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Reflecting a Langevin Process at an Absorbing Boundary

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Summary. We consider a Langevin process with white noise random forcing. We suppose that the energy of the particle is instantaneously absorbed when it hits some fixed obstacle. We show that nonetheless, the particle can be instantaneously reflected, and study some properties of this reflecting solution.

Key words. Langevin process, absorbing boundary, reflection, excursions.

A.M.S. Classification. `Primary 60 J 50, 60 J 55. Secondary 60 G 18, 60 G 52`

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1 Introduction

Langevin [8] introduced a probabilistic model to describe the evolution of a particle under random external forcing. Langevin's motion has smooth trajectories, in the sense that if Y_t stands for the position of the particle at time t, then the velocity $\dot{Y}_t := \mathrm{d}Y_t/\mathrm{d}t$ is well-defined and finite everywhere. We assume that the external force (i.e. the derivative of the velocity) is a white noise, so that

$$\dot{Y}_t = \dot{Y}_0 + W_t\,,$$

where $W = (W_t, t \geq 0)$ is a standard Wiener process[1], and Y_t can then be expressed in terms of the integral of the latter. Plainly the Langevin process $Y = (Y_t, t \geq 0)$ is not Markovian, however the pair $(Y, \dot{Y})$, which is often called Kolmogorov's process, is Markov. We refer to Lachal [7] for an interesting introduction to this field, historical comments, and a long list of references.

Recently, Maury [9] has considered the situation when the particle may hit an obstacle and

[1] We mention that Langevin considered more generally the case when the velocity is given by an Ornstein-Uhlenbeck process.

then its velocity is instantaneously absorbed. Roughly, this arises in a model for the motion of individuals during the formation of a crowd. For the sake of simplicity, we shall work in dimension 1 and assume that the particle evolves in the half-line $[0, \infty[$. As long as the particle remains in $]0, \infty[$, its dynamics are those described above. When the particle reaches the boundary point 0, we suppose that its energy is absorbed, in the sense that the velocity instantaneously drops to 0.

There is an obvious way to construct a process that fulfills these requirements. Namely, given an initial position $x \geq 0$ and an initial velocity $v \in \mathbb{R}$, if we consider the free Langevin process (i.e. when there is no obstacle) started from x with velocity v

$$Y_t = x + \int_0^t (v + W_s)\mathrm{d}s$$

and define

$$\zeta = \inf\{t \geq 0 : Y_t = 0\}\,,$$

then we can take $(Y_{t\wedge\zeta}, t \geq 0)$. In the sequel, we shall refer to this solution as the Langevin process stopped at 0.

It is easy to show that when $x = v = 0$, the free Langevin process $Y = (Y_t, t \geq 0)$ returns to 0 at arbitrarily small times, and that its velocity at such return times is never 0. In the presence of an obstacle, this suggests that reseting the velocity to 0 at every hitting time of the boundary might impede the particle to ever exit 0, so that the Langevin process stopped at 0 might be the unique solution.

The purpose of this work is to point at the rather surprising fact (at least for the author) that there exists a remarkable reflecting solution which exits instantaneously from the boundary point 0. Constructing recurrent extensions of a given Markov process killed when it reaches some boundary is a classical problem in the theory of Markov processes; see in particular Itô [4], Rogers [11, 12], Salisbury [14], Blumenthal [2] and references therein (note that in these works, the boundary is reduced to a single point whereas here, the Markov process $(Y, \dot{Y})$ is killed when it reaches $\{0\} \times \mathbb{R}$). However it does not seem easy to apply directly this theory in our setting, as technically, it would require the construction of a so-called entrance law from $(0, 0)$ for the killed Kolmogorov process. Some explicit information about the transitions of latter are available in the literature, see in particular Lachal [5, 7] and references therein. Unfortunately, the expressions are quite intricate and tedious to manipulate.

We shall first construct a reflecting solution by combining the classical idea of Skorohod with techniques of time substitution. Then, we shall establish uniqueness in distribution. In this direction, we have to obtain *a priori* information about the measure of the excursions away from 0 for any reflected solution. More precisely, we will first observe that the velocity of a reflecting Langevin process does not remain strictly positive immediately after the beginning of an excursion away from 0, which may be a rather surprising fact. Then, combining a fundamental connection between the Langevin process and the symmetric stable process of index 1/3 with a result of Rogozin [13] on the two-sided exit problem for stable Lévy processes, we shall compute explicitly certain distributions under the excursion measure of any reflecting solution. This will enable us to characterize the excursion measure and hence to establish uniqueness in law. Finally, we turn our attention to statistical self-similarity properties of the

reflected Langevin process; in particular we shall establish that the Hausdorff dimension of the set of times at which the particle reaches the boundary is 1/4 a.s.

2 Construction of the reflecting process

The purpose of this section is to provide an explicit construction of a Langevin process with white noise external forcing, such that the energy of the particle is absorbed each time it hits the boundary point 0, and which is nonetheless instantaneously reflected at 0. In order to focus on the most interesting situation, we shall suppose thereafter that the initial location and velocity are both zero, although the construction obviously extends to arbitrary initial conditions.

