Algèbre des structures de Dirichlet
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THE ALGEBRA OF DIRICHLET STRUCTURES

Nicolas BOULEAU and Francis HIRSCH

I. Some notations and definitions
I.1. Dirichlet Structures

\((\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})\)

\((\Omega, \mathcal{F}, m)\) : measured space with \(m\) σ-finite and positive \n\mathcal{E} : Dirichlet form with domain \(\mathbb{D}\)
i.e. Quadratic positive form with dense domain \(\mathbb{D}\) in \(L^2(m)\) which is
. closed : \(\mathbb{D}\) is a Hilbert space under the norm
\[
\|f\|_\mathbb{D} = [\|f\|_{L^2(m)}^2 + \mathcal{E}(f, f)]^{1/2}
\]
. and s.t. \(f \in \mathbb{D} \implies (f \wedge 1) \in \mathbb{D}\) and \(\mathcal{E}(f \wedge 1, f \wedge 1) \leq \mathcal{E}(f, f)\).

Notations for different hypotheses:
(P) (Probability) \(m(\Omega) = 1\)
(M) (Markovianity) \(1 \in \mathbb{D}\) and \(\mathcal{E}(1, 1) = 0\)
(Γ) (Existence of a Carré du Champ Operator):
\[
\forall f \in \mathbb{D} \cap L^\infty, \exists \tilde{f} \in L^1, \forall h \in \mathbb{D} \cap L^\infty,
2\mathcal{E}(fh, f) - \mathcal{E}(h, f^2) = \int \tilde{f}h \, dm
\]
(L) (Locality)
\[
\forall f \in \mathbb{D}, \forall F, G \in \mathcal{D}(\mathbb{R})\]
\[
\text{supp} F \cap \text{supp} G = \emptyset \implies \mathcal{E}(F(f) - F(0), G(f) - G(0)) = 0
\]
(W) (Wiener space)
\[
\Omega = \{\omega \in C(\mathbb{R}_+, \mathbb{R}^d), \omega(0) = 0\}
\]
\(\mathcal{F} = \) borelian σ-field of \(\Omega\) with compact convergence
\(m = \) Wiener measure
\((\mathcal{E}, \mathbb{D}) = \) form associated with the Ornstein-Uhlenbeck semi-group.
I.2. Basic properties

There is a sub-Markov semigroup associated with a Dirichlet structure.

**Theorem 1.** Let a Dirichlet structure $(\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ be given. There exists a strongly continuous contraction semi-group $(P_t)_{t \geq 0}$ symmetric on $L^2(m)$ and unique such that

\[
\begin{aligned}
\{ \mathbb{D} &= \{ f \in L^2(m) : \lim_{t \to 0} (\frac{f-P_t f}{t})_{L^2(m)} \text{exists} \} \\
\forall f \in \mathbb{D} &\quad \mathcal{E}(f, f) = \lim_{t \to 0} (\frac{f-P_t f}{t})_{L^2(m)}
\end{aligned}
\]

this semi-group is sub-Markov.

Conversely, if $(P_t)$ is a symmetric strongly continuous contraction semi-group on $L^2(m)$, and sub-Markov, the positive quadratic form associated with $(P_t)$ by $(\ast)$ is a Dirichlet form.

**Definition 2.** A function $F$ from $\mathbb{R}^n$ into $\mathbb{R}$ is a contraction [resp. a normal contraction] if

\[
\forall x, y \quad |F(x) - F(y)| \leq \sum_{i=1}^{n} |x_i - y_i|
\]

[resp. and $F(0) = 0$]

**Theorem 3.** $\forall f \in \mathbb{D}$, if $F$ is a normal contraction from $\mathbb{R}^n$ to $\mathbb{R}$ then

\[
F \circ f \in \mathbb{D} \quad \text{and} \quad (\mathcal{E}(F \circ f, F \circ f))^{1/2} \leq \sum_{i=1}^{n} (\mathcal{E}(f_i, f_i))^{1/2}.
\]

Under $(\mathcal{P})$ the word normal can be cancelled.

The hypothesis $(\Gamma)$ gives rise to a carré du champ operator:

**Theorem 4.** Under $(\Gamma)$ there exists a unique continuous symmetric positive bilinear map from $\mathbb{D} \times \mathbb{D}$ into $L^1(m)$, denoted by $\Gamma$ such that

\[
\forall f, g, h \in \mathbb{D} \cap L^\infty \\
\mathcal{E}(fh, g) + \mathcal{E}(gh, f) - \mathcal{E}(h, fg) = \int h \Gamma(f, g) \, dm
\]

$\Gamma$ is the Carré du Champ Operator (CCO) associated with $\mathcal{E}$, if $F$ is a normal contraction from $\mathbb{R}$ to $\mathbb{R}$

\[
\forall f \in \mathbb{D} \quad \Gamma(F \circ f, F \circ f) \leq \Gamma(f, f) \quad m - \text{a.e.}
\]
I.3. About hypothesis $(\Gamma)$

Equivalent hypotheses:

**Theorem 5.**

a) Let $P_t^{(1)}$ be the extension of $P_t \big|_{L^1 \cap L^\infty}$ to $L^1(m)$. $(P_t^{(1)})_{t \geq 0}$ is a strongly continuous contraction semi-group in $L^1(m)$ with generator $(A^{(1)}, \mathcal{D}A^{(1)})$. It is the smallest closed extension of the restriction of the generator $A$ of $P_t$ to the set $\{f \in \mathcal{D}A \cap L^1 : Af \in L^1\}$

b) $(\Gamma) \iff (\Gamma') \iff (\Gamma'')$

$(\Gamma') \forall f \in \mathcal{D}A \ f^2 \in \mathcal{D}A^{(1)}$

$(\Gamma'')$ There is a sub-space $H$ of $\mathcal{D}A$, dense in $\mathbb{D}$ such that $\forall f \in H \ f^2 \in \mathcal{D}A^{(1)}$

c) Under $(\Gamma)$ it holds

$$\forall f, g \in \mathcal{D}A \quad \Gamma(f, g) = A^{(1)}(fg) - fA(g) - gA(f) \quad m - a.e.$$ 

About the relationship between hypothesis $(\Gamma)$ and the existence of a C.C.O. for Markov processes, we have:

**Theorem 6.** Suppose $\Omega$ be l.c.d., $\mathcal{F}$ its borelian $\sigma$-field. Let $(Q_t)$ be a Feller semi-group which is symmetric on $C_c(\Omega)$ with respect to a Radon positive measure $m$, and $(P_t)$ the symmetric associated semi-group on $L^2(m)$.

