots, d_0 - 1$, $T_i \geq T_{i+1} > 0$. We endow M with the metric d induced by a right-translations invariant riemanian metric on $SL_{d_0}(\mathbb{R})$.
- (iii) (M, \mathcal{F}, ν, T) is the time-discrete dynamical system given by the discrete Sinai billiard (corresponding to the reflection times on a scatterer). We suppose that the billiard domain is $\mathcal{D} := \mathbb{T}^2 \setminus \left(\bigcup_{i=1}^I O_i \right)$, where the scatterers O_i are open convex subsets of \mathbb{T}^2 , the closures of which are pairwise disjoint and the boundaries of which are C^3 smooth with non-null curvature. We use the parametrisation by (r, φ) introduced by Sinai in [15] and we denote by d the natural corresponding metric.

Let $\eta > 0$. We can define $g \mapsto K_g^{(1)}$ and $g \mapsto K_g^{(2)}$ such that hypothesis 2 is true and such that, for any bounded $g : M \rightarrow \mathbb{C}$, $K_g^{(1)}$ and $K_g^{(2)}$ are dominated by the Hölder constant $C_g^{(\eta)}$ of g of order η (eventually multiplied by some constant).

In the case (iii), this is still true if we replace $C_g^{(\eta)}$ by :

$$C_g^{(\eta,m)} := \sup_{C \in \mathcal{C}_m} \sup_{x,y \in C, x \neq y} \frac{|g(x) - g(y)|}{\max(d(T^k(x), T^k(y)); k = -m, \dots, m)^\eta},$$

for some integer $m \geq 0$, with $\mathcal{C}_m = \{A \cap B; A \in \xi_m^u, B \in \xi_m^s\}$ with ξ_m^u and ξ_m^s as in [5] (page 7). (We recall that, for any $k = -m, \dots, m$, the map T^k is C^1 on each atom of \mathcal{C}_m).

Proof. Let $\eta > 0$.

- In the cases (i) and (ii), we denote by $\Gamma^{(s,e)}$ the set of stable-central manifolds and by Γ^u the set of unstable manifolds. In [12], each $\gamma^u \in \Gamma^u$ is endowed with some metric d^u and each $\gamma^{(s,e)} \in \Gamma^{(s,e)}$ is endowed with some metric $d^{(s,e)}$ such that there exist $\tilde{c}_0 > 0$, $\delta_0 \in]0; 1[$ and $\beta > 0$ such that, for any integer $n \geq 0$, for any $\gamma^u \in \Gamma^u$ and any $\gamma^{(s,e)} \in \Gamma^{(s,e)}$, we have :
 - For any $y, z \in \gamma^u$, $d^u(y, z) \geq d(y, z)$ and for any $y', z' \in \gamma^{(s,e)}$, $d^{(s,e)}(y', z') \geq d(y', z')$.
 - For any $y, z \in \gamma^u$, there exists $\gamma_{(n)}^u$ such that $T^{-n}(y)$ and $T^{-n}(z)$ belong to $\gamma_{(n)}^u$ and we have : $d^u(T^{-n}(y), T^{-n}(z)) \leq \tilde{c}_0(\delta_0)^n d^u(y, z)$.
 - For any $y, z \in \gamma^{(s,e)}$, there exists $\gamma_{(n)}^{(s,e)}$ such that $T^n(y)$ and $T^n(z)$ belong to $\gamma_{(n)}^{(s,e)}$ and we have : $d^{(s,e)}(T^n(y), T^n(z)) \leq \tilde{c}_0(1 + n^\beta) d^{(s,e)}(y, z)$.

Let us define :

$$K_f^{(1)} := \sup_{\gamma^u \in \Gamma^u} \sup_{y,z \in \gamma^u: y \neq z} \frac{|f(y) - f(z)|}{(d^u(y, z))^\eta} \quad \text{and} \quad K_f^{(2)} := \sup_{\gamma^{(s,e)} \in \Gamma^{(s,e)}} \sup_{y,z \in \gamma^{(s,e)}: y \neq z} \frac{|f(y) - f(z)|}{(d^{(s,e)}(y, z))^\eta}.$$

Hence, the points 2, 3, 4 and 5 of hypothesis 2 are satisfied with $\kappa_n = n^\beta$. Moreover, these two quantities are less than the Hölder constant of order η of f .

In [12], the point 1 of hypothesis 2 is proved in the particular case (ii). The same proof can be used in the case (i). We get a sequence $(\varphi_n)_n$ decreasing exponentially fast (cf. lemme 1.3.1 in [12]).

- Let us now consider the case (iii). Let us consider an integer $m \geq 0$. Let us consider the set Γ^s of homogeneous stable curves and the set Γ^u of homogeneous unstable curves (see [5] page 7 for the definition of these curves). We recall that there exist two constants $c_1 > 0$ and $\delta_1 \in]0; 1[$ such that :
 - let y and z belonging to the same homogeneous unstable curve. Then, for any integer $n \geq 0$, $T^{-n}(y)$ and $T^{-n}(z)$ belong to a same homogeneous unstable curve and we have : $d(T^{-n}(y), T^{-n}(z)) \leq c_1 \delta_1^n$. Moreover, for any integer $p \geq 0$, y and z belong to the same atom of ξ_p^u . Moreover, if y and z belong to the same atom of ξ_m^s , then $T^m(y)$ and $T^m(z)$ belong to a same homogeneous unstable curve.
 - let y and z belonging to the same homogeneous stable curve. Then, for any integer $n \geq 0$, $T^n(y)$ and $T^n(z)$ belong to a same homogeneous stable curve and we have : $d(T^n(y), T^n(z)) \leq c_1 \delta_1^n$. Moreover, for any integer $p \geq 0$, y and z belong to the same atom of ξ_p^s . Moreover, if y and z belong to the same atom of ξ_m^u , then $T^{-m}(y)$ and $T^{-m}(z)$ belong to a same homogeneous stable curve.

In [5], for any y, z , Chernov defines : $s_+(x, y) := \min\{n \geq 0 : y \notin \xi_n^s(x)\}$ and $s_-(x, y) := \min\{n \geq 0 : y \notin \xi_n^u(x)\}$, where $\xi_n^s(x)$ (resp. $\xi_n^u(x)$) is the atom of ξ_n^s (resp. ξ_n^u) containing the point x .

Following Chernov in [5] (page 15), let us introduce the following quantities :

$$\tilde{K}_f^{(1)} := \sup_{\gamma^u \in \Gamma^u} \sup_{y,z \in \gamma^u: y \neq z} \frac{|f(y) - f(z)|}{(\delta_1)^{\eta s_+(y,z)}}$$

and

$$\tilde{K}_f^{(2)} := \sup_{\gamma^s \in \Gamma^s} \sup_{y,z \in \gamma^s: y \neq z} \frac{|f(y) - f(z)|}{(\delta_1)^{\eta s_-(y,z)}}.$$

In the definition of [5], the suprema are taken over all unstable and stable curves instead of homogeneous unstable and stable curves. However, in the proofs of theorems 4.1, 4.2 and 4.3 of [5], Chernov only uses Hölder continuity on homogeneous stable and unstable curves. We observe that we have : $\tilde{K}_f^{(i)} \leq 2\|f\|_\infty \delta_1^{-\eta m} + K_f^{(i)}$, with :

$$K_f^{(1)} := \sup_{\gamma^u \in \Gamma^u} \sup_{y, z \in \gamma^u; y \neq z; s_+(y, z) \geq m+1} \frac{|f(y) - f(z)|}{(\delta_1)^{\eta s_+(y, z)}}$$

and

$$\tilde{K}_f^{(2)} := \sup_{\gamma^s \in \Gamma^s} \sup_{y, z \in \gamma^s; y \neq z; s_-(y, z) \geq m+1} \frac{|f(y) - f(z)|}{(\delta_1)^{\eta s_-(y, z)}}.$$

With these definitions, we have :

$$K_f^{(1)} \leq (\delta_1)^{-\eta(m+1)} (c_1)^\eta C_f^{(\eta, m)} \quad \text{and} \quad K_f^{(2)} \leq (\delta_1)^{-\eta(m+1)} (c_1)^\eta C_f^{(\eta, m)}.$$

Let us prove the first inequality. Let two points y and z belonging to the same homogeneous unstable curve such that $s^+(y, z) \geq m+1$. Then $y' := T^{s^+(y, z)-1}(y)$ and $z' := T^{s^+(y, z)-1}(z)$ belong to the same homogeneous unstable curve. Therefore, for any $k = -m, \dots, m$, we have :

$$\begin{aligned} d(T^k(y), T^k(z)) &= d(T^{-(s^+(y, z)-1-k)}(y'), T^{-(s^+(y, z)-1-k)}(z')) \\ &\leq c_1 \delta_1^{s^+(y, z)-1-k} \\ &\leq c_1 \delta_1^{s^+(y, z)-(m+1)}. \end{aligned}$$

Hence, since y and z belong to the same atom of \mathcal{C}_m , we have :

$$|f(y) - f(z)| \leq C_f^{(\eta, m)} (c_1)^\eta \delta_1^{\eta s^+(y, z)} \delta_1^{-\eta(m+1)}.$$

The proof of the second inequality is analogous.