By this, we mean that we are looking for a càdlàg process $(X, V) = ((X_t, V_t), t \geq 0)$ with values in $[0, \infty[\times\mathbb{R}$, which starts from $X_0 = V_0 = 0$ and fulfills the following five requirements. First, V is the velocity process of X, that is

$$X_t \;=\; \int_0^t V_s \mathrm{d}s\,. \tag{1}$$

Second, we require that the energy of the particle is absorbed at the boundary, viz.

$$V_t = 0 \qquad \text{for every } t \geq 0 \text{ such that } X_t = 0, \quad \text{a.s.}, \tag{2}$$

and, third, that the process X spends zero time at 0 :

$$\int_0^\infty \mathbf{1}_{\{X_t=0\}} \mathrm{d}t \;=\; 0 \qquad \text{a.s.} \tag{3}$$

Fourth, the process X evolves in $]0, \infty[$ as a free Langevin process, that is if we define for every $t \geq 0$ the first return time to the boundary after t,

$$\zeta_t = \inf\{s \geq t : X_s = 0\}\,,$$

then for every stopping time S in the natural filtration (after the usual completions) $(\mathcal{F}_t)_{t\geq 0}$ of X :

$$\begin{aligned}&\text{conditionally on } X_S = x > 0 \text{ and } V_S = v, \text{ the process } (X_{(S+t)\wedge\zeta_S}, t \geq 0)\\ &\text{is independent of } \mathcal{F}_S, \text{ and has the distribution of a Langevin process}\\ &\text{started with velocity } v \text{ from the location } x \text{ and stopped at 0.}\end{aligned} \tag{4}$$

Roughly, the final condition is a natural requirement of regeneration at return times to the boundary. Specifically, for every stopping time S such that $X_S = 0$ a.s., we have that

$$\text{the process } (X_{S+t}, t \geq 0) \text{ has the same law as } (X_t, t \geq 0) \text{ and is independent of } \mathcal{F}_S. \tag{5}$$

Conditions (4) and (5) entail that (X, V) enjoys the strong Markov property; note however the latter has jumps at predictable stopping times and thus is not a Hunt process. Note also that, since a free Langevin process started from an arbitrary position with an arbitrary velocity eventually reaches the boundary point 0 a.s., $(0, 0)$ is necessarily a recurrent point for (X, V).

Let $W = (W_t, t \geq 0)$ be a standard Wiener process started from $W_0 = 0$. Recall that W possesses a local time at 0 which is defined by

$$L_t := \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{0 < W_s < \varepsilon\}} \mathrm{d}s \,, \qquad t \geq 0 \,.$$

The process $L = (L_t, t \geq 0)$ has continuous paths and the support of the Stieltjes measure $\mathrm{d}L_t$ coincides with the set of times t at which $W_t = 0$, a.s. We write

$$\tau_t := \inf\{s \geq 0 : L_s > t\} \,, \qquad t \geq 0$$

for the right-continuous inverse of L.

Next, we define the free Langevin process

$$Y_t \;=\; \int_0^t W_s \mathrm{d}s \,, \qquad t \geq 0 \,.$$

The scaling and strong Markov properties of the Wiener process readily entail the following lemma which will be crucial for our analysis.

Lemma 1 *The time-changed process*

$$\sigma_t := Y_{\tau_t} \,, \qquad t \geq 0$$

is a symmetric stable Lévy process with index $1/3$. *Moreover,* Y_s *lies between* σ_t *and* σ_{t-} *for every* $s \in [\tau_{t-}, \tau_t]$.

We refer to Biane and Yor [1] for the first claim. The second follows from the fact that Y is monotone on every time-interval $[\tau_{t-}, \tau_t]$ on which the Wiener process W makes an excursion away from 0. It will play an important part in our study as it enables us to relate exit properties of the Langevin process to that of the symmetric stable process σ, and useful information on the latter can be gleaned from the literature.

We also consider the non-increasing infimum process

$$I_t = \inf\{Y_s : 0 \leq s \leq t\} \,.$$

Plainly, I is absolutely continuous (because so is Y) and $\mathrm{d}I_t = 0$ whenever $Y_t > I_t$. We observe that $W_t \leq 0$ for every t such that $Y_t = I_t$. Indeed, if we had $W_t > 0$ for such a time t, then Y would be strictly increasing on some neighborhood of t, which is incompatible with the requirement that $Y_t = I_t$. Now for every $t \geq 0$ such that $Y_t = I_t$ and $W_t < 0$, Y is strictly decreasing on some interval $[t, t']$ with $t' > t$, and thus $Y = I$ on $[t, t']$. Since the total time that W spends at 0 is zero, we conclude that

$$I_t \;=\; \int_0^t \mathbf{1}_{\{Y_s = I_s\}} W_s \mathrm{d}s \,.$$

Next, consider the process $\tilde{X}_t = Y_t - I_t$ which can thus be expressed in the form

$$\tilde{X}_t = \int_0^t \tilde{V}_s \mathrm{d}s \,,$$

where

$$\tilde{V}_t = \mathbf{1}_{\{Y_t > I_t\}} W_t = \mathbf{1}_{\{\tilde{X}_t > 0\}} W_t \,.$$

We observe that the velocity process $\tilde{V} = (\tilde{V}_t, t \geq 0)$ has càdlàg paths a.s. Indeed, if $\tilde{V}_t \neq 0$, then $\tilde{X}_t > 0$, so the indicator function $s \to \mathbf{1}_{\{\tilde{X}_s > 0\}}$ remains equal to 1 on some neighborhood of t and $s \to \mathbf{1}_{\{\tilde{X}_s > 0\}} W_s$ is thus continuous at t. Plainly, $s \to \mathbf{1}_{\{\tilde{X}_s > 0\}} W_s$ is also continuous at t when $W_t = 0$. Suppose now that $\tilde{V}_t = 0$ and $W_t \neq 0$, and recall that necessarily $W_t < 0$. Then, Y is strictly decreasing on some neighborhood of t, say $]t - \varepsilon, t + \varepsilon[$. As $Y_t = I_t$, either $\mathbf{1}_{\{\tilde{X}_s > 0\}} = 1$ for $t - \varepsilon < s < t$ and $\mathbf{1}_{\{\tilde{X}_s > 0\}} = 0$ for $t \leq s < t + \varepsilon$, or the indicator function $\mathbf{1}_{\{\tilde{X}_s > 0\}}$ remains equal to 0 on some neighborhood of t. In both cases, this implies that $s \to \mathbf{1}_{\{\tilde{X}_s > 0\}} W_s$ is càdlàg at t.

