HAL open science

Effective model of a finite group action

Matthieu Romagny

To cite this version:

Matthieu Romagny. Effective model of a finite group action. 2006. hal-00017906

HAL Id: hal-00017906

https://hal.science/hal-00017906v1

Preprint submitted on 26 Jan 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Effective model of a finite group action

Matthieu Romagny

January 20, 2006

ccsd-00017906, version 1 - 26 Jan 2006

Let R be a discrete valuation ring with fraction field K, uniformizer π and residue field k of characteristic $p > 0$. Let X_K be a flat K-scheme of finite type endowed with a faithful action of a finite group scheme G. Given an R-model X to which the action extends, it may happen that the reduced action on the special fibre acquires a kernel, especially if p divides $|G|$. Typically, this happens if K has characteristic 0 and $X_K = A_K$ is an abelian variety of dimension g with rational p-torsion, $G = (\mathbb{Z}/p\mathbb{Z})^{2g}$ acts by translations, and $X = A$ is the Néron model. However, in this example we feel that there is a better group acting : namely the p-torsion subgroup $A[p]$ acts faithfully on both fibres of A. In general, under the natural assumptions below, we will prove that there always exists a group $\mathcal{G}$ faithful on the special fibre, coming with a dominant map $G \to \mathcal{G}$ which is an isomorphism on the generic fibre. In our setting, we shall actually consider that a model X with an action of G is given from the start, and call $\mathcal{G}$ the *effective model* for the action.

Theorem : *Let G be a finite flat R-group scheme. Let X be a flat R-scheme of finite type with an action of G. Assume that X is covered by G-stable open affines U_i with function ring separated for the π-adic topology, such that G_K acts faithfully on the generic fibre $U_{i,K}$. Then if X has reduced special fibre, there exists an effective model for the action of G.*

This is corollary 1.2.3 below. We now briefly put our result in perspective by describing its original motivation and some of its corollaries related to other current work.

Our main motivation comes from Galois covers of curves. Assume that $K = \mathbb{Q}_p$ is the field of p-adic numbers. Then there is a nice smooth (nonconnected) proper stack classifying admissible G-Galois covers of stable curves, with fixed ramification invariants. At the moment, the question of understanding its reduction at p seems wide open ; one explanation is the following. A natural R-model for a smooth curve X_K is the stable model. If G acts on X, the above phenomenon where the group action degenerates shows up locally on components of the special fibre X_k. This perturbates the usual local-global principle saying that deformations are localized at singular points and ramification points... Thus the theorem is an attempt to remedy this pathology.

There are other useful notions of effective models. In fact let $X \to Y = X/G$ be a finite cover of stable R-curves. If $p^2 \not| |G|$ and the p-Sylow is normal, then Dan Abramovich has shown in [Ab] that there exists a finite flat group scheme $\mathcal{G}' \to Y_{\mathrm{sm}}$ on the smooth locus of the base, acting on X_{sm} in such a way that $X_{\mathrm{sm}} \to Y_{\mathrm{sm}}$ is a torsor away from some relative divisor. Abramovich calls it *Raynaud's group scheme* and shows how one may twist Y, in the sense of twisted curves, so as to extend $\mathcal{G}$ to the whole curve. Note that since $\mathrm{Aut}_R(X)$ is unramified, the effective model obtained from our theorem is $\mathcal{G} = G$, so $\mathcal{G}'$ is not just the pullback to Y_{sm} of ours. However, one salient feature of our construction is that no properness of X is required, so that we may perform it locally on Y and glue to recover the model $\mathcal{G}'$. We obtain some additional information that can not be reached from [Ab] : for example we prove that Raynaud's group

Key words: cover of stable curves, reduction mod p, group scheme, moduli.
Mathematics Subject Classification: 14L15, 14L30, 14H10, 11G25, 11G30

scheme is constant on irreducible components of Y_k, and also that it extends to nodes $p \in Y_k$ lying on one single component (without the need of a twisting), see corollary 1.3.2.

The main theorem occupies section 1 of the paper. Finally we remark that the construction of Raynaud's group scheme in [Ab] can be carried out without assumptions on $|G|$, as well as ours. The trouble is that in general one does not get a torsor structure on the special fibre, as is known to the experts (see Saïdi [Sa1] for an example). Our effective model gives an explanation for this, illustrated by several examples exposed in section 2.

