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A CONNECTEDNESS THEOREM FOR REAL SPECTRA OF POLYNOMIAL RINGS

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Abstract

Let R be a real closed field. The Pierce-Birkhoff conjecture says that any piecewise polynomial function f on R^n can be obtained from the polynomial ring $R[x_1, \dots, x_n]$ by iterating the operations of maximum and minimum. The purpose of this paper is twofold. First, we state a new conjecture, called the Connectedness conjecture, which asserts the existence of connected sets in the real spectrum of $R[x_1, \dots, x_n]$ satisfying certain conditions. We prove that the Connectedness conjecture implies the Pierce-Birkhoff conjecture.

Secondly, we construct a class of connected sets in the real spectrum which, though not in itself enough for the proof of the Pierce-Birkhoff conjecture, is the first and simplest example of the sort of connected sets we really need, and which constitutes a crucial step on the way to a proof of the Pierce-Birkhoff conjecture in dimension greater than 2, to appear in a subsequent paper.

1 Introduction

All the rings in this paper will be commutative with 1.

Throughout this paper, R will denote a real closed field and A the polynomial ring $R[x_1, \dots, x_n]$, unless otherwise specified.

The Pierce-Birkhoff conjecture asserts that any piecewise-polynomial function $f : R^n \rightarrow R$ can be expressed as a maximum of minima of a finite family of polynomials (see below for the definitions and a precise statement of the conjecture). This is the first in a series of three papers whose purpose is to prove the Pierce-Birkhoff conjecture in its full generality (the best result up to now is due to Louis Mahé [11], who proved the conjecture for $n = 2$).

We start by stating the Pierce-Birkhoff conjecture in its original form as it first appeared in the 1962 paper [5] by M. Henriksen and H. Isbell.

Definition 1.1 *A function $f : R^n \rightarrow R$ is said to be piecewise polynomial if R^n can be covered by a finite collection of closed semi-algebraic sets P_i such that for all i there exists a polynomial $f_i \in A$ satisfying $f|_{P_i} = f_i|_{P_i}$.*

Clearly, such a function is continuous. Piecewise polynomial functions form a ring, containing A , which is denoted by $PW(A)$.

On the other hand, one can consider the (lattice-ordered) ring of all the functions obtained from A by iterating the operations of sup and inf. Since applying the operations of sup and inf to polynomials produces functions which are piecewise polynomial, this ring is contained in $PW(A)$ (this latter ring is closed under sup and inf). It is natural to ask whether the two rings coincide. The precise statement of the conjecture is:

Conjecture 1 (Pierce-Birkhoff) *If $f : R^n \rightarrow R$ is in $PW(A)$, then there exists a finite family of polynomials $g_{ij} \in A$ such that $f = \sup_i \inf_j (g_{ij})$ (in other words, for all $x \in R^n$, $f(x) = \sup_i \inf_j (g_{ij}(x))$).*

In 1989 J.J. Madden [9] reformulated this conjecture in terms of the real spectrum of A and separating ideals. We will now recall Madden's formulation together with the relevant definitions.

Let B be a ring. A point α in the real spectrum of B is, by definition, the data of a prime ideal \mathfrak{p} of B , and a total ordering \leq of the quotient ring B/\mathfrak{p} , or, equivalently, of the field of fractions of B/\mathfrak{p} . Another way of defining the point α is as a homomorphism from B to a real closed field, where two homomorphisms are identified if they have the same kernel \mathfrak{p} and induce the same total ordering on B/\mathfrak{p} .

The ideal \mathfrak{p} is called the support of α and denoted by \mathfrak{p}_α , the quotient ring B/\mathfrak{p}_α by $B[\alpha]$, its field of fractions by $B(\alpha)$ and the real closure of $B(\alpha)$ by $k(\alpha)$. The total ordering of $B(\alpha)$ is denoted by \leq_α . Sometimes we write $\alpha = (\mathfrak{p}_\alpha, \leq_\alpha)$.

Definition 1.2 *The real spectrum of B , denoted by $\text{Sper } B$, is the collection of all pairs $\alpha = (\mathfrak{p}_\alpha, \leq_\alpha)$, where \mathfrak{p}_α is a prime ideal of B and \leq_α is a total ordering of B/\mathfrak{p}_α .*

The real spectrum $\text{Sper } B$ is endowed with two natural topologies. The first one, called the **spectral (or Harrison) topology**, has basic open sets of the form

$$U(f_1, \dots, f_n) = \{\alpha \mid f_1(\alpha) > 0, \dots, f_n(\alpha) > 0\}$$

with $f_1, \dots, f_n \in B$. Here and below, we commit the following standard abuse of notation: for an element $f \in B$, $f(\alpha)$ stands for the natural image of f in $B[\alpha]$ and the inequality $f(\alpha) > 0$ really means $f(\alpha) >_\alpha 0$.

The second is the **constructible topology** whose basic open sets are of the form

$$V(f_1, \dots, f_n, g) = \{\alpha \mid f_1(\alpha) > 0, \dots, f_n(\alpha) > 0, g(\alpha) = 0\},$$

where $f_1, \dots, f_n, g \in B$. Sets of the form $V(f_1, \dots, f_n, g)$ are called **constructible sets** of $\text{Sper } B$.

In what follows, it will be important to distinguish between two types of points in $\text{Sper } B$:

Definition 1.3 *Let B be an R -algebra and α a point of $\text{Sper } B$. We say that α is **bounded** if for any $y \in B[\alpha]$ there exists $N \in R$ such that $|y| < N$. Otherwise, we say that α is **unbounded**.*

Notation: The subspace of $\text{Sper } B$ consisting of all the bounded points will be denoted by $\text{Sper}^* B$.

Next, we recall the notion of **separating ideal**, introduced by Madden in [9].

Definition 1.4 *Let B be a ring. For $\gamma, \delta \in \text{Sper } B$, the **separating ideal** of γ and δ , denoted by $\langle \gamma, \delta \rangle$, is the ideal of B generated by all the elements $f \in B$ which change sign between γ and δ , that is, all the f such that $f(\gamma) \geq 0$ and $f(\delta) \leq 0$.*

Let f be a piecewise polynomial function on R^n and $\alpha \in \text{Sper } A$. Let the notation be as in Definition 1.1. The covering $R^n = \bigcup_i P_i$ induces a corresponding covering $\text{Sper } A = \bigcup_i \tilde{P}_i$ of the real spectrum. Pick and fix an i such that $\alpha \in \tilde{P}_i$. We set $f_\alpha := f_i$. We refer to f_α as a **local polynomial representative of f at α** . In general, the choice of i is not uniquely determined by α . Implicit in the notation f_α is the fact that one such choice has been made.

In [9], Madden reduced the Pierce–Birkhoff conjecture to a purely local statement about separating ideals and the real spectrum. Namely, he showed that the Pierce–Birkhoff conjecture is equivalent to

Conjecture 2 (Pierce–Birkhoff conjecture, the abstract version) *Let f be a piecewise polynomial function and α, β points in $\text{Sper } A$. Let $f_\alpha \in A$ be a local representative of f at α and $f_\beta \in A$ a local representative of f at β . Then $f_\alpha - f_\beta \in \langle \alpha, \beta \rangle$.*

The following statement, nominally weaker than the Pierce–Birkhoff conjecture, has proved to be an extremely useful stepping stone on the way to its proof (we give the contrapositive of Madden’s original statement, since it is better adapted to our needs).

Conjecture 3 (the separation conjecture) *Let $g \in A$ and let $\alpha, \beta \in \text{Sper } A$ be two points such that $g \notin \langle \alpha, \beta \rangle$. Then α and β lie in the same connected component of the set $\text{Sper } A \setminus \{g = 0\}$.*

(this statement follows from Conjecture 2: it suffices to let f be the piecewise polynomial function which is equal to g on the connected component of $\text{Sper } A \setminus \{g = 0\}$ containing α and $f = 0$ elsewhere).

We now state

Conjecture 4 (the Connectedness conjecture) *Let $\alpha, \beta \in \text{Sper } A$ and let g_1, \dots, g_s be a finite collection of elements of A , not belonging to $\langle \alpha, \beta \rangle$. Then there exists a connected set $C \subset \text{Sper } A$ such that $\alpha, \beta \in C$ and $C \cap \{g_i = 0\} = \emptyset$ for $i \in \{1, \dots, s\}$ (in other words, α and β belong to the same connected component of the set $\text{Sper } A \setminus \{g_1 \dots g_s = 0\}$).*

The advantage of the Connectedness conjecture is that it is a statement about polynomials which makes no mention of piecewise polynomial functions.

The purpose of this paper is twofold. First, we prove (§2) that the Connectedness conjecture implies the Pierce–Birkhoff conjecture. This reduces the Pierce–Birkhoff conjecture to constructing, for each $\alpha, \beta \in \text{Sper } A$, connected sets in $\text{Sper } A$ having certain properties.

Secondly, we construct a class of connected subsets of $\text{Sper } A$ which, though not in itself enough for the proof of the Pierce–Birkhoff conjecture, is the first and simplest example of the sort of connected sets we really need, and which constitutes a crucial step in our proof of the Pierce–Birkhoff conjecture in dimension greater than 2, the subject of the forthcoming paper [8]. The precise relation of the main connectedness theorem of the present paper (Theorem 1.1) to the Pierce–Birkhoff conjecture, that is, the part of the proof of the conjecture relegated to [8], is explained in more detail later in this introduction.

The rest of this paper is organized as follows. In §3 we define the valuation ν_α associated to a point α of the real spectrum. We also explain the geometric interpretation of a point of the real spectrum as a semi-curvette.

In §4 we study the behaviour of certain subsets of the real spectrum under blowing up.

In §5 we recall and adapt to our context some known results on resolution of singularities of a purely combinatorial nature. These results can be considered as a special case of the desingularization of toric varieties or Hironaka’s game [14]. Since they are easy to prove, we chose to include complete proofs. The conclusion of this section is an algorithm for resolving

singularities of any binomial by iterating combinatorial (toric) blowings up along non-singular centers.

Finally, §6 is devoted to the proof of the main theorem. Let $A = R[x_1, \dots, x_n]$ be a polynomial ring and let $\omega_{ij}, \theta_{il} \in \mathbb{Q}, i \in \{1, \dots, n\}, j \in \{1, \dots, q\}, l \in \{1, \dots, u\}$. Let

$$h_j(\nu_\delta(x)) = \sum_{i=1}^n \omega_{ij} \nu_\delta(x_i) \text{ for } j \in \{1, \dots, q\},$$

and

$$z_l(\nu_\delta(x)) = \sum_{i=1}^n \theta_{il} \nu_\delta(x_i) \text{ for } l \in \{1, \dots, u\}.$$

Let ν_δ be the valuation associated to the point $\delta \in \text{Sper}(A)$, defined in §3.

Theorem 1.1 *The sets*

$$S = \{\delta \in \text{Sper}(A) \mid x_i > 0, i \in \{1, \dots, n\}, h_j(\nu_\delta(x)) > 0, j \in \{1, \dots, q\}, \\ z_l(\nu_\delta(x)) = 0, l \in \{1, \dots, u\}\}$$

and

$$S^* = S \cap \text{Sper}^* A \tag{1}$$

are connected in the spectral topology.

In other words, subsets of $\text{Sper} A$ and $\text{Sper}^* A$ defined by finitely many \mathbb{Q} -linear equations and strict inequalities on $\nu(x_1), \dots, \nu(x_n)$ are connected.

