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Algèbre des structures de Dirichlet

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THE ALGEBRA OF DIRICHLET STRUCTURES

Nicolas BOULEAU and Francis HIRSCH

I. Some notations and definitions

I.1. Dirichlet Structures

$$(\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$$

(Ω, \mathcal{F}, m) : measured space with m σ -finite and positive

\mathcal{E} : Dirichlet form with domain \mathbb{D}

i.e. Quadratic positive form with dense domain \mathbb{D} in $L^2(m)$ which is

. closed : \mathbb{D} is an Hilbert space under the norm

$$\|f\|_{\mathbb{D}} = [\|f\|_{L^2(m)}^2 + \mathcal{E}(f, f)]^{1/2}$$

. and s.t. $f \in \mathbb{D} \implies (f \wedge 1) \in \mathbb{D}$ and $\mathcal{E}(f \wedge 1, f \wedge 1) \leq \mathcal{E}(f, f)$.

Notations for different hypotheses:

(**P**) (Probability) $m(\Omega) = 1$

(**M**) (Markovianity) $1 \in \mathbb{D}$ and $\mathcal{E}(1, 1) = 0$

(**\Gamma**) (Existence of a Carré du Champ Operator):

$$\forall f \in \mathbb{D} \cap L^\infty, \exists \tilde{f} \in L^1, \forall h \in \mathbb{D} \cap L^\infty,$$

$$2\mathcal{E}(fh, f) - \mathcal{E}(h, f^2) = \int \tilde{f}h \, dm$$

(**L**) (Locality)

$$\forall f \in \mathbb{D}, \forall F, G \in \mathcal{D}(\mathbb{R})$$

$$\text{supp}F \cap \text{supp}G = \emptyset \implies \mathcal{E}(F(f) - F(0), G(f) - G(0)) = 0$$

(**W**) (Wiener space)

$$\Omega = \{\omega \in C(\mathbb{R}_+, \mathbb{R}^d), \omega(0) = 0\}$$

\mathcal{F} = borelian σ -field of Ω with compact convergence

m = Wiener measure

$(\mathcal{E}, \mathbb{D})$ = form associated with the Ornstein-Uhlenbeck semi-group.

I.2. Basic properties

There is a sub-Markov semigroup associated with a Dirichlet structure.

Theorem 1 . *Let a Dirichlet structure $(\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ be given. There exists a strongly continuous contraction semi-group $(P_t)_{t \geq 0}$ symmetric on $L^2(m)$ and unique such that*

$$(\star) \left\{ \begin{array}{l} \mathbb{D} = \{f \in L^2(m) : \lim_{t \rightarrow 0} (\frac{f - P_t f}{t}, f)_{L^2(m)} \text{ exists} \} \\ \forall f \in \mathbb{D} \quad \mathcal{E}(f, f) = \lim_{t \rightarrow 0} (\frac{f - P_t f}{t}, f)_{L^2(m)} \end{array} \right.$$

this semi-group is sub-Markov.

Conversely, if (P_t) is a symmetric strongly continuous contraction semi-group on $L^2(m)$, and sub-Markov, the positive quadratic form associated with (P_t) by (\star) is a Dirichlet form.

Definition 2 . *A function F from \mathbb{R}^n into \mathbb{R} is a contraction [resp. a normal contraction] if*

$$\forall x, y \quad |F(x) - F(y)| \leq \sum_{i=1}^n |x_i - y_i|$$

[resp. and $F(0) = 0$]

Theorem 3 . *$\forall f \in \mathbb{D}$, if F is a normal contraction from \mathbb{R}^n to \mathbb{R} then*

$$F \circ f \in \mathbb{D} \quad \text{and} \quad (\mathcal{E}(F \circ f, F \circ f))^{1/2} \leq \sum_{i=1}^n (\mathcal{E}(f_i, f_i))^{1/2}.$$

Under $(\mathbf{P})(\mathbf{M})$ the word normal can be cancelled.

The hypothesis (Γ) gives rise to a carré du champ operator:

Theorem 4 . *Under (Γ) there exists a unique continuous symmetric positive bilinear map from $\mathbb{D} \times \mathbb{D}$ into $L^1(m)$, denoted by Γ such that*

$$\forall f, g, h \in \mathbb{D} \cap L^\infty$$

$$\mathcal{E}(fh, g) + \mathcal{E}(gh, f) - \mathcal{E}(h, fg) = \int h \Gamma(f, g) dm$$

Γ is the Carré du Champ Operator (CCO) associated with \mathcal{E} , if F is a normal contraction from \mathbb{R} to \mathbb{R}

$$\forall f \in \mathbb{D} \quad \Gamma(F \circ f, F \circ f) \leq \Gamma(f, f) \quad m - \text{a.e.}$$

I.3. About hypothesis (Γ)

Equivalent hypotheses:

Theorem 5 .

a) Let $P_t^{(1)}$ be the extension of $P_t \Big|_{L^1 \cap L^\infty}$ to $L^1(m)$. $(P_t^{(1)})_{t \geq 0}$ is a strongly continuous contraction semi-group in $L^1(m)$ with generator $(A^{(1)}, \mathcal{D}A^{(1)})$. It is the smallest closed extension of the restriction of the generator A of P_t to the set $\{f \in \mathcal{D}A \cap L^1 : Af \in L^1\}$

b) $(\Gamma) \iff (\Gamma') \iff (\Gamma'')$

$(\Gamma') \forall f \in \mathcal{D}A \quad f^2 \in \mathcal{D}A^{(1)}$

(Γ'') There is a sub-space H of $\mathcal{D}A$, dense in \mathbb{D} such that $\forall f \in H \quad f^2 \in \mathcal{D}A^{(1)}$

c) Under (Γ) it holds

$$\forall f, g \in \mathcal{D}A \quad \Gamma(f, g) = A^{(1)}(fg) - fA(g) - gA(f) \quad m - \text{a.e.}$$

About the relationship between hypothesis (Γ) and the existence of a C.C.O. for Markov processes, we have:

Theorem 6 . Suppose Ω be l.c.d., \mathcal{F} its borelian σ -field. Let (Q_t) be a Feller semi-group which is symmetric on $C_c(\Omega)$ with respect to a Radon positive measure m , and (P_t) the symmetric associated semi-group on $L^2(m)$.