Acknowledgements. I wish to thank José Bertin for raising this question in the context of my thesis and for reading a first version of this note. Laurent Moret-Bailly suggested the lines of a strategy of proof and is also warmly acknowledged. Finally I am thankful to Dan Abramovich for inviting me to Brown University in march 2005, and for several remarks and counterexamples leading to more accurate statements.

1 Models for finite group actions

Throughout we fix a discrete valuation ring R with fraction field K, uniformizer π and residue field k of characteristic $p > 0$. For G a group scheme over R, we denote by $R[G]$ or simply RG its ring of functions.

1.1 Definitions, basic properties

We first recall some well-known properties of the scheme-theoretic image for morphisms over a discrete valuation ring R, and we fix some terminology.

(i) Let $f\colon W \to X$ be a morphism of R-schemes such that $f_*\mathcal{O}_W$ is quasi-coherent. Then there exists a smallest closed subscheme $X' \subset X$ such that f factors through X'. We call it the schematic image of f. If it is equal to X we say that f is dominant.

(ii) If $W = Z_K$ is a closed subscheme of the generic fibre of X and f is the canonical immersion, then the schematic image is called the schematic closure of Z_K in X. It is the unique closed subscheme $Z \subset X$ which is flat over R and satisfies $Z \otimes K = Z_K$.

(iii) Let X be a scheme over R. A family of closed subschemes $Z_\alpha \subset X$ with ideal sheaves $\mathcal{I}_\alpha$ is called schematically dense in X if the morphism $\amalg Z_\alpha \to X$ is dominant. In other words this means that $\cap \mathcal{I}_\alpha = 0$ in $\mathcal{O}_X$. Any other family $Z'_\beta \subset X$ such that for any α there exists one β with $Z_\alpha \subset Z'_\beta$, is again schematically dense in X.

(iv) Let G be a finite, flat group scheme over R and let X be an R-scheme. Let $\mu\colon G \times X \to X$ be an action. Then μ is a finite, flat morphism of finite presentation. If $Z \subset X$ is a closed subscheme we denote by $G.Z$ the schematic image of $G \times Z$ under μ. If Z is finite (resp. flat) over R, then $G.Z$ also is.

We now recall from [Ra2], § 2 the relation of *domination* between models of a group :

Definition 1.1.1 (i) Let G_1 and G_2 be finite flat group schemes over R with an isomorphism $u_K\colon G_{1,K} \to G_{2,K}$. We say that G_1 *dominates* G_2 and we write $G_1 \geqslant G_2$, if we are given an R-morphism $u\colon G_1 \to G_2$ which restricts to u_K on the generic fibre. If moreover G_1 and G_2 act on X, we say that G_1 dominates G_2 *compatibly* (with the actions) if $\mu_1 = \mu_2 \circ (u \times \mathrm{id})$.

(ii) Let G be a finite flat group scheme over R. Let X a flat scheme over R. Let $\mu\colon G\times X\to X$ be an action, faithful on the generic fibre. An *effective model* for μ is a finite flat R-group scheme $\mathcal{G}$ acting on X, dominated by G compatibly, such that $\mathcal{G}$ acts universally faithfully on X, that is to say, faithfully both on the generic and the special fibre.

Lemma 1.1.2 *An effective model is unique up to unique isomorphism, if it exists.*

Proof : Giving the action is the same as giving a morphism of fppf sheaves $f\colon G\to \mathrm{Aut}_R(X)$. The requirements of universal effectivity and of flatness say that if it exists, $\mathcal{G}$ must be the scheme-theoretic closure of G_K in the fppf sheaf $\mathrm{Aut}_R(X)$, as defined in [Ra1], § 3. $\square$

We recall that an action $\mu\colon G\times X\to X$ is called *admissible* if X can be covered by G-stable open affine subschemes.