In conclusion, $(\tilde{X}, \tilde{V})$ is a càdlàg process which fulfills the requirements (1) and (2), and an immediate application of the Markov property shows that (4) also holds. However, (3) clearly fails, as the closed set

$$\mathcal{I} := \{t \geq 0 : \tilde{X}_t = 0\}$$

has a strictly positive (as a matter of fact, infinite) Lebesgue measure. This incites us to study $\mathcal{I}$ in further details, and in this direction, the following notion appears naturally. For every $t \geq 0$, we define the random set

$$\mathcal{J} := \left\{ t \geq 0 : W_t < 0, \tilde{X}_t = 0 \text{ and } \tilde{X}_{t-\varepsilon} > 0 \text{ for all } \epsilon > 0 \text{ sufficiently small} \right\}.$$

In other words, $\mathcal{J}$ is the set of times at which Y reaches its infimum for the first time during a negative excursion of W. Observe that each point $s \in \mathcal{J}$ is necessarily isolated in $\mathcal{J}$, and in particular $\mathcal{J}$ is countable. Further, we introduce the notation

$$d_t := \inf\{s > t : W_s = 0\}$$

for the first return time of the Wiener process W to 0 after time t. We may now state the following lemma.

Lemma 2 *The canonical decomposition of the interior $\mathcal{I}^o$ of $\mathcal{I}$ as the union of disjoint open intervals is given by*

$$\mathcal{I}^o = \bigcup_{s \in \mathcal{J}}]s, d_s[\,.$$

Moreover, the boundary $\partial\mathcal{I} = \mathcal{I} \backslash \mathcal{I}^o$ has zero Lebesgue measure.

Proof: We start by recalling that the free Langevin process Y oscillates at the initial time, in the sense that

$$\inf\{t > 0 : Y_t > 0\} = \inf\{t > 0 : Y_t < 0\} = 0 \qquad \text{a.s.}$$

Indeed, for every $t > 0$, Y_t is a centered Gaussian variable, so $\mathbb{P}(Y_t > 0) = \mathbb{P}(Y_t < 0) = 1/2$, and the claim follows from Blumenthal's 0-1 law for the Wiener process.

Now pick an arbitrary rational number $a > 0$ and work conditionally on $a \in \mathcal{I}^o$. Then we know that $W_a \leq 0$, and as $\mathbb{P}(W_a = 0) = 0$, we may thus assume that $W_a < 0$. Recall that

$d_a = \inf\{s > a : W_s = 0\}$ denotes the first hitting time of 0 by W after time a. Plainly Y decreases on $[a, d_a]$, and thus $]a, d_a[\subseteq \mathcal{I}^o$. An application of the strong Markov property of the Wiener process W at the stopping time d_a, combined with the oscillation property of the free Langevin process mentioned above, shows that d_a is the right extremity of the interval component of $\mathcal{I}^o$ which contains a.

Next, consider $g_a = \sup\{t < a : W_t = 0\}$, the left-extremity of the interval containing a on which W makes an excursion away from 0. We shall show that $]g_a, d_a[$ intersects $\mathcal{J}$ at a single point s. Recall that $L = (L_t, t \geq 0)$ denotes the process of the local time of W at 0 and $\tau_t = \inf\{s \geq 0 : L_s > t\}$ its right-continuous inverse, and that the compound process $\sigma := Y \circ \tau$ is a symmetric stable Lévy process with index 1/3 (see Lemma 1). We write $\iota_t := \inf_{0\leq s\leq t} \sigma_s$ for the infimum process of σ. The excursion interval $]g_a, d_a[$ corresponds to a jump of the inverse local time, that is $]g_a, d_a[=]\tau_{t-}, \tau_t[$ for some $t > 0$. Because Y is monotone on every excursion interval of W, the identities

$$\iota_v \;=\; I(\tau_v) \quad \text{and} \quad \iota_{v-} \;=\; I(\tau_{v-})$$

hold for every v, a.s. Since $Y_{d_a} = I_{d_a}$, the stable process σ reaches a new infimum at time t. On the other hand, we know that a symmetric stable process never jumps at times v such that $\sigma_{v-} = \iota_{v-}$, see for instance Rogers [10], so

$$I_{\tau_{t-}} = \iota_{t-} \;< \sigma_{t-} = Y_{\tau_{t-}}\,.$$

That is $I_{g_a} < Y_{g_a}$, and as the Langevin process Y decreases on $]g_a, d_a[$ and $Y_a = I_a$, this implies that $s := \inf\{u > g_a : Y_u = I_u\} \in \mathcal{J}$. More precisely, we have that $]g_a, d_a[\cap\mathcal{J} = \{s\}$, and it should now be plain that the interval component of $\mathcal{I}^o$ which contains a is $]s, d_a[=]s, d_s[$ a.s. Taking the union over the set of rational a's with $a \in \mathcal{I}^o$ establishes the inclusion

$$\mathcal{I}^o \;\subseteq \bigcup_{s\in\mathcal{J}}]s, d_s[\qquad \text{a.s.},$$

and the converse inclusion is obvious.

Finally, we turn our attention to the second assertion. Pick any $s \in \partial\mathcal{I}$; we then know that $W_s \leq 0$. Note that if $W_s < 0$, then $s \in \mathcal{J}$. Since the set of times at which the Wiener process is 0 has zero Lebesgue measure and $\mathcal{J}$ is countable, this shows the second claim. □

We are now able to complete the construction. Introduce

$$T_t := \inf\left\{s \geq 0 : \int_0^s \mathbf{1}_{\{\tilde{X}_u>0\}}\mathrm{d}u > t\right\}, \qquad t \geq 0\,,$$

and set

$$X_t = \tilde{X} \circ T_t \qquad \text{and} \qquad V_t = \tilde{V} \circ T_t \qquad \text{for every } t \geq 0\,.$$

Informally, the time-substitution amounts to suppressing the pieces of the paths on which the free Langevin process is at its minimum, and this makes the following statement rather intuitive. Observe that the process $t \to T_t$ is increasing and right-continuous, so the sample paths of $((X_t, V_t), t \geq 0)$ are càdlàg a.s.