1) If $(Q_t)$ has a C.C.O. in the sense of Meyer, then the Dirichlet structure associated to $(P_t)$ satisfies $(\Gamma)$.

2) Conversely, if the Dirichlet structure associated to $(P_t)$ satisfies $(\Gamma)$ and if the sets of zero potential are $m$-negligible, then $(Q_t)$ has a C.C.O. in the sense of Meyer.
I.4. The locality hypothesis, the functional calculi
and the absolute continuity criterion for image measures

**Theorem 7.** $\mathbf{(L)} \iff (\mathbf{L}') \iff (\mathbf{L}'')$

(\mathbf{L}') $\mathcal{E}(|f + 1| - 1, |f + 1| - 1) = \mathcal{E}(f, f)$

(\mathbf{L}'') $\forall f, g \in \mathbb{D}, \forall a \in \mathbb{R} \quad (f + a)g = 0 \implies \mathcal{E}(f, g) = 0$

and under $(\mathbf{P})$ it is enough to take $a = 0$

**Theorem 8.** Suppose $(\mathbf{\Gamma})(\mathbf{L})$:

a) $\forall f \in \mathbb{D} \quad \mathcal{E}(f, f) = \frac{1}{2} \int \Gamma(f, f) \, dm$

b) $\forall f \in \mathbb{D}^m, g \in \mathbb{D}^n, \forall F, G \text{ Lipschitz } C^1\text{-maps from } \mathbb{R}^m[\mathbb{R}^n] \text{ into } \mathbb{R}$:

$$\Gamma(F(f) - F(0), G(g) - G(0)) = \sum_{i=1}^{m} \sum_{j=1}^{n} F_i'(f)G_j'(g)\Gamma(f_i, g_j) \quad m - \text{a.e.}$$

There is a stronger result in one dimension: the Lipschitz functional calculus:

**Theorem 9.** Suppose $(\mathbf{\Gamma})(\mathbf{L})$

a) $\forall f \in \mathbb{D} \quad f_*(\Gamma(f, f).m) \ll \lambda_1$ ($\lambda_1$ = Lebesgue measure on $\mathbb{R}$)

b) Let be $f, g \in \mathbb{D}$ and $F, G$ Lipschitz map from $\mathbb{R}$ to $\mathbb{R}$ and let $F', G'$ be versions of their derivatives:

$$\Gamma(F(f) - F(0), G(g) - G(0)) = F'(f)G'(g)\Gamma(f, g) \quad m - \text{a.e.}$$

There is also a criterion of absolute continuity of image laws in the multivariate case:

**Theorem 10.** Suppose $(\mathbf{\Gamma})(\mathbf{L})$, if $f \in \mathbb{D}^n$ and if $\forall 1 \leq i, j \leq n$

$$\Gamma(f_i, f_j) \in \mathbb{D} \text{ then } f_*[\det \Gamma(f, f^*).m] \ll \lambda_n$$ ($\lambda_n$ = Lebesgue measure on $\mathbb{R}^n$)

**Theorem 11.** Suppose $(\mathbf{W})$

if $f \in \mathbb{D}^n$

$$f_*[\det \Gamma(f, f^*).m] \ll \lambda_n$$

This result can be extended to $\mathbb{D}_{loc}$ with a suitable definition.
II. Image structures

II.A. Finite dimensional images.

II.A.1. Definition and basic properties.

**Proposition 12.** Let \( S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D}) \) be a Dirichlet structure satisfying (P), and \( 1 \in \mathbb{D} \).

For \( U \in \mathbb{D}^d \), let us define

\[
\widetilde{\mathcal{D}}_U = \{ f \in L^2(U, m) : f \circ U \in \mathbb{D} \}
\]

\[
\mathcal{E}_U(f, f) = \mathcal{E}(f \circ U, f \circ U)
\]

then \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), U, m, \mathcal{E}_U, \widetilde{\mathcal{D}}_U)\) is a Dirichlet structure and the set \( \mathcal{L}_d \) of Lipschitz functions from \( \mathbb{R}^d \) into \( \mathbb{R} \) is in \( \widetilde{\mathcal{D}}_U \).

Let \( \mathcal{D}_U \) be the closure of \( \mathcal{L}_d \) in \( \widetilde{\mathcal{D}}_U \) and \( \mathcal{E}_U = \mathcal{E}_U \big|_{\mathcal{D}_U \times \mathcal{D}_U} \)
then \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), U, m, \mathcal{E}_U, \mathcal{D}_U)\) is a regular Dirichlet structure (satisfying again (P), and \( 1 \in \mathcal{D}_U \)).

**Definition 13.** The structure \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), U, m, \mathcal{E}_U, \mathcal{D}_U)\) will be called the image structure of \( S \) and will be denoted by \( U_S \).

**Notations.** For \( \phi \in L^1(m) \) we set

\[
\mathcal{E}_m[\phi|U = x] := \frac{dU_S(\phi, m)}{dU_S m}(x) \quad U_S m - \text{a.e.}
\]

then we have

**Proposition 14.**

1) If \( S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D}) \) possesses a C.C.O. \( \Gamma \), the same holds for \( U_S \) and \( U_S \) and their C.C.O. is given by

\[
\Gamma_U(f, f)(x) = \mathcal{E}_m[\Gamma(f \circ U, f \circ U)|U = x] \quad \forall f \in \mathcal{D}_U
\]

2) If \( S \) is local, so are \( U_S \) and \( U_S \) and if \( S \) satisfies both (L) and (F), \( \forall f \in \mathcal{L}_1 \cap C^1(\mathbb{R}^d) \) it holds

\[
\Gamma_U(f, f)(x) = \sum_{i,j} \mathcal{E}_m[\Gamma(U_i, U_j)|U = x] \frac{\partial f}{\partial x_i}(x) \frac{\partial f}{\partial x_j}(x)
\]
Remark. There are explicit examples in which

$$\bar{U}_* S \neq U_* S.$$
II.A.2 The Energy Image Density Property.