Proposition 1.1.3 *Let G be a finite flat group scheme over R. Let X a flat scheme over R and $\mu\colon G\times X\to X$ an admissible action, faithful on the generic fibre. Assume that there exists an effective model $G\to\mathcal{G}$. Then,*

(i) *If H is a finite flat subgroup of G, the restriction of the action to H has an effective model $\mathcal{H}$ which is the schematic image of H in $\mathcal{G}$. If H is normal in G, then $\mathcal{H}$ is normal in $\mathcal{G}$.*

(ii) *The identity of X induces an isomorphism $X/G\simeq X/\mathcal{G}$.*

(iii) *Assume that G is étale and note $p=\mathrm{char}(k)$. Let $N\lhd G$ be the (unique) subgroup of G such that N_k is the kernel of the action on X_k. Then, its effective model $\mathcal{N}$ is a connected p-group.*

(iv) *Assume that there is an open subset $U\subset X$ which is schematically dense in any fibre of $X\to\mathrm{Spec}(R)$, such that $\mathcal{G}$ acts freely on U. Then for any closed normal subgroup $H\lhd G$, the effective model of G/H acting on X/H is $\mathcal{G}/\mathcal{H}$.*

(v) *Under assumptions (iii)+(iv) the model $\mathcal{G}$ has a connected-étale sequence*

$$1\to\mathcal{N}\to\mathcal{G}\to G/N\to 1$$

Proof : (i) is clear.

(ii) Locally, $X=\mathrm{Spec}(A)$ and X/G is defined as the spectrum of the ring $A^G=\{\,a\in A,\ \mu_G^\sharp(a)=1\otimes a\,\}$ where $\mu_G^\sharp$ is the coaction. Now $\mu_G^\sharp$ factors through the coaction $\mu_{\mathcal{G}}^\sharp$ corresponding to the action of $\mathcal{G}$:

$$A\to R\mathcal{G}\otimes A\hookrightarrow RG\otimes A$$

Therefore, $A^{\mathcal{G}}=\{\,a\in A,\ \mu_{\mathcal{G}}^\sharp(a)=1\otimes a\,\}=A^G$.

(iii) Since the composition $N_k\to\mathcal{N}_k\hookrightarrow\mathrm{Aut}_k(X_k)$ is trivial (as a morphism of sheaves), the morphism $N_k\to\mathcal{N}_k$ also is. Moreover, $N\to\mathcal{N}$ is dominant and closed hence surjective. Hence $\mathcal{N}_k$ is infinitesimal so $\mathcal{N}$ is a p-group. Let us show that it is connected. We may and do assume that R is complete. Then $\mathcal{N}$ has a connected-étale sequence, we denote the étale quotient by $\mathcal{N}_{\text{ét}}$. The composite map $t:N\to\mathcal{N}\to\mathcal{N}_{\text{ét}}$ is trivial on the special fibre. Moreover, t is determined by its restriction to the special fibre because it is a morphism between étale schemes. So it is globally trivial. As t is dominant we get $\mathcal{N}_{\text{ét}}=1$ thus $\mathcal{N}$ is connected.

(iv) Clearly H acts admissibly, and $X/H\xrightarrow{\sim}X/\mathcal{H}$ by (ii). We just have to show that $\mathcal{G}/\mathcal{H}$ acts universally faithfully on $X/\mathcal{H}$. Under the assumption of existence of $U\subset X$, one checks easily that this follows from the fact that $\mathcal{G}$ acts universally faithfully on X.

(v) Apply (iv) to $H=N$. $\square$

1.2 Existence

In this section, our aim is to prove that an effective model exists when the scheme X is of finite type over R with reduced special fibre (corollary 1.2.3 below). For the moment let G be a finite flat group scheme over R and let X be a flat R-scheme on which G acts (no reducedness assumption). As always we assume that the action μ is faithful on the generic fibre.

Lemma 1.2.1 *If X is proper, there exists an effective model for the action of G.*

Proof : In this case, Artin has shown that the fpqc sheaf of automorphisms $\mathrm{Aut}_R(X)$ is an algebraic space locally of finite presentation. Moreover, X being flat over R, it is separated. It follows that $\mathrm{Aut}_R(X)$ is actually a scheme ([An], chapter IV). By assumption we have a morphism $u : G \to \mathrm{Aut}_R(X)$ and G_K is a closed subscheme of $\mathrm{Aut}_K(X_K)$. We call $\mathcal{G}$ the schematic image of u. It is finite and flat over R. By its definition, $\mathcal{G}$ acts universally faithfully on X. $\square$

We denote by $\mathcal{F}$ the family of all closed subschemes $Z \subset X$ which are finite and flat over R.