Theorem 1 *The process (X, V) constructed above fulfills the requirements (1), (2), (3), (4) and (5).*

Proof: The argument relies essentially on a change of variables formula for the increasing process T that we shall first justify with details.

For every $t \geq 0$, we consider the partition of the interval $[0, T_t]$ into three Borel sets

$$B_1(t) := [0, T_t] \cap \mathcal{I}^c \quad , \quad B_2(t) := [0, T_t] \cap \partial\mathcal{I} \quad , \quad B_3(t) := [0, T_t] \cap \mathcal{I}^o$$

where $\mathcal{I}^c = \{s \geq 0 : \tilde{X}_s > 0\}$. Recall Lemma 2, and in particular that $\partial\mathcal{I}$ has zero Lebesgue measure. We deduce that the canonical decomposition of the increasing process T into its continuous component and its jump component is

$$T_t \;=\; \int_0^{T_t} \mathbf{1}_{\{\tilde{X}_s > 0\}} \mathrm{d}s + \sum_{s \in \mathcal{J}, s \leq T_t} (d_s - s) \;=\; t + \sum_{s \in \mathcal{J}, s \leq T_t} (d_s - s)\,,$$

where the second equality stems from the very definition of the time-substitution.

The classical change of variables formula gives

$$f(T_t) \;=\; f(0) + \int_0^t f'(T_s)\mathrm{d}s + \sum_{s \in \mathcal{J}, s \leq T_t} (f(d_s) - f(s))\,, \tag{6}$$

where $f : \mathbb{R}_+ \to \mathbb{R}$ stands for a generic function which is continuously differentiable. A standard argument using the monotone class theorem shows that (6) extends to the case when f has a derivative f' in Lebesgue's sense which is locally bounded. Apply the change of variables formula for

$$f(s) \;=\; \tilde{X}_s \;=\; \int_0^s \tilde{V}_u \mathrm{d}u \quad , \quad f'(s) = \tilde{V}_s\,.$$

Since $f(d_s) - f(s) = 0$ for every $s \in \mathcal{J}$, this gives

$$X_t = f(T_t) = \int_0^t \tilde{V}_{T_s} \mathrm{d}s = \int_0^t V_s \mathrm{d}s\,,$$

which proves (1).

Next, as the pair of processes $(\tilde{X}, \tilde{V})$ fulfills (2), the time-substitution shows that the same holds for (X, V). Then, we apply the change of variables formula (6) to $f'(s) = g_n(\tilde{X}_s)$, where $g_n(x) = nx$ for $0 \leq x \leq 1/n$ and $g_N(x) = 1$ for $x > 1/n$. Since $g_n(\tilde{X}_u) = 0$ for every $u \in]s, d_s[$ when $s \in \mathcal{J}$, we get

$$\int_0^{T_t} g_n(\tilde{X}_u)\mathrm{d}u \;=\; \int_0^t g_n(X_u)\mathrm{d}u\,.$$

Letting n tend to ∞, we deduce by monotone convergence that

$$\int_0^{T_t} \mathbf{1}_{\{\tilde{X}_u > 0\}} \mathrm{d}u \;=\; \int_0^t \mathbf{1}_{\{X_u > 0\}} \mathrm{d}u\,.$$

Since the left-hand side equals t, this shows that (3) holds.

Finally, note that for every $t \geq 0$, there is the identity

$$\zeta_t - t := \inf\{u \geq 0 : X_{t+u} = 0\} = \inf\{u \geq 0 : \tilde{X}_{T_t+u} = 0\}\,.$$

More precisely, either $X_t = 0$ and then the two quantities above must be zero, or $X_t = \tilde{X}_{T_t} > 0$ and then it is immediately seen that $T_{t+u} = T_t + u$ for every $0 \leq u < \zeta_t - t$. That (X, V) verifies the requirement (4) now readily stems from the fact that if S is an $(\mathcal{F}_t)$-stopping time, then T_S is a stopping time in the natural filtration of the Wiener process W and the strong Markov property of W applied at time T_S, as $X_{t+S} = \tilde{X}_{T_S+t}$ for every $0 \leq t < \zeta_S - S$.

We then observe the identity

$$V_t = W \circ T_t\,, \qquad \text{for all } t \geq 0, \text{ a.s.} \tag{7}$$

Indeed, in the case when $X_t > 0$, we have $V_t = \tilde{V}_{T_t} = W_{T_t}$. In the case when $X_t = 0$, since by (3), X cannot stay at 0 in any neighborhood of t, V must take strictly positive values at some times arbitrarily closed to t. Because W is continuous and the time-change T right-continuous, this forces $W_{T_t} = 0$. By (2), we conclude that (7) also holds in that case.

The regeneration property (5) follows from an argument similar to that for (4). Let S be a stopping time in the natural filtration of X with $X_S = 0$ a.s. It is easily seen that T_S is a stopping time for the Wiener process W, and by (2) and (7), $V_S = W_{T_S} = 0$. By the strong Markov property, $W' = (W'_t := W_{T_S+t}, t \geq 0)$ is thus a standard Wiener process, which is independent of the stopped Wiener process $(W_{u\wedge T_S}, u \geq 0)$. Let Y' denote the free Langevin process started from the location 0 with zero velocity and driven by W', and by I' its infimum process. As the free Langevin process Y coincides with its infimum I at time T_S, we have the identities

$$Y'_t = Y_{T_{S+t}} - Y_{T_S} \quad , \quad I'_t = I_{T_{S+t}} - Y_{T_S}$$

for every $t \geq 0$. It should now be plain that $X' = (X'_t := X_{S+t}, t \geq 0)$ is the reflected Langevin process constructed from W', which establishes the regeneration property (5). □

We now conclude this section by pointing at an interesting connection between the reflecting Langevin process and a reflecting stable process, which completes Lemma 1. Recall that L denotes the local time at 0 of the Wiener process and introduce

$$\lambda_t := L \circ T_t\,, \qquad t \geq 0\,.$$

Proposition 1 (i) *The process $\lambda = (\lambda_t, t \geq 0)$ serves as a local time at 0 for the velocity V of the reflecting Langevin process X, in the sense that*

$$\lambda_t = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{0<V_s<\varepsilon\}} \mathrm{d}s\,, \qquad t \geq 0\,.$$

Moreover the non-decreasing process $\lambda = (\lambda_t, t \geq 0)$ has continuous paths and the support of the Stieltjes measure $\mathrm{d}\lambda_t$ coincides with $\{t \geq 0 : V_t = 0\}$ a.s.