Let two points y and z . If y and z belong to the same homogeneous unstable curve, then, for any integer $n \geq 0$, we have $s_+(T^{-n}(y), T^{-n}(z)) = s_+(y, z) + n$. In the same way, if y and z belong to the same homogeneous stable curve, then for any integer $n \geq 0$, we have $s_-(T^n(y), T^n(z)) = s_-(y, z) + n$. Hence, we get points 2, 3, 4 and 5 of hypothesis 2 with $\kappa_n = 1$.

Moreover, Chernov establishes the existence of $c_3 > 0$ and of $\alpha_3 \in]0; 1[$ such that, for any integer $n \geq 0$, for any bounded \mathbb{C} -valued functions f and g , we have :

$$|Cov(f, g \circ T^n)| \leq c_3 \left(\|f\|_\infty \|g\|_\infty + \|f\|_\infty K_g^{(2)} + \|g\|_\infty K_f^{(1)} \right) (\alpha_3)^n$$

(cf. theorem 4.3 in [5] and the remark after theorem 4.3 in [5]). This gives the points 1 and 5 of our hypothesis 2,

qed.

Theorem 4 *Let us suppose hypothesis 2. Let $f : M \rightarrow \mathbb{R}$ be a bounded function.*

(a) *Let us suppose that f is ν -centered, that $K_f^{(1)} < +\infty$ and $K_f^{(2)} < +\infty$. We suppose that there exists some real number $c_1 > 0$ such that, for any real number α , we have : $K_{\exp(i\alpha f)}^{(1)} \leq c_1 |\alpha|$ and $K_{\exp(i\alpha f)}^{(2)} \leq c_1 |\alpha|$. Then $(\xi_k := f \circ T^k)_{k \in \mathbb{Z}}$ satisfies the points 3, 4 and 5 of our theorem.*

(b) *Let us suppose that f takes its values in $[0; 1]$. Moreover let us suppose that there exists some $c_1 > 0$ such that, for any $a, b \in \mathbb{C}$, we have $K_{af+b}^{(1)} \leq c_1 |a|$ and $K_{af+b}^{(2)} \leq c_1 |a|$.*

Let $(\Omega_1 :=]0; 1]^{\mathbb{Z}}, \mathcal{F}_1 := (\mathcal{B}(]0; 1])^{\times \mathbb{Z}}, \nu_1 := \lambda^{\otimes \mathbb{Z}}$ where λ is the Lebesgue measure on $]0; 1[$. We define $(\xi_k)_{k \in \mathbb{Z}}$ on the product $(\Omega_2 := M \times \Omega_1, \mathcal{F}_2 := \mathcal{F} \otimes \mathcal{F}_1, \nu_2 := \nu \otimes \nu_1)$ as follows :

$$\xi_k(\omega, (z_m)_{m \in \mathbb{Z}}) := 2 \cdot \mathbf{1}_{\{z_k \leq f \circ T^k(\omega)\}} - 1.$$

Then $(\xi_k)_{k \in \mathbb{Z}}$ satisfies points 3, 4 and 5 of our theorem.

(c) Let us fix an integer $p \geq 2$. Let us fix p real numbers $\theta_1, \dots, \theta_p$ (and $\theta_0 := 0$) and p non-negative functions $f_1, \dots, f_p : M \rightarrow [0; 1]$ such that $\int_M (\theta_1 f_1 + \dots + \theta_p f_p) d\nu = 0$ and $f_1 + \dots + f_p = 1$ and such that there exists $c_2 > 0$ such that, for all complex numbers a_1, \dots, a_{p-1}, b , we have

$$\max(K_{a_1 f_1 + \dots + a_{p-1} f_{p-1} + b}^{(1)}, K_{a_1 f_1 + \dots + a_{p-1} f_{p-1} + b}^{(2)}) \leq c_2(|a_1| + \dots + |a_{p-1}|).$$

Let $(\Omega_1 :=]0; 1[^\mathbb{Z}, \mathcal{F}_1 := (\mathcal{B}(]0; 1[))^\otimes \mathbb{Z}, \nu_1 := \lambda^{\otimes \mathbb{Z}})$ where λ is the Lebesgue measure on $]0; 1[$. We define $(\xi_k)_{k \in \mathbb{Z}}$ on the product $(\Omega_2 := M \times \Omega_1, \mathcal{F}_2 := \mathcal{F} \otimes \mathcal{F}_1, \nu_2 := \nu \otimes \nu_1)$ as follows :

$$\xi_k(\omega, (z_m)_{m \in \mathbb{Z}}) = \sum_{l=1}^p (\theta_l - \theta_{l-1}) \mathbf{1}_{\{z_k \leq \sum_{j=1}^l f_j(T^k(\omega))\}},$$

Then $(\xi_k)_{k \in \mathbb{Z}}$ satisfies points 3, 4 and 5 of our theorem.

Let us make some comments on the point (b). Conditionally to $\omega \in M$, $(\tilde{\xi}_k(\omega, \cdot))_{k \in \mathbb{Z}}$ is a sequence of independent random variables with values in $\{-1; 1\}$ and $\tilde{\xi}_k(\omega, \cdot)$ is equal to 1 with probability $f \circ T^k(\omega)$. This model is envisaged by Guillotin-Plantard and Le Ny in [8].

The case (c) is a generalization of the case (b) to the case when $\tilde{\xi}_k$ takes p values (conditionally to $\omega \in M$, $\tilde{\xi}_k(\omega, \cdot)$ is equal to θ_j with probability $f_j \circ T^k(\omega)$).

A direct consequence of proposition 3 and of theorem 4 is :

Theorem 5 Let (M, \mathcal{F}, ν, T) be as in proposition 3. Let $\eta > 0$. Let $p \geq 2$. Let $f, f_1, \dots, f_p : M \rightarrow \mathbb{R}$ be $(p+1)$ bounded Hlder continuous function of order η (or, in the case (iii) of proposition 3, we suppose that these functions are bounded and such that $C_f^{(\eta, m)} < +\infty$ and $\sup_{i=1, \dots, p} C_{f_i}^{(\eta, m)} < +\infty$ for some integer $m \geq 0$).

We suppose that f_1, \dots, f_p are non-negative functions satisfying $f_1 + \dots + f_p = 1$.

(a) Let us suppose that f is ν -centered. Then $(\xi_k := f \circ T^k)_{k \in \mathbb{Z}}$ satisfies points 3, 4 and 5 of our theorem.

(b) Let us suppose that f takes its values in $[0; 1]$ and that we have $\int_M f d\nu = \frac{1}{2}$.

Let $(\Omega_1 :=]0; 1[^\mathbb{Z}, \mathcal{F}_1 := (\mathcal{B}(]0; 1[))^\otimes \mathbb{Z}, \nu_1 := \lambda^{\otimes \mathbb{Z}})$ where λ is the Lebesgue measure on $]0; 1[$. We define $(\xi_k)_{k \in \mathbb{Z}}$ on the product $(\Omega_2 := M \times \Omega_1, \mathcal{F}_2 := \mathcal{F} \otimes \mathcal{F}_1, \nu_2 := \nu \otimes \nu_1)$ as follows :

$$\tilde{\xi}_k(\omega, (z_m)_{m \in \mathbb{Z}}) := 2 \cdot \mathbf{1}_{\{z_k \leq f \circ T^k(\omega)\}} - 1.$$

Then $(\xi_k)_{k \in \mathbb{Z}}$ satisfies points 3, 4 and 5 of our theorem.