Theorem 1.2.2 *Let X be a flat R-scheme and $\mu\colon G \times X \to X$ an action. Assume that X is covered by G-stable open affines U_i with function ring separated for the π-adic topology, such that G acts faithfully on the generic fibre $U_{i,K}$. Assume that $\mathcal{F}_k$ (the family of all Z_k) is schematically dense in X_k. Then there exists an effective model for the action of G.*

Proof : The proof is divided into three steps :

(1) The subfamily $\mathcal{F}^*$ of all Z such that G stabilizes Z and acts faithfully on Z_K is again schematically dense in the special fibre X_k.

The point here is that, by lemma 1.2.1, for any $Z \in \mathcal{F}^*$ there is an effective model $G \to \mathcal{G}_Z$.

(2) There exists $\mathcal{Z} \in \mathcal{F}^*$ such that $\mathcal{G} := \mathcal{G}_{\mathcal{Z}}$ acts on all subschemes $Z \in \mathcal{F}^*$.

(3) The group $\mathcal{G}$ is the effective model of the action of G.

Let us prove (1). If $Z \in \mathcal{F}$ then the orbit $G.Z$ (see 1.1(iv)) is again in $\mathcal{F}$. Also, $\mathcal{F}$ is stable under union of subschemes, in the sense of intersection of defining ideals. Moreover, $\mathcal{F}^*$ contains at least one element Z_0 : consider a finite K-subscheme of X_K, on which G_K acts faithfully, and look at its schematic closure in X. Therefore $\mathcal{F}^*$ contains all $G.Z \cup Z_0$, for $Z \in \mathcal{F}$, so a fortiori $\mathcal{F}^*_k$ is schematically dense in X_k (see 1.1(iii)).

We now prove (2). As already said, $\mathcal{F}$ is a filtering set with the order given by inclusion. If $Z_1 \subset Z_2$ then $\mathcal{G}_{Z_2}$ stabilizes Z_1 (because this is true on the generic fibre). Hence the effective model for $\mathcal{G}_{Z_2}$ acting on Z_1 can be identified with $\mathcal{G}_{Z_1}$, and there is a map

$$\mathcal{G}_{Z_2} \to \mathcal{G}_{Z_1}$$

This gives inclusions of structure rings $R\mathcal{G}_{Z_1} \subset R\mathcal{G}_{Z_2} \subset RG$. As RG is a finite R-module there exists $\mathcal{Z} \in \mathcal{F}^*$ such that $R\mathcal{G}_{\mathcal{Z}}$ contains all the other $R\mathcal{G}_Z$. Therefore $\mathcal{G}_{\mathcal{Z}}$ acts on all Z's.

It only remains to prove (3). The point is to see that $\mathcal{G}$ acts on X. In view of the assumptions of the theorem, in order to construct this action we can assume that X is affine and G-stable, equal to $\mathrm{Spec}(A)$ with A separated for the π-adic topology. For $Z \in \mathcal{F}^*$ denote by I_Z its ideal in A. Consider the flat R-algebra

$$\widehat{A} := \varprojlim_{Z \in \mathcal{F}^*} A/I_Z$$

The kernel of the natural map $A \to \widehat{A}$ is the intersection of all I_Z. Under reduction modulo π, this intersection maps inside $\cap (I_Z \otimes k)$. By step (1) this is zero, so $\cap I_Z \subset \pi A$. As the quotients A/I_Z have no π-torsion, it follows that $\cap I_Z \subset \pi^n A$ for all n. As A is separated for the π-adic topology, we get $\cap I_Z = 0$. So we have a diagram with all morphisms injective

$$\begin{array}{ccc} \widehat{A} & \longrightarrow & \widehat{A}_K \\ \uparrow & & \uparrow \\ A & \longrightarrow & A_K \end{array}$$

Moreover we have $\widehat{A} \cap A_K = A$. Indeed an element in $\widehat{A} \cap A_K$ is written a/π^d with $a \in A$ and d minimal, such that there exists a system $(a_Z)_{Z \in \mathcal{F}}$ with $a \equiv \pi^d a_Z$ modulo I_Z for all Z. If $d \geqslant 1$, reducing modulo π we get $a \equiv 0$ modulo $I_Z \otimes k$. From $\cap I_Z \otimes k = 0$ it follows that $a \in \pi A$, which is a contradiction. Hence $a/\pi^d \in A$.