**Definition 15.** A Dirichlet structure \( S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathcal{D}) \) satisfying \((P), 1 \in \mathcal{D}, (\Gamma), (L)\) is said to satisfy the \((EID)\) property if \( \forall n \in \mathbb{N}^*, \forall F \in \mathcal{D}^n, \)

\[
F_n (\det[\Gamma(F, F^*)].m) << \lambda_n.
\]

A natural question is whether the \((EID)\) property is preserved by image.

**Proposition 16.** Let \( S \) satisfying \((P), 1 \in \mathcal{D}, (\Gamma), (L)\) and \((EID)\). Let \( U \in \mathbb{D}^d \) such that one of the following hypotheses holds:

a) the matrix \( \Gamma(U, U^*) \) is \( \sigma(U) \)-measurable

b) \( \det[\Gamma(U, U^*)] > 0 \) \( m \)-a.e.

then the image structure \( U_* S \) satisfies \((P), 1 \in \mathcal{D}, (\Gamma), (L)\) and \((EID)\).

With hypothesis a) the proof comes straightforward from the definitions. With hypothesis b) the result is a consequence of the following two lemmas:

**Lemma 17.** Let \( M \) be a random matrix which is symmetric and non-negative definite, then

\[
\{ \det[\mathbb{E}(M|\mathcal{F})] = 0 \} \subset \{ \det[M] = 0 \}
\]

**Lemma 18.** If \( \det[\Gamma(U, U^*)] > 0 \), for all \( \phi \in (\mathbb{D}_U)^n \) there exists an \( n \times d \)-matrix \( J \) which is \( \sigma(U) \)-measurable (up to \( m \)-negligible sets) such that

\[
\Gamma(\phi \circ U, \phi \circ U^*) = J\Gamma(U, U^*)J^* \quad m \text{ a.e.}
\]
II.A.2 The Image Generator.

There is in general no simple relationship between the initial semi-group and the semi-group of the image structure. Not better for the associated Markov process.

Nevertheless, the generator \((A_U, DA_U)\) of the image structure can be put in relation with the generator \((A, DA)\) of the initial structure:

If \(f, g \in \mathbb{D}_U\), and \(f \circ U \in DA\), we have

\[
\mathcal{E}_U(f, g) = \mathcal{E}(f \circ U, g \circ U) = -(A(f \circ U), g \circ U)_{L^2(m)}
\]

\[
= -\int \mathbb{E}_m[A(f \circ U) | U = x] g(x) \, dU_* m(x)
\]

hence \(f \in DA_U\) and \(A_U f = \mathbb{E}_m[A(f \circ U) | U = x]\).

But, hypotheses are needed to ensure the space \(\mathbb{D}_U \cap \{ f : f \circ U \in DA \}\) contains other functions than constants:

**Proposition 19.** Suppose \(S\) satisfies \((\mathbf{P})\), \(1 \in \mathbb{D}\), \((\Gamma)\), \((\mathbf{L})\). Let \(U \in (DA)^d\) such that \(\Gamma(U_i, U_j) \in L^2(m) \ \forall i, j = 1, \ldots, d\) then

\[
DA_U \supset \{ f \in C^2(\mathbb{R}^d) : \frac{\partial f}{\partial x_i}, \frac{\partial^2 f}{\partial x_i \partial x_j} \text{- bounded} \}
\]

and for such an \(f\)

\[
A_U f(x) = \mathbb{E}_m[A(f \circ U) | U = x]
\]

\[
= \frac{1}{2} \sum_{i,j} \alpha_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_i \beta_i \frac{\partial f}{\partial x_i}
\]

with \(\alpha_{ij}(x) = \mathbb{E}[\Gamma(U_i, U_j) | U = x] \ (\in L^2(U, m))\)

and \(\beta_i(x) = \mathbb{E}[AU_i | U = x] \ (\in L^2(U, m))\)

If further, \(\det[\Gamma(U, U^*)] > 0 \ \text{m-a.e.}\) then the function \(k = \frac{dU_* m}{d\lambda_n}\) satisfies

\[
2\beta_i k - \sum_j \frac{\partial}{\partial x_j} (\alpha_{ij} k) = 0 \quad \forall i = 1, \ldots, n
\]

in the sense of distributions.
H. Airault and P. Malliavin have studied the case of Wiener space with
\[ U \in W_\infty = \cap_{p,n} D_{p,n} \]
and
\[ [\det \Gamma(U, U^*)]^{1/2} \in W_\infty \]
and they show in this case that
\[ A_U = \Delta + \nabla \tilde{u} \]
where \( \Delta \) is the Laplace-Beltrami operator associated to the Riemannian metric with matrix \( [(\alpha_{ij})]^{-1} \) and where \( \tilde{u} = \frac{1}{2} \nabla \log \rho \) with \( \rho = \frac{dU_* m}{dv} \) and where \( dv = \sqrt{\det(\alpha_{ij})} \). \( \lambda_n \) is the associated area measure.
II.B. General Images.

II.B.1. $U$ does not need to be supposed in $\mathbb{D}$ or $\mathbb{D}^d$ for defining an image structure, whenever there is enough functions $f \circ U$ in $\mathbb{D}$.

Let $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ satisfy (P) and $1 \in \mathbb{D}$, let $(X, \mathcal{G})$ be a measurable space and let $U$ be a measurable map from $(\Omega, \mathcal{F})$ into $(X, \mathcal{G})$. Let us suppose that there exists a set $\mathcal{A}$ of measurable applications from $X$ into $\mathbb{R}$ such that

. $\mathcal{A}$ is a vector space containing the constants
. $\forall f \in \mathcal{A}, \ f \circ U \in \mathbb{D}$
. $\mathcal{A}$ is dense in $L^2(U,m)$

then the form $(\mathcal{E}_A, \mathcal{A})$ defined by $\mathcal{E}_A(f, f) = \mathcal{E}(f \circ U, f \circ U)$ is closable in $L^2(U,m)$, let $(\mathcal{E}_U, \mathbb{D}_U)$ its closure, we put

$$U_s S = (X, \mathcal{G}, U_s m, \mathcal{E}_U, \mathbb{D}_U).$$

II.B.2. Example.

