



HAL
open science

Strong disorder implies strong localization for directed polymers in a random environment

Philippe Carmona, Yueyun Hu

► **To cite this version:**

Philippe Carmona, Yueyun Hu. Strong disorder implies strong localization for directed polymers in a random environment. 2006. hal-00018048v1

HAL Id: hal-00018048

<https://hal.science/hal-00018048v1>

Preprint submitted on 27 Jan 2006 (v1), last revised 21 Jun 2006 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

STRONG DISORDER IMPLIES STRONG LOCALIZATION FOR DIRECTED POLYMERS IN A RANDOM ENVIRONMENT

PHILIPPE CARMONA AND YUEYUN HU

ABSTRACT. In this note we show that in any dimension d , the strong disorder property implies the strong localization property. This is established for a continuous time model of directed polymers in a random environment : the parabolic Anderson Model.

1. INTRODUCTION

Let $\omega = (\omega(t))_{t \geq 0}$ be the simple continuous time random walk on the d -dimensional lattice \mathbb{Z}^d , with jump rate $\kappa > 0$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider an *environment* $B = (B_x(t), t \geq 0, x \in \mathbb{Z}^d)$ made of independent standard Brownian motions B_x defined on another probability space $(H, \mathcal{G}, \mathbf{P})$.

For any $t > 0$ the (random) *polymer measure* μ_t is the probability defined on the path space (Ω, \mathcal{F}) by

$$(1) \quad \mu_t(d\omega) = \frac{1}{Z_t} e^{\beta H_t(\omega) - t\beta^2/2} \mathbb{P}(d\omega)$$

where $\beta \geq 0$ is the inverse temperature, the Hamiltonian is

$$(2) \quad H_t(\omega) = \int_0^t dB_{\omega(s)}(s)$$

and the *partition function* is

$$(3) \quad Z_t = Z_t(\beta) = \mathbb{E} \left[e^{\beta H_t(\omega) - t\beta^2/2} \right],$$

where $\mathbb{E}[\cdot]$ denotes expectation with respect to \mathbb{P} .

Erwin Bolthausen [2] was the first to establish that $(Z_t)_{t \geq 0}$ was a positive martingale, converging almost surely to a finite random variable Z_∞ , satisfying a zero-one law : $\mathbf{P}(Z_\infty > 0) \in \{0, 1\}$. We shall say that there is *strong disorder* if $Z_\infty = 0$ almost surely, and *weak disorder* if $Z_\infty > 0$ almost surely.

Another martingale argument, based on a supermartingale decomposition of $\log Z_t$, enabled Carmona-Hu [5], then Comets-Shiga-Yoshida [6,

Date: January 27, 2006.

1991 Mathematics Subject Classification. primary 60K35, secondary 60J30, 60H30, 60H15 .

Key words and phrases. Directed polymers, Random environment, strong disorder, localization, parabolic Anderson.

[7], and Rovira-Tindel [9], to show the equivalence between strong disorder and *weak-localization* :

$$(4) \quad Z_\infty = 0 \text{ a.s.} \iff \int_0^\infty \mu_t^{(2)}(\omega_1(t) = \omega_2(t)) dt = +\infty \text{ a.s.},$$

where ω_1, ω_2 are two independent copies of the random walk ω , considered under the product polymer measure $\mu_t^{(2)}$:

$$(5) \quad \mu_t^{(2)}(d\omega_1, d\omega_2) = \frac{1}{Z_t^2} e^{\beta(H_t(\omega_1) + H_t(\omega_2)) - t\beta^2} \mathbb{P}^{(2)}(d\omega_1, d\omega_2).$$

Let us define *strong localization* as the existence of a constant $c > 0$ such that

$$(6) \quad \limsup_{t \rightarrow +\infty} \sup_x \mu_t(\omega(t) = x) \geq c \text{ a.s.}$$

This property implies the existence of highly favored sites, in contrast to the simple random walk ($\beta = 0$) for which $\sup_x \mathbb{P}(X_t = x) \sim Ct^{-d/2} \rightarrow 0$. Carmona-Hu [5], and then Comets-Shiga-Yoshida [7], showed that in dimension $d = 1, 2$, for any $\beta > 0$, there was not only strong disorder but also strong localization.