1) If (Q_t) has a C.C.O. in the sense of Meyer, then the Dirichlet structure associated to (P_t) satisfies (Γ) .

2) Conversely, if the Dirichlet structure associated to (P_t) satisfies (Γ) and if the sets of zero potential are m -negligible, then (Q_t) has a C.C.O. in the sense of Meyer.

I.4. The locality hypothesis, the functional calculi
and the absolute continuity criterion for image measures

Theorem 7 . $(\mathbf{L}) \iff (\mathbf{L}') \iff (\mathbf{L}'')$
 $(\mathbf{L}') \mathcal{E}(|f+1| - 1, |f+1| - 1) = \mathcal{E}(f, f)$
 $(\mathbf{L}'') \forall f, g \in \mathbb{D}, \forall a \in \mathbb{R} \quad (f+a)g = 0 \implies \mathcal{E}(f, g) = 0$
and under $(\mathbf{P})(\mathbf{M})$ it is enough to take $a = 0$

Theorem 8 . *Suppose $(\Gamma)(\mathbf{L})$:*
a) $\forall f \in \mathbb{D} \quad \mathcal{E}(f, f) = \frac{1}{2} \int \Gamma(f, f) dm$
b) $\forall f \in \mathbb{D}^m, g \in \mathbb{D}^n, \quad \forall F, G$ Lipschitz C^1 -maps from $\mathbb{R}^m [\mathbb{R}^n]$ into \mathbb{R} :

$$\Gamma(F(f) - F(0), G(g) - G(0)) = \sum_{i=1}^m \sum_{j=1}^n F'_i(f) G'_j(g) \Gamma(f_i, g_j) \quad m - \text{a.e.}$$

There is a stronger result in one dimension : the Lipschitz functional calculus:

Theorem 9 . *Suppose $(\Gamma) (\mathbf{L})$*
a) $\forall f \in \mathbb{D} \quad f_(\Gamma(f, f).m) \ll \lambda_1 \quad (\lambda_1 = \text{Lebesgue measure on } \mathbb{R})$*
b) Let be $f, g \in \mathbb{D}$ and F, G Lipschitz map from \mathbb{R} to \mathbb{R} and let F', G' be versions of their derivatives :

$$\Gamma(F(f) - F(0), G(g) - G(0)) = F'(f) G' \Gamma(f, g) \quad m - \text{a.e.}$$

There is also a criterion of absolute continuity of image laws in the multivariate case:

Theorem 10 . *Suppose $(\Gamma) (\mathbf{L})$, if $f \in \mathbb{D}^n$ and if $\forall 1 \leq i, j \leq n$ $\Gamma(f_i, f_j) \in \mathbb{D}$ then*
 $f_*[\det \Gamma(f, f^*).m] \ll \lambda_n \quad (\lambda_n = \text{Lebesgue measure on } \mathbb{R}^n)$

Theorem 11 *Suppose (\mathbf{W})*
if $f \in \mathbb{D}^n$

$$f_*[\det \Gamma(f, f^*).m] \ll \lambda_n$$

This result can be extended to \mathbb{D}_{loc} with a suitable definition.

II. Image structures

II.A. Finite dimensional images.

II.A.1. Definition and basic properties.

Proposition 12 . *Let $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ be a Dirichlet structure satisfying (\mathbf{P}) , and $1 \in \mathbb{D}$.*

For $U \in \mathbb{D}^d$, let us define

$$\begin{aligned}\widetilde{\mathbb{D}}_U &= \{f \in L^2(U_*m) : f \circ U \in \mathbb{D}\} \\ \widetilde{\mathcal{E}}_U(f, f) &= \mathcal{E}(f \circ U, f \circ U)\end{aligned}$$

*then $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), U_*m, \widetilde{\mathcal{E}}_U, \widetilde{\mathbb{D}}_U)$ is a Dirichlet structure and the set \mathcal{L}_d of Lipschitz functions from \mathbb{R}^d into \mathbb{R} is in $\widetilde{\mathbb{D}}_U$.*

*Let \mathbb{D}_U be the closure of \mathcal{L}_d in $\widetilde{\mathbb{D}}_U$ and $\mathcal{E}_U = \widetilde{\mathcal{E}}_U \Big|_{\mathbb{D}_U \times \mathbb{D}_U}$ then $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), U_*m, \mathcal{E}_U, \mathbb{D}_U)$ is a regular Dirichlet structure (satisfying again (\mathbf{P}) , and $1 \in \mathbb{D}_U$).*

Definition 13 . *The structure $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), U_*m, \mathcal{E}_U, \mathbb{D}_U)$ will be called the image structure of S and will be denoted by U_*S .*

Notations. For $\phi \in L^1(m)$ we set

$$\mathbb{E}_m[\phi|U=x] := \frac{dU_*(\phi \cdot m)}{dU_*m}(x) \quad U_*m\text{-a.e.}$$

then we have

Proposition 14 .

*1) If $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ possesses a C.C.O. Γ , the same holds for $\widetilde{U_*S}$ and U_*S and their C.C.O. is given by*

$$\Gamma_U(f, f)(x) = \mathbb{E}_m[\Gamma(f \circ U, f \circ U)|U=x] \quad \forall f \in \widetilde{\mathbb{D}}_U$$

*2) If S is local, so are $\widetilde{U_*S}$ and U_*S and if S satisfies both (\mathbf{L}) and $(\mathbf{\Gamma})$, $\forall f \in \mathcal{L}_\uparrow \cap \mathcal{C}^1(\mathbb{R}^d)$ it holds*

$$\Gamma_U(f, f)(x) = \sum_{i,j}^d \mathbb{E}_m[\Gamma(U_i, U_j)|U=x] \frac{\partial f}{\partial x_i}(x) \frac{\partial f}{\partial x_j}(x)$$

Remark. There are explicit examples in which

$$\widetilde{U}_*S \neq U_*S.$$

II.A.2 The Energy Image Density Property.