(c) Let us fix p real numbers $\theta_1, \dots, \theta_p$ (and $\theta_0 = 0$) such that $\int_M (\theta_1 f_1 + \dots + \theta_p f_p) d\nu = 0$ and Let $(\Omega_1 :=]0; 1[^\mathbb{Z}, \mathcal{F}_1 := (\mathcal{B}(]0; 1[))^\otimes \mathbb{Z}, \nu_1 := \lambda^{\otimes \mathbb{Z}})$ where λ is the Lebesgue measure on $]0; 1[$. We define $(\xi_k)_{k \in \mathbb{Z}}$ on the product $(\Omega_2 := M \times \Omega_1, \mathcal{F}_2 := \mathcal{F} \otimes \mathcal{F}_1, \nu_2 := \nu \otimes \nu_1)$ as follows :

$$\xi_k(\omega, (z_m)_{m \in \mathbb{Z}}) = \sum_{l=1}^p (\theta_l - \theta_{l-1}) \mathbf{1}_{\{z_k \leq \sum_{j=1}^l f_j(T^k(\omega))\}},$$

Then $(\xi_k)_{k \in \mathbb{Z}}$ satisfies points 3, 4 and 5 of our theorem.

Let us observe that, in the case (iii) of proposition 3, we can take the function f constant on each atom of \mathcal{C}_m for some integer $m \geq 0$. For example $f = \mathbf{1}_{\cup_{k \geq k_0} \mathbb{H}_k} - \mathbf{1}_{\cup_{k \geq k_0} \mathbb{H}_{-k}}$ satisfies the case (a) of theorem 5 for the Sinai billiard (with the notations k_0 and \mathbb{H}_k of [5] page 5). In the case (c) of theorem 5, we can take $p = 3$, $\theta_1 = 1$, $\theta_2 = -1$, $\theta_3 = 0$, $f_1 = \mathbf{1}_{\cup_{k \geq k_0} \mathbb{H}_k}$, $f_2 = \mathbf{1}_{\cup_{k \geq k_0} \mathbb{H}_{-k}}$, $f_3 = \mathbf{1} - f_1 - f_2$ in the case of the Sinai billiard (with again the notations of [5] page 5).

4 Proof of theorem 4

In the cases (a), (b) and (c), it is easy to see that $(\xi_k)_k$ is a stationary sequence of bounded random variables

4.1 Proof of (a)

We have :

$$\begin{aligned} \sum_{p \geq 0} \sqrt{1+p} |\mathbb{E}[\xi_0 \xi_p]| &= \sum_{p \geq 0} \sqrt{1+p} |\mathbb{E}_\nu[f \cdot f \circ T^p]| \\ &\leq c_0 \|f\|_\infty \left(\|f\|_\infty + K_f^{(1)} + K_f^{(2)} \right) \sum_{p \geq 0} \sqrt{1+p} \varphi_p < +\infty. \end{aligned}$$

Let us consider an integer $N \geq 1$. We have :

$$\frac{1}{N^2} \sum_{k_1, k_2, k_3, k_4=0, \dots, N-1} |\mathbb{E}[\xi_{k_1} \xi_{k_2} \xi_{k_3} \xi_{k_4}]| \leq \frac{24}{N^2} \sum_{0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq N-1} |\mathbb{E}[\xi_{k_1} \xi_{k_2} \xi_{k_3} \xi_{k_4}]|.$$

Let us consider the set $E_N^{(1)}$ of (k_1, k_2, k_3, k_4) such that $0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq N-1$ and $k_4 - k_3 \geq N^{\frac{1}{3}}$. We have :

$$\begin{aligned} \sum_{(k_1, k_2, k_3, k_4) \in E_N^{(1)}} |\mathbb{E}[\xi_{k_1} \xi_{k_2} \xi_{k_3} \xi_{k_4}]| &= \sum_{(k_1, k_2, k_3, k_4) \in E_N^{(1)}} |Cov_\nu(f \circ T^{k_1 - k_3} f \circ T^{k_2 - k_3} f, f \circ T^{k_4 - k_3})| \\ &\leq c_0 N^4 \left(\|f\|_\infty^4 + \|f\|_\infty^3 (K_f^{(2)} + 3c_0 K_f^{(1)}) \right) \varphi_{\lceil N^{\frac{1}{3}} \rceil} \\ &\leq c_0 N^2 \left(\|f\|_\infty^4 + \|f\|_\infty^3 (K_f^{(2)} + 3c_0 K_f^{(1)}) \right) \sup_{n \geq 1} n^6 \varphi_n. \end{aligned}$$

Let us consider the set $E_N^{(2)}$ of (k_1, k_2, k_3, k_4) such that $0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq N-1$ and $k_4 - k_3 < N^{\frac{1}{3}}$ and $k_3 - k_2 \geq rN^{\frac{1}{3}}$. We have :

$$\begin{aligned} \sum_{(k_1, k_2, k_3, k_4) \in E_N^{(2)}} |Cov(\xi_{k_1} \xi_{k_2}, \xi_{k_3} \xi_{k_4})| &= \sum_{(k_1, k_2, k_3, k_4) \in E_N^{(2)}} |Cov_\nu(f \circ T^{k_1 - k_2} f, (f \circ T^{k_4 - k_3}) \circ T^{k_3 - k_2})| \\ &\leq c_0 N^4 \left(\|f\|_\infty^4 + 2c_0 \|f\|_\infty^3 (K_f^{(2)} + K_f^{(1)}) \right) (1 + \kappa_{\frac{1}{r} \lceil rN^{\frac{1}{3}} \rceil}) \varphi_{\lceil rN^{\frac{1}{3}} \rceil} \\ &\leq c_0 N^2 \left(\|f\|_\infty^4 + 2c_0 \|f\|_\infty^3 (K_f^{(2)} + K_f^{(1)}) \right) \sup_{n \geq 1} n^6 (1 + \kappa_n) \varphi_{rn}. \end{aligned}$$

Moreover, we have :

$$\begin{aligned} \sum_{(k_1, k_2, k_3, k_4) \in E_N^{(2)}} |\mathbb{E}[\xi_{k_1} \xi_{k_2}] \mathbb{E}[\xi_{k_3} \xi_{k_4}]| &\leq \left(\sum_{0 \leq k_1 \leq k_2 \leq N-1} |\mathbb{E}[\xi_{k_1} \xi_{k_2}]| \right)^2 \\ &\leq \left(N \sum_{k \geq 0} |\mathbb{E}_\nu[f \cdot f \circ T^k]| \right)^2 \\ &\leq N^2 \left(c_0 \left(\|f\|_\infty^2 + \|f\|_\infty (K_f^{(1)} + K_f^{(2)}) \right) \sum_{k \geq 0} \varphi_k \right)^2. \end{aligned}$$

Let us consider the set $E_N^{(3)}$ of (k_1, k_2, k_3, k_4) such that $0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq N-1$ and $k_4 - k_3 < N^{\frac{1}{3}}$ and $k_3 - k_2 < rN^{\frac{1}{3}}$ and $k_2 - k_1 \geq r(1+r)N^{\frac{1}{3}}$. By the same method, we get :

$$\sum_{(k_1, k_2, k_3, k_4) \in E_N^{(3)}} |\mathbb{E} [\xi_{k_1} \xi_{k_2} \xi_{k_3} \xi_{k_4}]| \leq N^2 \frac{c_0}{(1+r)^6} \left(\|f\|_\infty^4 + 3c_0 \|f\|_\infty^3 (K_f^{(2)} + K_f^{(1)}) \right) \sup_{n \geq 1} n^6 (1 + \kappa_n) \varphi_{rn}.$$

Since the number of (k_1, k_2, k_3, k_4) such that $0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq N-1$ and that do not belong to $E_N^{(1)} \cup E_N^{(2)} \cup E_N^{(3)}$ is bounded by $N^2 2(r+1)^3$, we get :

$$\sup_{N \geq 1} \frac{1}{N^2} \sum_{k_1, k_2, k_3, k_4=0, \dots, N-1} |\mathbb{E} [\xi_{k_1} \xi_{k_2} \xi_{k_3} \xi_{k_4}]| < +\infty.$$

Now, let us prove the point 5. Let n_1, n_2, n_3 and n_4 be four integers such that $0 \leq n_1 \leq n_2 \leq n_3 \leq n_4$. Let us consider any real numbers $\alpha_{n_1}, \dots, \alpha_{n_2}$ and $\beta_{n_3}, \dots, \beta_{n_4}$. We have :

$$\begin{aligned} \left| Cov \left(e^{i \sum_{k=n_1}^{n_2} \alpha_k \xi_k}, e^{i \sum_{k=n_3}^{n_4} \beta_k \xi_k} \right) \right| &= \left| Cov_\nu \left(e^{i \sum_{k=n_1}^{n_2} \alpha_k f \circ T^{-(n_2-k)}}, \left(e^{i \sum_{k=n_3}^{n_4} \beta_k f \circ T^{k-n_3}} \right) \circ T^{n_3-n_2} \right) \right| \\ &\leq c_0 \left(1 + K_{\exp(i \sum_{k=n_1}^{n_2} \alpha_k f \circ T^{-(n_2-k)})}^{(1)} + K_{\exp(i \sum_{k=n_3}^{n_4} \beta_k f \circ T^{k-n_3})}^{(2)} \right) \varphi_{n_3-n_2} \\ &\leq c_0 \left(1 + \sum_{k=n_1}^{n_2} K_{\exp(i \alpha_k f \circ T^{-(n_2-k)})}^{(1)} + \sum_{k=n_3}^{n_4} K_{\exp(i \beta_k f \circ T^{k-n_3})}^{(2)} \right) \varphi_{n_3-n_2} \\ &\leq c_0 \left(1 + \sum_{k=n_1}^{n_2} c_0 c_1 |\alpha_k| + \sum_{k=n_3}^{n_4} c_0 c_1 |\beta_k| (1 + \kappa_{n_4-n_3}) \right) \varphi_{n_3-n_2}. \end{aligned}$$

We conclude by taking $\varphi_{p,s} := (1 + \kappa_s) \varphi_p$.