We shall prove in this note the

Theorem 1. *In any dimension d , strong disorder implies strong localization.*

For sake of completeness, let us state yet another localization property. The *free energy* is the limit

$$(7) \quad p(\beta) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log Z_t,$$

where the limit can be shown to hold almost surely in every L^p , $p \geq 1$ (see e.g. [7, 3]). The function $p(\beta)$ is continuous, decreasing, $p(\beta) \leq 0$, $p(0) = 0$, so there exists a critical inverse temperature $\beta_c \in [0, +\infty]$ such that:

$$(8) \quad \begin{cases} p(\beta) = 0 & \text{if } 0 \leq \beta \leq \beta_c \\ p(\beta) < 0 & \text{if } \beta > \beta_c \end{cases}$$

When $p(\beta) < 0$ we say that the system enjoys the *very strong disorder* property. It is easily proved that (see equation (19)):

$$(9) \quad p(\beta) < 0 \iff \exists c > 0, \quad \liminf \frac{1}{t} \int_0^t \mu_s^{(2)}(\omega_1(s) = \omega_2(s)) ds \geq c.$$

The recent beautiful result of Comets-Vargas [8], that is $\beta_c = 0$ in dimension $d = 1$, strengthen our belief in the

$$(10) \quad \text{Conjecture : } \text{very strong disorder} \iff \text{strong disorder}$$

Proving this conjecture would unify all these notions of disorder and localization.

Eventually, let us end this rather lengthy introduction by making clearer the connection with the parabolic Anderson model. The point to point partition functions

$$(11) \quad Z_t(x, y) = \mathbb{E}_x \left[e^{\beta H_t(\omega) - t\beta^2/2} \mathbf{1}_{(\omega(t)=y)} \right]$$

satisfy the stochastic partial differential equation (see section 2)

$$(12) \quad dZ_t(0, x) = LZ_t(0, \cdot)(x) dt + Z_t(0, x) dB_x(t),$$

where $L = \kappa\Delta$ is the generator of the simple random walk ω with jump rate κ , that is Δ is the discrete Laplacian.

2. THE PARTITION FUNCTION

Without loss in generality we can work on the canonical path space Ω made of $\omega : \mathbb{R}^+ \rightarrow \mathbb{Z}^d$, càdlàg, with a finite number of jumps in each finite interval $[0, t]$. We endow Ω with the canonical sigma-field \mathcal{F} and the family of laws $\mathbb{P}_x, x \in \mathbb{Z}^d$ such that under $\mathbb{P}_x, (\omega(t))_{t \geq 0}$ is the simple random walk starting from x , with generator $L = \kappa\Delta$. With these notations, we consider, attached to each path $\omega \in \Omega$, the exponential martingale

$$(13) \quad M_t^\omega = \exp(\beta H_t(\omega) - t\beta^2/2) = 1 + \beta \int_0^t M_s^\omega dB_{\omega(s)}(s),$$

with respect to the filtration $\mathcal{G}_t = \sigma(B_x(s), s \leq t, x \in \mathbb{Z}^d)$. We have $Z_t = \mathbb{E}[M_t^\omega]$ and thus the

Proposition 2. *The process $(Z_t)_{t \geq 0}$ is a continuous positive \mathcal{G}_t martingale with quadratic variation*

$$(14) \quad d\langle Z, Z \rangle_t = Z_t^2 \beta^2 I_t dt, \quad \text{with } I_t = \mu_t^{(2)}(\omega_1(t) = \omega_2(t)).$$

Proof. We know that linear combinations of martingales are martingales. this extends easily to probability mixtures of martingales. Indeed, let $0 \leq s \leq t$ and let U be positive bounded and \mathcal{G}_s measurable. Then, by Fubini-Tonelli's theorem :

$$\begin{aligned} \mathbf{E}[Z_t U] &= \mathbf{E}[\mathbb{E}[M_t^\omega] U] = \mathbf{E}[\mathbf{E}[M_t^\omega U]] \\ &= \mathbf{E}[\mathbf{E}[M_s^\omega U]] && (M^\omega \text{ is a martingale}) \\ &= \mathbf{E}[\mathbb{E}[M_s^\omega] U] = \mathbf{E}[Z_s U]. \end{aligned}$$

Observe that if ω_1, ω_2 are paths, then we can compute the quadratic covariation

$$(15) \quad d\langle M^{\omega_1}, M^{\omega_2} \rangle_t = M_t^{\omega_1} M_t^{\omega_2} \beta^2 \mathbf{1}_{(\omega_1(t)=\omega_2(t))} dt.$$