We write $\mu_Z^\sharp \colon A/I_Z \to R\mathcal{G} \otimes A/I_Z$ for the coactions (step (2)). As $R\mathcal{G}$ is finite free over R, tensor product and inverse limit commute so we can pass to the limit and obtain

$$\widehat{\mu}^\sharp := \varprojlim \mu_Z^\sharp \colon \widehat{A} \to R\mathcal{G} \otimes \widehat{A}$$

On the generic fibre, all actions are induced from μ so that $\widehat{\mu}^\sharp \otimes K$ maps A_K into $R\mathcal{G} \otimes A_K$. From $\widehat{A} \cap A_K = A$ it follows that $\widehat{\mu}^\sharp$ maps A into $R\mathcal{G} \otimes A$. Thus $\widehat{\mu}^\sharp$ extends to an action of $\mathcal{G}$ on X, universally faithful by construction. $\square$

For algebraic schemes, we will use theorem 1.2.2 in the following more convenient version.

Corollary 1.2.3 *Let G be a finite flat group scheme over R. Let X be a flat scheme of finite type over R and let $\mu \colon G \times X \to X$ be an action. We assume that X is covered by G-stable open affines U_i with function ring separated for the π-adic topology, such that G acts faithfully on the generic fibre $U_{i,K}$. Then, if X has reduced special fibre, there exists an effective model for the action of G.*

Proof : If we have an effective model $\mathcal{G}^h$ after extension to the henselization R^h, then by unicity it descends to R as well as the action. So we may assume that R is henselian. We will prove that the family $\mathcal{F}_k$ (notation of 1.2.2) contains the set of Cohen-Macaulay points, which is dense open. Since X_k is reduced, it follows that $\mathcal{F}_k$ is schematically dense and we can apply theorem 1.2.2. So let $x \in X_k$ be Cohen-Macaulay and let $(r_i)_{1 \leqslant i \leqslant m}$ be a system of parameters for the ring $\mathcal{O}_{X_k,x}$. On an affine neighbourhood $U = \mathrm{Spec}(A)$ of x in X, pick sections $s_1, \ldots, s_m \in A$ whose germ at x map to r_i in $\mathcal{O}_{X_k,x}$. Let Y be the closed subscheme of U defined by the vanishing of the s_i. As (r_i) is a regular sequence, it follows that $\mathcal{O}_{Y,x}$ is flat over R. Furthermore, $\mathcal{O}_{Y_k,x}$ is artinian (by the Cohen-Macaulay assumption) hence $\mathcal{O}_{Y,x}$ is also quasi-finite over R. Since R is henselian $\mathcal{O}_{Y,x}$ is even finite over R. Hence the schematic image of $\mathrm{Spec}(\mathcal{O}_{Y,x})$ in X is a subscheme $Z \in \mathcal{F}$ which contains x. $\square$

1.3 Applications to Raynaud's group scheme

Here we explain the link between our effective model and the constructions in [Ab], in particular we derive some new features of the so-called Raynaud's group scheme presented therein.

Construction 1.3.1 Let X and G be as in corollary 1.2.3. Let $f\colon X \to Y = X/G$ be the quotient morphism. Let $Z_1, \ldots, Z_r$ be the orbits of irreducible components of X_k under the action of G. Let U_i denote the open subscheme of X obtained by removing all the components of X_k except those in the orbit Z_i. Let

$$X^* := \cup U_i$$

In Y we consider $V_i := f(U_i)$ and $Y^* := f(X^*)$. By theorem 1.2.3 there is a finite flat group scheme $\mathcal{G}_i \to \mathrm{Spec}(R)$ which is an effective model for the action of G on U_i. Let $\mathcal{G}$ be the flat group scheme over Y^* obtained by glueing the schemes $\mathcal{G}_i \times V_i$ along X_K. We call it *Raynaud's group scheme.* □

Note that X^* and Y^* strictly contain the smooth loci of X and Y : actually the only singularities that are omitted are the intersections of components of the special fibre. Moreover, under the assumptions of theorem 3.1.1 of [Ab], by unicity it is clear that our group scheme $\mathcal{G} \to Y^*$ coincides with Raynaud's group scheme as defined in [Ab] over the smooth locus. Thus we obtain :