4.2 Proof of (b) and of (c)

Let us consider (c) which is an extension of the case (b) (by taking $p = 2, \theta_1 = 1, \theta_2 = -1, f_1 = f$ and $f_2 = 1 - f$). Let us define the function $g := \sum_{j=1}^p \theta_j f_j$ (in the case (b), we have : $g = 2f - 1$). This function is ν -centered and satisfies $K_g^{(1)} + K_g^{(2)} < +\infty$. We observe that, conditionally to $\omega \in M$, the expectation of $\xi_k(\omega, \cdot)$ is equal to $g \circ T^k(\omega)$. Using the Fubini theorem and starting by integrating over Ω_1 , we observe that, for any integers k and l , we have : $\mathbb{E} [\xi_k \xi_l] = \mathbb{E}_\nu [g \circ T^k \cdot g \circ T^l]$ and that, for any integers k_1, k_2, k_3, k_4 , we have : $\mathbb{E} [\xi_{k_1} \xi_{k_2} \xi_{k_3} \xi_{k_4}] = \mathbb{E}_\nu \left[\prod_{j=1}^4 g \circ T^{k_j} \right]$. Hence, we can prove the point 4 of theorem 1 as we proved it for (a).

Now, let us prove the point 5 of theorem 1. We observe that, conditionally to $\omega \in M$, the expectation of $\exp(iu \xi_k(\omega, \cdot))$ is $h_u \circ T^k$ with $(h_u := \sum_{l=1}^p e^{i\theta_l u} f_l)$. This function can be rewritten : $h_u = e^{i\theta_p u} + \sum_{l=1}^{p-1} (e^{i\theta_l u} - e^{i\theta_p u}) f_l$. The modulus of this function is bounded by 1 and we have :

$$\max \left(K_{h_u}^{(1)}, K_{h_u}^{(2)} \right) \leq c_2 2p \max_{j=0, \dots, p} |\theta_j| |u|.$$

Let n_1, n_2, n_3 and n_4 be four integers such that $0 \leq n_1 \leq n_2 \leq n_3 \leq n_4$. Let us consider any real numbers $\alpha_{n_1}, \dots, \alpha_{n_2}$ and $\beta_{n_3}, \dots, \beta_{n_4}$. We have :

$$\begin{aligned} \left| Cov \left(e^{i \sum_{k=n_1}^{n_2} \alpha_k \xi_k}, e^{i \sum_{k=n_3}^{n_4} \beta_k \xi_k} \right) \right| &= \\ &= \left| Cov_\nu \left(\prod_{k=n_1}^{n_2} h_{\alpha_k} \circ T^k, \prod_{k=n_3}^{n_4} h_{\beta_k} \circ T^k \right) \right| \\ &\leq c_0 \left(1 + c_0 c_2 2p \max_{j=0, \dots, p} |\theta_j| \left(\sum_{k=n_1}^{n_2} |\alpha_k| + \sum_{k=n_3}^{n_4} |\beta_k| \right) \right) (1 + \kappa_{n_4-n_3}) \varphi_{n_3-n_2}. \end{aligned}$$

5 Proof of theorem 1

To prove our result of convergence in distribution, we use characteristic functions. Let us fix some real number t . We will show that :

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{k=1}^n \xi_{S_k} \right) \right] = \mathbb{E} \left[\exp \left(it \sqrt{\sum_{p \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_p]} \Delta_1 \right) \right].$$

Let us notice that we have (cf [13] lemma 5, for example) :

$$\mathbb{E} [\exp(iu\Delta_1)] = \mathbb{E} \left[\exp \left(-\frac{u^2}{2} \int_{\mathbb{R}} (L_1(x))^2 dx \right) \right].$$

Hence, it is enough to prove that :

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{k=1}^n \xi_{S_k} \right) \right] = \mathbb{E} \left[\exp \left(-\frac{t^2}{2} \sum_{p \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_p] \int_{\mathbb{R}} (L_1(x))^2 dx \right) \right].$$

In the following, for any integer $m \geq 1$ and any integer k , we define :

$$N_m(k) := \text{Card}\{j = 1, \dots, m : S_j = k\}.$$

We notice that, for any integer $n \geq 1$, we have :

$$\sum_{j=1}^n \xi_{S_j} = \sum_{k \in \mathbb{Z}} \xi_k N_n(k).$$

In the step 1 of our proof, we will use the following facts :

$$C_0 := \sup_{n \geq 1} \sup_{K > 0} K^2 n^{-1} \mathbb{P} \left(\max_{m=1, \dots, n} |S_m| \geq K \right) < +\infty,$$

$$C_1 := \sup_{n \geq 1} \sup_{k \in \mathbb{Z}} n^{-\frac{1}{2}} \|N_n(k)\|_6 < +\infty,$$

$$C_2 := \sup_{n \geq 1} \sup_{k, \ell \in \mathbb{Z}} \frac{\|N_n(\ell) - N_n(k)\|_2}{\sqrt{1 + |\ell - k|} n^{\frac{1}{4}}} < +\infty.$$

The first fact comes from the Kolmogorov inequality. We refer to [13] lemmas 1, 2, 3 and 4 for the proof of the other facts.

5.1 Step 1 : Technical part

This is the big part of our proof. In this part, we prove that the following quantity goes to zero as n goes to $+\infty$:

$$\left| \mathbb{E} \left[\exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell \in \mathbb{Z}} \xi_{\ell} N_n(\ell) \right) \right] - \mathbb{E} \left[\exp \left(-\frac{t^2}{2n^{\frac{3}{2}}} \sum_{\ell, k \in \mathbb{Z}} \mathbb{E}[\xi_{\ell} \xi_k] N_n(\ell)^2 \right) \right] \right|.$$

Let us fix $\varepsilon > 0$. We will prove that, if n is large enough, this quantity is less than ε .

Our proof is inspired by a method used by Jan to establish central limit theorem with rate of convergence (cf. [11], [9], method also used in [14]). More precisely, we adapt the idea of the first step of the inductive method of Jan.

- For any $K \geq 1$ and any integer $n \geq 1$, we have :

$$\mathbb{P} \left(\max_{m=1, \dots, n} |S_m| \geq K\sqrt{n} \right) \leq \frac{C_0 n}{K^2 n} = \frac{C_0}{K^2}.$$

Let us fix $K \geq 1$ such that $2\frac{C_0}{K^2} < \frac{\varepsilon}{10}$. Then, we have

$$\left| \mathbb{E} \left[\exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell \in \mathbb{Z}} \xi_\ell N_n(\ell) \right) \right] - \mathbb{E} \left[\exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell=-\lceil K\sqrt{n} \rceil}^{\lceil K\sqrt{n} \rceil} \xi_\ell N_n(\ell) \right) \right] \right| \leq 2\frac{C_0}{K^2} < \frac{\varepsilon}{10} \quad (1)$$

and :

$$\left| \mathbb{E} \left[\exp \left(-\frac{t^2}{2n^{\frac{3}{2}}} \sum_{\ell, k \in \mathbb{Z}} \mathbb{E}[\xi_\ell \xi_k] N_n(\ell)^2 \right) \right] - \mathbb{E} \left[\exp \left(-\frac{t^2}{2n^{\frac{3}{2}}} \sum_{\ell=-\lceil K\sqrt{n} \rceil}^{\lceil K\sqrt{n} \rceil} \sum_{k \in \mathbb{Z}} \mathbb{E}[\xi_\ell \xi_k] N_n(\ell)^2 \right) \right] \right| < \frac{\varepsilon}{10}. \quad (2)$$