Therefore, we have formally:

$$\begin{aligned}
d\langle Z, Z \rangle_t &= d\left\langle \int \mathbb{P}(d\omega_1) M^{\omega_1}, \int \mathbb{P}(d\omega_2) M^{\omega_2} \right\rangle_t \\
&= \int \mathbb{P}^{(2)}(d\omega_1, d\omega_2) d\langle M^{\omega_1}, M^{\omega_2} \rangle_t \\
&= \beta Z_t^2 \frac{1}{Z_t^2} \int \mathbb{P}^{(2)}(d\omega_1, d\omega_2) M_t^{\omega_1} M_t^{\omega_2} \mathbf{1}_{(\omega_1(t)=\omega_2(t))} dt \\
&= Z_t^2 \beta^2 I_t dt.
\end{aligned}$$

This again can be made rigorous by writing $N_t = Z_t^2 - \beta^2 \int_0^t Z_s^2 I_s ds$ as a probability mixture of martingales:

(16)

$$N_t = \int \mathbb{P}^{(2)}(d\omega_1, d\omega_2) (M_t^{\omega_1} M_t^{\omega_2} - \beta^2 \int_0^t M_s^{\omega_1} M_s^{\omega_2} \mathbf{1}_{(\omega_1(s)=\omega_2(s))} ds).$$

□

The positive martingale Z_t converges almost surely to a positive finite random variable Z_∞ . We refer to any of [2, 7, 4, 3] for a proof of the zero-one law

Proposition 3.

$$\mathbf{P}(Z_\infty = 0) \in [0, 1].$$

We can now show the equivalence between strong disorder and weak localization

Proposition 4. *The supermartingale $\log Z_t$ has the decomposition*

$$(17) \quad \log Z_t = M_t - \frac{1}{2} A_t$$

with $(M_t)_{t \geq 0}$ a continuous martingale of quadratic variation

$$(18) \quad \langle M, M \rangle_t = A_t = \beta^2 \int_0^t I_s ds.$$

Consequently:

- either $Z_\infty = 0$ and $\int_0^\infty I_s ds = +\infty$ almost surely;
- or $Z_\infty > 0$ and $\int_0^\infty I_s ds < +\infty$ almost surely.

In both cases the free energy is given by

$$(19) \quad p(\beta) = -\frac{\beta^2}{2} \lim \frac{1}{t} \int_0^t I_s ds = -\frac{\beta^2}{2} \lim \frac{1}{t} \int_0^t \mathbf{E}[I_s] ds.$$

Proof. One can even prove (see [4]) that weak disorder is equivalent to the uniform integrability of the martingale $(Z_t)_{t \geq 0}$.

Itô's formula yields :

$$(20) \quad \log Z_t = \int_0^t \frac{dZ_s}{Z_s} - \frac{1}{2} \int_0^t \frac{d\langle Z, Z \rangle_s}{Z_s^2} = M_t - \frac{1}{2} \beta^2 \int_0^t I_s ds = M_t - \frac{1}{2} A_t.$$

Therefore,

- On $\{A_\infty = \langle M, M \rangle_\infty < +\infty\}$ the martingale M_t converges almost surely, $M_t \rightarrow M_\infty$ so $\log Z_t \rightarrow M_\infty - \frac{1}{2}A_\infty$ and $Z_\infty > 0$ almost surely, and $p(\beta) = \lim \frac{1}{t} \log Z_t = 0$.
- On $\{A_\infty = \langle M, M \rangle_\infty = +\infty\}$, we have almost surely $\frac{M_t}{\langle M, M \rangle_t} \rightarrow 0$ so $\frac{\log Z_t}{A_t} \rightarrow -\frac{1}{2}$ and $\log Z_t \rightarrow -\infty$, so $Z_\infty = 0$. Furthermore, $p(\beta) = \lim \frac{1}{t} \log Z_t = -\frac{1}{2} \lim \frac{1}{t} A_t$.