In the forthcoming paper [8] we develop the theory of approximate roots of a valuation. Given a ring A and a valuation ν , non-negative on A , a family of approximate roots is a collection $\{Q_i\}$, finite or countable, of elements of A . A **generalized monomial** (with respect to a given collection $\{Q_i\}$ of approximate roots) is, by definition, an element of A of the form $\prod_j Q_j^{\gamma_j}$,

$\gamma_j \in \mathbb{N}$. The main defining properties of the approximate roots are the fact that every ν -ideal I in A is generated by generalized monomials contained in it, that is, generalized monomials $\prod_j Q_j^{\gamma_j}$ satisfying

$$\sum_j \gamma_j \nu(Q_j) \geq \nu(I),$$

and the fact that for each i , Q_i is described by an explicit formula in terms of Q_1, \dots, Q_{i-1} . In particular, the valuation ν is completely determined by the set $\{Q_i\}$ and the values $\nu(Q_i)$. In [8] we give an explicit description of the set of generalized monomials which generate the separating ideal $\langle \alpha, \beta \rangle$ for a pair of points $\alpha, \beta \in \text{Sper} A_m$, where A_m is the localization of A at the common center m of the valuations ν_α and ν_β .

We then show that every element $g \in A$ can be written as a finite sum of the form

$$g = c \mathbf{Q}^\theta + \sum_{j=1}^N c_j \mathbf{Q}^{\delta_j}, \tag{2}$$

where c and c_j are units of A_m and \mathbf{Q}^θ and \mathbf{Q}^{δ_j} are generalized monomials such that

$$\nu(\mathbf{Q}^\theta) < \nu(\mathbf{Q}^{\delta_j}) \text{ for } 1 \leq j \leq N. \tag{3}$$

Then the sign of g is determined by the sign of its leading coefficient c . In particular, if the inequalities (3) hold on a certain set $C \subset \text{Sper } A$ and c does not change sign on C then g does not change sign on C . Saying that $g \notin \langle \alpha, \beta \rangle$ is equivalent to saying that

$$\nu(\mathbf{Q}^\theta) < \nu(\mathbf{Q}^\gamma) \tag{4}$$

for any generalized monomial $\mathbf{Q}^\gamma \in \langle \alpha, \beta \rangle$. Both types of inequalities (3) and (4) can be viewed as linear inequalities on $\nu(Q_1), \dots, \nu(Q_t)$ with integer coefficients.

In [8] we construct the connected set C required in the Connectedness conjecture; this set has the form

$$C = \left\{ \delta \in \text{Sper}(A) \mid Q_i > 0, i \in \{1, \dots, t\}, \sum_{i=1}^n \omega_{ij} \nu_\delta(Q_i) > 0, j \in \{1, \dots, q\} \right\},$$

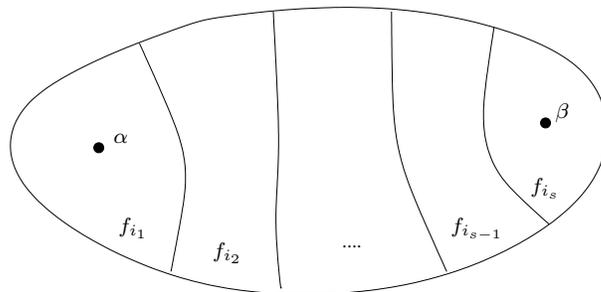
where ω_{ij} are certain explicitly given integers and $\{Q_1, \dots, Q_t\}$ is an (also explicitly given) finite subset of the set of approximate roots. The inequalities $\sum_{i=1}^n \omega_{ij} \nu_\delta(Q_i) > 0$ are nothing but the inequalities of the form (3) and (4), applied to each of the elements g_1, \dots, g_s , appearing in the statement of the Connectedness conjecture.

The only delicate part of the proof is proving the connectedness of C . The connectedness theorem of the present paper is the special case of the desired result in which the finite set $\{Q_1, \dots, Q_t\}$ is a subset of the set of variables $\{x_1, \dots, x_n\}$. In [8] we reduce the general case to the special one using sequences of blowings up of the form described in §5: we construct a sequence π of blowings up such that the total transform of each $Q_i, i \in \{1, \dots, t\}$ is a usual monomial with respect to new coordinates times a unit. The preimage of C under π has the form (1) in the new coordinates. This will reduce the connectedness of C to that of sets of the form S^* proved in this paper, completing our proof of the Connectedness and the Pierce-Birkhoff conjectures.

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2 The Connectedness conjecture implies the Pierce-Birkhoff conjecture

Theorem 2.1 *The Connectedness conjecture implies the Pierce-Birkhoff conjecture.*



Proof: We will assume the Connectedness conjecture and deduce the Pierce–Birkhoff conjecture in the form of Conjecture 2. Let $f \in PW(A)$ and let $\{f_i\}_{i=1}^r$ denote the elements of A which

represent f on the various closed semi-algebraic subsets $P_i \subset \text{Sper } A$. Let $\alpha, \beta \in \text{Sper } A$ and let

$$T = \{\{i, j\} \subset \{1, \dots, r\} \mid f_i - f_j \notin \langle \alpha, \beta \rangle\}. \quad (5)$$

We apply the Connectedness conjecture to the finite collection $\{f_i - f_j \mid (i, j) \in T\}$ of elements of A . By the Connectedness conjecture, there exists a connected subset $C \subset \text{Sper } A$ such that $\alpha, \beta \in C$ and

$$C \cap \{f_i - f_j = 0\} = \emptyset \text{ for all } (i, j) \in T. \quad (6)$$

Let $J \subset \{1, \dots, r\}$ be the set of all indices j having the following property: there exist $i_1, \dots, i_s \in \{1, \dots, r\}$ such that

$$f_\alpha = f_{i_1}, \quad (7)$$

$$f_j = f_{i_s} \quad (8)$$

and for each $q \in \{1, \dots, s-1\}$, we have

$$C \cap \{f_{i_q} - f_{i_{q+1}} = 0\} \neq \emptyset. \quad (9)$$

Let $F = \bigcup_{j \in J} (P_j \cap C)$. We have $\alpha \in F$ by definition.

Claim: $F = C$; in particular, $\beta \in F$.

Proof of Claim. Let $J^c = \{1, \dots, r\} \setminus J$ and $G = \bigcup_{j \in J^c} (P_j \cap C)$. Clearly

$$C = F \cup G \quad (10)$$

and both sets F and G are closed in the induced topology of C (since all the P_j are closed). Moreover, $F \cap G = \emptyset$, so the union in (10) is a disjoint union. Indeed, if $\delta \in F \cap G$ then there exist $j \in J$ and $j' \in J^c$ such that $\delta \in P_j \cap P_{j'}$. But then $f_j(\delta) = f_{j'}(\delta)$, so $\delta \in C \cap \{f_j - f_{j'} = 0\}$, hence $j' \in J$, a contradiction.

Now, since C is connected and $F \neq \emptyset$ (since $\alpha \in F$), the expression (10) of C as a disjoint union of closed sets implies that $G = \emptyset$. Hence $\beta \in F$, which completes the proof of the Claim.

Let $j \in J$ be such that $\beta \in P_j$, so that $f_j = f_\beta$. Let i_1, \dots, i_s be as in (7)–(9), expressing the fact that $j \in J$. Together, (5), (6) and (9) imply that $f_{i_q} - f_{i_{q+1}} \in \langle \alpha, \beta \rangle$ for all $q \in \{1, \dots, s-1\}$. In view of (7) and (8), we obtain $f_\alpha - f_\beta \in \langle \alpha, \beta \rangle$, as desired. \square

3 The valuation associated to a point in the real spectrum

Let B be a ring and α a point in $\text{Sper } B$. In this section we define the valuation ν_α of $B(\alpha)$, associated to α . We also give a geometric interpretation of points in $\text{Sper } B$ as semi-curvettes.

First, we define the valuation ring R_α by

$$R_\alpha = \{x \in B(\alpha) \mid \exists z \in B[\alpha], |x| \leq_\alpha z\}.$$

That R_α is, in fact, a valuation ring, follows because for any $x \in B(\alpha)$, either $x \in R_\alpha$ or $\frac{1}{x} \in R_\alpha$. The maximal ideal of R_α is $M_\alpha = \{x \mid |x| < \frac{1}{z} \forall z \in B[\alpha]\}$; its residue field k_{ν_α} comes equipped with a total ordering, induced by \leq_α . By definition, we have a natural ring homomorphism

$$B \rightarrow R_\alpha \quad (11)$$

whose kernel is \mathfrak{p}_α .

Remark: Conversely, the point α can be reconstructed from the ring R_α by specifying a certain number of sign conditions (finitely many conditions when B is noetherian), as we now explain. Take a prime ideal $\mathfrak{p} \subset B$ and a valuation ν of $\kappa(\mathfrak{p}) := \frac{B_{\mathfrak{p}}}{\mathfrak{p}B_{\mathfrak{p}}}$, with value group Γ . Let

$$r = \dim_{\mathbb{F}_2}(\Gamma/2\Gamma)$$

(if B is not noetherian, it may happen that $r = \infty$). Let x_1, \dots, x_r be elements of $\kappa(\mathfrak{p})$ such that $\nu(x_1), \dots, \nu(x_r)$ induce a basis of the \mathbb{F}_2 -vector space $\Gamma/2\Gamma$. Then for every $x \in B(\alpha)$, there exists $f \in B(\alpha)$, and a unit u of R_α such that $x = ux_1^{\epsilon_1} \cdots x_r^{\epsilon_r} f^2$ with $\epsilon_i \in \{0, 1\}$ (to see this, note that for a suitable choice of f and ϵ_j the value of the quotient u of x by the product $x_1^{\epsilon_1} \cdots x_r^{\epsilon_r} f^2$ is 0, hence u is invertible in R_α). Now, specifying a point $\alpha \in \text{Sper } B$ supported at \mathfrak{p} amounts to specifying a valuation ν of $\frac{B}{\mathfrak{p}}$, whose residue field k_ν comes equipped with a total ordering, and the sign data $\text{sgn } x_1, \dots, \text{sgn } x_r$. For $x \notin \mathfrak{p}$, the sign of x is given by the product $\text{sgn}(x_1)^{\epsilon_1} \cdots \text{sgn}(x_r)^{\epsilon_r} \text{sgn}(u)$, where $\text{sgn}(u)$ is determined by the ordering of k_ν .

Remark: Assume that $\alpha \in \text{Sper}^* B$ (Definition 1.3). Then

$$R_\alpha = \{x \in B(\alpha) \mid \exists N \in R, |x| \leq_\alpha N\}. \quad (12)$$

Thus for points in $\text{Sper}^* B$ the valuation ν_α of $B(\alpha)$ depends on the ordering \leq_α but not on the ring $B[\alpha]$ (this is precisely the reason why bounded points of $\text{Sper } A$ will be important in this paper).

Points of $\text{Sper } B$ admit the following geometric interpretation.

Definition 3.1 *Let k be a field and Γ an ordered abelian group. The generalized formal power series ring $k[[t^\Gamma]]$ is the ring formed by elements of the form $\sum_{\gamma} a_\gamma t^\gamma$, $a_\gamma \in k$ such that the set $\{\gamma \mid a_\gamma \neq 0\}$ is well ordered.*

The ring $k[[t^\Gamma]]$ is equipped with the natural t -adic valuation v with values in Γ , defined by $v(f) = \inf\{a_\gamma \mid a_\gamma \neq 0\}$ for $f = \sum_{\gamma} a_\gamma t^\gamma \in k[[t^\Gamma]]$. Specifying a total ordering on k and $\dim_{\mathbb{F}_2}(\Gamma/2\Gamma)$ sign conditions defines a total ordering on $k[[t^\Gamma]]$. In this ordering $|t|$ is smaller than any element of k . For example, if $t^\gamma > 0$ for any $\gamma \in \Gamma$ then $f > 0$ if and only if $a_{v(f)} > 0$.

For an ordered field k , let \bar{k} denote the real closure of k . The following result is a slight variation on a theorem of Kaplansky for valued fields equipped with a total ordering, whose proof carries over almost verbatim from Kaplansky's original proof.

