



# Stochastic approach for the subordination in Bochner sense

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# Stochastics techniques for the subordination in Bochner sense

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It is possible to construct (cf [1] and [2]) a positive process  $(Y_{\psi,t})$  indexed by Bernstein functions and time which, for fixed  $t$ , is Markovian in  $\psi$  with respect to composition of Bernstein functions and which, for fixed  $\psi$  is the subordinator associated with  $\psi$ . In the same manner a realisation  $(X_{Y_{\psi,t}})$  can be obtained of all the subordinated processes of a Markov process  $(X_t)$ .

This probabilistic interpretation of the initial idea of Bochner [3] allows to construct martingales and to apply stochastic calculus to questions related to symbolic calculus for operators semi-groups.

We study here a particular branch of the subordination process: the homographic branch which has the advantage of being connected with the works on positive infinitely divisible diffusions and brings on them a different point of view. Nevertheless, what follows would apply, except technical difficulties, to any branch of the subordination process.

It is easy to see that the functions

$$f_a(x) = \frac{xe^a}{1 + x(e^a - 1)}, \quad a \geq 0, x \geq 0$$

satisfy  $f_a \circ f_b = f_{a+b}$  and are the Bernstein functions (cf [4]) associated with the subordinators

$$Y_t^a = (e^a - 1) \sum_{k=1}^{N_{t/(1-e^{-a})}} E_k \quad (1)$$

where  $(N_t)$  is a standard Poisson process and the  $E_k$ 's are independent exponential variables independent of  $(N_t)$ . The Lévy measure of  $(Y_t^a)$  is therefore

$$\nu_a(dy) = \frac{1}{(e^a - 1)^2} e^{-y/(e^a - 1)} dy \text{ on } \mathbb{R}_+.$$

The relation  $P_a u(x) = \mathbb{E}[u(Y_x^a)]$  for  $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  defines a Markovian semi-group  $(P_a)_{a \geq 0}$  with generator given by  $Au(x) = xu'(x) + xu''(x)$  which is the transition semi-group of the diffusion  $(Z_a)_{a \geq 0}$  solution of the SDE

$$dZ_a = \sqrt{2Z_a} dB_a + Z_a da, \quad Z_0 = z \geq 0. \quad (2)$$

Now we define a two parameter process  $(Y_{a,t})_{a \geq 0, t \geq 0}$  by choosing, for  $a_1 < \dots < a_n$ , the joint law of the processes

$$(Y_{a_1,t})_{t \geq 0}, \dots, (Y_{a_n,t})_{t \geq 0}$$

to be that of the processes

$$(Y_t^{a_1})_{t \geq 0}, \left( Y_{Y_t^{a_1}}^{a_2 - a_1} \right)_{t \geq 0}, \dots, \left( Y_{Y_{\dots Y_t^{a_1}}^{a_{n-1} - a_{n-2}}}^{a_n - a_{n-1}} \right)_{t \geq 0}$$

where  $(Y_t^{a_1})_{t \geq 0}, \dots, (Y_t^{a_n})_{t \geq 0}$  are independent subordinators of the type (1).

A version of the process  $(Y_{a,t})$  can be chosen such that  $Y_t = Y_{.,t}$  be right continuous and increasing with values in  $\mathcal{W} = \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$  and with independent stationary increments, on the other hand for fixed  $t$ ,  $(Y_{a,t})_{a \geq 0}$  has the same law as  $(Z_a)$  with  $Z_0 = t$ , this law is denoted by  $\mathbb{E}_t$ .

The Lévy measure of  $(Y_t)$  is the law of the process  $(Z_a)_{a \geq 0}$  when built with the entrance law  $m_a = (+\infty)\delta_0 + \nu_a$  and this can be written symbolically

$$\mathbb{E}_t[e^{-F}] = \exp\{-t\mathbb{E}_{\delta'_0}(1 - e^{-F})\} \quad (3)$$

for  $F : \mathcal{W} \rightarrow \mathbb{R}_+$  such that  $F(\underline{0}) = 0$ .

The law of the subordinators  $\int Y_{a,y} \mu(da)$  where  $\mu$  is a positive measure with compact support on  $\mathbb{R}_+$  can be studied by using the fact that if we decompose  $\mu$  in

$$\mu = 1_{[0,x]} \cdot \mu + 1_{(x,\infty)} \cdot \mu = \mu_1 + \mu_2$$

the following representation holds

$$\int Y_{a,y} \mu(da) = \int Y_{a,y} \mu_1(da) + \int Y_{Y_{x,t}}^{b-x} \mu_2(db)$$

where  $(Y_t^\beta)_{\beta \geq 0, t \geq 0}$  is independent of  $(Y_{a,t}, a \leq x, t \geq 0)$ .

One gets for  $\lambda \geq 0$ ,

$$\mathbb{E} \exp\{-\lambda \int Y_{a,t} \mu(da)\} = \exp\{-t(\frac{1}{2} - g'_x(0, \lambda))\}$$

where  $g$  is the positive solution decreasing in  $x$  of

$$\begin{cases} g''_{x^2} &= g \cdot (\frac{1}{4} + \lambda \cdot \mu) \\ g(0, \lambda) &= 1 \end{cases}$$

and if  $\xi(x, \lambda) = \frac{1}{2} - \frac{g'_x(x, \lambda)}{g(x, \lambda)}$  the process

$$M_{a,t} = \exp\{-\xi(a, \lambda)Y_{a,t} + \xi(a, \lambda)t - \lambda \int_0^a Y_{\alpha,t} \mu(d\alpha)\} \quad (4)$$

is a two parameters martingale for the filtration of  $(Y_{a,t})$ .

A similar study can be made with the family  $g_a(x) = \frac{x}{1+ax}$  which corresponds to the diffusion

$$Z_a = z + \int_0^a \sqrt{2Z_b} dB_b \quad (5)$$

or with  $h_a(x) = f_a(x+1) - 1$  which corresponds to

$$Z_a = z + \int_0^a \sqrt{2Z_b} dB_b - \int_0^a Z_b db \quad (6)$$

The three associated processes  $(Y_{a,t})$  being gotten from each other by absolutely continuous change of probability measure.

These results were obtained by Pitman and Yor [5] by a direct study of the diffusions (2)(5)(6) and by interpreting the process  $(Y_{a,t})$  of case (5) as  $l_{\tau_t}^a$  where  $l_t^a$  is the family of brownian local times and  $\tau_t = \inf\{s : l_s^0 = t\}$ , and using the Ray-Knight theorems. The measure  $\mathbb{E}_{\delta_0'}$  of (3) is, in this frame work, an image of the Ito measure of brownian excursions. Among the works in this direction let us quote [6], [7], [8]. Narrow connexions exist also with the stream of research on measured valued branching processes (cf [9] and references in it)

Continuous positive martingales such as (4) can be used for subordinating semi-groups of operators, among the results in this direction is a variant of Girsanov theorem involving symmetric semi-groups.

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