We conclude this proof by taking expectations:

$$p(\beta) = \lim \frac{1}{t} \mathbf{E}[\log Z_t] = -\frac{1}{2} \lim \frac{1}{t} \mathbf{E}[A_t] = -\frac{\beta^2}{2} \lim \frac{1}{t} \int_0^t \mathbf{E}[I_s] ds.$$

□

The connection with the parabolic Anderson model is contained in the

Proposition 5. *The point to point partition functions $(Z_t(0, x), t \geq 0, x \in \mathbb{Z}^d)$*

satisfy the stochastic partial differential equation

$$dZ_t(0, x) = LZ_t(0, \cdot)(x) dt + Z_t(0, x) dB_x(t),$$

Proof. Let $p_t(x) = \mathbb{P}(X_t = x)$ be the probability function at time t of simple Random Walk. By Fubini's stochastic theorem and Markov property:

$$\begin{aligned} Z_t(0, x) &= \int \mathbb{P}(d\omega) M_t^\omega \mathbf{1}_{(\omega(t)=x)} \\ &= \int \mathbb{P}(d\omega) \mathbf{1}_{(\omega(t)=x)} (1 + \beta \int_0^t M_s^\omega dB_{\omega(s)}(s)) \\ &= p_t(x) + \beta \int_0^t \int \mathbb{P}(d\omega) \mathbf{1}_{(\omega(t)=x)} M_s^\omega dB_{\omega(s)}(s) \\ &= p_t(x) + \beta \int_0^t \int \mathbb{P}(d\omega) p_{t-s}(\omega(s) - x) M_s^\omega dB_{\omega(s)}(s) \\ &= p_t(x) + \beta \int_0^t Z_s \mu_s(p_{t-s}(\omega(s) - x) dB_{\omega(s)}(s)). \end{aligned}$$

We conclude by differentiating with respect to t , taking into account that

$$\frac{d}{dt} p_t(x) = Lp_t(x)$$

In other words, we combine

$$p_{t-s}(y) = \mathbf{1}_{(y=0)} + \int_s^t Lp_{u-s}(y) du$$

and Fubini's stochastic theorem. (This result is just Feynman-Kac formula combined with time reversal of the continuous time random walk). □

3. ITÔ'S FORMULA FOR THE POLYMER MEASURE

Let $(P_t^{(n)})_{t \geq 0}$ be the semi-group of the Markov process $\boldsymbol{\omega}(t) = (\omega_1(t), \dots, \omega_n(t))$ constructed from n independent copies of the simple Random Walk $(\omega(t))_{t \geq 0}$: if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a bounded borel function, then

$$P_t^{(n)} f(x_1, \dots, x_n) = \mathbb{E}_{x_1, \dots, x_n} [f(\omega_1(t), \dots, \omega_n(t))].$$

Theorem 6. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a bounded borel function, and $t \geq t_0 \geq 0$. Then,*

$$\begin{aligned} \mu_t[f(\boldsymbol{\omega}(t))] &= \mu_{t_0} \left[P_{t-t_0}^{(n)} f(\boldsymbol{\omega}(t_0)) \right] \\ &+ \beta^2 \sum_{i < j} \int_{t_0}^t \mu_s^{(n)} \left[\mathbf{1}_{(\omega_i(s)=\omega_j(s))} P_{t-s}^{(n)} f(\boldsymbol{\omega}(s)) \right] ds \\ &- n\beta^2 \sum_i \int_{t_0}^t \mu_s^{(n+1)} \left[\mathbf{1}_{(\gamma(s)=\omega_i(s))} P_{t-s}^{(n)} f(\boldsymbol{\omega}(s)) \right] ds \\ &+ \frac{n(n+1)}{2} \beta^2 \int_{t_0}^t \mu_s^{(n)} \left[P_{t-s}^{(n)} f(\boldsymbol{\omega}(s)) \right] I_s ds \\ &+ \int_{t_0}^t \mu_s^{(n)} \left[P_{t-s}^{(n)} f(\boldsymbol{\omega}(s)) \left(\beta \sum_i dB_{\omega_i(s)}(s) - n \frac{dZ(s)}{Z(s)} \right) \right], \end{aligned}$$

where γ is an extra independent copy of ω .