Theorem 3.1 (Kaplansky, [6]) *Let K be a real valued field, with residue field k and value group Γ . There exists an injection $K \hookrightarrow \bar{k}((t^\Gamma))$ of real valued fields.*

Let $\alpha \in \text{Sper } B$ and let Γ be the value group of ν_α . In view of (11) and the Remark above, specifying a point $\alpha \in \text{Sper } B$ is equivalent to specifying a morphism

$$B[\alpha] \rightarrow \bar{k}_{\nu_\alpha}[[t^\Gamma]],$$

and $\dim_{\mathbb{F}_2}(\Gamma/2\Gamma)$ sign conditions as above.

We may pass to usual spectra to obtain a morphism

$$\text{Spec } (\bar{k}_{\nu_\alpha}[[t^\Gamma]]) \rightarrow \text{Spec } B[\alpha] \rightarrow \text{Spec } B.$$

In particular, if $\Gamma = \mathbb{Z}$, we obtain a **formal curve** in $\text{Spec } B$ (an analytic curve if the series are convergent). This motivates the following definition:

Definition 3.2 A k -curvette in A is a morphism $\alpha : B \rightarrow k[[t^\Gamma]]$ (where Γ is an ordered group). A k -semi-curvette is a k -curvette α together with a choice of the sign data $\text{sgn } x_1, \dots, \text{sgn } x_r$, where x_1, \dots, x_r are elements of B whose t -adic values induce an \mathbb{F}_2 -basis of $\Gamma/2\Gamma$.

We have thus shown how to associate to a point α of $\text{Sper } B$ a $\overline{\kappa(\mathfrak{p}_\alpha)}$ -semi-curvette. Conversely, given an ordered field k , a k -semi-curvette α determines a prime ideal \mathfrak{p}_α (the ideal of all the elements of B which vanish identically on α) and a total ordering on B/\mathfrak{p}_α induced by the ordering of the ring $k[[t^\Gamma]]$ of formal power series. These two operations are inverse to each other. This establishes a one-to-one correspondence between semi-curvettes and points of $\text{Sper } B$.

Below, we will often describe points in the real spectrum by specifying corresponding curvettes.

Example: Consider the curvette $\mathbb{R}[x, y] \rightarrow \mathbb{R}[[t]]$ defined by $x \mapsto t^2$, $y \mapsto t^3$, and the semi-curvette given by declaring, in addition, that t is positive. We obtain the upper branch of the cusp.

4 Affine monomial blowings up

In this section we define one of our main technical tools — affine monomial blowing up — and show that **bounded** points of the real spectrum behave particularly well under such blowings up, more precisely, that the valuation ν_α is preserved under blowing up whenever $\alpha \in \text{Sper}^* A$.

Notation. For a subset $J \subset \{1, \dots, n\}$, x_J will stand for the set $\{x_q \mid q \in J\}$.

Let G be an ordered group. For an n -tuple $a = (a_1, \dots, a_n) \in G^n$ of elements of G , we define

$$\text{rat.rk } a = \dim_{\mathbb{Q}} \sum_{j=1}^n \mathbb{Q}a_j \subset G \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Consider a set $J \subset \{1, \dots, n\}$. Fix an element $j \in J$. Let

$$x'_q = x_q \quad \text{if } q = j \text{ or } q \notin J \quad (13)$$

$$= \frac{x_q}{x_j} \quad \text{if } q \in J \setminus \{j\}. \quad (14)$$

Let $x = (x_1, \dots, x_n)$, $x' = (x'_1, \dots, x'_n)$ and $A' = R[x']$. We have a natural ring homomorphism $\pi : A \rightarrow A'$ and the corresponding maps of real spectra $\pi^* : \text{Sper } A' \rightarrow \text{Sper } A$.

Remark: Since the variables x are monomials in the x' and vice versa, we have

$$\text{rat.rk}(\nu(x_1), \dots, \nu(x_n)) = \text{rat.rk}(\nu(x'_1), \dots, \nu(x'_n)). \quad (15)$$

Definition 4.1 The map π is called an **affine monomial blowing up** of $\text{Sper } A$ (along the ideal (x_J)). The choice of $j \in J$ is referred to as **the choice of a coordinate chart**. Finally, let \mathfrak{p} be a prime ideal of A , not containing any of x_1, \dots, x_n . Let ν be a valuation of $\frac{A}{\mathfrak{p}}$. We say that π is **affine monomial blowing up with respect to ν** if

$$\nu(x_j \text{ mod } \mathfrak{p}) = \min\{\nu(x_q \text{ mod } \mathfrak{p}) \mid q \in J\}. \quad (16)$$

Condition (16) is equivalent to saying that ν is non-negative on $\frac{A'}{\mathfrak{p}'}$, where \mathfrak{p}' is the strict transform of \mathfrak{p} in $\text{Spec } A$.

We consider the coordinates x' as part of the data of the affine monomial blowing up π . An affine monomial blowing up is completely determined by the choice of J and j as above.

Let \mathcal{OGM} be the following category. An object in \mathcal{OGM} is an ordered abelian group G together with n fixed generators a_1, \dots, a_n (such an object will be denoted by (G, a_1, \dots, a_n)). A morphism from (G, a_1, \dots, a_n) to (G', a'_1, \dots, a'_n) is a homomorphism $G \rightarrow G'$ of ordered group sending a_j to a'_j for each j .

Notation: Let us denote by Γ the ordered group \mathbb{R}_{lex}^n . The reason for this definition is that by Abhyankar's inequality we have $\text{rank } \nu_\alpha \leq \dim A = n$ for all $\alpha \in \text{Sper } A$, so the value group Γ_α can be embedded into Γ as an ordered subgroup (of course, this embedding is far from being unique). Let Γ_+ be the semigroup of non-negative elements of Γ .

Take an element

$$a = (a_1, \dots, a_n) \in \Gamma_+^n.$$

Let $G \subset \Gamma$ be the ordered group generated by (a_1, \dots, a_n) . Then $(G, a_1, \dots, a_n) \in \text{Ob}(\mathcal{OGM})$. For each $\delta \in \text{Sper}(A)$, let Γ_δ denote the value group of the associated valuation ν_δ and Γ_δ^* the subgroup of Γ_δ generated by $\nu_\delta(x_1), \dots, \nu_\delta(x_n)$. In this way, we associate to δ the object $(\Gamma_\delta^*, \nu_\delta(x_1), \dots, \nu_\delta(x_n)) \in \text{Ob}(\mathcal{OGM})$.

We will use the following notation. For a set $E \subset \Gamma_+^n$, let

$$S_E = \{\delta \in \text{Sper}(A) \mid x_i >_\delta 0, i \in \{1, \dots, n\}, \exists a = (a_1, \dots, a_n) \in E \text{ such that} \\ (\Gamma_\delta^*, \nu_\delta(x_1), \dots, \nu_\delta(x_n)) \cong (G, a_1, \dots, a_n)\}.$$

and

$$S_E^* = \{\delta \in \text{Sper}^*(A) \mid x_i >_\delta 0, i \in \{1, \dots, n\}, \exists a = (a_1, \dots, a_n) \in E \text{ such that} \\ (\Gamma_\delta^*, \nu_\delta(x_1), \dots, \nu_\delta(x_n)) \cong (G, a_1, \dots, a_n)\}$$

In particular, for $a \in \Gamma_+^n$ we will write

$$S_a := \{\delta \in \text{Sper}(A) \mid x_i >_\delta 0, i \in \{1, \dots, n\}, (\Gamma_\delta^*, \nu_\delta(x_1), \dots, \nu_\delta(x_n)) \cong (G, a_1, \dots, a_n)\}$$

and

$$S_a^* = \{\delta \in \text{Sper}^*(A) \mid x_i >_\delta 0, i \in \{1, \dots, n\}, (\Gamma_\delta^*, \nu_\delta(x_1), \dots, \nu_\delta(x_n)) \cong (G, a_1, \dots, a_n)\}.$$

We will need the following comparison result which says that blowing up induces a homeomorphism on sets of the form S_E^* .

Let E be a subset of Γ_+^n . Take a subset $J \subset \{1, \dots, n\}$. Pick a $j \in J$ and consider the affine monomial blowing up

$$\pi : A \rightarrow A' = R[x'],$$

determined by J and j . Assume that π is an affine monomial blowing up with respect to all ν_δ , $\delta \in S_E^*$ (in other words, $\nu_\delta(x_j) = \min\{\nu_\delta(x_q)\}_{q \in J}$ for all $\delta \in S_E^*$; we have $\nu_\delta(x'_i) \geq 0$ for all $\delta \in S_E^*$ and $i \in \{1, \dots, n\}$).

For $a \in \Gamma_+^n$, let a' be the element of Γ_+^n defined by

$$a'_q = a_q \quad \text{if } q \notin J \text{ or } q = j \tag{17}$$

$$a'_q = a_q - a_j \quad \text{if } q \in J \setminus \{j\}. \tag{18}$$

Let $E' = \{a' \mid a \in E\}$ and let $S_{E'}^*$ denote the corresponding subset of $\text{Sper}^* A'$. Let

$$\pi^* : \text{Sper} A' \rightarrow \text{Sper} A$$

be the map of real spectra induced by π . It is well known and easy to see that π^* is a homeomorphism away from the zero set $V(x_J)$ of the ideal (x_J) . Since S_E^* is disjoint from $V(x_J)$, π^* induces a homeomorphism $\pi^*|_{S_{E'}^*} : (\pi^*)^{-1}(S_E^*) \xrightarrow{\sim} S_E^*$. For $\delta \in \text{Sper} A \setminus V(x_J)$, let δ' denote the unique preimage of δ in $\text{Sper} A'$.

Proposition 4.1 *Take $a \in \Gamma_+^n$ and $\delta \in \text{Sper}^* A$. Then*

$$\nu_\delta = \nu_{\delta'} \tag{19}$$

and

$$\delta' \in S_{a'}^* \iff \delta \in S_a^*. \tag{20}$$

Proof: Since δ is bounded, (19) follows immediately from (12): in fact, (12) characterizes ν_δ (resp. $\nu_{\delta'}$) purely in terms of the ordering \leq_δ , which is the same as $\leq_{\delta'}$, without any reference to the rings A or A' . (20) follows from the equations (13)–(14) and (17)–(19). \square

Corollary 4.1 *We have $\pi^*(S_{E'}^*) = S_E^*$ and the restriction of π^* to $S_{E'}^*$ is a homeomorphism.*

Definition 4.2 *The point δ' is called the **transform** of δ . Similarly, a' , E' and $S_{E'}^*$ are called **transforms** of a , E and S_E^* , respectively.*

For future reference, we will also define the transform of a \mathbb{Q} -linear relation on $\nu_\delta(x_1), \dots, \nu_\delta(x_n)$. Consider a \mathbb{Q} -linear equality of the form

$$\sum_{i=1}^n \theta_i a_i = 0. \tag{21}$$

Definition 4.3 *The transform of (21) under π is the equality $\sum_{i=1}^n \theta'_i a'_i = 0$, where*

$$\theta'_i = \theta_i \quad \text{if } i \neq j \tag{22}$$

$$= \sum_{q \in J} \theta_q \quad \text{if } i = j. \tag{23}$$

The transforms of all the above objects under sequences of blowings up are defined in the obvious way (that is, as iterated transforms) by induction on the length of the blowing up sequence.