(ii) *Introduce the right-continuous inverse*

$$\lambda_t^{-1} := \inf\{s \geq 0 : \lambda_s > t\}\,, \qquad t \geq 0\,.$$

The time-changed process $X \circ \lambda^{-1}$ can be expressed as $X \circ \lambda_t^{-1} = \sigma_t - \iota_t$, where σ is the symmetric stable process which appears in Lemma 1 and $\iota_t = \inf_{0\leq s\leq t} \sigma_s$. In other words, $X \circ \lambda^{-1}$ is a symmetric stable process with index $1/3$ reflected at its infimum.

Proof: (i) Introduce for every $\varepsilon > 0$ and $t \geq 0$

$$L_t^{(\varepsilon)} := \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{0<W_s<\varepsilon\}} \mathrm{d}s \,.$$

The change of variables formula (6) yields

$$L_{T_t}^{(\varepsilon)} := \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{0<W_{T_s}<\varepsilon\}} \mathrm{d}s \,.$$

Then recall from (7) that $W_{T_s} = V_s$ for all $s \geq 0$, so taking the limit as $\varepsilon \to 0+$ establishes the first assertion. Further, as $L_\cdot^{(\varepsilon)}$ converges to $L_\cdot$ uniformly on every compact time-interval (this follows from the fact that the occupation densities of the Wiener process are jointly continuous), we see that $t \to L_{T_t}$ is continuous a.s. The last assertion can then be proved by a routine argument.

(ii) Introduce $A_t = \int_0^t \mathbf{1}_{\{\tilde{X}_s>0\}} \mathrm{d}s$, so that $T_\cdot$ is the right-continuous inverse of $A_\cdot$. Then $\lambda_t^{-1} = A \circ \tau_t$ and thus $X \circ \lambda^{-1} = \tilde{X} \circ T \circ A \circ \tau$. It is easily checked from the strong Markov property of W at τ_t that $T \circ A \circ \tau_t = \tau_t$ a.s. for each $t \geq 0$, and as both processes are càdlàg, the identity holds simultaneously for all $t \geq 0$, a.s. We conclude that $X \circ \lambda_t^{-1} = Y \circ \tau_t - I \circ \tau_t = \sigma_t - \iota_t$. $\square$

3 Uniqueness in distribution

The purpose of this section is to show that the requirements (1-5) characterize the law of the reflecting Langevin process. Our approach is based on excursion theory for Markov processes (see e.g. Blumenthal [2] for details and references); let us now explain roughly the main steps.

We shall denote here by (X, V) any càdlàg process which fulfills the conditions (1-5) of Section 2; recall from (4) and (5) that the latter has the strong Markov property, and is regenerated at every stopping time S at which $X_S = 0$ a.s. It is easy to deduce from the right-continuity of the sample paths and elementary properties of the free Langevin process that X returns to 0 at arbitrarily small times a.s. (recall from (2) that necessarily $V = 0$ at such times); one says that the point $(0, 0)$ is regular for (X, V). This enables us to construct a *local time process* $\ell = (\ell_t, t \geq 0)$, that is a continuous non-decreasing additive functional which increases exactly on the set of times at which X visits 0, in the sense that the support of the random measure $\mathrm{d}\ell_t$ coincides with $\mathcal{Z} := \{t \geq 0 : X_t = 0\}$ a.s. The local time process is unique up to some deterministic factor; the choice of this multiplicative constant is unimportant and just a matter of convention. **In the sequel, we shall denote generically by c or c' a positive finite constant whose value may be different in different expressions.**

Next, introduce the (right-continuous) inverse local time

$$\ell_t^{-1} := \inf\{s \geq 0 : \ell_s > t\} \,, \qquad t \geq 0 \,.$$

The jumps of ℓ^{-1} correspond to the time-intervals on which X makes an excursion away from 0, in the sense that the canonical decomposition of the random open set $\mathcal{Z}^c$ on which $X > 0$ into interval components, is given by

$$\mathcal{Z}^c = \bigcup]\ell_{t-}^{-1}, \ell_t^{-1}[\,,$$

where the union is taken over the set of times t at which ℓ^{-1} jumps. The point process

$$t \to \left(X_{(\ell^{-1}_{t-}+s)\wedge \ell^{-1}_t}, s \geq 0 \right) , \qquad t \text{ jump time of } \ell^{-1} ,$$

is a Poisson point process. It takes values in the space of excursions, viz. continuous paths in $[0, \infty[$ which start from 0, immediately leave 0, and are absorbed at 0 after their first return. Its intensity $\mathbf{n}$ is called the excursion measure (of X away from 0). The measure $\mathbf{n}$ does not need to be finite; however it always assigns a finite mass to the set of excursion with height greater than η, for any $\eta > 0$. As, by (3), X spends no time at 0, one can reconstruct X from the excursion process (this is known as Itô's program), and thus the law of X is determined by the excursion measure $\mathbf{n}$. The latter is only defined up to a multiplicative constant depending on the normalization of the local time. Thus, in order to establish that the requirements (1-5) characterize the law of X, it suffices to show that these conditions specify $\mathbf{n}$ up to some deterministic factor, which is the main goal of this section.