Proof. Given paths $\omega_1, \dots, \omega_n$, we let

$$U_t = U_t(\omega_1, \dots, \omega_n) = \frac{M_t^{\omega_1} \dots M_t^{\omega_n}}{Z_t^n}.$$

We use the following easy computations of quadratic variations:

$$\begin{aligned} d\langle M^\gamma, M^\gamma \rangle_t &= M_t^\gamma M_t^\gamma \beta^2 \mathbf{1}_{(\gamma(t)=\gamma(t))} dt \\ d\langle M^\gamma, Z \rangle_t &= \beta^2 M_t^\gamma Z_t \mu_t \left[\mathbf{1}_{(\omega(t)=\gamma(t))} \right] dt, \quad d\langle Z, Z \rangle_t = Z_t^2 \beta^2 I_t dt, \end{aligned}$$

The classical Itô's formula yields:

$$\begin{aligned} U_t &= U_{t_0} + \int_{t_0}^t U(s) \left(\sum_{i=1}^n \beta dB_{\omega_i(s)}(s) - nU(s) \frac{dZ(s)}{Z(s)} \right) \\ &+ \beta^2 \int_{t_0}^t U(s) \left(\sum_{i < j} \mathbf{1}_{(\omega_i(s)=\omega_j(s))} - n \sum_i \mu_s \left[\mathbf{1}_{(\gamma(s)=\omega_i(s))} \right] + \frac{n(n+1)}{2} \right) I_s ds \end{aligned}$$

Since,

$$\mu_t[f(\boldsymbol{\omega}(t))] = \int \mathbb{P}^{(n)} f(\boldsymbol{\omega}(t)) U_t(\boldsymbol{\omega})$$

we conclude this proof by applying Fubini's theorem and Markov's property. For example,

$$\begin{aligned}
 \int \mathbb{P}^{(n)} f(\boldsymbol{\omega}(t)) U_{t_0} &= \mathbb{E} \left[f(\boldsymbol{\omega}(t)) \frac{M_{t_0}^{\omega_1} \cdots M_{t_0}^{\omega_n}}{Z(t_0)^n} \right] \\
 &= \frac{1}{Z(t_0)^n} \mathbb{E} \left[P_{t-t_0}^{(n)} f(\boldsymbol{\omega}(t_0)) M_{t_0}^{\omega_1} \cdots M_{t_0}^{\omega_n} \right] \\
 &= \mu_{t_0} \left[P_{t-t_0}^{(n)} f(\boldsymbol{\omega}(t_0)) \right].
 \end{aligned}$$

□

4. PROOF OF THE MAIN RESULT

We assume that there is strong disorder so almost surely, $Z_\infty = 0$ and $\int_0^\infty I_s ds = +\infty$, and we shall show that for a certain $c_0 > 0$, $\limsup V_t \geq c_0$ almost surely, with $V_t = \sup_x \mu_t(\omega(t) = x)$.

Let $r(t) = \mathbb{P}(\omega_1(t) = \omega_2(t))$ and $R(t) = \int_0^t r(s) ds$. In dimension $d = 1, 2$, $R(\infty) = +\infty$ so certainly $\beta^2 R(\infty) > 1$. In dimension $d \geq 3$, $R(\infty) < +\infty$ and Markov's property implies that $L_\infty = \int_0^\infty \mathbf{1}_{(\omega_1(s) = \omega_2(s))} ds$ is under $\mathbb{P}_{0,0}$ an exponential random variable of expectation $R(\infty)$. Since, by Fubini's theorem,

$$\begin{aligned}
 \mathbf{E}[Z_t^2] &= \mathbb{E} \left[\mathbf{E} \left[e^{\beta(H_t(\omega_1) + H_t(\omega_2)) - t\beta^2} \right] \right] \\
 &= \mathbb{E} \left[e^{\frac{\beta^2}{2} \text{Var}(H_t(\omega_1) + H_t(\omega_2)) - t\beta^2} \right] \\
 &= \mathbb{E} \left[e^{\beta^2 \int_0^t \mathbf{1}_{(\omega_1(s) = \omega_2(s))} ds} \right],
 \end{aligned}$$

the second moment method yields that if $\beta^2 R(\infty) < 1$, then $\sup_t \mathbf{E}[Z_t^2] = \mathbb{E} \left[e^{\beta^2 L_\infty} \right] < +\infty$, so Z_t is an L^2 bounded martingale, hence $\mathbb{E}[Z_\infty] = 1$ and $Z_\infty > 0$ almost surely. Birkner [1] improved this result by using a conditional moment method : if $R(\infty) < +\infty$, then there exists $\beta_c^- > \frac{1}{\sqrt{R(\infty)}}$ such that for $\beta < \beta_c^-$, $Z_\infty > 0$ almost surely. Hence, since we assumed strong disorder, we can assume that $\beta^2 R(\infty) > 1$.