Definition 15 . A Dirichlet structure $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ satisfying (\mathbf{P}) , $1 \in \mathbb{D}$, $(\mathbf{\Gamma})$, (\mathbf{L}) is said to satisfy the (\mathbf{EID}) property if $\forall n \in \mathbb{N}^*, \forall F \in \mathbb{D}^n$,

$$F_* (\det[\Gamma(F, F^*)].m) \ll \lambda_n.$$

A natural question is whether the (\mathbf{EID}) property is preserved by image.

Proposition 16 . Let S satisfying (\mathbf{P}) , $1 \in \mathbb{D}$, $(\mathbf{\Gamma})$, (\mathbf{L}) and (\mathbf{EID}) . Let $U \in \mathbb{D}^d$ such that one of the following hypotheses holds:

- a) the matrix $\Gamma(U, U^*)$ is $\sigma(U)$ -measurable
- b) $\det[\Gamma(U, U^*)] > 0$ m -a.e.

then the image structure U_*S satisfies (\mathbf{P}) , $1 \in \mathbb{D}$, $(\mathbf{\Gamma})$, (\mathbf{L}) and (\mathbf{EID}) .

With hypothesis a) the proof comes straightforward from the definitions. With hypothesis b) the result is a consequence of the following two lemmas:

Lemma 17 . Let M be a random matrix which is symmetric and non-negative definite, then

$$\{\det[\mathbb{E}(M|\mathcal{F})] = 0\} \subset \{\det[M] = 0\}$$

Lemma 18 . If $\det[\Gamma(U, U^*)] > 0$, for all $\phi \in (\mathbb{D}_U)^n$ there exists an $n \times d$ -matrix J which is $\sigma(U)$ -measurable (up to m -negligible sets) such that

$$\Gamma(\phi \circ U, \phi \circ U^*) = J\Gamma(U, U^*)J^* \quad m - \text{a.e.}$$

II.A.2 The Image Generator.

There is in general no simple relationship between the initial semi-group and the semi-group of the image structure. Not better for the associated Markov process.

Nevertheless, the generator (A_U, DA_U) of the image structure can be put in relation with the generator (A, DA) of the initial structure:

If $f, g \in \mathbb{D}_U$, and $f \circ U \in DA$, we have

$$\begin{aligned} \mathcal{E}_U(f, g) &= \mathcal{E}(f \circ U, g \circ U) = -(A(f \circ U), g \circ U)_{L^2(m)} \\ &= - \int \mathbb{E}_m[A(f \circ U)|U = x]g(x) dU_*m(x) \end{aligned}$$

hence $f \in DA_U$ and $A_U f = \mathbb{E}_m[A(f \circ U)|U = x]$.

But, hypotheses are needed to ensure the space $\mathbb{D}_U \cap \{f : f \circ U \in DA\}$ contains other functions than constants:

Proposition 19 . *Suppose S satisfies (\mathbf{P}) , $1 \in \mathbb{D}$, $(\mathbf{\Gamma})$, (\mathbf{L}) . Let $U \in (DA)^d$ such that $\Gamma(U_i, U_j) \in L^2(m) \quad \forall i, j = 1, \dots, d$ then*

$$DA_U \supset \{f \in \mathcal{C}^2(\mathbb{R}^d) : \frac{\partial f}{\partial x_i}, \frac{\partial^2 f}{\partial x_i \partial x_j} \text{ bounded}\}$$

and for such an f

$$\begin{aligned} A_U f(x) &= \mathbb{E}_m[A(f \circ U)|U = x] \\ &= \frac{1}{2} \sum_{i,j} \alpha_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_i \beta_i \frac{\partial f}{\partial x_i} \end{aligned}$$

with $\alpha_{ij}(x) = \mathbb{E}[\Gamma(U_i, U_j)|U = x] \quad (\in L^2(U_*m))$

and $\beta_i(x) = \mathbb{E}[AU_i|U = x] \quad (\in L^2(U_*m))$

If further, $\det[\Gamma(U, U^*)] > 0$ m -a.e. then the function $k = \frac{dU_*m}{d\lambda_n}$ satisfies

$$2\beta_i k - \sum_j \frac{\partial}{\partial x_j} (\alpha_{ij} k) = 0 \quad \forall i = 1, \dots, n$$

in the sense of distributions.

H. Airault and P. Malliavin have studied the case of Wiener space with

$$U \in W_\infty = \cap_{p,n} D_{p,n}$$

and

$$[\det \Gamma(U, U^*)]^{1/2} \in W_\infty$$

and they show in this case that

$$A_U = \Delta + \nabla_{\vec{u}}$$

where Δ is the Laplace-Beltrami operator associated to the Riemannian metric with matrix $[(\alpha_{ij})]^{-1}$ and where $\vec{u} = \frac{1}{2} \overrightarrow{\text{grad}} \log \rho$ with $\rho = \frac{dU_* m}{dv}$ and where $dv = \sqrt{\det(\alpha_{ij})} \cdot \lambda_n$ is the associated area measure.

II.B. General Images.

II.B.1. U does not need to be supposed in \mathbb{D} or \mathbb{D}^d for defining an image structure, whenever there is enough functions $f \circ U$ in \mathbb{D} .

Let $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ satisfy **(P)** and $1 \in \mathbb{D}$, let (X, \mathcal{G}) be a measurable space and let U be a measurable map from (Ω, \mathcal{F}) into (X, \mathcal{G}) . Let us suppose that there exists a set \mathcal{A} of measurable applications from X into \mathbb{R} such that

- . \mathcal{A} is a vector space containing the constants
- . $\forall f \in \mathcal{A}, \quad f \circ U \in \mathbb{D}$
- . \mathcal{A} is dense in $L^2(U_*m)$

then the form $(\mathcal{E}_{\mathcal{A}}, \mathcal{A})$ defined by $\mathcal{E}_{\mathcal{A}}(f, f) = \mathcal{E}(f \circ U, f \circ U)$ is closable in $L^2(U_*m)$, let $(\mathcal{E}_U, \mathbb{D}_U)$ its closure, we put

$$U_*S = (X, \mathcal{G}, U_*m, \mathcal{E}_U, \mathbb{D}_U).$$

II.B.2. Example.