Corollary 1.3.2 *Let X be a stable curve over R with smooth generic fiber, G a finite group acting on X, and $Y = X/G$. Assume that the closure of fixed points of G in X_K are disjoint sections lying in the smooth locus X_{sm}. Assume $p^2 \not| |G|$ and the p-Sylow subgroup of G is normal. Let $\mathcal{G} \to Y_{\mathrm{sm}}$ be Raynaud's group scheme as in theorem 3.1.1 of* [Ab] *(notations X and Y are inverted there). Then $\mathcal{G}$ is constant on (orbits of) irreducible components of the special fibre, and it extends to nodes $p \in Y_k$ lying on one single component.* □

We stress that the fact that $\mathcal{G}$ extends to some nodes does not mean that the extension of $\mathcal{G}$ defined in [Ab], § 3.2 is representable by a scheme at these points, because the latter is actually endowed with a supplementary stack structure so as to make the covering into a torsor (always assuming that p^2 does not divide $|G|$).

When $|G|$ is arbitrary, $X^* \to Y^*$ will not be a torsor under $\mathcal{G}$, because general points in the special fibre of X^* may have nontrivial stabilizers. Note that this prevents any hope of commutation between quotient and base change. We will see an example of this for $G = \mathbb{Z}/p^2\mathbb{Z}$ in the next section.

2 Examples

Computations of effective models provide a whole zoo of finite flat group schemes. Effective models of $(\mathbb{Z}/p\mathbb{Z})_R$ appear in [OSS], I. § 2 or [He], § 1 for degenerations of μ_p-torsors in unequal characteristics and in [Ma], § 3.2 for degenerations of $\mathbb{Z}/p\mathbb{Z}$-torsors in equal characteristic $p > 0$.

Now we give examples in degree p^2. Recently Mohamed Saïdi studied degenerations of torsors under $\mathbb{Z}/p^2\mathbb{Z}$ in equal characteristics [Sa2]. He computed equations for such degenerations ; here we will provide the group scheme which extends the $\mathbb{Z}/p^2\mathbb{Z}$ action on this set of equations. We will study one case where one gets a torsor structure, and one where this fails to happen (of course all cases in theorem 2.4.3. of [Sa2] could be treated similarly).

Note that it is very likely that one could give similar examples in the case of mixed characteristics, using the Kummer-to-Artin-Schreier isogeny of Sekiguchi and Suwa in degree p^2. The computations would just be (possibly substantially) more complicated.

2.1 Witt vectors of length 2

We assume that R is complete and has equal characteristics $p > 0$, so $R \simeq k[[\pi]]$. Under this assumption, torsors under $\mathbb{Z}/p^2\mathbb{Z}$ are described by Witt theory.

2.1.1 Classical Witt theory. First we briefly recall the notations of Witt theory in degree p^2 (see [DG], chap. V). The group scheme of Witt vectors of length 2 over R has underlying scheme $W_{2,R} = \mathrm{Spec}(R[u_1, u_2]) \simeq \mathbb{A}^2_R$ with multiplication law

$$(u_1, u_2) + (v_1, v_2) = \big(u_1 + v_1, u_2 + v_2 + \sum_{k=1}^{p-1} \{{p \atop k}\}\, u_1^k v_1^{p-k}\big)$$

Here we put once for all $\{{p \atop k}\} := \frac{1}{p}\binom{p}{k}$ where $\binom{p}{k}$ is the binomial coefficient. The Frobenius morphism of W_2 is denoted by $F(u_1, u_2) = (u_1^p, u_2^p)$. Put $\phi := F - \mathrm{id}$. From the exact sequence

$$0 \to (\mathbb{Z}/p^2\mathbb{Z})_R \to W_{2,R} \xrightarrow{\phi} W_{2,R} \to 0$$

it follows that any étale torsor $f\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ under $(\mathbb{Z}/p^2\mathbb{Z})_R$ is given by an equation

$$F(X_1, X_2) - (X_1, X_2) = (a_1, a_2)$$

where $(a_1, a_2) \in W_2(A)$ is a Witt vector and the substraction is that of Witt vectors. Furthermore, (a_1, a_2) is well-defined up to addition of elements of the form $F(c_1, c_2) - (c_1, c_2)$. Note that

$$F(X_1, X_2) - (X_1, X_2) = \big(X_1^p - X_1, X_2^p - X_2 + \sum_{k=1}^{p-1} \{{p \atop k}\}\, (X_1)^{pk}(-X_1)^{p-k}\big)$$

We emphasize that the Hopf algebra of $(\mathbb{Z}/p^2\mathbb{Z})_R$ is

$$R[\mathbb{Z}/p^2\mathbb{Z}] = \frac{R[u_1, u_2]}{(u_1^p - u_1, u_2^p - u_2)}$$

with comultiplication that of W_2.