Observe that since $V_t = \sup_x U_t(x)$ with $U_t(x) = \mu_t(\omega(t) = x)$, we have

$$\begin{aligned}
 I_t &= \mu_t^{(2)}(\omega_1(t) = \omega_2(t)) = \sum_x \mu_t^{(2)}(\omega_1(t) = x = \omega_2(t)) \\
 &= \sum_x U_t(x)^2 \leq V_t \sum_x U_t(x) = V_t
 \end{aligned}$$

and $I_t \geq V_t^2$. Therefore we shall show that almost surely, $\limsup I_t \geq c_0$. It is sufficient to prove that if $J_t = I_t \mathbf{1}_{(I_t \geq c_0)}$ then for a constant $c_1 > 0$,

$$(21) \quad \liminf_{t \rightarrow +\infty} \frac{\int_0^t J_s ds}{\int_0^t I_s ds} \geq c_1 \quad \text{almost surely}$$

(indeed recall that $\int_0^\infty I_s ds = +\infty$ almost surely).

We now have to choose $c_0 > 0$. Since $\beta^2 R(\infty) > 1$, there exists $\epsilon_0 \in (0, \frac{1}{16})$ and $t_0 > 0$ such that $\beta^2 R(t_0)(1 - 4\sqrt{\epsilon_0}) > 1$. We let $c_0 = \epsilon_0 \inf_{0 \leq t \leq t_0} r(t)$.

Let us apply now Itô's formula of Theorem 6, between $t - t_0$ and t , to the function $f(x_1, x_2) = \mathbf{1}_{(x_1=x_2)}$:

$$\begin{aligned} I_t &= \mu_t(f(\boldsymbol{\omega}(t))) = N_{t_0, t} + \mu_{t-t_0} \left[P_{t_0}^{(2)} f(\boldsymbol{\omega}(t-t_0)) \right] \\ &+ \beta^2 \int_{t-t_0}^t \mu_s^{(2)}(P_{t-s}^{(2)} f(\boldsymbol{\omega}(s)) \mathbf{1}_{(\omega_1(s)=\omega_2(s))}) ds \\ &- 2\beta^2 \int_{t-t_0}^t \mu_s^{(3)}(P_{t-s}^{(2)} f(\boldsymbol{\omega}(s)) (\mathbf{1}_{(\gamma(s)=\omega_1(s))} + \mathbf{1}_{(\gamma(s)=\omega_2(s))})) ds \\ &+ 3\beta^2 \int_{t-t_0}^t \mu_s^{(2)}(P_{t-s}^{(2)} f(\boldsymbol{\omega}(s))) I_s ds, \end{aligned}$$

where

$$N_{t_0, t} = \int_{t-t_0}^t \mu_s^{(2)}(P_{t-s}^{(2)} f(\boldsymbol{\omega}(s))) (\beta \sum_i dB_{\omega_i(s)}(s) - 2 \frac{dZ_s}{Z_s}).$$

Since $0 \leq P_t^{(2)} f(x_1, x_2) \leq r(t) = P_t^{(2)} f(x, x) \leq 1$, we have

$$(22) \quad I_t \geq N_{t_0, t} + \beta^2 \int_{t-t_0}^t r(t-s) I_s ds - 4\beta^2 \int_{t-t_0}^t \mu_s^{(3)}(P_{t-s}^{(2)} f(\boldsymbol{\omega}(s)) \mathbf{1}_{(\gamma(s)=\omega_1(s))}) ds.$$

Claim 1 :

$$(23) \quad \mu_s^{(3)}(P_{t-s}^{(2)} f(\boldsymbol{\omega}(s)) \mathbf{1}_{(\gamma(s)=\omega_1(s))}) \leq I_s \inf(\sqrt{I_s r(t-s)}, r(t-s)).$$

Indeed with $U_s(x) = \mu_s(\omega(s) = x)$ we have $I_s = \sum_x U_s(x)^2$, therefore

$$\begin{aligned} \mu_s^{(3)}(P_{t-s}^{(2)} f(\boldsymbol{\omega}(s)) \mathbf{1}_{(\gamma(s)=\omega_1(s))}) &= \sum_x U_s(x)^2 \mu_s(P_{t-s}^{(2)} f(x, \omega(s))) \\ \mu_s(P_{t-s}^{(2)} f(x, \omega(s))) &= \sum_y U_s(y) P_{t-s}^{(2)} f(x, y) \leq r(t-s) \sum_y U_s(y) = r(t-s). \end{aligned}$$