Let $S = (\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}), \nu^{\otimes \mathbb{N}}, \mathbb{D}, \mathcal{E})$ be a Dirichlet structure such that

- . $\nu = N(0, 1)$ is the standard Gaussian measure on \mathbb{R}
- . $(\mathbb{D}, \mathcal{E})$ is any Dirichlet form on $L^2(\nu^{\otimes \mathbb{N}})$ such that the coordinates χ_n belong to \mathbb{D} .

Let be $X = \mathcal{C}[0, 1]$ and \mathcal{G} be its borelian σ -field. Let (\dot{h}_n) be a C.O.N.S. of $L^2([0, 1])$ and put

$$h_n(t) = \int_0^t \dot{h}_n(s) ds$$

Let us consider the map U from $\mathbb{R}^{\mathbb{N}}$ into X defined by

$$(\star) \quad U(x) = \sum_{n=0}^{\infty} \chi_n(x) h_n \text{ if the serie converges in } \mathcal{C}[0, 1],$$

$$U(x) = 0 \quad \text{elsewhere} \quad .$$

A vector valued martingale argument shows that

Lemma 20 . *The serie (\star) converges almost surely and in $L^p_{\mathcal{C}[0,1]}(\nu^{\otimes \mathbb{N}})$ ($1 < p < \infty$), and the law of its sum is the Wiener measure $\mu : \mu = U_*(\nu^{\otimes \mathbb{N}})$.*

Let us denote by (B_t) the brownian motion defined by this Wiener measure on $\mathcal{C}[0, 1]$, and let be

$$\widetilde{h}_n = \int_0^1 \dot{h}_n(s) dB_s$$

then it can be shown that

$$\widetilde{h}_n \circ U(x) = \chi_n(x) \quad \text{for } \nu^{\otimes \mathbb{N}} - \text{a.e. } x.$$

Hence by hypothesis $\widetilde{h}_n \circ U \in \mathbb{ID}$, therefore if $f = F(\widetilde{h}_1, \dots, \widetilde{h}_n)$ with F Lipschitz, we have $f \circ U \in \mathbb{ID}$. But $F(\chi_1, \dots, \chi_n)$ is dense in $L^2(\nu^{\otimes \mathbb{N}})$ hence $F(\widetilde{h}_1, \dots, \widetilde{h}_n)$ is dense in $L^2(\mu)$.

So, the image structure

$$(X, \mathcal{G}, \mu, \mathcal{E}_U, \mathbb{ID}_U)$$

is well defined and contains $\{F(\widetilde{h}_1, \dots, \widetilde{h}_n)\}$ for Lipschitz F .

III. Tensor products and projective limits.

III.A. Finite products.

Let $S_1 = (\Omega_1, \mathcal{F}_1, m_1, \mathcal{E}_1, \mathbb{D}_1)$ and $S_2 = (\Omega_2, \mathcal{F}_2, m_2, \mathcal{E}_2, \mathbb{D}_2)$ be Dirichlet structures.

Definition 21 .

$$S_1 \otimes S_2 = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, m_1 \times m_2, \mathcal{E}, \mathbb{D})$$

with

$$\mathbb{D} = \left\{ f \in L^2(m_1 \times m_2) : \begin{array}{ll} \text{for } m_2\text{-a.e. } y & f(\cdot, y) \in \mathbb{D}_1 \\ \text{for } m_1\text{-a.e. } x & f(x, \cdot) \in \mathbb{D}_2 \end{array} \right.$$

$$\text{and } \int \mathcal{E}_1(f(\cdot, y), f(\cdot, y)) dm_1(y) + \int \mathcal{E}_2(f(x, \cdot), f(x, \cdot)) dm_2(x) < \infty \}$$

and $\forall f \in \mathbb{D}$,

$$\mathcal{E}(f, f) = \int \mathcal{E}_1(f(\cdot, y), f(\cdot, y)) dm_1(y) + \int \mathcal{E}_2(f(x, \cdot), f(x, \cdot)) dm_2(x).$$

It is indeed easy to see that this form is closed and that contractions operate.

If S_1 and S_2 satisfy **(P)** and **(M)** the same holds for $S_1 \otimes S_2$.

If S_1 and S_2 are local, $S_1 \otimes S_2$ is local.

If S_1 and S_2 satisfy **(Γ)**, the same holds for $S_1 \otimes S_2$ and its OCC is given by

$$\Gamma(f, f) = \Gamma_1(f(\cdot, y), f(\cdot, y))(x) + \Gamma_2(f(x, \cdot), f(x, \cdot))(y)$$

Concerning the associated semi-group we have the following :

Let $(P_t^1), (P_t^2)$ be the semi-groups associated with S_1 and S_2 , and let \widehat{P}_t^1 and \widehat{P}_t^2 be the semi-groups on $L^2(m_1 \times m_2)$ defined by

$$\begin{aligned} \widehat{P}_t^1 f(x, y) &= P_t^1(f(\cdot, y))(x) \\ \widehat{P}_t^2 f(x, y) &= P_t^2 f(x, \cdot)(y) \end{aligned}$$

which are symmetric, strongly continuous and sub-Markov.

Proposition 22 . *a) The semi-group associated with $S_1 \otimes S_2$ is*

$$P_t = \widehat{P}_t^1 \widehat{P}_t^2 = \widehat{P}_t^2 \widehat{P}_t^1$$

b) its generator is the smallest closed extension of the operator defined on $DA_1 \otimes DA_2$ by $A(\phi \otimes \psi) = A_1\phi \otimes \psi + \phi \otimes A_2\psi$

c) $\mathbb{D}_1 \otimes \mathbb{D}_2$ is dense in \mathbb{D} .

III.B. Infinite tensor products.

The preceding construction extends without any problem to the infinite tensor products (countable or not) :

$$\bigotimes_{i \in I} (E_i, \mathcal{F}_i, \mu_i, \mathcal{E}_i, \mathbb{D}_i)$$

where the factors are supposed to satisfy **(P)**.