Hence we have to estimate :

$$A_n := \left| \mathbb{E} \left[\exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell=-\lceil K\sqrt{n} \rceil}^{\lceil K\sqrt{n} \rceil} \xi_\ell N_n(\ell) \right) \right] - \mathbb{E} \left[\exp \left(-\frac{t^2}{2n^{\frac{3}{2}}} \sum_{\ell=-\lceil K\sqrt{n} \rceil}^{\lceil K\sqrt{n} \rceil} \sum_{k \in \mathbb{Z}} \mathbb{E}[\xi_\ell \xi_k] N_n(\ell)^2 \right) \right] \right|. \quad (3)$$

- In the following, L will be some real number bigger than 8 and large enough and n any integer bigger than 1 and large enough such that : $\frac{2K\sqrt{n}}{L} \geq L$. We will have : $\frac{K\sqrt{n}}{L} \leq \left\lfloor \frac{2\lceil K\sqrt{n} \rceil + 1}{L} \right\rfloor \leq \frac{5K\sqrt{n}}{L}$.
- We split our sums $\sum_{\ell=-\lceil K\sqrt{n} \rceil}^{\lceil K\sqrt{n} \rceil}$ in L sums over $\left\lfloor \frac{2\lceil K\sqrt{n} \rceil + 1}{L} \right\rfloor$ terms and one sum over less than L terms and so over less than $\left\lfloor \frac{2\lceil K\sqrt{n} \rceil + 1}{L} \right\rfloor$ terms.

For any $k = 0, \dots, L-1$, we define :

$$a_{k,n,L} = \exp \left(-\frac{t^2}{2n^{\frac{3}{2}}} \sum_{\ell=-\lceil K\sqrt{n} \rceil + k}^{-\lceil K\sqrt{n} \rceil + (k+1) \left\lfloor \frac{2\lceil K\sqrt{n} \rceil + 1}{L} \right\rfloor - 1} \sum_{k \in \mathbb{Z}} \mathbb{E}[\xi_\ell \xi_k] N_n(\ell)^2 \right)$$

and

$$b_{k,n,L} = \exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell=-\lceil K\sqrt{n} \rceil + k}^{-\lceil K\sqrt{n} \rceil + (k+1) \left\lfloor \frac{2\lceil K\sqrt{n} \rceil + 1}{L} \right\rfloor - 1} \xi_\ell N_n(\ell) \right).$$

Moreover, we define :

$$a_{L,n,L} = \exp \left(-\frac{t^2}{2n^{\frac{3}{2}}} \sum_{\ell=-\lceil K\sqrt{n} \rceil + L}^{\lceil K\sqrt{n} \rceil} \sum_{k \in \mathbb{Z}} \mathbb{E}[\xi_\ell \xi_k] N_n(\ell)^2 \right)$$

and

$$b_{L,n,L} = \exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell=-\lceil K\sqrt{n} \rceil + L}^{\lceil K\sqrt{n} \rceil} \xi_\ell N_n(\ell) \right).$$

Let us notice that, for any $k = 0, \dots, L$, we have :

$$|a_{k,n,L}| \leq 1 \quad \text{and} \quad |b_{k,n,L}| \leq 1.$$

We have :

$$\begin{aligned}
|A_n| &= \left| \mathbb{E} \left[\prod_{k=0}^L b_{k,n,L} - \prod_{k=0}^L a_{k,n,L} \right] \right| \\
&= \left| \sum_{k=0}^L \mathbb{E} \left[\left(\prod_{m=0}^{k-1} b_{m,n,L} \right) (b_{k,n,L} - a_{k,n,L}) \prod_{m'=k+1}^L a_{m',n,L} \right] \right|. \tag{4}
\end{aligned}$$

- Now we explain how we can restrict our study to the sum over the k such that $(r+1)^3 \leq k \leq L-1$. Indeed, the number of k that do not satisfy this is equal to $(r+1)^3 + 1$. Let us consider any $k = 0, \dots, L$. We have :

$$\mathbb{E} [|b_{k,n,L} - 1|] \leq \frac{|t|}{n^{\frac{3}{4}}} \mathbb{E} \left[\left| \sum_{\ell=\dots}^{\dots} \xi_\ell N_n(\ell) \right| \right]$$

and :

$$\mathbb{E} [|a_{k,n,L} - 1|] \leq \frac{t^2}{2n^{\frac{3}{2}}} \mathbb{E} \left[\left| \sum_{\ell=\dots}^{\dots} \sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_\ell \xi_m] N_n(\ell)^2 \right| \right].$$

But, for any integers α and β with $\beta \geq 1$, we have :

$$\begin{aligned}
\mathbb{E} \left[\left(\sum_{\ell=\alpha+1}^{\alpha+\beta} \xi_\ell N_n(\ell) \right)^2 \right] &\leq \sum_{\ell=\alpha+1}^{\alpha+\beta} \sum_{m=\alpha+1}^{\alpha+\beta} |\mathbb{E}[\xi_\ell \xi_m]| |\mathbb{E}[N_n(\ell) N_n(m)]| \\
&\leq \sum_{\ell=\alpha+1}^{\alpha+\beta} \sum_{m=\alpha+1}^{\alpha+\beta} |\mathbb{E}[\xi_\ell \xi_m]| \|N_n(\ell)\|_2 \|N_n(m)\|_2 \\
&\leq (C_1)^2 n \beta \sum_{m \in \mathbb{Z}} |\mathbb{E}[\xi_0 \xi_m]|.
\end{aligned}$$

From which, we get :

$$\begin{aligned}
\mathbb{E} [|b_{k,n,L} - 1|] &\leq \frac{|t|}{n^{\frac{3}{4}}} \sqrt{(C_1)^2 n \frac{5K\sqrt{n}}{L} \sum_{m \in \mathbb{Z}} |\mathbb{E}[\xi_0 \xi_m]|} \\
&\leq \frac{|t|}{\sqrt{L}} C_1 \sqrt{5K \sum_{m \in \mathbb{Z}} |\mathbb{E}[\xi_0 \xi_m]|}.
\end{aligned}$$

Moreover we have :

$$\begin{aligned}
\mathbb{E} [|a_{k,n,L} - 1|] &\leq \frac{t^2}{2n^{\frac{3}{2}}} \mathbb{E} \left[\sum_{\ell=\dots}^{\dots} \sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_\ell \xi_m] N_n(\ell)^2 \right] \\
&\leq \frac{t^2}{2n^{\frac{3}{2}}} \frac{5K\sqrt{n}}{L} \sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_m] (C_1)^2 n \\
&\leq \frac{5t^2}{2} \frac{K(C_1)^2}{L} \sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_m].
\end{aligned}$$

Let $L_1 \geq 8$ be such that for all $L \geq L_1$, we have :

$$((r+1)^3 + 1) \frac{|t|}{\sqrt{L}} C_1 \sqrt{5K \sum_{m \in \mathbb{Z}} |\mathbb{E}[\xi_0 \xi_m]|} < \frac{\varepsilon}{20}$$

and

$$((r+1)^3 + 1) \frac{5t^2}{2} \frac{K(C_1)^2}{L} \sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_m] < \frac{\varepsilon}{20}.$$

Then, if we have $L \geq L_1$ and $n \geq 1$ such that $\frac{2K\sqrt{n}}{L} \geq L$, we have :

$$\mathbb{E} [|b_{L,n,L} - a_{L,n,L}|] + \sum_{k=0}^{(r+1)^3-1} \mathbb{E} [|b_{k,n,L} - a_{k,n,L}|] < \frac{\varepsilon}{10}. \quad (5)$$

It remains to estimate :

$$\sum_{k=(r+1)^3}^{L-1} \left| \mathbb{E} \left[\left(\prod_{m=0}^{k-1} b_{m,n,L} \right) (b_{k,n,L} - a_{k,n,L}) \prod_{m'=k+1}^L a_{m',n,L} \right] \right|. \quad (6)$$

• We estimate :

$$\begin{aligned} B_{n,L} &:= \sum_{k=(r+1)^3}^{L-1} \left| \mathbb{E} \left[\left(\prod_{m=0}^{k-(r+1)^3} b_{m,n,L} \right) \left(\left(\prod_{m=k-(r+1)^3+1}^{k-(r+1)^2} b_{m,n,L} \right) - 1 \right) \times \right. \right. \\ &\times \left. \left. \left(\left(\prod_{m=k-(r+1)^2+1}^{k-r-1} b_{m,n,L} \right) - 1 \right) \left(\prod_{m'=k-r}^{k-1} b_{m',n,L} \right) (b_{k,n,L} - a_{k,n,L}) \prod_{m'=k+1}^L a_{m',n,L} \right] \right|. \end{aligned}$$