We also have, by Cauchy Schwarz,

$$\begin{aligned} \mu_s(P_{t-s}^{(2)} f(x, \omega(s))) &\leq \left(\sum_y U_s(y)^2 \sum_y (P_{t-s}^{(2)} f(x, y))^2 \right)^{\frac{1}{2}} \\ &= \sqrt{I_s r(2(t-s))} \leq \sqrt{I_s r(t-s)}, \end{aligned}$$

since if $\tilde{\omega}(t) = \omega_1(t) - \omega_2(t)$ we have, thanks to Markov property and symmetry,

$$\begin{aligned}
 r(2t) &= \mathbb{P}(\tilde{\omega}(t) = 0) = \sum_y \mathbb{P}_0(\tilde{\omega}(t) = y) \mathbb{P}_y(\tilde{\omega}(t) = 0) = \sum_y \mathbb{P}_0(\tilde{\omega}(t) = y)^2 \\
 &= \sum_y P_t^{(2)} f(0, y)^2 = \sum_y P_t^{(2)} f(x, y)^2
 \end{aligned}$$

Claim 2 :

(24)

$$4\beta^2 R(t_0) \int_0^T J(s) ds + \int_{t_0}^T I(s) ds \geq \int_{t_0}^T N_{t_0,t} dt + \beta^2(1-4\sqrt{\epsilon_0})R(t_0) \int_{t_0}^{T-t_0} I(s) ds.$$

Observe that when $I_s \leq c_0$ and $t - t_0 \leq s \leq t$, we have $I_s \leq \epsilon_0 r(t - s)$, therefore, from Claim 1 we deduce that,

$$\begin{aligned}
 \int_{t-t_0}^t \mu_s^{(3)}(P_{t-s}^{(2)} f(\boldsymbol{\omega}(s)) \mathbf{1}_{(\gamma(s)=\omega_1(s))}) ds &\leq \int_{t-t_0}^t I_s \sqrt{I_s r(t-s)} \mathbf{1}_{(I_s \leq c_0)} ds \\
 &\quad + \int_{t-t_0}^t r(t-s) I_s \mathbf{1}_{(I_s > c_0)} ds \\
 &\leq \sqrt{\epsilon_0} \int_{t-t_0}^t r(t-s) I(s) ds \\
 &\quad + \int_{t-t_0}^t r(t-s) J(s) ds.
 \end{aligned}$$

Plugging this inequality into (22) yields

(25)

$$I_t \geq N_{t_0,t} + \beta^2(1-4\sqrt{\epsilon_0}) \int_{t-t_0}^t r(t-s) I(s) ds - 4\beta^2 \int_{t-t_0}^t r(t-s) J(s) ds.$$

Given $T \geq t_0$, we are going to integrate this inequality between t_0 and T . On the one hand,

$$\int_{t_0}^T \int_{t-t_0}^t r(t-s) J(s) ds \leq \int_0^T J(s) ds \int_0^{t_0} r(u) du = R(t_0) \int_0^T J(s) ds.$$

On the other hand,

$$\int_{t_0}^T \int_{t-t_0}^t r(t-s) I(s) ds \geq \int_{t_0}^{T-t_0} I(s) ds \int_0^{t_0} r(u) du = R(t_0) \int_{t_0}^{T-t_0} I(s) ds.$$

The claim follows immediately.

Claim 3 : let $\mathcal{N}_T = \int_{t_0}^T N_{t_0,t} dt$. Then

$$\frac{\mathcal{N}_T}{\int_0^T I_s ds} \rightarrow 0 \quad \text{in probability.}$$

Let us defer the proof of this claim. Since $0 \leq I_s \leq 1$ and $\int_0^\infty I_s ds = +\infty$, we have,

$$\lim \frac{\int_{t_0}^T I_s ds}{\int_0^T I_s ds} = \lim \frac{\int_{t_0}^{T-t_0} I_s ds}{\int_0^T I_s ds} = 1$$

if we divide (24) by $\int_0^T I_s ds$ and let $T \rightarrow +\infty$, we obtain

$$R(t_0)4\beta^2 \liminf \frac{\int_0^T J(s)ds}{\int_0^T I_s ds} + 1 \geq \beta^2(1 - 4\sqrt{\epsilon_0})R(t_0)$$

this yields

$$\liminf \frac{\int_0^T J(s)ds}{\int_0^T I_s ds} \geq c_1 \quad a.s.$$

with $c_1 = \frac{\beta(1-4\sqrt{\epsilon_0})R(t_0)-1}{4\beta^2 R(t_0)}$.