That comes mainly from the fact that the limit of an increasing sequence of Dirichlet forms is a Dirichlet form:

Lemma 23 . *Let (Ω, \mathcal{F}, m) be a measured space equipped with Dirichlet forms $\mathcal{E}^{(n)}, \mathbb{D}^{(n)}$ such that*

. $\mathbb{D}^{(n)} \downarrow$ as $n \uparrow$

. $\mathcal{E}^{(n)} \uparrow$ as $n \uparrow$: $\forall f \in \mathbb{D}^{(n)} \quad \mathcal{E}^{(n+1)}(f, f) \geq \mathcal{E}^{(n)}(f, f)$

then $\mathbb{D} = \bigcap \mathbb{D}^{(n)}$, $\mathcal{E}(f, f) = \lim \mathcal{E}^{(n)}(f, f)$ is a Dirichlet form.

If the S_i 's are local, so is $\bigotimes_i S_i$.

if each S_i possesses a CCO, the same holds for $\bigotimes_i S_i$.

Suppose now that the family S_i is countable and that each finite product

$$\bigotimes_{i=0}^n S_i$$

satisfies the **(EID)** property, then $S = \bigotimes_{i=0}^{\infty} S_i$ satisfies **(EID)**.

That comes directly from the definitions.

As an example let us take

$$S_i = (\mathbb{R}, \mathcal{B}(\mathbb{R}), h_i(x)dx, \int \nabla^2 h_i(x)dx, \mathbb{D}_i)$$

where h_i satisfies the Hamza condition and $\int h_i(x)dx = 1$.

Then by the coarea formula of Federer, the finite products satisfy **(EID)** and therefore the infinite product structure (which is in general non Gaussian) satisfies **(EID)**.

Remark. In this example, putting

$$\mu^i = \bigotimes_{\substack{j \in \mathbb{N} \\ j \neq i}} (h_j dx)$$

and

$$\mathcal{E}_i(f, f) = \int_{\mathbb{R}} (\nabla f)^2 h_i dx$$

we have

$$\mathbb{D} = \{f \in L^2(m) : \forall i \in \mathbb{N} \quad f(\cdot, y) \in \mathbb{D}_i \text{ for } \mu^i - \text{a.e. } y$$

$$\text{and } \sum_{i=0}^{\infty} \int \mathcal{E}_i(f(\cdot, y), f(\cdot, y)) d\mu^i(y) < \infty\}$$

and for $f \in \mathbb{D}$

$$\Gamma(f, f) = \sum_{i=0}^{\infty} \Gamma_i(f, f) = \sum_{i=0}^{\infty} f_i'^2$$

where Γ_i acts only on the i -th variable.

Therefore if for $f \in \mathbb{D}$ we put

$$Df = (f_i')_{i \in \mathbb{N}}$$

this defines a continuous operator from \mathbb{D} into $L^2(m, \ell^2)$ and we have

$$\Gamma(f, f) = \langle Df, Df \rangle_{\ell^2}$$

We shall see later that this allows to develop a conditional Dirichlet calculus. These product structures are examples of Classical Dirichlet forms in the sense of Albeverio-Röckner.

III.C. Projective limits.

1. General projective system of Dirichlet structures can be defined in an obvious way.

But there is a difficulty for passing to the limit unless some uniform closability property is known (which is the case for products). Here is an example of projective system without limit :

Example

Let μ be the Gauss measure on \mathbb{R} . Let us consider the structures

$$S^{(n)} = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu^{\otimes n}, \mathcal{E}^{(n)}, \mathbb{D}^n)$$

with

$$\mathbb{D}^{(n)} = \left\{ f \in L^2(\mu^{\otimes n}) : \frac{\partial f}{\partial x_i} \in L^1_{loc}(\mathbb{R}^n) \text{ and } \int \left(\sum_{i=0}^n \frac{1}{a_i} \frac{\partial f}{\partial x_i} \right)^2 d\mu^{\otimes n} < \infty \right\}$$

and

$$\mathcal{E}^{(n)}(f, f) = \int \left(\sum_{i=0}^n \frac{1}{a_i} \frac{\partial f}{\partial x_i} \right)^2 d\mu^{\otimes n}$$

where the numbers a_i are chosen such that

$$a_i > 0, \quad \lim_{i \rightarrow \infty} a_i = 0$$

The $S^{(n)}$'s define a compatible system of Dirichlet structures, but if $h_n(x) = a_n x_n$ we have $\|h_n\|_{L^2(\mu^{\otimes \mathbb{N}})} \rightarrow 0$ and the candidate $\tilde{\mathcal{E}}$ satisfies

$$\tilde{\mathcal{E}}(h_n - h_m, h_n - h_m) = \int \left(\frac{1}{a_n} a_n - \frac{1}{a_m} a_m \right)^2 d\mu^{\otimes \mathbb{N}} = 0$$

and $\tilde{\mathcal{E}}(h_n, h_n) = 1$ therefore $\tilde{\mathcal{E}}$ is not closable.

2. An important special case where the limit exists

Consider a Dirichlet structure $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ and a family (U_n) of applications such that

$$U_n : \Omega \longrightarrow \mathbb{R}^d \quad \text{and} \quad U_n \in \mathbb{D}^{d_n} \quad \forall n$$

then the image structure of S by (U_0, \dots, U_n) defines a Dirichlet structure $S^{(n)}$ with state space

$$\prod_{i=0}^n \mathbb{R}^{d_i}$$

These structures $S^{(n)}$ define a projective system which always possesses a limit. This comes easily from the fact that the initial form $(\mathcal{E}, \mathbb{D})$ is closed.

The limit can be called the image structure by the process $(U_n)_{n \in \mathbb{N}}$.

The same would be true, *mutatis mutandis*, for uncountable families.

IV. D-independence.

IV.A. Definition and examples.

Let $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ be a Dirichlet structure satisfying **(P)** and $1 \in \mathbb{D}$.

If $U \in \mathbb{D}^p$ the image structure U_*S will be called the D-law of U .