Let $S = (\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N), \nu^\otimes N, \mathbb{D}, \mathcal{E})$ be a Dirichlet structure such that

. $\nu = N(0, 1)$ is the standard Gaussian measure on $\mathbb{R}$
. $(\mathbb{D}, \mathcal{E})$ is any Dirichlet form on $L^2(\nu^\otimes N)$ such that the coordinates $\chi_n$ belong to $\mathbb{D}$.

Let be $X = C[0, 1]$ and $\mathcal{G}$ be its borelian $\sigma$-field. Let $(\hat{h}_n)$ be a C.O.N.S. of $L^2([0, 1])$ and put

$$h_n(t) = \int_0^t \hat{h}_n(s) \, ds$$

Let us consider the map $U$ from $\mathbb{R}^N$ into $X$ defined by

$$(\star) \quad U(x) = \sum_{n=0}^{\infty} \chi_n(x) h_n$$

if the serie converges in $C[0, 1]$,

$$U(x) = 0$$

elsewhere.

A vector valued martingale argument shows that

**Lemma 20.** The serie $(\star)$ converges almost surely and in $L^p_C([0, 1], \nu^\otimes N)$ ($1 < p < \infty$), and the law of its sum is the Wiener measure $\mu : \mu = U_s(\nu^\otimes N)$. 
Let us denote by \((B_t)\) the brownian motion defined by this Wiener measure on \(\mathcal{C}[0,1]\), and let be

\[
\tilde{h}_n = \int_0^1 \dot{h}_n(s) \, dB_s
\]

then it can be shown that

\[
\tilde{h}_n \circ U(x) = \chi_n(x) \quad \text{for } \nu^{\otimes \mathbb{N}} - \text{a.e. } x.
\]

Hence by hypothesis \(\tilde{h}_n \circ U \in \mathcal{D}\), therefore if \(f = F(\tilde{h}_1, \ldots, \tilde{h}_n)\) with \(F\) Lipschitz, we have \(f \circ U \in \mathcal{D}\). But \(F(\chi_1, \ldots, \chi_n)\) is dense in \(L^2(\nu^{\otimes \mathbb{N}})\) hence \(F(\tilde{h}_1, \ldots, \tilde{h}_n)\) is dense in \(L^2(\mu)\).

So, the image structure

\[
(X, \mathcal{G}, \mu, \mathcal{E}_U, \mathcal{D}_U)
\]

is well defined and contains \(\{F(\tilde{h}_1, \ldots, \tilde{h}_n)\}\) for Lipschitz \(F\).
III. Tensor products and projective limits.

III.A. Finite products. Let $S_1 = (\Omega_1, \mathcal{F}_1, m_1, \mathcal{E}_1, \mathbb{D}_1)$ and $S_2 = (\Omega_2, \mathcal{F}_2, m_2, \mathcal{E}_2, \mathbb{D}_2)$ be Dirichlet structures.

**Definition 21.**

\[ S_1 \otimes S_2 = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, m_1 \times m_2, \mathcal{E}, \mathbb{D}) \]

with

\[ \mathbb{D} = \{ f \in L^2(m_1 \times m_2) : \text{ for } m_2-\text{a.e. } y \quad f(., y) \in \mathbb{D}_1 \]

\[ \text{for } m_1-\text{a.e. } x \quad f(x, .) \in \mathbb{D}_2 \]

and \[ \int \mathcal{E}_1(f(., y), f(., y)) \, dm_1(y) + \int \mathcal{E}_2(f(x, .), f(x, .)) \, dm_2(x) < \infty \]

and \( \forall f \in \mathbb{D} \),

\[ \mathcal{E}(f, f) = \int \mathcal{E}_1(f(., y), f(., y)) \, dm_1(y) + \int \mathcal{E}_2(f(x, .), f(x, .)) \, dm_2(x). \]

It is indeed easy to see that this form is closed and that contractions operate.

If $S_1$ and $S_2$ satisfy (P) and (M) the same holds for $S_1 \otimes S_2$.

If $S_1$ and $S_2$ are local, $S_1 \otimes S_2$ is local.

If $S_1$ and $S_2$ satisfy (T), the same holds for $S_1 \otimes S_2$ and its OCC is given by

\[ \Gamma(f, f) = \Gamma_1(f(., y), f(., y))(x) + \Gamma_2(f(x, .), f(x, .))(y) \]

Concerning the associated semi-group we have the following:

Let $(P^1_t), (P^2_t)$ be the semi-groups associated with $S_1$ and $S_2$, and let \( \overline{P}^1_t \) and \( \overline{P}^2_t \) be the semi-groups on $L^2(m_1 \times m_2)$ defined by

\[ \overline{P}^1_t f(x, y) = P^1_t(f(., y))(x) \]

\[ \overline{P}^2_t f(x, y) = P^2_t(f(., .))(y) \]

which are symmetric, strongly continuous and sub-Markov.
Proposition 22. a) The semi-group associated with $S_1 \otimes S_2$ is

\[ P_t = \hat{P}_t^1 \hat{P}_t^2 = \hat{P}_t^2 \hat{P}_t^1 \]

b) its generator is the smallest closed extension of the operator defined on $DA_1 \otimes DA_2$ by $A(\phi \otimes \psi) = A_1 \phi \otimes \psi + \phi \otimes A_2 \psi$

c) $\mathcal{D}_1 \otimes \mathcal{D}_2$ is dense in $\mathcal{D}$. 
III.B. Infinite tensor products.

The preceding construction extends without any problem to the infinite tensor products (countable or not):

$$\bigotimes_{i \in I} (E_i, \mathcal{F}_i, \mu_i, \mathcal{E}_i, \mathbb{D}_i)$$

where the factors are supposed to satisfy (P).

That comes mainly from the fact that the limit of an increasing sequence of Dirichlet forms is a Dirichlet form:

**Lemma 23.** Let $(\Omega, \mathcal{F}, m)$ be a measured space equipped with Dirichlet forms $\mathcal{E}^{(n)}$, $\mathbb{D}^{(n)}$ such that

1. $\mathbb{D}^{(n)} \downarrow$ as $n \uparrow$
2. $\mathcal{E}^{(n)} \uparrow$ as $n \uparrow$: $\forall f \in \mathbb{D}^{(n)}$, $\mathcal{E}^{(n+1)}(f, f) \geq \mathcal{E}^{(n)}(f, f)$

then $\mathbb{D} = \cap \mathbb{D}^{(n)}$, $\mathcal{E}(f, f) = \lim \mathcal{E}^{(n)}(f, f)$ is a Dirichlet form.