Proof of Claim 3

By Fubini's theorem,

$$\begin{aligned} \mathcal{N}_T &= \int_{t_0}^T dt \int_{t-t_0}^t \mu_s \left[P_{t-s}^{(2)} f(\omega_1(s), \omega_2(s)) \left(\sum_i \beta dB_{\omega_i(s)}(s) - 2 \frac{dZ(s)}{Z(s)} \right) \right] \\ &= \int_0^T \mu_s \left[G(s, \omega_1(s), \omega_2(s)) \left(\sum_i \beta dB_{\omega_i(s)}(s) - 2 \frac{dZ(s)}{Z(s)} \right) \right], \end{aligned}$$

with

$$G(s, x_1, x_2) := \int_{(t_0-s)^+}^{(T-s)^+ \wedge t_0} P_{t-s}^{(2)}(x_1, x_2) \leq t_0, \quad \forall x_1, x_2 \in \mathbb{Z}^d.$$

A direct computation of the variation of the martingale \mathcal{N}_T gives that

$$\langle \mathcal{N}, \mathcal{N} \rangle_T \leq 4\beta^2 \int_0^T \mu_s \left[G(s, \omega_1(s), \omega_2(s)) G(s, \omega_3(s), \omega_4(s)) (1_{(\omega_1(s)=\omega_3(s))} + I_s) \right] ds,$$

which yields that

$$\langle \mathcal{N}, \mathcal{N} \rangle_T \leq 8\beta^2 t_0^2 \int_0^T I_s ds.$$

Since $\int_0^\infty I(s) ds = +\infty$ we have $\mathcal{N}_T / (\int_0^T I(s) ds) \rightarrow 0$, almost surely as $T \rightarrow +\infty$.

REFERENCES

- [1] M. BIRKNER, *A condition for weak disorder for directed polymers in random environment*, Electron. Comm. Probab., 9 (2004), pp. 22–25 (electronic).
- [2] E. BOLTHAUSEN, *A note on the diffusion of directed polymers in a random environment*, Comm. Math. Phys., 123 (1989), pp. 529–534.
- [3] P. CARMONA, *Minicours sur les polymères dirigés en milieu aléatoire*, tech. rep., Laboratoire de Mathématiques Jean Leray, 2005. Cours de DEA.

- [4] P. CARMONA AND Y. HU, *On the partition function of a directed polymer in a Gaussian random environment*, Probab. Theory Related Fields, 124 (2002), pp. 431–457.
- [5] ———, *Fluctuation exponents and large deviations for directed polymers in a random environment*, Stochastic Process. Appl., 112 (2004), pp. 285–308.
- [6] F. COMETS, T. SHIGA, AND N. YOSHIDA, *Directed polymers in a random environment: path localization and strong disorder*, Bernoulli, 9 (2003), pp. 705–723.
- [7] ———, *Probabilistic analysis of directed polymers in a random environment: a review*, in Stochastic analysis on large scale interacting systems, vol. 39 of Adv. Stud. Pure Math., Math. Soc. Japan, Tokyo, 2004, pp. 115–142.
- [8] F. COMETS AND V. VARGAS, *Majorizing multiplicative cascades for directed polymers in random media*. arXiv:math.PR/0510525.
- [9] C. ROVIRA AND S. TINDEL, *On the Brownian-directed polymer in a Gaussian random environment*, J. Funct. Anal., 222 (2005), pp. 178–201.

PHILIPPE CARMONA, LABORATOIRE JEAN LERAY, UMR 6629, UNIVERSITÉ DE NANTES, BP 92208, F-44322 NANTES CEDEX 03 BP
E-mail address: philippe.carmona@math.univ-nantes.fr

YUEYUN HU, DÉPARTEMENT DE MATHÉMATIQUES, (INSTITUT GALILÉE, L.A.G.A. UMR 7539), UNIVERSITÉ PARIS 13 99 Av. J-B CLÉMENT, 93430 VILLETANEUSE, FRANCE
E-mail address: yueyun@math.univ-paris13.fr