Definition 24 . *If $U \in \mathbb{D}^p$, $V \in \mathbb{D}^q$, U and V will be said to be D-independent if the D-law of (U, V) is the product of the D-laws of U and V .*

Proposition 25 . *A necessary and sufficient condition for independent U and V to be D-independent is*

$$\begin{aligned} & \forall f_1, f_2 \in \mathcal{C}_c^1(\mathbb{R}^p), \forall g_1, g_2 \in \mathcal{C}_c^1(\mathbb{R}^q) \\ & \mathcal{E}(f_1 \circ U g_1 \circ V, f_2 \circ U g_2 \circ V) \\ & = \mathcal{E}(f_1 \circ U, f_2 \circ U)(g_1 \circ V, g_2 \circ V)_{L^2(m)} + \mathcal{E}(g_1 \circ V, g_2 \circ V)(f_1 \circ U, f_2 \circ U)_{L^2(m)} \end{aligned}$$

If E is local and possesses a CCO we have the more explicit result

Proposition 26 . *If S satisfies **(P)**, $1 \in \mathbb{D}, (\mathbf{L}), (\mathbf{\Gamma})$, for $U \in \mathbb{D}^p$ and $V \in \mathbb{D}^q$ to be D-independent it is necessary and sufficient that*

- 1) U and V are independent,
- 2) $\forall i, k \quad \mathbb{E}[\Gamma(U_i, V_k)|U, V] = 0 \quad m\text{-a.e.}$
- 3) $\forall i, j \quad \mathbb{E}[\Gamma(U_i, U_j)|U, V] = \mathbb{E}[\Gamma(U_i, U_j)|U] \quad m\text{-a.e.}$
- 4) $\forall l, k \quad \mathbb{E}[\Gamma(V_k, V_l)|U, V] = \mathbb{E}[\Gamma(V_k, V_l)|V] \quad m\text{-a.e.}$

Remark. These conditions are fulfilled as soon as

- . $\Gamma(U_i, V_k) = 0$ for all i, k
- . $(U, \Gamma(U_i, U_j))$ is independent of V for all i, j
- . $(V, \Gamma(V_k, V_l))$ is independent of U for all k, l .

Examples.

1) If U and V are random variables in the first chaos on the Wiener space, they are D-independent as soon as they are independent i.e. orthogonal.

2) Let $f \in L_{sym}^2(\mathbb{R}_+^p)$ and $g \in L_{sym}^2(\mathbb{R}_+^q)$. By a result of Ustunel and Zakai if the multiple Wiener integrals $I_p(f)$ and $I_q(g)$ are independent so are the σ -fields

$$\sigma(I_p(f), \langle DI_p(f), h_{1,1} \rangle, \dots, \langle D^{p-1}I_p(f), h_{p-1,1} \otimes \dots \otimes h_{p-1,p-1} \rangle)$$

and

$$\sigma(I_q(g), \langle DI_q(g), k_{1,1} \rangle, \dots, \langle D^{q-1}I_q(g), k_{q-1,1} \otimes \dots \otimes k_{q-1,q-1} \rangle)$$

Therefore $I_p(f)$ and $I_q(g)$ are D-independent as soon as they are independent and $\Gamma(I_p(f), I_q(g)) = 0$.

3) This extends to the case of multivariate random variables whose components are multiple Wiener integrals.

IV.B. Convergence in D-law.

Let as before $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ be a Dirichlet structure satisfying **(P)** and $1 \in \mathbb{D}$.

Definition 27 . Let (U_n) be a sequence in \mathbb{D}^d and $V \in \mathbb{D}^d$. The sequence (U_n) is said to converge in D-law to V

$$U_n \xrightarrow{D-L} V$$

if

- . $U_{n*}m$ converges to the law of V in the narrow sense
- . $\forall f \in \mathcal{L} \cap \mathcal{C}^1(\mathbb{R}^d) \quad \mathcal{E}(f \circ U_n, f \circ U_n) \rightarrow \mathcal{E}(f \circ V, f \circ V)$

in other words that means convergence of the D-laws on bounded continuous functions for the measures, on \mathcal{C}^1 -Lipschitz functions for the forms.

The central limit theorem becomes the following :

Theorem 28 . Let us suppose S satisfies **(P)**, $1 \in \mathbb{D}$, **(L)**, **(Γ)**. Let (U_n) be a sequence of functions in \mathbb{D}^d which are centered, with the same D-law, and D-independent, then

$$V_n = \frac{1}{\sqrt{n}}(U_1 + \dots + U_n)$$

converges in D-law and the limit Dirichlet structure is

$$(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \nu, \hat{\mathcal{E}}, \hat{\mathbb{D}})$$

with

$$\nu = N_d(0, \Sigma)$$

$$\forall f \in \mathcal{L} \cap \mathcal{C}^1(\mathbb{R}^d) \quad \hat{\mathcal{E}}(f, f) = \sum_{i,j} \int \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} a_{ij} d\nu$$

where $\Sigma_{ij} = \int x_i x_j d\mu$ (μ being the common law of the U_n 's) and $a_{ij} = \mathcal{E}(U_{n,i}, U_{n,j}) = \mathcal{E}_{U_n}(x_i, x_j)$ (which does not depend on n).

The main step of the proof is the following lemma

Lemma 29 . Let U_1, \dots, U_n be in \mathbb{D}^d and D -independent then $\forall f \in \mathcal{L} \cap \mathcal{C}^1(\mathbb{R}^d)$

$$\mathcal{E}(f(U_1 + \dots + U_n), f(U_1 + \dots + U_n))$$

$$= \frac{1}{2} \sum_{ij} \int \frac{\partial f}{\partial x_i}(y_1 + \dots + y_n) \frac{\partial f}{\partial x_j}(y_1 + \dots + y_n) \left(\sum_{\ell=1}^n a_{ij}^\ell(y_\ell) \right) d\mu_1(y_1) \dots d\mu_n(y_n)$$

where $\mu_n = U_{n*}m$ is the law of U_n and

$$a_{ij}^\ell(y_\ell) = \mathbb{E}[\Gamma(U_{\ell,i}, U_{\ell,j}) | U_\ell = y_\ell] \quad (= \Gamma_{U_\ell}(x_i, x_j)(y_\ell)).$$

Let $(Z^{(n)})_{n \in \mathbb{N}}$ be a sequence of discrete time processes

$$Z^{(n)} = (Z_1^{(n)}, \dots, Z_k^{(n)}, \dots)$$

defined on a Dirichlet structure S , we shall say that $(Z^{(n)})$ converges in D-law to the process Z , if the marginal D-laws of $Z^{(n)}$ converge to those of Z .