In this direction, we shall denote by $e = (e_t, t \geq 0)$ a generic excursion of X away from 0, by $\zeta = \inf\{t > 0 : e_t = 0\}$ the lifetime of the excursion, and by $v = (v_t, t \geq 0)$ its velocity, so

$$e_t = \int_0^t v_s \mathrm{d}s , \qquad t \geq 0 .$$

The first step in the analysis of the excursion measure $\mathbf{n}$ concerns the behavior of the velocity immediately after the initial time. More precisely, we are interested in the first instant at which the velocity v is 0 again, viz

$$d := \inf\{t > 0 : v_t = 0\} .$$

The requirements $v_0 = 0$ and $e_t > 0$ for every $0 < t < \zeta$ might suggest that the velocity could remain strictly positive immediately after the initial time, that is that $d > 0$. However we shall see that this intuition is actually wrong.

Lemma 3 *We have* $\mathbf{n}(d > 0) = 0$.

Proof: Roughly, the reason why the velocity cannot remain strictly positive immediately after the initial time under the excursion measure $\mathbf{n}$, is that otherwise the duration ζ of the excursion e would be too long to allow the reconstruction of the process X from its excursions; in other words the conditions for Itô's program would fail. More precisely, recall that the distribution of the duration ζ of the generic excursion must fulfill

$$\int_0^\infty (t \wedge 1) \mathbf{n}(\zeta \in \mathrm{d}t) \; < \; \infty ; \tag{8}$$

see e.g. Condition (iii) on page 133 in [2]. We shall show that (8) can only hold if $\mathbf{n}(d > 0) = 0$.

We suppose hereafter that $\mathbf{n}(d > 0) > 0$. It is well-known that the strong Markov property of (X, V) can be shifted to (e, v) under the excursion measure $\mathbf{n}$. It follows from (4) that under the truncated measure $\mathbf{1}_{\{d>0\}}\mathbf{n}$, the process $(v_{t\wedge d}, t > 0)$ is Markovian with semigroup given by that of the Wiener process in $[0, \infty[$ stopped at its first hitting time of 0. As under $\mathbf{n}$, the velocity process $v = (v_t, t \geq 0)$ is right-continuous (in fact, continuous on $[0, \zeta[$) with $v_0 = 0$,

it is well-known that this entails that the distribution of $(v_{t\wedge d}, t \geq 0)$ under $\mathbf{1}_{\{d>0\}}\mathbf{n}$ must be proportional to the Itô measure of positive Brownian excursions.

It then follows from the scaling property of Itô measure that for some constant $c > 0$,

$$\mathbf{n}(e_d \in \mathrm{d}x) = cx^{-4/3}\mathrm{d}x\,, \qquad x > 0\,, \tag{9}$$

see Biane and Yor [1]. Further, as d is a stopping time, the strong Markov property entails that conditionally on $e_d = x > 0$, the evolution of e after time d is given by that of the Langevin process $Y^{(x)}$ started from x with zero velocity and stopped at 0, that is at time $\zeta^{(x)} := \inf\{t \geq 0 : Y_t^{(x)} = 0\}$.

It is easily seen from the scaling property of the Wiener process that there is the identity in distribution

$$(Y_t^{(x)}, t \geq 0) \stackrel{\mathcal{L}}{=} (xY^{(1)}_{x^{-2/3}t}, t \geq 0)\,,$$

and as a consequence

$$\zeta^{(x)} \stackrel{\mathcal{L}}{=} x^{2/3}\zeta^{(1)}\,.$$

Using (9), we now see that

$$\begin{aligned}\int_0^\infty (t\wedge 1)\mathbf{n}(\zeta \in \mathrm{d}t) &\geq c\int_0^\infty x^{-4/3}\mathbb{E}(\zeta^{(x)}\wedge 1)\mathrm{d}x \\ &= c\int_0^\infty x^{-2/3}\mathbb{E}(\zeta^{(1)}\wedge x^{-2/3})\mathrm{d}x\,.\end{aligned}$$

The exact distribution of $\zeta^{(1)}$ has been determined by Goldman [3], see also Lachal [5]. We recall that the asymptotic behavior of its tail distribution is given by

$$\mathbb{P}(\zeta^{(1)} > t) \sim c't^{-1/4}\,, \qquad t \to \infty\,,$$

where $c' > 0$ is some constant, see Proposition 2 in [3]. As a consequence, we have

$$\mathbb{E}(\zeta^{(1)}\wedge x^{-2/3}) \geq \int_0^{x^{-2/3}} \mathbb{P}(\zeta^{(1)} > t)\mathrm{d}t \sim c'\int_0^{x^{-2/3}} t^{-1/4}\mathrm{d}t = \frac{4}{3}c'x^{-1/2}\,, \quad x \to 0+\,.$$

This shows that (8) fails and thus $\mathbf{n}(d > 0)$ must be zero. □

The next step of our analysis is provided by Rogozin's solution of the two-sided exit problem for stable Lévy processes, that we now specify for the symmetric stable process with index 1/3 σ which arises in Lemma 1. Recall that single points are polar for σ, so the latter always exits from an interval by a jump.

Lemma 4 (Rogozin [13]) *For every $\varepsilon > 0$ and $x \in]0, \varepsilon[$, write*

$$\varrho_{x,\varepsilon} := \inf\{t \geq 0 : x + \sigma_t \notin [0, \varepsilon]\}\,.$$

Then the following identity between sub-probability measures on $[\varepsilon, \infty[$, holds :

$$\mathbb{P}(x + \sigma_{\varrho_{x,\varepsilon}} \in \mathrm{d}y) = \frac{1}{2\pi}x^{1/6}(\varepsilon - x)^{1/6}(y - \varepsilon)^{-1/6}y^{-1/6}(y - x)^{-1}\mathrm{d}y\,.$$

In particular

$$\mathbb{P}\left(x+\sigma \text{ exits from } [0,\varepsilon] \text{ at the upper boundary}\right) \geq c^{-1}(x/\varepsilon)^{1/6}\,,$$

where $c>0$ is some constant.