We have :

$$B_{n,L} \leq \sum_{k=(r+1)^3}^{L-1} \left\| \left(\prod_{m=k-(r+1)^3+1}^{k-(r+1)^2} b_{m,n,L} \right) - 1 \right\|_3 \left\| \left(\prod_{m=k-(r+1)^2+1}^{k-r-1} b_{m,n,L} \right) - 1 \right\|_3 \|b_{k,n,L} - a_{k,n,L}\|_3.$$

– We have :

$$\|b_{k,n,L} - 1\|_3 \leq \frac{|t|}{n^{\frac{3}{4}}} \left\| \sum_{\ell=\dots}^{\dots} \xi_{\ell} N_n(\ell) \right\|_3.$$

For any integers α and β with $\beta \geq 1$, we have :

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{\ell=\alpha+1}^{\alpha+\beta} \xi_{\ell} N_n(\ell) \right)^4 \right] &\leq \sum_{\ell_1, \ell_2, \ell_3, \ell_4=\alpha+1}^{\alpha+\beta} |\mathbb{E} [\xi_{\ell_1} \xi_{\ell_2} \xi_{\ell_3} \xi_{\ell_4}]| (C_1)^4 n^2 \\ &\leq (C_1)^4 n^2 C_2' \beta^2. \end{aligned} \quad (7)$$

with $C_2' := \sup_{N \geq 1} N^{-2} \sum_{k_1, k_2, k_3, k_4=0, \dots, N-1} |\mathbb{E} [\xi_{k_1} \xi_{k_2} \xi_{k_3} \xi_{k_4}]|$. Hence, we have :

$$\begin{aligned} \|b_{k,n,L} - 1\|_3 &\leq \frac{|t|}{n^{\frac{3}{4}}} \left((C_1)^4 n^2 C_2' \left(\frac{5K\sqrt{n}}{L} \right)^2 \right)^{\frac{1}{4}} \\ &\leq |t| C_1 (C_2')^{\frac{1}{4}} \sqrt{\frac{5K}{L}}. \end{aligned}$$

– We have :

$$\begin{aligned} \|a_{k,n,L} - 1\|_3 &\leq \frac{t^2}{2n^{\frac{3}{2}}} \sum_{k \in \mathbb{Z}} \mathbb{E} [\xi_0 \xi_k] \left\| \sum_{\ell=\dots}^{\dots} N_n(\ell)^2 \right\|_3 \\ &\leq \frac{t^2}{2n^{\frac{3}{2}}} \sum_{k \in \mathbb{Z}} \mathbb{E} [\xi_0 \xi_k] \sum_{\ell=\dots}^{\dots} \|N_n(\ell)\|_6^2 \\ &\leq \frac{t^2}{2n^{\frac{3}{2}}} \sum_{k \in \mathbb{Z}} \mathbb{E} [\xi_0 \xi_k] \frac{5K\sqrt{n}}{L} (C_1)^2 n \\ &\leq \frac{5t^2}{2} \sum_{k \in \mathbb{Z}} \mathbb{E} [\xi_0 \xi_k] \frac{K}{L} (C_1)^2. \end{aligned}$$

– Using formula (7), we get :

$$\begin{aligned} \left\| \left(\prod_{m=k-(r+1)^3+1}^{k-(r+1)^2} b_{m,n,L} \right) - 1 \right\|_3 &\leq \frac{|t|}{n^{\frac{3}{4}}} \left\| \sum_{\ell=-\lceil K\sqrt{n} \rceil + (k-(r+1)^3+1)}^{\lceil \frac{2\lceil K\sqrt{n} \rceil + 1}{L} \rceil - 1} \xi_\ell N_n(\ell) \right\|_3 \\ &\leq \frac{|t|}{n^{\frac{3}{4}}} \left((C_1)^4 n^2 C_2' (r(r+1)^2)^2 \left(\frac{5K\sqrt{n}}{L} \right)^2 \right)^{\frac{1}{4}} \\ &\leq |t| C_1 (C_2')^{\frac{1}{4}} \sqrt{r(r+1)} \sqrt{\frac{5K}{L}}. \end{aligned}$$

– Analogously, we get :

$$\left\| \left(\prod_{m=k-(r+1)^2+1}^{k-r-1} b_{m,n,L} \right) - 1 \right\|_3 \leq |t| C_1 (C_2')^{\frac{1}{4}} \sqrt{r(r+1)} \sqrt{\frac{5K}{L}}.$$

Hence, we have :

$$\begin{aligned} B_{n,L} &\leq L \left(|t| C_1 (C_2')^{\frac{1}{4}} \sqrt{r(r+1)} \sqrt{\frac{5K}{L}} \right)^2 \left(|t| C_1 (C_2')^{\frac{1}{4}} \sqrt{\frac{5K}{L}} + \frac{5t^2}{2} \sum_{k \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_k] \frac{K}{L} (C_1)^2 \right) \\ &\leq |t|^2 (C_1)^2 (C_2')^{\frac{1}{2}} r(r+1)^2 5K \left(|t| C_1 (C_2')^{\frac{1}{4}} \sqrt{\frac{5K}{L}} + \frac{5t^2}{2} \sum_{k \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_k] \frac{K}{L} (C_1)^2 \right). \end{aligned}$$

Let $L'_1 \geq L_1$ be such that, for all $L \geq L_1$, the right term of this last inequality is less than $\frac{\varepsilon}{10}$.

Then, for any $L \geq L'_1$ and any $n \geq 1$ such that $\frac{2K\sqrt{n}}{L} \geq L$, we have : $B_{n,L} \leq \frac{\varepsilon}{10}$.

- In the following, we suppose $L \geq L'_1$ and $\frac{2K\sqrt{n}}{L} \geq L$. It remains to estimate :

$$\sum_{k=(r+1)^3+1}^{L-1} C_{n,k,L,1,3} + C_{n,k,L,1,2} + C_{n,k,L,2,3}$$

where C_{n,k,L,j_0,j_1} is the following quantity :

$$\left| \mathbb{E} \left[\left(\prod_{m=0}^{k-(r+1)^{j_1}} b_{m,n,L} \right) \left(\prod_{m=k-(r+1)^{j_0}+1}^{k-1} b_{m,n,L} \right) (b_{k,n,L} - a_{k,n,L}) \prod_{m'=k+1}^L a_{m',n,L} \right] \right|$$

- Let j_0, j_1 be fixed. We estimate C_{n,k,L,j_0,j_1} . We have :

$$C_{n,k,L,j_0,j_1} \leq D_{n,k,L,j_0,j_1} + E_{n,k,L,j_0,j_1},$$

where :

$$D_{n,k,L,j_0,j_1} := \left| \mathbb{E} \left[\text{Cov}_{(S_p)_p} (\Delta_{n,k,L,j_1}, \Gamma_{n,k,L,j_0}) \prod_{m'=k+1}^L a_{m',n,L} \right] \right|$$

and

$$E_{n,k,L,j_0,j_1} := \left| \mathbb{E} \left[\mathbb{E} [\Delta_{n,k,L,j_1} | (S_p)_p] \mathbb{E} [\Gamma_{n,k,L,j_0} | (S_p)_p] \prod_{m'=k+1}^L a_{m',n,L} \right] \right|.$$

with $\Delta_{n,k,L,j_1} := \prod_{m=0}^{k-(r+1)^{j_1}} b_{m,n,L}$ and $\Gamma_{n,k,L,j_0} := \left(\prod_{m=k-(r+1)^{j_0}+1}^{k-1} b_{m,n,L} \right) (b_{k,n,L} - a_{k,n,L})$.

- Control of the terms with the product of the expectations.