If the $S_i$'s are local, so is $\bigotimes_i S_i$.

if each $S_i$ possesses a CCO, the same holds for $\bigotimes_i S_i$.

Suppose now that the family $S_i$ is countable and that each finite product

$$\bigotimes_{i=0}^{n} S_i$$

satisfies the (EID) property, then $S = \bigotimes_{i=0}^{\infty} S_i$ satisfies (EID).

That comes directly from the definitions.

As an example let us take

$$S_i = (\mathbb{R}, \mathcal{B}(\mathbb{R}), h_i(x)dx, \int \nabla^2 h_i(x)dx, \mathbb{D}_i)$$

where $h_i$ satisfies the Hamza condition and $\int h_i(x)dx = 1$.

Then by the coarea formula of Federer, the finite products satisfy (EID) and therefore the infinite product structure (which is in general non Gaussian) satisfies (EID).

**Remark.** In this example, putting

$$\mu^i = \bigotimes_{j \in \mathbb{N}, j \neq i} (h_idx)$$
and
\[ \mathcal{E}_i(f, f) = \int_{\mathbb{R}} (\nabla f)^2 h_i \, dx \]
we have
\[ \mathbb{D} = \{ f \in L^2(m) : \forall i \in \mathbb{N} \quad f(., y) \in \mathbb{D}_i \text{ for } \mu^i \text{ - a.e. } y \]
and
\[ \sum_{i=0}^{\infty} \int \mathcal{E}_i(f(., y), f(., y)) \, d\mu^i(y) < \infty \]
and for \( f \in \mathbb{D} \)
\[ \Gamma(f, f) = \sum_{i=0}^{\infty} \Gamma_i(f, f) = \sum_{i=0}^{\infty} f_i^2 \]
where \( \Gamma_i \) acts only on the \( i \)-th variable.

Therefore if for \( f \in \mathbb{D} \) we put
\[ Df = (f_i^i)_{i \in \mathbb{N}} \]
this defines a continuous operator from \( \mathbb{D} \) into \( L^2(m, \ell^2) \) and we have
\[ \Gamma(f, f) = \langle Df, Df \rangle_{\ell^2} \]

We shall see later that this allows to develop a conditional Dirichlet calculus. These product structures are examples of Classical Dirichlet forms in the sense of Albeverio-Röckner.
III.C. Projective limits.

1. General projective system of Dirichlet structures can be defined in an obvious way.

But there is a difficulty for passing to the limit unless some uniform closability property is known (which is the case for products). Here is an example of projective system without limit:

Example

Let $\mu$ be the Gauss measure on $\mathbb{R}$. Let us consider the structures

$$S^{(n)} = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu^\otimes n, \mathcal{E}^{(n)}, \mathbb{D}^n)$$

with

$$\mathbb{D}^{(n)} = \{ f \in L^2(\mu^\otimes n) : \frac{\partial f}{\partial x_i} \in L^{1}_{loc}(\mathbb{R}^n) \text{ and } \int (\sum_{i=0}^{n} \frac{1}{a_i} \frac{\partial f}{\partial x_i})^2 \, d\mu^\otimes n < \infty \}$$

and

$$\mathcal{E}^{(n)}(f, f) = \int (\sum_{i=0}^{n} \frac{1}{a_i} \frac{\partial f}{\partial x_i})^2 \, d\mu^\otimes n$$

where the numbers $a_i$ are chosen such that

$$a_i > 0, \quad \lim_{i \to \infty} a_i = 0$$

The $S^{(n)}$’s define a compatible system of Dirichlet structures, but if $h_n(x) = a_n x_n$ we have $\|h_n\|_{L^2(\mu^\otimes \mathbb{N})} \to 0$ and the candidate $\tilde{\mathcal{E}}$ satisfies

$$\tilde{\mathcal{E}}(h_n - h_m, h_n - h_m) = \int (\frac{1}{a_n} - \frac{1}{a_m})^2 \, d\mu^\otimes \mathbb{N} = 0$$

and $\tilde{\mathcal{E}}(h_n, h_n) = 1$ therefore $\tilde{\mathcal{E}}$ is not closable.
2. An important special case where the limit exists
Consider a Dirichlet structure \( S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{I}) \) and a family \((U_n)\) of applications such that
\[
U_n : \Omega \rightarrow \mathbb{R}^d \quad \text{and} \quad U_n \in \mathbb{I}^{d_n} \quad \forall n
\]
then the image structure of \( S \) by \((U_0, \ldots, U_n)\) defines a Dirichlet structure \( S^{(n)} \) with state space
\[
\prod_{i=0}^{n} \mathbb{R}^{d_i}
\]
These structures \( S^{(n)} \) define a projective system which always possesses a limit. This comes easily from the fact that the initial form \((\mathcal{E}, \mathbb{I})\) is closed.

The limit can be called the image structure by the process \((U_n)_{n \in \mathbb{N}}\).

The same would be true, mutatis mutandis, for uncountable families.
IV. D-independence.

IV.A. Definition and examples.

Let $S = (\Omega, F, m, \mathcal{E}, D)$ be a Dirichlet structure satisfying $(P)$ and $1 \in D$.

If $U \in D^p$ the image structure $U_\ast S$ will be called the D-law of $U$.

**Definition 24.** If $U \in D^p$, $V \in D^q$, $U$ and $V$ will be said to be D-independent if the D-law of $(U, V)$ is the product of the D-laws of $U$ and $V$.