Example: This example shows that Proposition 4.1 and Corollary 4.1 are false for unbounded points of $\text{Sper} A$. Let $n = 2$ and $a = (1, 1)$, where we view 1 as an element of Γ via the embedding $\mathbb{R} \hookrightarrow (0) \oplus \mathbb{R} = \Gamma$. Consider the points $\alpha, \delta \in \text{Sper} A$ given by the curvettes $\delta = (t, t)$ and $\alpha = (t^{(1,0)}, t^{(0,1)})$. Let $J = \{1, 2\}$, $j = 1$, $x'_1 = x_1$, $x'_2 = \frac{x_2}{x_1}$. Since $\nu_\delta(x_1) = \nu_\delta(x_2)$, the corresponding blowing up $A \rightarrow A'$ is a blowing up with respect to ν_δ (but not with respect to ν_α). The point α' is defined by the ordering of $R(x_1, x_2)$ in which $0 < x'_1 < c < x'_2$ for any positive real constant c . We have $R_{\alpha'} = R(x'_2)[x_1]_{(x_1)}$, $\nu_{\alpha'}(x'_2) = 0$, $\nu_{\alpha'}(x'_1) = 1$, so $\nu_{\alpha'} \neq \nu_\alpha$. We have $\alpha' \in S_{a'}$ but $\alpha' \notin S_a$, so the analogues of (20) and Corollary 4.1 do not hold for unbounded points.

5 Desingularization of binomials by monomial blowings up

In this section, we recall and adapt to our context a result from the theory of resolution of singularities.

Proposition 5.1 *Consider two n -tuples $\alpha, \gamma \in \mathbb{N}^n$ and the corresponding monomials x^α and x^γ . Then there exist finitely many sequences $\pi_i : A \rightarrow A'_i$, $1 \leq i \leq s$, of affine monomial blowings up, having the following properties:*

(1) *for each $i \in \{1, \dots, s\}$, one of x^α and x^γ divides each other in A'_i*

(2) *for any prime ideal \mathfrak{p} of A not containing x_1, \dots, x_n , and any valuation ν of $\frac{A}{\mathfrak{p}}$, there exists $i \in \{1, \dots, s\}$ such that π_i is a sequence of affine monomial blowings up with respect to ν .*

Remarks: (1) Proposition 5.1 is a special case of [14]. We give a proof here since it is much simpler than that of [14].

(2) Let ν be a valuation as in Proposition 5.1, and $i \in \{1, \dots, s\}$ and index satisfying the conclusion (2) of the Proposition for ν . Then ν is non-negative on A'_i . If $\nu(x^\alpha) < \nu(x^\gamma)$, we know *which* of x^α, x^γ divides the other in A'_i , namely, $x^\alpha \mid x^\gamma$ (and not the other way around).

Proof of Proposition 5.1: We will define a numerical character $\tau(\alpha, \gamma)$, associated to the *unordered* pair (α, γ) and consisting of a pair of non-negative integers. If one of x^α, x^γ divides the other, there is nothing to prove. Assume that neither of x^α, x^γ divides the other. We will describe a subset $J \subset \{1, \dots, n\}$, such that for any choice of $j \in J$ and the corresponding affine monomial blowing up $A \rightarrow A'$, writing $x^\alpha = x'^{\alpha'}$, $x^\gamma = x'^{\gamma'}$ in the new coordinates, we have

$$\tau(\alpha', \gamma') < \tau(\alpha, \gamma) \quad (24)$$

in the lexicographical ordering. Proposition 5.1 will follow immediately by iterating the above procedure.

We start by defining the numerical character $\tau(\alpha, \gamma)$. Let

$$\alpha = (\alpha_1, \dots, \alpha_n) \text{ and} \quad (25)$$

$$\gamma = (\gamma_1, \dots, \gamma_n). \quad (26)$$

Let $\delta_q = \min\{\alpha_q, \gamma_q\}$, $1 \leq q \leq n$; let $\delta := (\delta_1, \dots, \delta_n)$. Let $\tilde{\alpha} := \alpha - \delta$, $\tilde{\gamma} = \gamma - \delta$. Interchanging α and γ , if necessary, we may assume that $|\tilde{\alpha}| \leq |\tilde{\gamma}|$ (here and below, $|\cdot|$ stands for the sum of the components). Put

$$\tau(\alpha, \gamma) := (|\tilde{\alpha}|, |\tilde{\gamma}|).$$

Note that if $\tilde{\alpha} = (0, \dots, 0)$ ($\iff |\tilde{\alpha}| = 0$) then $x^\alpha \mid x^\gamma$ in A . Assume that $|\tilde{\alpha}| > 0$. We will now describe a subset $J \subset \{1, \dots, n\}$, such that for any choice of $j \in J$ and the corresponding affine monomial blowing up $\pi : A \rightarrow A'$ along (x_j) , the inequality (24) holds.

Write $\tilde{\alpha} = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_n)$, $\tilde{\gamma} = (\tilde{\gamma}_1, \dots, \tilde{\gamma}_n)$. Renumbering the variables, we may assume that there exists a , $1 \leq a < n$, such that $\tilde{\alpha}_j = 0$ for $a < j \leq n$ and $\tilde{\gamma}_j = 0$ for $1 \leq j \leq a$. In other words,

$$\tilde{\alpha} = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_a, \underbrace{0, \dots, 0}_{n-a \text{ zeroes}}) \quad (27)$$

$$\tilde{\gamma} = (\underbrace{0, \dots, 0}_a \text{ zeroes}, \tilde{\gamma}_{a+1}, \dots, \tilde{\gamma}_n). \quad (28)$$

We may also assume that

$$\tilde{\alpha}_i > 0 \quad \text{for } 1 \leq i \leq a. \quad (29)$$

Let J denote a minimal subset of $\{1, \dots, n\}$ (in the sense of inclusion), having the following properties:

$$\{1, \dots, a\} \subset J \quad \text{and} \quad (30)$$

$$\sum_{q \in J} \tilde{\gamma}_q \geq |\tilde{\alpha}| \quad (31)$$

Remark: (31) means that the numerical character $|\tilde{\alpha}|$ is constant on the subscheme

$$Y_J \subset \text{Spec } A,$$

defined by (x_J) . This is equivalent to saying that the hypersurface of $\text{Spec } A$, defined by the binomial $x^\alpha - x^\gamma$, is normally flat along Y_J (we will not need this in the sequel).

Pick a $j \in J$ subject to condition (16). Let $\pi : A \rightarrow A'$ be the affine monomial blowing up along (x_J) , associated to j and J . We will now write out the monomials x^α and x^γ in the new coordinates and observe that the numerical character τ has decreased. Define the non-negative integers $\tilde{\alpha}'_q$ and $\tilde{\gamma}'_q$, $1 \leq q \leq n$, as follows:

$$\tilde{\alpha}'_q = \tilde{\alpha}_q \quad \text{if } q \neq j \quad (32)$$

$$= 0 \quad \text{if } q = j \quad (33)$$

$$\tilde{\gamma}'_q = \tilde{\gamma}_q \quad \text{if } q \neq j \quad (34)$$

$$= \sum_{q \in J} \tilde{\gamma}_q - |\tilde{\alpha}| \quad \text{if } q = j. \quad (35)$$

Put $\tilde{\alpha}' = (\tilde{\alpha}'_1, \dots, \tilde{\alpha}'_n)$, $\tilde{\gamma}' = (\tilde{\gamma}'_1, \dots, \tilde{\gamma}'_n)$. Let δ' denote the n -vector obtained from δ by adding $|\tilde{\alpha}|$ to the j -th component; that is,

$$\delta' = (\delta_1, \dots, \delta_{j-1}, \delta_j + |\tilde{\alpha}|, \delta_{j+1}, \dots, \delta_n).$$

With these definitions, we have

$$x^\alpha = (x')^{\delta' + \tilde{\alpha}'} \quad (36)$$

$$x^\gamma = (x')^{\delta' + \tilde{\gamma}'}. \quad (37)$$

Put $\alpha' = \delta' + \tilde{\alpha}'$, $\gamma' = \delta' + \tilde{\gamma}'$.

Lemma 5.1 *We have $\tau(\alpha', \gamma') < \tau(\alpha, \gamma)$ in the lexicographical ordering.*

Proof: There are two possibilities: either $j \in \{1, \dots, a\}$ or $j \in \{a+1, \dots, n\}$. If $j \in \{1, \dots, a\}$ then (29), (32) and (33) imply that

$$|\tilde{\alpha}'| = |\tilde{\alpha}| - \tilde{\alpha}_j < |\tilde{\alpha}|. \quad (38)$$

Suppose that $j \in \{a+1, \dots, n\}$. Then by (32),

$$|\tilde{\alpha}'| = |\tilde{\alpha}|. \quad (39)$$

We will prove that $|\tilde{\gamma}'| < |\tilde{\gamma}|$. Indeed, by the minimality of J ,

$$\sum_{q \in J \setminus \{j\}} \tilde{\gamma}_q < |\tilde{\alpha}| \quad (40)$$

(otherwise we could replace J by $J \setminus \{j\}$). Now, by (34), (35) and (40),

$$\begin{aligned}
|\tilde{\gamma}'| &\equiv \sum_{\substack{q=a+1 \\ q \neq j}}^n \tilde{\gamma}'_q + \tilde{\gamma}'_j = \sum_{\substack{q=a+1 \\ q \neq j}}^n \tilde{\gamma}_q + \sum_{q \in J} \tilde{\gamma}_q - |\tilde{\alpha}| \\
&= \sum_{q=a+1}^n \tilde{\gamma}_q + \left(\sum_{q \in J \setminus \{j\}} \tilde{\gamma}_q - |\tilde{\alpha}| \right) < \sum_{q=a+1}^n \tilde{\gamma}_q \equiv |\tilde{\gamma}|.
\end{aligned} \tag{41}$$

To summarize, (38), (39) and (41) say that

$$(|\tilde{\alpha}'|, |\tilde{\gamma}'|) < (|\tilde{\alpha}|, |\tilde{\gamma}|) \equiv \tau(\alpha, \gamma) \tag{42}$$

in the lexicographical ordering. If $|\tilde{\alpha}'| \leq |\tilde{\gamma}'|$ then according to our definitions

$$\tau(\alpha', \gamma') = (|\tilde{\alpha}'|, |\tilde{\gamma}'|),$$

and the Lemma follows from (42). If $|\tilde{\gamma}'| < |\tilde{\alpha}'|$ then $\tau(\alpha', \gamma') = (|\tilde{\gamma}'|, |\tilde{\alpha}'|) < (|\tilde{\alpha}'|, |\tilde{\gamma}'|)$, and, again, the Lemma follows from (42). This completes the proof of Lemma 5.1. \square

Unless one of x^α, x^γ divides the other, we can iterate the construction of Lemma 5.1. Since τ cannot decrease indefinitely, this process must stop after finitely many steps. Therefore after a finite number of steps we will arrive at the situation when one of $x^{\alpha'}, x^{\gamma'}$ divides the other. In other words, one of x^α, x^γ divides the other in A' .

Of course, the above construction is not unique: at each step we made an arbitrary choice of a coordinate chart. The inequality (24) and hence the final conclusion that one of x^α, x^γ divides the other hold for all the possible choices of j . Now let $\{\pi_i : A \rightarrow A'_i\}_{1 \leq i \leq s}$ be the totality of all the blowing up sequences constructed above for all the possible choices of coordinate charts, such that one of x^α, x^γ divides the other in A'_i for all $i \in \{1, \dots, s\}$ (since the number of choices of coordinate charts is finite at each step and each sequence stops after finitely many steps, the overall set is finite). Since for each valuation ν there always exists a choice of coordinate chart satisfying (16) at each step, the set $\{\pi_i : A \rightarrow A'_i\}_{1 \leq i \leq s}$ satisfies the conclusion of the Proposition. \square

Let E be a subset of $\text{Sper } A$, such that all $\delta \in E$ satisfy u \mathbb{Q} -linearly independent equations of the form

$$\sum_{i=1}^n \theta_{il} \nu_\delta(x_i) = 0, l \in \{1, \dots, u\}, \tag{43}$$

where $\theta_{il} \in \mathbb{Q}$. Let $r := n - u$.