Let j_0, j_1 be fixed. Let $k = (r+1)^3, \dots, L-1$. We can notice that E_{n,k,L,j_0,j_1} is bounded from away by the following quantity :

$$F_{n,k,L,j_0,j_1} := \mathbb{E} \left[\left| \mathbb{E} \left[\prod_{m=k-(r+1)^{j_0}+1}^k b_{m,n,L} - \left(\prod_{m=k-(r+1)^{j_0}+1}^{k-1} b_{m,n,L} \right) a_{k,n,L} \middle| (S_p)_p \right] \right| \right].$$

We use the Taylor expansions of the exponential function. To simplify expressions, we will use the following notation :

$$\forall m \geq 0, \quad \alpha_{(m)} := -\lceil K\sqrt{n} \rceil + m \left\lfloor \frac{2\lceil K\sqrt{n} \rceil + 1}{L} \right\rfloor.$$

- Let us show that, in F_{n,k,L,j_0,j_1} , we can replace

$$\prod_{m=k-(r+1)^{j_0}+1}^k b_{m,n,L} = \exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}^{\alpha_{(k+1)}-1}} \xi_\ell N_n(\ell) \right)$$

by the formula given by the Taylor expansion of the exponential function at the second order :

$$1 + \frac{it}{n^{\frac{3}{4}}} \sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}^{\alpha_{(k+1)}-1}} \xi_\ell N_n(\ell) - \frac{t^2}{2n^{\frac{3}{2}}} \left(\sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}^{\alpha_{(k+1)}-1}} \xi_\ell N_n(\ell) \right)^2. \quad (8)$$

Indeed the L^1 -norm of the error between these two quantities is less than :

$$\frac{|t|^3}{6n^{\frac{9}{4}}} \mathbb{E} \left[\left| \sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}^{\alpha_{(k+1)}-1} \xi_\ell N_n(\ell) \right|^3 \right]$$

which, according to formula (7), is less than :

$$\frac{|t|^3}{6n^{\frac{9}{4}}} \left((C_1)^4 n^2 C_2' \left(((r+1)^{j_0}) \frac{5K\sqrt{n}}{L} \right)^2 \right)^{\frac{3}{4}} = \frac{|t|^3}{6} (C_1)^3 (C_2')^{\frac{3}{4}} \left((r+1)^{j_0} \frac{5K}{L} \right)^{\frac{3}{2}}.$$

Hence, the sum over $k = (r+1)^3, \dots, L-1$ of the L^1 -norm of these errors is less than :

$$\frac{1}{\sqrt{L}} \frac{|t|^3}{6} (C_2')^{\frac{3}{4}} (C_1)^3 ((r+1)^{j_0} 5K)^{\frac{3}{2}}.$$

Let us consider $L_2 \geq L_1'$ such that, for all $L \geq L_2$, this last quantity is less than $\frac{\varepsilon}{10}$.

- Let us introduce $Y_k := \sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}^{\alpha_{(k)}-1}} \xi_\ell N_n(\ell)$ and $Z_k := \sum_{\ell=\alpha_{(k)}^{\alpha_{(k+1)}-1}} \sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_\ell \xi_m] N_n(\ell)^2$. We show that, in F_{n,k,L,j_0,j_1} , we can replace

$$\left(\prod_{m=k-(r+1)^{j_0}+1}^{k-1} b_{m,n,L} \right) a_{k,n,L} = \exp \left(\frac{it}{n^{\frac{3}{4}}} Y_k - \frac{t^2}{2n^{\frac{3}{2}}} Z_k \right)$$

by the formula given by the Taylor expansion of the exponential function at the second order :

$$1 + \frac{it}{n^{\frac{3}{4}}} Y_k - \frac{t^2}{2n^{\frac{3}{2}}} Z_k + \frac{1}{2} \left(\frac{it}{n^{\frac{3}{4}}} Y_k - \frac{t^2}{2n^{\frac{3}{2}}} Z_k \right)^2, \quad (9)$$

Indeed, the L^1 -norm of the error between these two quantities is less than :

$$\frac{1}{6} \mathbb{E} \left[\left| \frac{it}{n^{\frac{3}{4}}} Y_k - \frac{t^2}{2n^{\frac{3}{2}}} Z_k \right|^3 \right] \leq \frac{4}{3} \mathbb{E} \left[\left| \frac{it}{n^{\frac{3}{4}}} Y_k \right|^3 + \left| \frac{t^2}{2n^{\frac{3}{2}}} Z_k \right|^3 \right].$$

According to formula (7), we have :

$$\frac{4}{3} \mathbb{E} \left[\left| \frac{it}{n^{\frac{3}{4}}} Y_k \right|^3 \right] \leq \frac{4|t|^3}{3} (C_1)^3 (C_2')^{\frac{3}{4}} \left((r+1)^{j_0} \frac{5K}{L} \right)^{\frac{3}{2}}.$$

Moreover, we have :

$$\begin{aligned} \frac{4}{3} \mathbb{E} \left[\left| \frac{t^2}{2n^{\frac{3}{2}}} Z_k \right|^3 \right] &= \frac{t^6}{6n^{\frac{3}{2}}} \left(\sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_m] \right)^3 \mathbb{E} \left[\sum_{\ell_1 \ell_2, \ell_3 = \alpha(k)}^{\alpha(k+1)-1} N_n(\ell_1)^2 N_n(\ell_2)^2 N_n(\ell_3)^2 \right] \\ &\leq \frac{t^6}{6n^{\frac{3}{2}}} \left(\sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_m] \right)^3 \left(\frac{5K\sqrt{n}}{L} \right)^3 (C_1)^6 n^3. \end{aligned}$$

The sum over $k = (r+1)^3, \dots, L-1$ of the L^1 -norm of these errors is less than :

$$\frac{1}{\sqrt{L}} \frac{4|t|^3}{3} (C_2')^{\frac{3}{4}} (C_1)^3 \left((r+1)^{j_0} 5K \right)^{\frac{3}{2}} + \frac{1}{L^2} \frac{t^6}{6} \left(\sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_m] \right)^3 (5K)^3 (C_1)^6.$$

Let us consider $L'_2 \geq L_2$ such that, for all $L \geq L'_2$, this last quantity is less than $\frac{\varepsilon}{10}$.

– Now we show that, in formula (9), we can omit the term with $(Z_k)^2$. Indeed, we have :

$$\frac{1}{2} \mathbb{E} \left[\left(\frac{t^2}{2n^{\frac{3}{2}}} Z_k \right)^2 \right] \leq \frac{t^4}{8n^3} \left(\sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_m] \right)^2 \left(\frac{5K\sqrt{n}}{L} \right)^2 (C_1)^4 n^2.$$

The sum over $k = (r+1)^3, \dots, L-1$ of the L^1 -norm of these errors is less than :

$$\frac{1}{L} \frac{t^4}{8} \left(\sum_{m \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_m] \right)^2 (5K)^2 (C_1)^4.$$

Let us consider $L''_2 \geq L'_2$ such that, for all $L \geq L''_2$, this last quantity is less than $\frac{\varepsilon}{10}$.

– From now, we fix $L := L''_2$ and we consider an integer $n \geq \frac{L^4}{4K^2}$.

– Hence, it remains to estimate the following quantity called G_{n,k,L,j_0,j_1} :

$$\begin{aligned} \mathbb{E} \left[\left| \mathbb{E} \left[\frac{it}{n^{\frac{3}{4}}} (Y_k + W_k) - \frac{t^2}{2n^{\frac{3}{2}}} (Y_k + W_k)^2 - \frac{it}{n^{\frac{3}{4}}} Y_k + \frac{t^2}{2n^{\frac{3}{2}}} Z_k + \right. \right. \right. \\ \left. \left. \left. + \frac{t^2}{2n^{\frac{3}{2}}} (Y_k)^2 + \frac{it}{n^{\frac{3}{4}}} Y_k \frac{t^2}{2n^{\frac{3}{2}}} Z_k \middle| (S_p)_p \right] \right| \right] \end{aligned}$$

with $W_k := \sum_{\ell=\alpha(k)}^{\alpha(k+1)-1} \xi_\ell N_n(\ell)$. Using the fact that the ξ_k are centered and independent of $(S_p)_p$, we get :

$$\begin{aligned} G_{n,k,L,j_0,j_1} &= \mathbb{E} \left[\left| \mathbb{E} \left[-\frac{t^2}{2n^{\frac{3}{2}}} (Y_k + W_k)^2 + \frac{t^2}{2n^{\frac{3}{2}}} Z_k + \frac{t^2}{2n^{\frac{3}{2}}} (Y_k)^2 \middle| (S_p)_p \right] \right| \right] \\ &= \frac{t^2}{2n^{\frac{3}{2}}} \mathbb{E} \left[\left| \mathbb{E} \left[(W_k)^2 + 2W_k Y_k - Z_k \middle| (S_p)_p \right] \right| \right]. \end{aligned}$$

Let us notice that we have :

$$Z_k := \sum_{\ell=\alpha(k)}^{\alpha(k+1)-1} \left(\mathbb{E}[(\xi_\ell)^2] N_n(\ell)^2 + 2 \sum_{m \leq \ell-1} \mathbb{E}[\xi_\ell \xi_m] N_n(\ell)^2 \right).$$