**Proposition 25.** A necessary and sufficient condition for independent $U$ and $V$ to be D-independent is

\[
\forall f_1, f_2 \in C_\nu^1(\mathbb{R}^p), \forall g_1, g_2 \in C_\nu^1(\mathbb{R}^q)
\]

\[
\mathcal{E}(f_1 \circ U, g_1 \circ V, f_2 \circ U, g_2 \circ V)
\]

\[
= \mathcal{E}(f_1 \circ U, f_2 \circ U, g_1 \circ V, g_2 \circ V)_{L^2(m)} + \mathcal{E}(g_1 \circ V, g_2 \circ V)(f_1 \circ U, f_2 \circ U)_{L^2(m)}
\]

If $E$ is local and possesses a CCO we have the more explicit result

**Proposition 26.** If $S$ satisfies $(P)$, $1 \in D, (L), (\Gamma)$, for $U \in D^p$ and $V \in D^q$ to be D-independent it is necessary and sufficient that

1) $U$ and $V$ are independent,

2) $\forall i, k \quad \mathbb{E}[\Gamma(U_i, V_k) | U, V] = 0$ m-a.e.

3) $\forall i, j \quad \mathbb{E}[\Gamma(U_i, U_j) | U, V] = \mathbb{E}[\Gamma(U_i, U_j) | U]$ m-a.e.

4) $\forall l, k \quad \mathbb{E}[\Gamma(V_l, V_i) | U, V] = \mathbb{E}[\Gamma(V_l, V_i) | V]$ m-a.e.

**Remark.** These conditions are fulfilled as soon as

- $\Gamma(U_i, V_k) = 0$ for all $i, k$
- $(U, \Gamma(U_i, U_j))$ is independent of $V$ for all $i, j$
- $(V, \Gamma(V_k, V_l))$ is independent of $V$ for all $k, l$.

**Examples.**

1) If $U$ and $V$ are random variables in the first chaos on the Wiener space, they are D-independent as soon as they are independent i.e. orthogonal.
2) Let $f \in L^2_{sym}(\mathbb{R}^p_+)$ and $g \in L^2_{sym}(\mathbb{R}^q_+)$. By a result of Ustunel and Zakaï if the multiple Wiener integrals $I_p(f)$ and $I_q(g)$ are independent so are the $\sigma$-fields

$$\sigma(I_p(f), <DI_p(f), h_{1,1}>, \ldots, <D^{p-1}I_p(f), h_{p-1,1} \otimes \ldots \otimes h_{p-1,p-1}>)$$

and

$$\sigma(I_q(g), <DI_q(g), k_{1,1}>, \ldots, <D^{q-1}I_q(g), k_{q-1,1} \otimes \ldots \otimes k_{q-1,q-1}>)$$

Therefore $I_p(f)$ and $I_q(g)$ are D-independent as soon as they are independent and $\Gamma(I_p(f), I_q(g)) = 0$.

3) This extends to the case of multivariate random variables whose components are multiple Wiener integrals.
IV.B. Convergence in D-law.

Let as before $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ be a Dirichlet structure satisfying (P) and $1 \in \mathbb{D}$.

**Definition 27.** Let $(U_n)$ be a sequence in $\mathbb{D}^d$ and $V \in \mathbb{D}^d$. The sequence $(U_n)$ is said to converge in D-law to $V$

$$U_n \overset{D-l}{\to} V$$

if

1. $U_{n \to m}$ converges to the law of $V$ in the narrow sense
2. $\forall f \in \mathcal{L} \cap C^1(\mathbb{R}^d)$, $\mathcal{E}(f \circ U_n, f \circ U_n) \longrightarrow \mathcal{E}(f \circ V, f \circ V)$

in other words that means convergence of the D-laws on bounded continuous functions for the measures, on $C^1$-Lipschitz functions for the forms.

The central limit theorem becomes the following:

**Theorem 28.** Let us suppose $S$ satisfies (P), $1 \in \mathbb{D}$, (L), (P). Let $(U_n)$ be a sequence of functions in $\mathbb{D}^d$ which are centered, with the same D-law, and D-independent, then

$$V_n = \frac{1}{\sqrt{n}}(U_1 + \ldots + U_n)$$

converges in D-law and the limit Dirichlet structure is

$$(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \nu, \hat{\mathcal{E}}, \hat{\mathbb{D}})$$

with

$$\nu = N_d(0, \Sigma)$$

$$\forall f \in \mathcal{L} \cap C^1(\mathbb{R}^d), \hat{\mathcal{E}}(f, f) = \sum_{i,j} \int \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} a_{ij} \, d\nu$$

where $\Sigma_{ij} = \int x_i x_j \, d\mu$ ($\mu$ being the common law of the $U_n$’s)

and $a_{ij} = \mathcal{E}(U_n,i, U_n,j) = \mathcal{E}_{U_n}(x_i, x_j)$ (which does not depend on $n$).

The main step of the proof is the following lemma
**Lemma 29.** Let $U_1, \ldots, U_n$ be in $\mathbb{D}^d$ and $D$-independent then $\forall f \in \mathcal{L} \cap C^1(\mathbb{R}^d)$ 

$$
\mathcal{E}(f(U_1 + \ldots + U_n), f(U_1 + \ldots + U_n))
$$

$$
= \frac{1}{2} \sum_{ij} \int \frac{\partial f}{\partial x_i}(y_1+\ldots+y_n) \frac{\partial f}{\partial x_j}(y_1+\ldots+y_n) \left( \sum_{\ell=1}^n a_{ij}^\ell(y_\ell) \right) d\mu_1(y_1) \ldots d\mu_n(y_n)
$$

where $\mu_n = U_n \ast m$ is the law of $U_n$ and

$$
a_{ij}^\ell(y_\ell) = \mathbb{E}[\Gamma(U_{\ell,i}, U_{\ell,j})|U_\ell = y_\ell] \quad (= \Gamma_{U_\ell}(x_i, x_j)(y_\ell)).
$$
Let \( (Z^{(n)})_{n \in \mathbb{N}} \) be a sequence of discrete time processes
\[
Z^{(n)} = (Z_1^{(n)}, \ldots, Z_k^{(n)}, \ldots)
\]
defined on a Dirichlet structure \( S \), we shall say that \( (Z^{(n)}) \) converges in D-law to the process \( Z \), if the marginal D-laws of \( Z^{(n)} \) converge to those of \( Z \).