Corollary 5.1 *There exist finitely many sequences $\pi_i : A \rightarrow A'_i = k[x_{i1}, \dots, x_{in}]$, $1 \leq i \leq s$, of affine monomial blowings up, such that for each $\delta \in E$ the following conditions hold:*

(1) *for each $i \in \{1, \dots, s\}$, we have*

$$\nu_\delta(x_{i,r+1}) = \dots = \nu_\delta(x_{in}) = 0. \tag{44}$$

(2) *there exists $i \in \{1, \dots, s\}$ such that π_i is a sequence of affine monomial blowings up with respect to ν_δ .*

Proof: If $r = n$, there is nothing to prove. Assume that $r < n$. Then the existence of a non-trivial \mathbb{Q} -linear relation

$$\sum_{i=1}^n \theta_{i1} \nu_\delta(x_i) = 0, \quad (45)$$

implies that there exist two monomials x^α and x^γ with non-negative integer exponents such that $\nu_\delta(x^\alpha) = \nu_\delta(x^\gamma)$ for all $\delta \in E$. Let $A \rightarrow A' = k[x'_1, \dots, x'_n]$ be one of the sequences of affine monomial blowings up, appearing in the conclusion of Proposition 5.1 for x^α and x^γ . We have either $\frac{x^\alpha}{x^\gamma} \in A'$ or $\frac{x^\gamma}{x^\alpha} \in A'$. Since $\frac{x^\alpha}{x^\gamma}$ is a monomial in the x' and since

$$\nu_\delta \left(\frac{x^\alpha}{x^\gamma} \right) = \nu_\delta \left(\frac{x^\gamma}{x^\alpha} \right) = 0,$$

one of the $\nu_\delta(x'_q)$ is equal to zero. Renumbering the x'_q , if necessary, we may assume that $\nu_\delta(x'_n) = 0$ for all $\delta \in E$. Moreover, $\nu_\delta(x'_1), \dots, \nu_\delta(x'_{n-1})$ satisfy $u - 1$ linearly independent relations \mathbb{Q} -linear relations (namely, the transforms of the relations (43)). Next, repeat the above procedure with the ring A replaced by $k[x'_1, \dots, x'_{n-1}]$. After u repetitions, we arrive at the situation where (44) holds. \square

Corollary 5.2 *Let the notation be as in Corollary 5.1. Take $\delta \in E$ such that*

$$r = \text{rat.rk}(\nu_\delta(x_1), \dots, \nu_\delta(x_n)) \quad (46)$$

and an index $i \in \{1, \dots, s\}$. Then $\nu_\delta(x_{i1}), \dots, \nu_\delta(x_{ir})$ are \mathbb{Q} -linearly independent.

Proof: This follows immediately from (15) and (46).

6 A connectedness theorem

As usual, let us denote by Γ the ordered group \mathbb{R}_{lex}^n and by Γ_+ the semigroup of non-negative elements of Γ . Let $\omega_{ij}, \theta_{il} \in \mathbb{Q}, i \in \{1, \dots, n\}, j \in \{1, \dots, q\}, l \in \{1, \dots, u\}$ and consider the subset E of Γ_+^n defined by

$$E = \left\{ (a_1, \dots, a_n) \in \Gamma_+^n \left| \sum_{i=1}^n \omega_{ij} a_i > 0, j \in \{1, \dots, q\}, \sum_{i=1}^n \theta_{il} a_i = 0, l \in \{1, \dots, u\} \right. \right\}. \quad (47)$$

Theorem 6.1 *Let $A = R[x_1, \dots, x_n]$ be a polynomial ring. Then the sets S_E and S_E^* are connected for the spectral topology of $\text{Sper}(A)$.*

A proof of Theorem 6.1 will be given after a few lemmas.

Lemma 6.1 *Let X be a topological space, normal and compact (not necessarily Hausdorff) and let \mathcal{F} be a filter of non-empty closed connected sets. Then the intersection $C = \bigcap_{F \in \mathcal{F}} F$ is non-empty, closed and connected.*

Proof: That C is non-empty and closed is well known and easy to see. To prove connectedness, suppose $C = X_1 \amalg X_2$, X_1, X_2 closed. By normality, there are two open sets $U_1 \supset X_1, U_2 \supset X_2, U_1 \cap U_2 = \emptyset$. For any $F \in \mathcal{F}$, Let $G = F \setminus (U_1 \cup U_2)$. We have $G \neq \emptyset$ because F is connected. By the compactness of X , we have $\bigcap_{F \in \mathcal{F}} G \neq \emptyset$. Take an element $x \in \bigcap_{F \in \mathcal{F}} G$. Then

$$x \in \bigcap_{F \in \mathcal{F}} G \subset \bigcap_{F \in \mathcal{F}} F = X_1 \amalg X_2,$$

but $x \notin U_1 \cup U_2 \supset X_1 \amalg X_2$, which is a contradiction. \square

Lemma 6.2 *Let B be any ring and S a closed connected subset of $\text{Sper}(B)$. Then S is the intersection of a filter of closed connected constructible sets.*

Proof: Consider the filter \mathcal{F} of all the closed connected constructible sets containing S . We want to show that $S = \bigcap_{F \in \mathcal{F}} F$. Suppose that $S \subsetneq \bigcap_{F \in \mathcal{F}} F$ and take a point $\alpha \in \bigcap_{F \in \mathcal{F}} F \setminus S$. Then there exists a basic open set $U \ni \alpha$ and such that $U \cap S = \emptyset$. Then the connected component C of the complement of U , containing S , is a constructible closed connected set containing S , hence a member of \mathcal{F} . But $\alpha \notin C$, contradicting $\alpha \in \bigcap_{F \in \mathcal{F}} F$. \square

Lemma 6.3 (the Projection lemma) *Let $1 \leq s \leq n - 1$ and let*

$$p : \text{Sper } R[x_1, \dots, x_n]_{x_1 \dots x_n} \rightarrow \text{Sper } R[x_1, \dots, x_s]_{x_1 \dots x_s}$$

be the canonical projection. Fix a strictly positive constant $N \in R$. If \tilde{C}_0 is a closed connected subset of $\text{Sper } R[x_1, \dots, x_s]_{x_1 \dots x_s}$, then the sets

$$\tilde{C}_N := \left\{ \delta \in p^{-1}(\tilde{C}_0) \mid 0 < x_i(\delta) \leq N, i \in \{1, \dots, n\} \right\},$$

and

$$\tilde{C} := p^{-1}(\tilde{C}_0) \cap \text{Sper}^* R[x_1, \dots, x_n]_{x_1 \dots x_n}$$

are connected in $\text{Sper } R[x_1, \dots, x_n]_{x_1 \dots x_n}$.

Remark: We will see later that the set $p^{-1}(\tilde{C}_0)$ is also connected. However, from the point of view of the Pierce-Birkhoff conjecture, $p^{-1}(\tilde{C}_0) \cap \text{Sper}^* R[x_1, \dots, x_n]_{x_1 \dots x_n}$ is the more natural object of the two.

Proof of Lemma 6.3: Let C_0 denote the preimage of \tilde{C}_0 in the maximal spectrum $R^s \setminus \{x_1 = \dots = x_s = 0\}$. According to the previous Lemma, we can write \tilde{C}_0 as the intersection of a filter of closed connected constructible sets $\tilde{F} \in \mathcal{F}$. Take an $\tilde{F} \in \mathcal{F}$. Since \tilde{F} is connected in the real spectrum, its preimage F in the maximal spectrum $R^s \setminus \{x_1 = \dots = x_s = 0\}$ is semi-algebraically connected ([3], Proposition 7.5.1, p. 130). By abuse of notation, we will denote the restriction of p to maximal spectra also by p . Let

$$C_N := \{(y_1, \dots, y_n) \in R^n \mid (y_1, \dots, y_s) \in C_0, 0 < y_i \leq N, i \in \{1, \dots, n\}\}, \quad (48)$$

$$\tilde{D}_N := \{\delta \in \text{Sper } R[x_1, \dots, x_n]_{x_1 \dots x_n} \mid 0 < x_i \leq N, i \in \{1, \dots, n\}\}, \quad (49)$$

and let D_N denote the preimage of \tilde{D}_N in the maximal spectrum $R^n \setminus \{x_1 = \dots = x_n = 0\}$. Let p_N be the restriction of p to both \tilde{D}_N and D_N .

We claim that the closed semi-algebraic subset $p_N^{-1}(F) \subset D_N$ is semi-algebraically connected. Indeed, $p_N|_{D_N}$ has semi-algebraically connected fibers (in fact, each fiber of $p_N|_{D_N}$ is the cube defined by the inequalities $0 < x_j \leq N$, $j \in \{1, \dots, n\}$). Suppose $p_N^{-1}(F)$ were not semi-algebraically connected. Then there would exist open semi-algebraic sets U, V such that $p_N^{-1}(F) = U \amalg V$. Since $p_N|_{D_N}$ has connected fibers, U and V can be written as $U = p_N^{-1}(U_0)$, $V = p_N^{-1}(V_0)$, where U_0, V_0 are disjoint open sets in $R^s \setminus \{x_1 = \dots = x_n = 0\}$, such that $F = U_0 \amalg V_0$. Then $\tilde{F} = \tilde{U}_0 \amalg \tilde{V}_0$ which contradicts the connectedness of \tilde{F} .

Since $p_N^{-1}(F)$ is closed in D_N , semi-algebraic and semi-algebraically connected, $p_N^{-1}(\tilde{F})$ is also connected for the spectral topology (applying Proposition 7.5.1 of [3] once again). Since

$$\tilde{C}_N = \bigcap_{\tilde{F} \in \mathcal{F}} p_N^{-1}(\tilde{F}),$$

\tilde{C}_N is connected by Lemma 6.1. The set \tilde{C} can be written as a nested union $\tilde{C} = \bigcup_{N=1}^{\infty} \tilde{C}_N$, hence it, too, is connected. \square

Lemma 6.4 (the fiber connectedness lemma) *Fix an element $a = (a_1, \dots, a_n) \in \Gamma_+^n$. The set S_a^* is connected in $\text{Sper}(A)$.*

Remark: We will see later that S_a is also connected. Again, S_a^* is the more important object of the two for the Pierce-Birkhoff conjecture.

Proof of Lemma 6.4: Let $r = \text{rat.rk } a$. Take a subset $J \subset \{1, \dots, n\}$ and let

$$x_J = \{x_q \mid q \in J\}.$$

Let $j \in J$ be such that $a_j = \min\{a_q\}_{q \in J}$ and consider the affine monomial blowing up

$$\pi : A \rightarrow A' = R[x'],$$

determined by J and j :

$$\begin{aligned} x'_q &= x_q, a'_q = a_q && \text{if } q \notin J \text{ or } q = j \\ x'_q &= \frac{x_q}{x_j}, a'_q = a_q - a_j && \text{if } q \in J \setminus \{j\}. \end{aligned}$$

(here and below, we will sometimes say that π is an affine monomial blowing up with respect to a).

Let (G', a'_1, \dots, a'_n) be the ordered group generated by the a'_i (actually $G' = G$) and let $S_{a'}$ be the corresponding subset of $\text{Sper}(A')$. Then $S_a^* \cong S_{a'}^*$ by Proposition 4.1 and Corollary 4.1.