The result of Rogozin combined with Lemma 1 enables us to determine a key distribution under the excursion measure $\mathbf{n}$. Introduce for every $\eta>0$ the passage time

$$\rho_\eta := \inf\{t\geq 0 : e_t \geq \eta \text{ and } v_t = 0\}\,,$$

with the usual convention that $\inf\emptyset = 0$. It should be plain that

$$\rho_\eta < \infty \Leftrightarrow \max_{0\leq t\leq \zeta} e_t \geq \eta\,,$$

and in this case, ρ_η is simply the first instant after the first passage time of e at η at which the velocity v vanishes. Observe that $\eta\to\rho_\eta$ is non-decreasing, and also from Lemma 3 that

$$\mathbf{n}(\rho_{0+}>0) = 0\,, \quad \text{where} \quad \rho_{0+} = \lim_{\eta\to 0+}\rho_\eta\,. \tag{10}$$

Corollary 1 *There is a finite constant $c>0$ such that for every $\varepsilon>0$,*

$$\mathbf{n}(e_{\rho_\varepsilon}\in \mathrm{d}y, \rho_\varepsilon<\infty) = c\varepsilon^{1/6}(y-\varepsilon)^{-1/6}y^{-7/6}\mathrm{d}y\,, \qquad y\geq\varepsilon\,.$$

Proof: For the sake of simplicity, we shall assume that $\varepsilon<1$. Fix $\eta>0$. By standard excursion theory, the condition (4) implies that the distribution of the shifted process $(e_{\rho_\eta+t}, t\geq 0)$ under the conditional law $\mathbf{n}(\cdot \mid e_{\rho_\eta}=x, \rho_\eta<\infty)$ is that of a Langevin process started from the location x with zero velocity and stopped at 0. It follows from Lemmas 1 and 4 that for every $x>0$,

$$\mathbf{n}(\rho_1<\infty \mid e_{\rho_\eta}=x, \rho_\eta<\infty) \;\geq\; c^{-1}(1\wedge x)^{1/6}\,,$$

and therefore

$$\int_{]0,\infty[}(1\wedge x)^{1/6}\mathbf{n}(e_{\rho_\eta}\in \mathrm{d}x, \rho_\eta<\infty) \;\leq\; c'\,.$$

Thus

$$\mu_\eta(\mathrm{d}x) := (1\wedge x)^{1/6}\mathbf{n}(e_{\rho_\eta}\in\mathrm{d}x, \rho_\eta<\infty)\,, \qquad \eta>0$$

is a family of bounded measures on $\mathbb{R}_+$, and we deduce from (10) that the following limit holds in the sense of weak convergence of finite measures on $\mathbb{R}_+$ as η tends to 0 along some sequence, say $(\eta_k, k\in\mathbb{N})$:

$$\lim_{k\to\infty}\mu_{\eta_k}(\mathrm{d}x) = a\delta_0(\mathrm{d}x), \tag{11}$$

where $a\geq 0$ is some constant.

Next, we observe from the strong Markov property (and again Lemma 1) that the measure $\mathbf{n}(e_{\rho_\epsilon}\in\mathrm{d}y, \rho_\epsilon<\infty)$ on $]\varepsilon,\infty[$ can be expressed in the form

$$\mathbf{1}_{\{y\geq\varepsilon\}}\mathbf{n}(e_{\rho_\eta}\in\mathrm{d}y, \rho_\eta<\infty) + \int_{]0,\varepsilon[}\mathbf{n}(e_{\rho_\eta}\in\mathrm{d}x, \rho_\eta<\infty)\mathbb{P}(x+\sigma_{\varrho_{x,\varepsilon}}\in\mathrm{d}y)\,.$$

On the one hand, (10) entails that the first term in the sum converges to 0 as $\eta \to 0+$. On the other hand, we may rewrite the second term in the form

$$\begin{aligned} & \int_{]0,\varepsilon[} \mu_\eta(\mathrm{d}x) x^{-1/6} \mathbb{P}(x + \sigma_{\varrho_{x,\varepsilon}} \in \mathrm{d}y) \\ = \; & \frac{1}{2\pi}(y-\varepsilon)^{-1/6} y^{-1/6} \left(\int_{]0,\varepsilon[} \mu_\eta(\mathrm{d}x)(\varepsilon - x)^{1/6}(y-x)^{-1} \right) \mathrm{d}y \,, \end{aligned}$$

where the equality stems from Lemma 4. Taking the limit as $\eta \to 0+$, we conclude from (11) that

$$\mathbf{n}(e_{\rho_\epsilon} \in \mathrm{d}y, \rho_\epsilon < \infty) = \frac{a}{2\pi} \varepsilon^{1/6} (y-\varepsilon)^{-1/6} y^{-7/6} \mathrm{d}y \,, \qquad y > \varepsilon \,,$$

which is our claim. □

Roughly speaking, Corollary 1 provides a substitute for the entrance law under the excursion measure. We are now able to establish the uniqueness in distribution for the reflecting Langevin process.

Theorem 2 *Any càdlàg process which fulfills the conditions (1-5) is distributed as the process constructed in Section 2.*

Proof: Recall that all that is needed is to check that the conditions (1-5) characterize the excursion measure $\mathbf{n}$ up to some deterministic factor.