Example. Let us take \( S = (\mathbb{IR}, \mathcal{B}(\mathbb{IR}), \mu, f \sqrt{\nabla^2 d\mu}, H^1(\mathbb{IR}, \mu)) \) with \( \mu = \mathcal{N}(0, 1) \) and let us consider the “standard Gaussian product space”
\[
S^{\otimes \mathbb{N}} = (\mathbb{IR}^\mathbb{N}, \mathcal{B}(\mathbb{IR}^\mathbb{N}), \mu^{\otimes \mathbb{N}}, \mathcal{E}, \mathbb{ID}).
\]

Let \( X_i \) be the coordinates and let us put
\[
Y_k^{(n)} = \frac{X_k \sqrt{n}}{\sqrt{X_1^2 + \ldots + X_n^2}}
\]
then the process
\[
Z^{(n)} = (Y_1^{(n)}, \ldots, Y_n^{(n)}, 0, 0, \ldots)
\]
converges in D-law toward
\[
X = (X_1, \ldots, X_n, \ldots)
\]

That is an extension of the Gateaux-Lévy theorem which states the same result with only probability structures.
V. Dirichlet sub-spaces and conditioning.

V.A. Dirichlet sub-spaces.

**Definition 30.** Let \( S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D}) \) be a Dirichlet structure satisfying (P) and \( 1 \in \mathbb{D} \), a sub-vector space \( \mathbb{D} \) of \( \mathbb{D} \) will be called a Dirichlet sub-space if it is closed in \( \mathbb{D} \) and stable under composition by Lipschitz functions on \( \mathbb{R} \).

**Proposition 31.** If \( \mathbb{D} \) is a Dirichlet sub-space, it holds

\[
\overline{\mathbb{D}}^{L^2(m)} = L^2(m, \sigma(\mathbb{D}))
\]

and so \( S_{\mathbb{D}} = (\Omega, \sigma(\mathbb{D}), m, \mathcal{E}|_{\mathbb{D} \times \mathbb{D}}, \mathbb{D}) \) is a Dirichlet structure.

In particular, \( \mathbb{D} \) is stable by composition by Lipschitz functions of several variables.

For example if \( X_i \in \mathbb{D}, \ \forall i \in I \), the space

\[
\mathbb{D}(X_i, i \in I) = \{G(X_i, \ldots, X_{i_k}) \ i_k \in I, \ G \in C^1(\mathbb{R}^n)\}^{\mathbb{D}}
\]

is a Dirichlet sub-space which will be called the Dirichlet sub-space generated by the family \( (X_i)_{i \in I} \).

**Remark.** If \( S \) satisfies (T), \( S_{\mathbb{D}} \) satisfies (T) and its CCO is given by

\[
\Gamma_{\mathbb{D}}(v, v) = \mathbb{E}[\Gamma(v, v)|\sigma(\mathbb{D})] \ \forall v \in \mathbb{D}
\]

**Example.** It is easily seen that the kernel of the form

\[
K = \{f \in \mathbb{D} : \mathcal{E}(f, f) = 0\}
\]

is a Dirichlet sub-space.
V.B. Conditional calculus.
We consider a $D$-structure $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ satisfying (P), (M), (L), (Γ).

Hypothesis (G). We shall say that $S$ admits a gradient if there exists a separable Hilbert space and a linear map $D$ from $\mathbb{D}$ into $L^2(m, H)$ such that

$$< Du, Du >_H = \Gamma(u, u) \quad \forall u \in \mathbb{D}.$$  

This is the case for the Wiener space, for some product spaces and some Classical Dirichlet space in the sense of Albeverio-Röckner.

From the functional calculus for $\Gamma$ we deduce

**Proposition 32**. $D$ is continuous and satisfies

a) $D(f \circ U) = f' \circ U.DU$, $f \in \mathcal{L}(\mathbb{R})$, $U \in \mathbb{D}$

b) $D(F \circ \tilde{U}) = \sum_i F_i \circ \tilde{U}.DU_i$, $F \in \mathcal{L} \cap \mathcal{C}^1(\mathbb{R}^d)$, $\tilde{U} \in \mathbb{D}^d$

Most of the features of the conditional calculus of Nualart-Zakaï extends to this situation:

Let $(X_i, i \in \mathbb{N})$ be a countable family in $\mathbb{D}$ nad let $\mathcal{H}$ be the following measurable field of sub-spaces of $H$

$$\mathcal{H} = (\mathcal{L}(DX_i, i \in \mathbb{N}))^\perp$$

For $F \in \mathbb{D}$, let us define

$$D^X(F) = \mathcal{P}^\mathcal{H}(DF)$$

$$\Gamma^X(F, F) = < \mathcal{P}^\mathcal{H}(DF), \mathcal{P}^\mathcal{H}(DF) >_H$$

$$\mathcal{E}^X(F, F) = \mathbb{E}[\Gamma^X(F, F)]$$

**Proposition 33**. a) $(D^X, \mathbb{D})$ is a closable operator in $L^2(m, H)$ iff the form $(\mathcal{E}^X, \mathbb{D})$ is closable.

b) This is the case if $P^\mathcal{H}h \in \mathbb{D}$ for all $h \in H$.

c) In this case the $D$-structure associated with $\mathcal{E} : (\Omega, \mathcal{F}, m, \mathcal{E}^X, \mathbb{D}^X)$ satisfies (P), (M), (L), (Γ) and (G) with gradient operator $D^X$ and will be called the conditional structure knowing $X$.

The main result in this theory is then:
Theorem 34. Suppose the conditional structure knowing $X = (X_i, i \in \mathbb{N})$ exists. Let $\tau_X = \sigma(X_i, i \in \mathbb{N})$.

a) Let $F : \mathbb{R} \times \Omega \to \mathbb{R}$, $\mathcal{B}(\mathbb{R}) \otimes \tau_X$-measurable, s.t. $F(x, \omega)$ is Lipschitz in $x$, bounded as well as $F_x'$ then for all $U \in \mathcal{D}$ (even for $U \in \mathcal{D}_X^X$)

$$(\omega \to F(U(\omega), \omega)) \in \mathcal{D}_X^X$$

and

$$\Gamma^X(F(U(.,.), F(U(.,.)))(\omega) = F'^2_x(U(\omega), \omega)\Gamma^X(U, U)(\omega) \text{ m-a.e.}$$

b) For all $U \in \mathcal{D}$ (even for $U \in \mathcal{D}_X^X$), the image of the measure $\Gamma^X(U, U).m$ by the map $\omega \to (U(\omega), \omega)$ from $(\Omega, \mathcal{F})$ into $(\mathbb{R}\times\Omega, \mathcal{B}(\mathbb{R}) \otimes \tau_X)$ is absolutely continuous w.r. to $dx \times m$.