We now iterate the above procedure. By Corollaries 5.1 and 5.2 (where we take $E = \{a\}$), after a succession of such transformations we may assume (after passing to the new coordinates) that a_1, \dots, a_r are \mathbb{Q} -linearly independent and $a_{r+1} = \dots = a_n = 0$. As a_1, \dots, a_r are \mathbb{Q} -linearly independent, there exists a unique point

$$\alpha = (t^{a_1}, \dots, t^{a_r}) \in \text{Sper } R[x_1, \dots, x_r]_{x_1 \dots x_r}$$

such that $x_i >_{\alpha} 0$ and $\nu_{\alpha}(x_i) = a_i$ for $i \in \{1, \dots, r\}$: because of the linear independence, the support of α is the zero ideal of $R[x_1, \dots, x_r]$ and the ordering \leq_{α} of $R(x_1, \dots, x_r)$ is completely described by the inequalities

$$x^{\gamma} < x^{\delta} \iff \sum \gamma_j a_j > \sum \delta_j a_j. \quad (50)$$

The fact that a_1, \dots, a_r are \mathbb{Q} -linearly independent implies that (50) imposes a total ordering on the set of monomials and hence on $R(x_1, \dots, x_r)$. This point α is closed in the relative topology of $\text{Sper } R[x_1, \dots, x_r]_{x_1 \dots x_r}$ because α has no non-trivial specializations. We have

$$S_a^* = p^{-1}(\{\alpha\}),$$

where p is the projection $\text{Sper}^* R[x_1, \dots, x_n]_{x_1 \dots x_n} \rightarrow \text{Sper}^* R[x_1, \dots, x_r]_{x_1 \dots x_r}$. Then S_a^* is connected by the Projection lemma. \square

Corollary 6.1 *Let S^{\dagger} be a connected component of S_E^* and a an element of Γ_+^n . Then either $S_a^* \subset S^{\dagger}$ or $S_a^* \cap S^{\dagger} = \emptyset$. There exists a subset $E^{\dagger} \subset E$ such that $S^{\dagger} = S_{E^{\dagger}}^*$.*

The main idea of the proof of Theorem 6.1 is the following. Taking two points α and β in S , let $a = (a_1, \dots, a_n) := (\nu_\alpha(x_1), \dots, \nu_\alpha(x_n))$ and $b = (b_1, \dots, b_n) := (\nu_\beta(x_1), \dots, \nu_\beta(x_n))$. We join α and β by a “staircase” in Γ^n where each stair lies completely in a connected component. Two examples at the end of the paper show that we have to do this with some care.

First, we reduce the Theorem to the case when $u = 0$, that is, when E is defined by strict inequalities (and no equalities). Indeed, without loss of generality, we may assume that the equalities

$$\sum_{i=1}^n \theta_{il} a_i = 0, \quad l \in \{1, \dots, u\} \quad (51)$$

are linearly independent.

Lemma 6.5 *If Theorem 6.1 is true for $u = 0$ (that is, no equalities in the definition of E) then it is true in general.*

Proof: Assume the Theorem for $u = 0$. Apply Corollary 5.1 to the set E . For each $i \in \{1, \dots, s\}$, let $E_i = \{a \in E \mid \pi_i \text{ is a sequence of blowings up with respect to } a\}$. By Corollary 5.1 (2), $E = \bigcup_{i=1}^s E_i$. Let E'_i denote the transform of E_i under π_i (by the choice of E_i , E'_i is well defined). By construction, E'_i is contained in the plane $a_{i,r+1} = \dots = a_{in} = 0$, and is defined in $\Gamma^n_+ \cap \{a_{i,r+1} = \dots = a_{in} = 0\}$ by finitely many strict \mathbb{Q} -linear inequalities — the transforms of the inequalities $\sum_{i=1}^n \omega_{ij} a_i > 0$. Let $\iota : \Gamma^r \rightarrow \Gamma^n$ denote the natural inclusion of the plane $a_{i,r+1} = \dots = a_{in} = 0$ in Γ^n . Let $E_i^\dagger = \iota^{-1}(E'_i)$. The set $S_{E_i^\dagger}^*$ is the preimage of $S_{E'_i}^*$ under the natural projection $\text{Sper } R[x_{i1}, \dots, x_{in}]_{x_{i1} \dots x_{in}} \rightarrow \text{Sper } R[x_{i1}, \dots, x_{ir}]_{x_{i1} \dots x_{ir}}$. Now, $S_{E_i^\dagger}^*$ is connected by the $u = 0$ case of Theorem 6.1. Its preimage $S_{E_i}^*$ is connected by the projection Lemma. Hence $S_{E_i}^*$ is connected by Corollary 4.1.

Let N_i denote the number of affine monomial blowings up composing π_i and let

$$N = \max_{1 \leq i \leq s} \{N_i\}.$$

To complete the proof of Lemma 6.5, we will now show that E is connected by induction on N . Let $J \subset \{1, \dots, n\}$ be such that the first blowing up in each of the sequences π_i is a blowing up along J . There are $\#J$ possible choices of coordinate charts, one for each element $j \in J$. Let $\pi_{0j} : A \rightarrow A_j$ denote the affine monomial blowing up, defined by J and j . Let $E_{0j} \subset E$ be defined by $E_{0j} = \{a \in E \mid \pi_{0j} \text{ is a blowing up with respect to } a\}$ and let E'_{0j} denote the transform of E_{0j} under π . By the induction assumption, $S_{E'_{0j}}^*$ is connected.

Take two indices $j, \tilde{j} \in J$. Assume that both E_{0j} and $E_{0\tilde{j}}$ are not empty. This means that there exist $a, \tilde{a} \in E$ such that

$$a_j \leq a_{\tilde{j}} \quad \text{and} \quad (52)$$

$$\tilde{a}_{\tilde{j}} \leq \tilde{a}_j. \quad (53)$$

(52) and (53) imply that there exists $b \in E$ such that $b_j = b_{\tilde{j}}$. Then both π_{0j} and $\pi_{0\tilde{j}}$ are blowings up with respect to b , so $b \in E_{0j} \cap E_{0\tilde{j}}$; in particular, $S_{E_{0j}}^* \cap S_{E_{0\tilde{j}}}^* \neq \emptyset$. To summarize, each $S_{E_{0j}}^*$ is connected and whenever two such sets are non-empty, they have a non-empty intersection. This implies that $\bigcup_{j \in J} S_{E_{0j}}^*$ is connected, as desired. This completes the proof of Lemma 6.5. \square

From now on, we will assume that $u = 0$ in Theorem 6.1.

The next two Lemmas are the basic building blocks of all the connectedness results of this paper.

Consider an element $b \in \Gamma_+^n$ such that b_1, \dots, b_n are \mathbb{Q} -linearly independent. Then S_b^* consists of a single point $\delta(b)$.

Lemma 6.6 *Let U be a basic open set for the spectral topology of $\text{Sper } A$, containing $\delta(b)$. There exists a subset $V \subset \Gamma_+^n$, defined by strict \mathbb{Q} -linear inequalities, such that $b \in V$ and for any $a \in V$, we have $S_a^* \subset U$.*

Proof: Let $U = \{\alpha \mid f_1(\alpha) > 0, \dots, f_s(\alpha) > 0\}$. Write

$$f_j = \sum_{\gamma \in \mathbb{N}^n} c_{j\gamma} x^\gamma \quad (54)$$

Let $M(j) = c_{j, \gamma(j)} x^{\gamma(j)}$ be the monomial of f_j of smallest valuation (which exists because b_1, \dots, b_n are \mathbb{Q} -linearly independent).

For any j , $1 \leq j \leq s$,

$$\nu_{\delta(b)}(x^{\gamma(j)}) < \nu_{\delta(b)}(x^\gamma) \quad (55)$$

for all γ such that $c_{j\gamma} \neq 0$. Writing $\gamma = (\gamma_1, \dots, \gamma_n)$, $\gamma(j) = (\gamma_1(j), \dots, \gamma_n(j))$, (55) is equivalent to saying that $\sum_{q=1}^n (\gamma_q - \gamma_q(j)) b_q > 0$. Let V be the subset of Γ_+^n , defined by the inequalities

$\sum_{q=1}^n (\gamma_q - \gamma_q(j)) a_q > 0$ for all γ with $c_{j\gamma} \neq 0$. Take an element $a \in V$ and a point $\delta \in S_a^*$. By construction, each f_j has the same dominant monomial at δ as at $\delta(b)$. Then $S_a^* \subset U$, as desired. \square

Let $r \in \{1, \dots, n-1\}$. Consider an element $b \in \Gamma_+^n$ such that $b_{r+1} = \dots = b_n = 0$ and b_1, \dots, b_r are \mathbb{Q} -linearly independent. For $1 \leq j \leq r$, write $b_j = (b_{j1}, \dots, b_{jn})$ with $b_{jq} \in \mathbb{R}$. Consider the $(r \times n)$ -matrix $(b_{jq})_{\substack{1 \leq j \leq r \\ 1 \leq q \leq n}}$. If this matrix contains a column \vec{v} consisting entirely of zeroes, permuting the columns so as to move \vec{v} to the right of the matrix does not change the set S_b^* . Moving all such zero columns to the right, if necessary, we will assume, in addition, that the $(r \times r)$ matrix $(b_{jq})_{\substack{1 \leq j \leq r \\ 1 \leq q \leq r}}$ is non-singular.

Define $b^\bullet = (b_1^\bullet, \dots, b_n^\bullet)$ by $b_i^\bullet = b_i, i \neq r+1$ and $b_{r+1}^\bullet = \underbrace{(0, \dots, 0)}_r, 1, 0, \dots, 0$. We have

$$b \in E \implies b^\bullet \in E \quad (56)$$

because, by the non-singularity of the matrix $(b_{jq})_{\substack{1 \leq j \leq r \\ 1 \leq q \leq r}}$, the presence of b_{r+1}^\bullet does not affect the strict inequalities defining E : any integer multiple of b_{r+1}^\bullet is infinitesimal in absolute value compared to any \mathbb{Z} -linear combination of b_1, \dots, b_r .

Lemma 6.7 *We have $S_b^* \cap \overline{S_{b^\bullet}^*} \neq \emptyset$. In particular, if b (and hence b^\bullet) belongs to E then S_b^* and $S_{b^\bullet}^*$ lie in the same connected component of S_E^* .*

Proof: Pick a point in $S_{b^\bullet}^*$, for example,

$$\alpha^\bullet = (t^{b_1}, \dots, t^{b_r}, t^{b_{r+1}^\bullet}, 1, \dots, 1)$$

(here we are representing α^\bullet by a parametrized semi-curvette in R^n - see §3). Let

$$U = \{\alpha \mid f_1(\alpha) > 0, \dots, f_s(\alpha) > 0\}$$

be a basic open set such that $U \ni \alpha^\bullet$. It remains to prove that $U \cap S_b^* \neq \emptyset$. Write

$$f_j = \sum_{\gamma=(\bar{\gamma}, \gamma_{r+1})} c_{j\gamma}(\bar{x}) \bar{x}^{\bar{\gamma}} x_{r+1}^{\gamma_{r+1}} \quad (57)$$

where $\bar{x} = (x_1, \dots, x_r)$ and $\bar{\bar{x}} = (x_{r+2}, \dots, x_n)$. Let $M(j) = c_{j, \gamma(j)} \bar{x}^{\bar{\gamma}(j)} x_{r+1}^{\gamma_{r+1}(j)}$ be the monomial of f_j of smallest valuation (which exists because $b_1^\bullet, \dots, b_{r+1}^\bullet$ are \mathbb{Q} -linearly independent).