In this direction, it is convenient to introduce the space $\mathcal{C}_b$ of bounded continuous paths $\omega : \mathbb{R}_+ \to \mathbb{R}$ endowed with the supremum norm, and to denote by $\theta_t : \mathcal{C}_b \to \mathcal{C}_b$ the usual shift operator. We write $\mathbb{P}_x^\partial$ for the probability measure on $\mathcal{C}_b$ induced by the Langevin process started from $x > 0$ with zero velocity and stopped at 0. For every $\eta > 0$ and every path ω, set

$$\varrho_\eta := \inf\{t \geq 0 : \bar{\omega}(t) > \eta, \text{ and } \bar{\omega}(t) > \omega(t)\} \,,$$

where $\bar{\omega}(t) := \max_{0 \leq s \leq t} \omega(s)$. Observe that for $\mathbf{n}$-almost every paths, $\rho_\eta = \varrho_\eta$. Recall also (10)

Consider a continuous bounded functional $F : \mathcal{C}_b \to \mathbb{R}$ which is identically 0 on some neighborhood of the degenerate path $\omega \equiv 0$. Then for $\mathbf{n}$-almost every paths ω, we have

$$\lim_{\eta \to 0+} \mathbf{1}_{\{\varrho_\eta < \infty\}} F(\theta_{\varrho_\eta}(\omega)) = F(\omega) \,,$$

and by dominated convergence

$$\mathbf{n}(F(\omega)) = \lim_{\eta \to 0+} \mathbf{n}(F(\theta_{\varrho_\eta}(\omega)), \varrho_\eta < \infty) \,.$$

Since, by the strong Markov property and Corollary 1,

$$\mathbf{n}(F(\theta_{\varrho_\eta}(\omega)), \varrho_\eta < \infty) = c\eta^{1/6} \int_\eta^\infty \mathbb{E}_x^\partial(F(\omega))(x-\eta)^{-1/6} x^{-7/6} \mathrm{d}x \,,$$

this determines $\mathbf{n}$ up to a multiplicative constant. □

We point out that a slight variation of the argument in the proof of Theorem 2 shows that for every continuous bounded functional $F : \mathcal{C}_b \to \mathbb{R}$ which is identically 0 on some neighborhood of $\omega \equiv 0$, there is the approximation

$$\mathbf{n}(F(e)) \; = \; \lim_{x \to 0+} c x^{-1/6} \mathbb{E}_x^\partial(F(\omega)) \,. \tag{12}$$

4 Scaling exponents

We now conclude this paper by answering some natural questions about the scaling exponents of the reflecting Langevin process.

Proposition 2 (i) *For every $a > 0$, there is the identity in distribution*

$$(a^{-3/2}X_{at}, a^{-1/2}V_{at})_{t\geq 0} \stackrel{\mathcal{L}}{=} (X_t, V_t)_{t\geq 0}\,.$$

(ii) *For every $a > 0$, the distribution of $(a^{-3/2}e_{at})_{t\geq 0}$ under the excursion measure $\mathbf{n}$ is $a^{-1/4}\mathbf{n}$.*

Proof: (i) It is immediately seen from the scaling property of the Wiener process that the free Langevin process Y started from 0 with zero initial velocity fulfills

$$(a^{-3/2}Y_{at}, a^{-1/2}\dot{Y}_{at})_{t\geq 0} \stackrel{\mathcal{L}}{=} (Y_t, \dot{Y}_t)_{t\geq 0}\,.$$

The first assertion now follows readily, as the time substitution $(T_t)_{t\geq 0}$ does not affect the scaling property.

(ii) Standard arguments of excursion theory combined with the scaling property (i) show that the distribution of $(a^{-3/2}e_{at})_{t\geq 0}$ under the excursion measure $\mathbf{n}$ must be proportional to $\mathbf{n}$. The value of the factor can be determined using Corollary 1. Indeed, if we denote by $h := \max_{t\geq 0} e_t$ the height of the excursion, then we see that

$$\mathbf{n}(h > x) = cx^{-1/6}\,, \qquad x > 0\,. \tag{13}$$

Thus $\mathbf{n}(a^{-3/2}h > x) = a^{-1/4}\mathbf{n}(h > x)$, which entails our claim. □

In the proof of the scaling property of the excursion measure, we have determined the law of the height of the excursion; cf. (13). More generally, the scaling property enables us to specify the law under the excursion measure of any variable which enjoys the self-similarity property. Here is an example of application.

Corollary 2 (i) *The tail-distribution of the lifetime ζ of an excursion is given by*

$$\mathbf{n}(\zeta > t) = ct^{-1/4}\,, \qquad t > 0\,.$$

As a consequence, the inverse local time process ℓ^{-1} is a stable subordinator with index $1/4$, and in particular the exact Hausdorff function of the zero set $\mathcal{Z} = \{t \geq 0 : X_t = 0\}$ of the reflecting Langevin process of is $H(\varepsilon) = \varepsilon^{1/4}(\ln\ln 1/\varepsilon)^{3/4}$ a.s.

(ii) *The tail-distribution of the velocity of the excursion immediately before its lifetime is given by*

$$\mathbf{n}(v_{\zeta-} < -x) = c'x^{-1/2}\,, \qquad x > 0\,.$$

Proof: The formulas for the tail-distributions follow immediately from Proposition 2. In particular, the first one entails that the inverse local time ℓ^{-1} of the reflecting Langevin process

is a stable subordinator with index 1/4. The assertion about the exact Hausdorff measure then derives from a well-known result due to Taylor and Wendel [15]. □

It may be interesting to compare Corollary 2(i) with the case when the boundary is reflecting, in the sense that a particle arriving at 0 with incoming velocity $-v < 0$ bounces back with velocity v. It is straightforward to check that if Y is a free Langevin process, then $|Y|$ describes the process that bounces at the boundary. The set of times $\{t \geq 0 : |Y_t| = 0\}$ at which the particle hits the obstacle is then countable a.s., and its only accumulation point is $t = 0$. We refer to Lachal [6] for much more on this topic.

To conclude, we also point out that Corollary 2(ii) entails that the total energy absorbed by the obstacle at time t,

$$\mathcal{E}_t := \frac{1}{2} \sum_{s \in \mathcal{Z} \cap [0,t]} V_{s-}^2 ,$$

is finite a.s. More precisely, the time-changed process $\mathcal{E} \circ \ell^{-1}$ is a stable subordinator with index 1/4.

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