Example. Let us take $S = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu, \int \nabla^2 d\mu, H^1(\mathbb{R}, \mu))$ with $\mu = N(0, 1)$ and let us consider the “standard Gaussian product space”

$$S^{\otimes \mathbb{N}} = (\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}), \mu^{\otimes \mathbb{N}}, \mathcal{E}, \mathbb{D}).$$

Let X_i be the coordinates and let us put

$$Y_k^{(n)} = \frac{X_k \sqrt{n}}{\sqrt{X_1^2 + \dots + X_n^2}}$$

then the process

$$Z^{(n)} = (Y_1^{(n)}, \dots, Y_n^{(n)}, 0, 0 \dots)$$

converges in D-law toward

$$X = (X_1, \dots, X_n, \dots)$$

.

That is an extension of the Gateaux-Lévy theorem which states the same result with only probability structures.

V. Dirichlet sub-spaces and conditioning.

V.A. Dirichlet sub-spaces.

Definition 30 . Let $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ be a Dirichlet structure satisfying **(P)** and $1 \in \mathbb{D}$, a sub-vector space $\mathfrak{d}\mathbb{I}$ of \mathbb{D} will be called a Dirichlet sub-space if it is closed in \mathbb{D} and stable under composition by Lipschitz functions on \mathbb{R} .

Proposition 31 . If $\mathfrak{d}\mathbb{I}$ is a Dirichlet sub-space, it holds

$$\overline{\mathfrak{d}\mathbb{I}}^{L^2(m)} = L^2(m, \sigma(\mathfrak{d}\mathbb{I}))$$

and so $S_{\mathfrak{d}\mathbb{I}} = (\Omega, \sigma(\mathfrak{d}\mathbb{I}), m, \mathcal{E}|_{\mathfrak{d}\mathbb{I} \times \mathfrak{d}\mathbb{I}}, \mathfrak{d}\mathbb{I})$ is a Dirichlet structure.

In particular, $\mathfrak{d}\mathbb{I}$ is stable by composition by Lipschitz functions of several variables.

For example if $X_i \in \mathbb{D}$, $\forall i \in I$, the space

$$\mathbb{D}(X_i, i \in I) = \overline{\{G(X_{i_1}, \dots, X_{i_n}) \mid i_k \in I, G \in \mathcal{C}^1(\mathbb{R}^n)\}}^{\mathbb{D}}$$

is a Dirichlet sub-space which will be called the Dirichlet sub-space generated by the family $(X_i)_{i \in I}$.

Remark. If S satisfies **(Γ)**, $S_{\mathfrak{d}\mathbb{I}}$ satisfies **(Γ)** and its CCO is given by

$$\Gamma_{\mathfrak{d}\mathbb{I}}(v, v) = \mathbb{E}[\Gamma(v, v) | \sigma(\mathfrak{d}\mathbb{I})] \quad \forall v \in \mathfrak{d}\mathbb{I}$$

Example. It is easily seen that the kernel of the form

$$K = \{f \in \mathbb{D} : \mathcal{E}(f, f) = 0\}$$

is a Dirichlet sub-space.

V.B. Conditional calculus.

We consider a D-structure $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ satisfying **(P)**, **(M)**, **(L)**, **(Γ)**.

Hypothesis (G). We shall say that S admits a gradient if there exists a separable Hilbert space and a linear map D from \mathbb{D} into $L^2(m, H)$ such that

$$\langle Du, Du \rangle_H = \Gamma(u, u) \quad \forall u \in \mathbb{D}.$$

This is the case for the Wiener space, for some product spaces and some Classical Dirichlet space in the sense of Albeverio-Röckner.

From the functional calculus for Γ we deduce

Proposition 32 . *D is continuous and satisfies*

- a) $D(f \circ U) = f' \circ U \cdot DU$, $f \in \mathcal{L}(\mathbb{R})$, $U \in \mathbb{D}$
- b) $D(F \circ \vec{U}) = \sum_i F' \circ \vec{U} \cdot DU_i$, $F \in \mathcal{L} \cap \mathcal{C}^1(\mathbb{R}^d)$, $\vec{U} \in \mathbb{D}^d$

Most of the features of the conditional calculus of Nualart-Zakai extends to this situation :

Let $(X_i, i \in \mathbb{N})$ be a countable family in \mathbb{D} and let \mathcal{H} be the following measurable field of sub-spaces of H

$$\mathcal{H} = (\mathcal{L}(DX_i, i \in \mathbb{N}))^\perp$$

For $F \in \mathbb{D}$, let us define

$$\begin{aligned} D^X(F) &= P^{\mathcal{H}}(DF) \\ \Gamma^X(F, F) &= \langle P^{\mathcal{H}}(DF), P^{\mathcal{H}}(DF) \rangle_H \\ \mathcal{E}^X(F, F) &= \mathbb{E}[\Gamma^X(F, F)] \end{aligned}$$

Proposition 33 . a) (D^X, \mathbb{D}) is a closable operator in $L^2(m, H)$ iff the form $(\mathcal{E}^X, \mathbb{D})$ is closable.

b) This is the case if $P^{\mathcal{H}}h \in \mathbb{D}$ for all $h \in H$.

c) In this case the D-structure associated with $\mathcal{E} : (\Omega, \mathcal{F}, m, \mathcal{E}^X, \mathbb{D}^X)$ satisfies **(P)**, **(M)**, **(L)**, **(Γ)** and **(G)** with gradient operator D^X and will be called the conditional structure knowing X .

The main result in this theory is then :

Theorem 34 . Suppose the conditional structure knowing $X = (X_i, i \in \mathbb{N})$ exists. Let $\tau_X = \sigma(X_i, i \in \mathbb{N})$.

a) Let $F : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$, $\mathcal{B}(\mathbb{R}) \otimes \tau_X$ -measurable, s.t. $F(x, \omega)$ is Lipschitz in x , bounded as well as F'_x then for all $U \in \mathbb{D}$ (even for $U \in \mathbb{D}^X$)

$$(\omega \longrightarrow F(U(\omega), \omega)) \in \mathbb{D}^X$$

and

$$\Gamma^X(F(U(\cdot), \cdot), F(U(\cdot), \cdot))(\omega) = F_x'^2(U(\omega), \omega) \Gamma^X(U, U)(\omega) \quad m\text{-a.e.}$$

b) For all $U \in \mathbb{D}$ (even for $U \in \mathbb{D}^X$), the image of the measure $\Gamma^X(U, U).m$ by the map $\omega \rightarrow (U(\omega), \omega)$ from (Ω, \mathcal{F}) into $(\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \tau_X)$ is absolutely continuous w.r. to $dx \times m$.