To say that $M(j)$ is the monomial of f_j of smallest valuation means that for any j , $1 \leq j \leq s$,

$$\nu(\bar{x}^{\bar{\gamma}(j)}) \leq \nu(\bar{x}^{\bar{\gamma}}) \quad (58)$$

for all $\gamma = (\bar{\gamma}, \gamma_{r+1})$ such that $c_{j\gamma} \neq 0$ and if equality holds in (58) then

$$\gamma_{r+1}(j) < \gamma_{r+1}. \quad (59)$$

Take $\epsilon \in R$ sufficiently small so that for all $j \in \{1, \dots, s\}$ we have

$$\left| c_{j, \gamma(j)}(1, \dots, 1) \epsilon^{\gamma_{r+1}(j)} \right| > \sum_{\substack{\gamma \\ \gamma_{r+1} > \gamma_{r+1}(j)}} \left| c_{j, (\bar{\gamma}(j), \gamma_{r+1})}(1, \dots, 1) \epsilon^{\gamma_{r+1}} \right|.$$

Now let $\alpha = (t^{b_1}, \dots, t^{b_r}, \epsilon, 1, \dots, 1)$, then $\alpha \in U \cap S_b^*$, as desired. \square

Lemma 6.8 *For any connected component $S_{E^\dagger}^*$ of S_E^* , there exists $b \in E^\dagger$ such that*

$$\text{rat.rk}(b) = n.$$

Proof: Take $b \in E^\dagger$ such that $\text{rat.rk}(b) = \max \{ \text{rat.rk}(b^\dagger) \mid b^\dagger \in E^\dagger \}$. Suppose

$$\text{rat.rk}(b) = r < n.$$

We will now construct $b^\bullet \in \Gamma_+^n$ such that

$$S_{b^\bullet}^* \cap \bar{S}_b^* \neq \emptyset \quad (60)$$

and

$$\text{rat.rk}(b^\bullet) = r + 1. \quad (61)$$

This will contradict the maximality of r .

Let $A \rightarrow A'$ be a sequence of affine monomial blowings up, constructed in §5, such that, denoting by b' the transform of b by π , b'_1, \dots, b'_r are \mathbb{Q} -linearly independent and

$$b'_{r+1} = \dots = b'_n = 0.$$

Since the natural map $\pi^* : \text{Sper } A \rightarrow A'$ is continuous and preserves rational rank, we may replace A by A' and b by b' : if the conditions (60) and (61) are satisfied in $\text{Sper } A'$, they will still be satisfied after applying π^* to everything in sight. From now on, we drop the primes and assume that b_1, \dots, b_r are \mathbb{Q} -linearly independent and $b_{r+1} = \dots = b_n = 0$.

Now, in view of (56), b^\bullet of Lemma 6.7 satisfies (60). This completes the proof of the Lemma. \square

For $a = (a_1, \dots, a_n)$, write $a_j = (a_{j1}, \dots, a_{jn})$. Take $a \in E$ such that (a_{11}, \dots, a_{n1}) are \mathbb{Q} -linearly independent, in particular, $\text{rat.rk } a = \dim_{\mathbb{Q}}(\Gamma_a^* \otimes \mathbb{Q}) = n$ (such an a exists because the n -tuples (a_{11}, \dots, a_{n1}) such that a_{11}, \dots, a_{n1} are \mathbb{Q} -linearly independent are dense in \mathbb{R}^n).

Now suppose that S_E^* is not connected and let $S^{(1)}, S^{(2)}$ be two open and closed sets such that $S_E^* = S^{(1)} \amalg S^{(2)}$ with $S^{(1)}$ containing S_a^* . Let $E^{(1)}, E^{(2)}$ be the subsets of E such that $S^{(1)} = S_{E^{(1)}}^*$ and $S^{(2)} = S_{E^{(2)}}^*$; $E^{(1)}$ and $E^{(2)}$ exist by Corollary 6.1.

Lemma 6.9 *There exists $b^\bullet \in E^{(2)}$ such that $a_{11}, \dots, a_{n1}, b_{11}^\bullet, \dots, b_{n1}^\bullet$ are \mathbb{Q} -linearly independent.*

Proof: According to the preceding lemma, we can find $b \in E^{(2)}$ such that $\text{rat.rk}(b) = n$. Then S_b^* consists of a single point. Let U be a basic open set containing S_b^* and such that

$$U \cap S^{(1)} = \emptyset. \quad (62)$$

Let $V \subset \Gamma_+^n$ be a set satisfying the conclusion of Lemma 6.6. Since the n -tuples $(b_{11}^\bullet, \dots, b_{n1}^\bullet)$ such that $(a_{11}, \dots, a_{n1}, b_{11}^\bullet, \dots, b_{n1}^\bullet)$ are \mathbb{Q} -linearly independent are dense in R^n , there exists $b^\bullet \in E \cap V$ such that $(a_{11}, \dots, a_{n1}, b_{11}^\bullet, \dots, b_{n1}^\bullet)$ are \mathbb{Q} -linearly independent. By Lemma 6.6, $S_{b^\bullet}^* \subset U$. By (62), $b^\bullet \in E^{(2)}$, as desired. \square

Let $a \in E^{(1)}$, $b \in E^{(2)}$ be such that $a_{11}, \dots, a_{n1}, b_{11}, \dots, b_{n1}$ are \mathbb{Q} -linearly independent.

Remark: If $\lambda, \mu \in \mathbb{Q}$, $\lambda \neq \mu$, then $\lambda a_{11} + (1 - \lambda)b_{11}, \dots, \lambda a_{n1} + (1 - \lambda)b_{n1}, \mu a_{11} + (1 - \mu)b_{11}, \dots, \mu a_{n1} + (1 - \mu)b_{n1}$ are \mathbb{Q} -linearly independent.

In the sequel, let N be a large natural number and $\lambda = \frac{i}{N}$, $\mu = \frac{i+1}{N}$.

Lemma 6.10 *For $N \in \mathbb{N}$ sufficiently large we have*

$$\left(\begin{array}{c} \left(\frac{i}{N}a_{11} + \left(1 - \frac{i}{N}\right)b_{11}, 0, \dots, 0 \right) \\ \vdots \\ \left(\frac{i}{N}a_{j-1,1} + \left(1 - \frac{i}{N}\right)b_{j-1,1}, 0, \dots, 0 \right) \\ \left(\frac{i+1}{N}a_{j1} + \left(1 - \frac{i+1}{N}\right)b_{j1}, 0, \dots, 0 \right) \\ \vdots \\ \left(\frac{i+1}{N}a_{n1} + \left(1 - \frac{i+1}{N}\right)b_{n1}, 0, \dots, 0 \right) \end{array} \right) \in E.$$

for all $i \in \{0, \dots, N-1\}$ and all $j \in \{1, \dots, n\}$.

Proof: Since a_{11}, \dots, a_{n1} are \mathbb{Q} -linearly independent, saying that $\sum_{i=1}^n \omega_{il} a_i > 0$, $l \in \{1, \dots, q\}$ is equivalent to saying that

$$\sum_{i=1}^n \omega_{il} a_{i1} > 0, \quad l \in \{1, \dots, q\}.$$

Similarly, we have

$$\sum_{i=1}^n \omega_{il} b_{i1} > 0, \quad l \in \{1, \dots, q\}.$$

The set of n -tuples $(c_{11}, \dots, c_{n1}) \in \mathbb{R}^n$ such that $\sum_{i=1}^n \omega_{il} c_{i1} > 0$, $l \in \{1, \dots, q\}$, is open and convex and contains both (a_{11}, \dots, a_{n1}) and (b_{11}, \dots, b_{n1}) . Then for N sufficiently large we have

$$\begin{aligned} \sum_{i=1}^{j-1} \omega_{il} \left(\frac{i}{N}a_{i1} + \left(1 - \frac{i}{N}\right)b_{i1} \right) + \sum_{i=j}^n \omega_{il} \left(\frac{i+1}{N}a_{i1} + \left(1 - \frac{i+1}{N}\right)b_{i1} \right) = \\ \sum_{i=1}^n \omega_{il} \left(\frac{i}{N}a_{i1} + \left(1 - \frac{i}{N}\right)b_{i1} \right) + \sum_{i=j}^n \omega_{il} \left(\frac{1}{N}a_{i1} - \frac{1}{N}b_{i1} \right) > 0, \end{aligned}$$

$l \in \{1, \dots, q\}$. This completes the proof of the Lemma. \square

For each $i \in \{0, \dots, N-1\}$ and $j \in \{1, \dots, n\}$, consider the pair

$$a(i, j-1) := \begin{pmatrix} \left(\frac{i}{N}a_{11} + \left(1 - \frac{i}{N}\right)b_{11}, 0, \dots, 0\right) \\ \vdots \\ \left(\frac{i}{N}a_{j-1,1} + \left(1 - \frac{i}{N}\right)b_{j-1,1}, 0, \dots, 0\right) \\ \left(\frac{i+1}{N}a_{j1} + \left(1 - \frac{i+1}{N}\right)b_{j1}, 0, \dots, 0\right) \\ \vdots \\ \left(\frac{i+1}{N}a_{n1} + \left(1 - \frac{i+1}{N}\right)b_{n1}, 0, \dots, 0\right) \end{pmatrix}, \quad (63)$$

$$a(i, j) := \begin{pmatrix} \left(\frac{i}{N}a_{11} + \left(1 - \frac{i}{N}\right)b_{11}, 0, \dots, 0\right) \\ \vdots \\ \left(\frac{i}{N}a_{j,1} + \left(1 - \frac{i}{N}\right)b_{j,1}, 0, \dots, 0\right) \\ \left(\frac{i+1}{N}a_{j+1,1} + \left(1 - \frac{i+1}{N}\right)b_{j+1,1}, 0, \dots, 0\right) \\ \vdots \\ \left(\frac{i+1}{N}a_{n1} + \left(1 - \frac{i+1}{N}\right)b_{n1}, 0, \dots, 0\right) \end{pmatrix}. \quad (64)$$

of elements of Γ_+^n . The point of the next Proposition is to show that for each $i \in \{0, \dots, N-1\}$ and $j \in \{1, \dots, n\}$ these 2 points belong to the same set $E^{(s)}$.

Take $c = (c_1, \dots, c_n), d = (d_1, \dots, d_n) \in E$ such that $c_j = d_j$ for $j \in \{1, \dots, n-1\}$. We

define the **segment** $[c, d]$ as $[c, d] = \left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \\ e_n \end{pmatrix} \mid c_n \leq e_n \leq d_n \right\}$. Let

$$S_{[c,d]}^* = \left\{ \alpha \in S_{(c_1, \dots, c_{n-1}, e_n)}^* \mid (c_1, \dots, c_{n-1}, e_n) \in [c, d] \right\}.$$

Proposition 6.1 *Take $c, d \in E$ such that :*

- (1) $c_j = d_j$ for $j \in \{1, \dots, n-1\}$
- (2) $c_{n1} < d_{n1}$
- (3) $c_{nl} = d_{nl} = 0$ for $l \in \{2, \dots, n\}$.
- (4) $c_{11}, \dots, c_{n1}, d_{n1}$ are \mathbb{Q} -linearly independent.

Then $S_{[c,d]}^*$ is connected.

For the moment, let us assume Proposition 6.1 and finish proving the connectedness of S_E^* . Take $i \in \{0, \dots, N-1\}$ and $j \in \{1, \dots, n\}$. Up to renumbering the components c_1, \dots, c_n , the pair of points $c = a(i, j-1)$ and $d = a(i, j)$ satisfy the hypotheses of Proposition 6.1. Then Proposition 6.1 says that $S_{a(i, j-1)}^*$ and $S_{a(i, j)}^*$ are contained in the same connected component of S_E^* , so $a(i, j-1)$ and $a(i, j)$ belong to the same set $E^{(s)}$, $s = 1, 2$.

Since $b \in E^{(2)}$ and $S_b^* = S_{b_{trunc}}^*$ by linear independence of b_{11}, \dots, b_{n1} , we must have

$$b_{trunc} := \begin{pmatrix} (b_{11}, 0, \dots, 0) \\ \vdots \\ (b_{n1}, 0, \dots, 0) \end{pmatrix} \in E^{(2)}.$$

Since $b_{trunc} = a(0, n)$ and $a(i, j-1)$ and $a(i, j)$ belong to the same set $E^{(s)}$ for all i, j , we have $a(i, j) \in E^{(2)}$ for all i, j by induction on $(i, n-j)$ in the lexicographical ordering (note that

$a(i, 0) = a(i + 1, n)$). Then

$$a(N - 1, 0) = a_{trunc} := \begin{pmatrix} (a_{11}, 0, \dots, 0) \\ \vdots \\ (a_{n1}, 0, \dots, 0) \end{pmatrix} \in E^{(2)}$$

so, finally, $a \in E^{(2)}$ which contradicts the fact that a was chosen in $E^{(1)}$. Thus, the connectedness of S_E^* is reduced to proving Proposition 6.1.