– Let us show that, in the last expression of G_{n,k,L,j_0,j_1} , we can replace Z_k by :

$$\tilde{Z}_k := \sum_{\ell=\alpha(k)}^{\alpha(k+1)-1} \left(\mathbb{E}[(\xi_\ell)^2] N_n(\ell)^2 + 2 \sum_{m \leq \ell-1} \mathbb{E}[\xi_\ell \xi_m] N_n(\ell) N_n(m) \right).$$

Indeed, we have :

$$\begin{aligned} \frac{t^2}{2n^{\frac{3}{2}}} \mathbb{E} \left[\left| Z_k - \tilde{Z}_k \right| \right] &\leq \frac{t^2}{n^{\frac{3}{2}}} \sum_{\ell=\alpha(k)}^{\alpha(k+1)-1} \sum_{m \leq \ell-1} |\mathbb{E}[\xi_\ell \xi_m]| \|N_n(\ell)\|_2 \|N_n(m) - N_n(\ell)\|_2 \\ &\leq \frac{t^2}{n^{\frac{3}{2}}} \frac{5K\sqrt{n}}{L} \sum_{p \geq 1} |\mathbb{E}[\xi_0 \xi_p]| C_1 \sqrt{n} C_2 n^{\frac{1}{4}} \sqrt{1+p}. \end{aligned}$$

The sum over $k = (r+1)^3, \dots, L-1$ of these quantities is less than :

$$\frac{t^2}{n^{\frac{1}{4}}} 5K C_1 C_2 \sum_{p \geq 1} \sqrt{1+p} |\mathbb{E}[\xi_0 \xi_p]|,$$

which goes to zero when n goes to infinity. Hence, there exists some $n_0 \geq \frac{L^4}{4K^2}$ such that, for any integer $n \geq n_0$, this sum is less than $\frac{\varepsilon}{10}$.

– Hence we have to estimate :

$$\tilde{G}_{n,k,L,j_0,j_1} = \frac{t^2}{2n^{\frac{3}{2}}} \mathbb{E} \left[\left| \mathbb{E} \left[(W_k)^2 + 2W_k Y_k \mid (S_p)_p \right] - \tilde{Z}_k \right| \right].$$

We have :

$$\mathbb{E} \left[(W_k)^2 \mid (S_p)_p \right] = \sum_{\ell=\alpha(k)}^{\alpha(k+1)-1} \left(\mathbb{E}[(\xi_\ell)^2] (N_n(\ell))^2 + 2 \sum_{m=\alpha(k)}^{\ell-1} \mathbb{E}[\xi_\ell \xi_m] N_n(\ell) N_n(m) \right).$$

Hence we have :

$$\mathbb{E} \left[(W_k)^2 + 2W_k Y_k \mid (S_p)_p \right] = \sum_{\ell=\alpha(k)}^{\alpha(k+1)-1} \left(\mathbb{E}[(\xi_\ell)^2] (N_n(\ell))^2 + 2 \sum_{m=\alpha(k-(r+1)j_0+1)}^{\ell-1} \mathbb{E}[\xi_\ell \xi_m] N_n(\ell) N_n(m) \right).$$

We get :

$$\begin{aligned} \tilde{G}_{n,k,L,j_0,j_1} &= \frac{t^2}{n^{\frac{3}{2}}} \mathbb{E} \left[\left| \sum_{\ell=\alpha(k)}^{\alpha(k+1)-1} \sum_{m \leq \alpha(k-(r+1)j_0+1)-1} \mathbb{E}[\xi_\ell \xi_m] N_n(\ell) N_n(m) \right| \right] \\ &\leq \frac{t^2}{n^{\frac{3}{2}}} \frac{5K\sqrt{n}}{L} \sum_{m \geq (r+1)j_0}^{\frac{K\sqrt{n}}{L}} |\mathbb{E}[\xi_0 \xi_m]| (C_1)^2 n. \end{aligned}$$

The sum over $k = (r+1)^3, \dots, L-1$ of these quantities is less than :

$$t^2 5K \sum_{m \geq (r+1)j_0}^{\frac{K\sqrt{n}}{L}} |\mathbb{E}[\xi_0 \xi_m]| (C_1)^2,$$

which goes to zero when n goes to infinity. Hence, there exists some $n'_0 \geq n_0$ such that, for any integer $n \geq n_0$, this sum is less than $\frac{\varepsilon}{10}$.

- Control of the covariance terms.

Let j_0, j_1 be fixed. Let $k = (r+1)^3, \dots, L-1$. We have :

$$D_{n,k,L,j_0,j_1} \leq \left| \mathbb{E} \left[\text{Cov}_{|(S_p)_p} \left(\prod_{m=0}^{k-(r+1)^{j_1}} b_{m,n,L}, \prod_{m=k-(r+1)^{j_0}+1}^k b_{m,n,L} \right) \prod_{m'=k+1}^L a_{m',n,L} \right] \right| + \left| \mathbb{E} \left[\text{Cov}_{|(S_p)_p} \left(\prod_{m=0}^{k-(r+1)^{j_1}} b_{m,n,L}, \prod_{m=k-(r+1)^{j_0}+1}^{k-1} b_{m,n,L} \right) \prod_{m'=k}^L a_{m',n,L} \right] \right|.$$

But we have :

$$\prod_{m=\alpha}^{\alpha+\beta} b_{m,n,L} = \exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell=-\lceil K\sqrt{n} \rceil + \alpha}^{-\lceil K\sqrt{n} \rceil + (\alpha+\beta+1) \lfloor \frac{2\lceil K\sqrt{n} \rceil + 1}{L} \rfloor - 1} \xi_\ell N_n(\ell) \right).$$

Therefore, according to point 4 of the hypothesis of our theorem, we have :

$$D_{n,k,L,j_0,j_1} \leq 2\mathbb{E} \left[C \left(1 + \frac{|t|}{n^{\frac{3}{4}}} \sum_{\ell \in \mathbb{Z}} N_n(\ell) \right) \left(\frac{rK\sqrt{n}}{2L} \right)^{-\frac{1}{2}} \sup_{s \geq r \frac{K\sqrt{n}}{2L}} \sqrt{s} \varphi_{r,s} \right].$$

Hence, we have :

$$\sum_{k=(r+1)^3}^{L-1} D_{n,k,L,j_0,j_1} \leq 2CL\sqrt{L}C(1 + |t|n^{\frac{1}{4}}) \frac{n^{-\frac{1}{4}}\sqrt{2}}{\sqrt{rK}} \sup_{s \geq r \frac{K\sqrt{n}}{2L}} \sqrt{s} \varphi_{r,s}.$$

which goes to zero as n goes to infinity. Hence, there exists some $N_0 \geq n'_0$ such that, for any integer $n \geq n_0$, this sum is less than $\frac{\varepsilon}{10}$.

Therefore, there exists N_0 (depending on t and on ε) such that, for any integer $n \geq N_0$, we have :

$$\left| \mathbb{E} \left[\exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell \in \mathbb{Z}} \xi_\ell N_n(\ell) \right) \right] - \mathbb{E} \left[\exp \left(-\frac{t^2}{2n^{\frac{3}{2}}} \sum_{\ell, k \in \mathbb{Z}} \mathbb{E}[\xi_\ell \xi_k] N_n(\ell)^2 \right) \right] \right| < \varepsilon.$$

This ends the step 1 of our proof.

5.2 Step 2 : Conclusion

In the previous section we proved that :

$$\lim_{n \rightarrow +\infty} \left| \mathbb{E} \left[\exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell \in \mathbb{Z}} \xi_\ell N_n(\ell) \right) \right] - \mathbb{E} \left[\exp \left(-\frac{t^2}{2n^{\frac{3}{2}}} \sum_{\ell, k \in \mathbb{Z}} \mathbb{E}[\xi_\ell \xi_k] N_n(\ell)^2 \right) \right] \right| = 0.$$

According to [13] lemma 6, we know that : $\left(\frac{1}{n^{\frac{3}{2}}} \sum_{\ell \in \mathbb{Z}} N_n(\ell)^2 \right)_{n \geq 1}$ converges in distribution to $Z_1 := \int_{\mathbb{R}} (L_1(x))^2 dx$. Hence, we get :

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\exp \left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell \in \mathbb{Z}} \xi_\ell N_n(\ell) \right) \right] = \mathbb{E} \left[\exp \left(-\frac{t^2}{2} \sum_{k \in \mathbb{Z}} \mathbb{E}[\xi_0 \xi_k] \int_{\mathbb{R}} (L_1(x))^2 dx \right) \right].$$

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