In particular if $\Gamma^X(U, U) > 0$ a.e., U possesses a conditional law knowing the σ -field $\tau_X = \sigma(X_i, i \in \mathbb{N})$.

There are two limitations for applying this theory in practice

1) The verification of the closability condition.

2) Most examples, especially from filtration problems, do allow a direct treatment because the conditional law is absolutely continuous with respect to the Wiener measure: The ordinary criterion applies.

Example. Let $(\eta_t(w))_{t \in \mathbb{R}_+}$ and $(\xi_t(w))_{t \in \mathbb{R}_+}$ be two processes defined on a probability space $(W, \mathcal{A}, \mathbb{P})$.

If the law of ξ knowing η (i.e. knowing $\tau = \sigma(\eta_s, s \in \mathbb{R}_+)$) is absolutely continuous w.r. to the Wiener measure, then a sufficient condition for a random variable $F : \Omega \rightarrow \mathbb{R}^d$ of the form

$$F = f(\eta, \xi)$$

to possess a conditional density knowing τ is that for \mathbb{P} -a.e. w , setting

$$F_w(\omega) = f(\eta(w), \omega),$$

1) $F_w \in \mathbb{D}$ ($= D_{2,1}$ here)

2) $\det \Gamma(F_w, F_w^*)(\omega) > 0 \quad dm(\omega)\text{-a.e.}$

V.C. A glance to stationary processes.

Let $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ be a D-structure satisfying **(P)**, **(M)**, **(L)**, **(Γ)**.

A map $X : t \rightarrow X_t$ from \mathbb{R} into \mathbb{D} will be called a D-stationary process if its marginal D-laws are invariant under translations of time.

Let $F \in \mathcal{L} \cap \mathcal{C}\mathbb{R}^n$, the relation

$$\mathcal{T}_t[F(X_{t_1}, \dots, X_{t_n})] = F(X_{t_1+t}, \dots, X_{t_n+t})$$

defines a group of isometries which extends to the Dirchlet sub-space generated by $X : \mathbb{D}_X = D(X_t, t \in \mathbb{R})$

It is easy to see that (\mathcal{T}_t) is strongly continuous on \mathbb{D}_X if and only if $t \rightarrow X_t$ is continuous from \mathbb{R} into \mathbb{D} .

If this is satisfied, we get a spectral representation : $\mathcal{T}_t = e^{itA}$, A self-adjoint on \mathbb{D}_X and if $E(d\lambda)$ is the resolution of the identity associated with A :

$$X_t = \int e^{i\lambda t} E(d\lambda) X_0 \quad \text{in } \mathbb{D}_X$$

Let Γ_X be the CCO of the sub-structure $(\Omega, \sigma(X), m, \mathcal{E}_X, \mathbb{D}_X)$ which is given by

$$\Gamma_X(U, V) = \mathbb{E}[\Gamma(U, V) | \sigma(X)].$$

Let us suppose moreover that $\Gamma(X_s, X_t)$ be deterministic $\forall s, t$.

(This happens often without any Gaussian hypothesis : for example for product spaces

$$\bigotimes_{n=0}^{\infty} (\mathbb{R}, \mathcal{B}(\mathbb{R}), h_n dx, \int \nabla^2 h_n dx, \mathbb{D}_n)$$

the h_n 's satisfying the Hamza condition and $\int x^2 h_n(x) dx < \infty$ if for all t , X_t belongs to the closure in \mathbb{D} of linear combinations of coordinates.)

Then $\Gamma(X_s, X_t) = \Gamma(X_0, X_{t-s})$ hence by Bochner theorem

$$\Gamma(X_{t+h}, X_t) = \gamma(h) = \int e^{i\lambda h} d\mu(\lambda)$$

for a finite positive measure μ since γ is continuous.

It follows that

- X_t has a density as soon as $\mu \neq 0$
- $(X_{t_1}, \dots, X_{t_n})$ has a density if the functions

$$e^{it_1 \bullet}, \dots, e^{it_n \bullet}$$

are linearly independent in $L^2(\mathbb{R}, \mu)$.

Let ν be the usual spectral measure of X , from the two spectral representations it follows that the space $\overline{\mathcal{L}(X_t, t \in \mathbb{R})}^{\mathbb{D}^X}$ is isomorphic to $L^2(\sigma(X), \mu + \nu)$.

Hence if $\nu \ll \mu$ with $\frac{d\nu}{d\mu}$ bounded, $\Gamma(Y, Y)^{1/2}$ is on $\mathcal{L}(X_t, t \in \mathbb{R})$ a norm equivalent to $\|Y\|_{\mathbb{D}^X}$.

If we project X_{t+h} on $\overline{\mathcal{L}(X_s, s \leq t)}^{\mathbb{D}^X}$ for this Hilbert scalar product we get

$$\widehat{X}_{t+h} \in \overline{\mathcal{L}(X_s, s \leq t)}^{\mathbb{D}^X}$$

and \widehat{X}_{t+h} is also the best estimate of X_{t+h} in the whole space $\mathbb{D}(X_s, s \leq t)$ in the sense of the Dirichlet form \mathcal{E} , because

$$\begin{aligned} \mathcal{E}(X_{t+h} - \widehat{X}_{t+h}, X_s) &= 0 \quad \forall s \leq t \\ \implies \mathcal{E}(X_{t+h} - \widehat{X}_{t+h}, F(X_{s_1}, \dots, X_{s_n})) &= 0 \quad \forall F \in \mathcal{L} \cap \mathcal{C}^1(\mathbb{R}^n) \\ &\quad s_i \leq t \quad i = 1, \dots, n \end{aligned}$$

This situation is similar to the Gaussian case for the filtrage of Wiener.