Proof of Proposition 6.1: Suppose $S_{[c,d]}^*$ is not connected. By Corollary 6.1 we can write

$$[c, d] = B_1 \coprod B_2 \tag{65}$$

and

$$S_{[c,d]}^* = X_1 \coprod X_2, \tag{66}$$

such that $X_1 = S_{B_1}^*$ and $X_2 = S_{B_2}^*$ are open and closed in the relative topology of $S_{[c,d]}^*$. In view of assumption (1) of Proposition 6.1, (65) induces a decomposition of the segment $[c_n, d_n] = B_{1n} \coprod B_{2n}$.

Consider the segment $[c_{n1}, d_{n1}]$ of the real line.

Lemma 6.11 *Every point $e_{n1} \in [c_{n1}, d_{n1}]$ can be covered by a set $V(1)$, open in the topology induced on $[c_{n1}, d_{n1}]$ by the Euclidean topology of \mathbb{R} , having the following property. Let V be the subset of Γ_+^n defined by*

$$V = \{e \in [c, d] \mid e_{n1} \in V(1)\}. \tag{67}$$

Then all of $S_e^, e \in V$ are contained in the same set $X_s, s = 1, 2$.*

Lemma 6.11 implies Proposition 6.1 and with it the connectedness of S_E^* . Indeed, assume Lemma 6.11 and consider the decomposition (66). By Lemma 6.11, (66) induces a decomposition $[c_{n1}, d_{n1}] = W_1 \coprod W_2$ into two disjoint open sets in the Euclidean topology of \mathbb{R} , which gives the desired contradiction since $[c_{n1}, d_{n1}]$ is connected.

It remains to prove Lemma 6.11.

Proof of Lemma 6.11:

Case 1 : $e_{n1} \notin \sum_{j=1}^{n-1} \mathbb{Q}c_{j1}$. Let $e_n = (e_{n1}, 0, \dots, 0)$ and $e = (c_1, \dots, c_{n-1}, e_n)$. Without loss of generality, we may assume that $S_e^* \subset X_1$ (the set S_E^* consists of a single point). Take a basic open set U of $\text{Sper } A$, containing S_e^* and disjoint from X_2 . The existence of $V(1)$ with the desired properties follows immediately from Lemma 6.6.

Case 2 : $e_{n1} \in \sum_{j=1}^{n-1} \mathbb{Q}c_{j1}$.

Let $\pi : A \rightarrow A'$ be a sequence of affine monomial blowings up, constructed in §5, such that, denoting by e' the transform of e by π , $e'_{11}, \dots, e'_{1,n-1}$ are \mathbb{Q} -linearly independent and

$$e'_{n1} = 0.$$

Moreover, assuming $e_n \neq d_n$, we can choose π to be a sequence of blowings up with respect to a small half-open interval of the form $I = [e, h) \subset [c, d]$; in other words, for any $b \in [e, h)$, and any $\alpha \in S_b^*$, we have that ν_α is non-negative on A' . For any such b , we have $S_{b'}^* \cong S_b^*$ by Proposition 4.1.

Since the natural map $\pi^* : \text{Sper } A \rightarrow \text{Sper } A'$ is continuous and preserves rational rank, we will work with A', c', d', e' instead of A, c, d, e .

The decomposition (65) induces a decomposition of I and hence also a decomposition

$$I' = B'_1 \coprod B'_2,$$

such that $S_{B'_1}^*$ and $S_{B'_2}^*$ are open and closed in the induced topology of $S_{I'}^*$.

Let $e_n^\bullet = (0, 1, 0, \dots, 0)$, $e^\bullet = (e'_1, \dots, e'_{n-1}, e_n^\bullet)$. Without loss of generality, assume that $e^\bullet \in B'_1$. Let U be a basic open set of $\text{Sper } A$ having non-empty intersection with $S_{e^\bullet}^*$ and disjoint from $S_{B'_2}^*$. Now Lemma 6.7 shows that $S_{e^\bullet}^* \cap \overline{S_{e'}^*} \neq \emptyset$ and Lemma 6.6 shows that for a sufficiently small interval $V^\dagger(1) = (0, \epsilon)$ of the real line, $\epsilon > 0$, if $b'_{n1} \in V^\dagger(1)$ then $S_{b'}^* \subset U$. Let $V_+(1) = [0, \epsilon)$ and let $V_+ = \{e' \in [c', d'] \mid e'_{n1} \in V_+(1)\}$. This proves that $\bigcup_{b' \in V_+} S_{b'}^* \subset S_{B'_1}^*$.

Coming back to the original interval $[c_{n1}, d_{n1}]$ (that is, before doing the blowing up sequence π), we have shown that all the S_b^* for $b \in [e, h)$ lie in the same connected component of S_E^* , provided we take h_{n1} sufficiently close and to the right of e_{n1} . Now, assuming $e_n \neq c_n$, let J be a small half-open interval of the form $(v, e]$ with $v < e$. Repeating the above reasoning with I replaced by J , we obtain that for $V = (v, h)$, $\bigcup_{b' \in V} S_{b'}^* \subset S_{B'_1}^*$, so the open interval $V(1) = (v_{n1}, h_{n1})$ satisfies the conclusion of the Lemma.

This completes the proof of Lemma 6.11, Proposition 6.1 and the connectedness of S_E^* . \square

To complete the proof of Theorem 6.1, it remains to prove the connectedness of S_E . Since the connectedness of S_E^* is already known, it is sufficient to prove the connectedness of S_a . If $a_j > 0$ for all $j \in \{1, \dots, n\}$ then $S_a^* = S_a$ and there is nothing to prove. Assume that some of the a_j are 0. Without loss of generality, we may assume that for some p , $1 \leq p \leq n$, $a_1 = \dots = a_p = 0$ and $a_{p+1} > 0, \dots, a_n > 0$. Let

$$y_j = x_j \text{ for } j \in \{p+1, \dots, n\}, \quad (68)$$

$$= \frac{1}{x_j} \text{ for } j \in \{1, \dots, p\}. \quad (69)$$

Let $B = R[y_1, \dots, y_n]$. For $a \in \Gamma_+^n$, let

$$S_{a,B}^* = \{\delta \in \text{Sper}^*(B) \mid y_i >_\delta 0, i \in \{1, \dots, n\}, (\Gamma_\delta^*, \nu_\delta(x_1), \dots, \nu_\delta(x_1)) \cong (G, a_1, \dots, a_n)\},$$

where ν_δ is the valuation associated to δ in $\text{Sper } B$. Consider the set $S_a \cap \text{Sper } B$. For any $\delta \in S_a \cap \text{Sper } B$, all of the y_i are bounded by some real constant, hence $S_a \cap \text{Sper } B = S_{a,B}^*$, which we already know to be connected. Since both S_a^* and $S_a \cap \text{Sper } B$ are connected,

$$S_a = S_a^* \cup (S_a \cap \text{Sper } B)$$

and since the union is not disjoint (it contains, for example, the curvette $(\underbrace{1, \dots, 1}_{l \text{ ones}}, t^{a_{p+1}}, \dots, t^{a_n})$), the connectedness of S_a follows. This completes the proof of Theorem 6.1. \square

The next two examples show that the intuition that “convex sets in Γ_+^n give rise to connected sets in $\text{Sper } A$ ” is not completely accurate.

Example 1: Let $n = 2$. Let $c = (0, 0), d = (0, 1) \in \mathbb{R}_{lex}^2$. Then $S_{[c,d]}^*$ is not connected: we have $S_{[c,d]}^* = S_c^* \coprod S_{[c,d]}^*$ and both S_c^* and $S_{[c,d]}^*$ are open and closed. In particular, Lemma 6.1 is false without the assumption that $c_{11}, \dots, c_{n1}, d_{n1}$ are linearly independent. Another way of interpreting this example is that intervals consisting only of rank 1 valuations do not give rise to connected sets in $\text{Sper } A$.

Example 2: In this example, higher rank valuations appear, but, again, $c_{11}, \dots, c_{n1}, d_{n1}$ are \mathbb{Q} -linearly independent. Let $n = 3$ and take $c_1 = d_1 = e_1 = (1, 0, 0), c_2 = d_2 = e_2 = (0, 1, 0)$,

$c_3 = (0, 0, 0)$, $d_3 = (1, 0, 0)$, $e_3 = (e_{31}, e_{32}, e_{33})$, $c_3 \leq e_3 \leq d_3$. Then $c_{11}, c_{21}, c_{31}, d_{31}$ are \mathbb{Q} -linearly dependent and the segment $[c_{31}, d_{31}]$ is a disjoint union of $V^{(1)} = \{e_{31} = 0\}$ and $V^{(2)} = \{e_{31} > 0\}$. The decomposition $[c_{31}, d_{31}] = V^{(1)} \amalg V^{(2)}$ induces a decomposition $S_{[c,d]}^* = S_{V^{(1)}}^* \amalg S_{V^{(2)}}^*$ into two disjoint sets which are both open and closed in the induced topology. To see this, note that $S_{V^{(1)}}^*$ may be written as the union of basic open sets $\{f > 0\}$ where f is of the form $x_3 - x_2^q$, $q \in \mathbb{Q}_+$, while $S_{V^{(2)}}^*$ is the union of basic open sets $\{f > 0\}$ where f is of the form $f = x_1 - x_3^q$, $q \in \mathbb{Q}_+$.

References

- [1] D. Alvis, B. Johnston, J.J. Madden, *Local structure of the real spectrum of a surface, infinitely near points and separating ideals*. Preprint.
- [2] G. Birkhoff and R. Pierce, *Lattice-ordered rings*. Annales Acad. Brasil Ciênc. 28, 41–69 (1956).
- [3] J. Bochnak, M. Coste, M.-F. Roy, *Géométrie algébrique réelle*. Springer–Verlag, Berlin 1987.
- [4] C. N. Delzell, *On the Pierce–Birkhoff conjecture over ordered fields*, Rocky Mountain J. Math. 19 (1989), no. 3, 651–668.
- [5] M. Henriksen and J. Isbell, *Lattice-ordered rings and function rings*. Pacific J. Math. 11, 533–566 (1962).
- [6] I. Kaplansky, *Maximal fields with valuations I*. Duke Math. J., 9:303–321 (1942).
- [7] I. Kaplansky, *Maximal fields with valuations II*. Duke Math. J., 12:243–248 (1945).
- [8] F. Lucas, J. J. Madden, D. Schaub, M. Spivakovsky *A proof of the Pierce–Birkhoff conjecture in dimension 3*, in preparation.
- [9] J. J. Madden, *Pierce–Birkhoff rings*. Arch. Math. 53, 565–570 (1989).
- [10] J. J. Madden, *The Pierce–Birkhoff conjecture for surfaces*, unpublished preprint.
- [11] L. Mahé, *On the Pierce–Birkhoff conjecture*. Rocky Mountain J. Math. 14, 983–985 (1984).
- [12] M. Marshall *The Pierce–Birkhoff conjecture for curves*. Can. J. Math. 44, 1262–1271 (1992).
- [13] N. Schwartz, *Real closed spaces*. Habilitationsschrift, München 1984.
- [14] M. Spivakovsky, *A solution to Hironaka’s polyhedra game*. Arithmetic and Geometry, Vol II, Papers dedicated to I. R. Shafarevich on the occasion of his sixtieth birthday, M. Artin and J. Tate, editors, Birkhäuser, 1983, pp. 419–432. Preprint.