In particular if $\Gamma^X(U, U) > 0$ a.e., $U$ possesses a conditional law knowing the $\sigma$-field $\tau_X = \sigma(X_i, i \in \mathbb{N})$.

There are two limitations for applying this theory in practice

1) The verification of the closability condition.

2) Most examples, especially from filtration problems, do allow a direct treatment because the conditional law is absolutely continuous with respect to the Wiener measure: The ordinary criterion applies.

Example. Let $(\eta_t(w))_{t \in \mathbb{R}_+}$ and $(\xi_t(w))_{t \in \mathbb{R}_+}$ be two processes defined on a probability space $(\mathcal{W}, \mathcal{A}, \mathbb{P})$.

If the law of $\xi$ knowing $\eta$ (i.e. knowing $\tau = \sigma(\eta_s, s \in \mathbb{R}_+)$) is absolutely continuous w.r. to the Wiener measure, then a sufficient condition for a random variable $F : \Omega \to \mathbb{R}^d$ of the form

$$F = f(\eta, \xi)$$

to possess a conditional density knowing $\tau$ is that for $\mathbb{P}$-a.e. $w$, setting $F_w(\omega) = f(\eta(w), \omega)$,

1) $F_w \in \mathcal{D}$ (= $D_{2,1}$ here)

2) $\det \Gamma(F_w, F_w^*)(\omega) > 0 \quad dm(\omega)$-a.e.
V.C. A glance to stationary processes.

Let $S = (\Omega, \mathcal{F}, m, \mathcal{E}, \mathbb{D})$ be a D-structure satisfying \((P)\), \((M)\), \((L)\), \((\Gamma)\).

A map $X : t \to X_t$ from $\mathbb{R}$ into $\mathbb{D}$ will be called a D-stationary process if its marginal D-laws are invariant under translations of time.

Let $F \in \mathcal{L} \cap \mathcal{C} [\mathbb{R}^n]$, the relation

$$
\mathcal{T}_t[F(X_{t_1}, \ldots, X_{t_n})] = F(X_{t_1+t}, \ldots, X_{t_n+t})
$$

defines a group of isometries which extends to the Dirichlet sub-space generated by $X$ : \(\mathbb{D}_X = D(X_t, t \in \mathbb{R})\).

It is easy to see that $\mathcal{T}_t$ is strongly continuous on $\mathbb{D}_X$ if and only if $t \to X_t$ is continuous from $\mathbb{R}$ into $\mathbb{D}$.

If this is satisfied, we get a spectral representation : $\mathcal{T}_t = e^{itA}$, \(A\) self-adjoint on $\mathbb{D}_X$ and if $E(d\lambda)$ is the resolution of the identity associated with $A$:

$$
X_t = \int e^{i\lambda t} E(d\lambda) X_0 \quad \text{in } \mathbb{D}_X
$$

Let $\Gamma_X$ be the CCO of the sub-structure $(\Omega, \sigma(X), m, \mathcal{E}_X, \mathbb{D}_X)$ which is given by

$$
\Gamma_X(U, V) = \mathbb{E} [\Gamma(U, V) | \sigma(X)].
$$

Let us suppose moreover that $\Gamma(X_s, X_t)$ be deterministic $\forall s, t$.

(This happens often without any Gaussian hypothesis : for example for product spaces

$$
\bigotimes_{n=0}^{\infty} (\mathbb{R}, \mathcal{B}(\mathbb{R}), h_n dx, \int \nabla^2 h_n dx, \mathbb{D}_n)
$$

the $h_n$’s satisfying the Hamza condition and $\int x^2 h_n(x) dx < \infty$ if for all \(t\), $X_t$ belongs to the closure in $\mathbb{D}$ of linear combinations of coordinates.)

Then $\Gamma(X_s, X_t) = \Gamma(X_0, X_{t-s})$ hence by Bochner theorem

$$
\Gamma(X_{t+h}, X_t) = \gamma(h) = \int e^{i\lambda h} d\mu(\lambda)
$$

for a finite positive measure $\mu$ since $\gamma$ is continuous.
It follows that
. $X_t$ has a density as soon as $\mu \neq 0$
. $(X_{t_1}, \ldots, X_{t_n})$ has a density if the functions
$$e^{it_1}, \ldots, e^{it_n}$$
are linearly independent in $L^2(\mathbb{R}, \mu)$.

Let $\nu$ be the usual spectral measure of $X$, from the two spectral representations it follows that the space $\overline{\mathcal{L}(X_t, t \in \mathbb{R})}^{D_x}$ is isomorphic to $L^2(\sigma(X), \mu + \nu)$.

Hence if $\nu << \mu$ with $\frac{d\nu}{d\mu}$ bounded, $\Gamma(Y, Y)^{1/2}$ is on $\mathcal{L}(X_t, t \in \mathbb{R})$ a norm equivalent to $\|Y\|_{D_x}$.

If we project $X_{t+h}$ on $\overline{\mathcal{L}(X_s, s \leq t)}^{D_x}$ for this Hilbert scalar product we get
$$\overline{X_{t+h}} \in \overline{\mathcal{L}(X_s, s \leq t)}^{D_x}$$
and $\overline{X_{t+h}}$ is also the best estimate of $X_{t+h}$ in the whole space $\mathbb{D}(X_s, s \leq t)$ in the sense of the Dirichlet form $\mathcal{E}$, because

$$\mathcal{E}(X_{t+h} - \overline{X_{t+h}}, X_s) = 0 \quad \forall s \leq t$$
$$\implies \mathcal{E}(X_{t+h} - \overline{X_{t+h}}, F(X_{s_1}, \ldots, X_{s_n})) = 0 \quad \forall F \in \mathcal{L} \cap \mathcal{C}^1(\mathbb{R}^n) \quad s_i \leq t \quad i = 1, \ldots, n$$

This situation is similar to the Gaussian case for the filtration of Wiener.