2.1.2 Twisted forms of W_2. Let λ, μ, ν be elements of R. We define a "twisted" group W_2^λ as the group with underlying scheme $\mathrm{Spec}(R[u_1, u_2])$ and multiplication law given by

$$(u_1, u_2) + (v_1, v_2) = \left(u_1 + v_1\,,\, u_2 + v_2 + \lambda \sum_{k=1}^{p-1} \{{p \atop k}\}\, u_1^k v_1^{p-k}\right)$$

We have the following analogues of the scalar multiplication and the Frobenius of W_2 :

$$\begin{aligned} I^\nu_{\lambda,\mu} :\ W_2^\lambda &\longrightarrow W_2^{\lambda\mu} \\ (u_1, u_2) &\longmapsto (\nu u_1, \mu\nu^p u_2) \end{aligned}$$

and

$$\begin{aligned} F_\lambda :\ W_2^\lambda &\longrightarrow W_2^{\lambda^p} \\ (u_1, u_2) &\longmapsto (u_1^p, u_2^p) \end{aligned}$$

In case $\mu = \lambda^{p-1}$ we define an isogeny

$$\phi_{\lambda,\nu} := F_\lambda - I^\nu_{\lambda,\lambda^{p-1}} \colon W_2^\lambda \to W_2^{\lambda^p}$$

We have

$$\phi_{\lambda,\nu}(u_1,u_2) = \left(u_1^p - \nu u_1 \, , \, u_2^p - \nu^p \lambda^{p-1} u_2 + \lambda^p \sum_{k=1}^{p-1} \tbinom{p}{k} u_1^{pk} (-\nu u_1)^{p-k} \right)$$

The kernel $\mathcal{K}_{\lambda,\nu} := \ker(\phi_{\lambda,\nu})$ is a finite flat group of rank p^2. If $p > 2$ its Hopf algebra is

$$R[\mathcal{K}_{\lambda,\nu}] = \frac{R[u_1,u_2]}{(u_1^p - \nu u_1, u_2^p - \nu^p \lambda^{p-1} u_2)}$$

2.2 Two examples

The examples come from the following situation studied by Saïdi. Denote by $G = \mathbb{Z}/p^2\mathbb{Z}$ the constant group, and by $Y = \mathbb{A}^1_R = \mathrm{Spec}(R[w])$ the affine line over R. Let $m_1, m_2 \in \mathbb{Z}$ be integers. Let $f_K \colon X_K \to Y_K$ be the $(\mathbb{Z}/p^2\mathbb{Z})_K$-torsor over $Y_K = \mathbb{A}^1_K$ given by the equations :

$$\begin{cases} T_1^p - T_1 &= \pi^{m_1} w \\ T_2^p - T_2 &= \pi^{m_2} w - \sum_{k=1}^{p-1} \tbinom{p}{k} (T_1)^{pk} (-T_1)^{p-k} \end{cases}$$

Depending on the values of the conductors m_1, m_2 this gives rise to different group degenerations.

Example 2.2.1 Assume $m_1 = 0$ and $m_2 = -p$. Then after the change of variables $Z_1 = T_1$, $Z_2 = \pi T_2$ the map f_K extends to a cover $X \to Y$ with equations

$$\begin{cases} Z_1^p - Z_1 &= w \\ Z_2^p - \pi^{(p-1)} Z_2 &= w - \pi^p \sum_{k=1}^{p-1} \tbinom{p}{k} (Z_1)^{pk} (-Z_1)^{p-k} \end{cases}$$

It is quickly seen that the action of $\mathbb{Z}/p^2\mathbb{Z}$ extends to X. As is obvious from the expression of the isogeny $\phi_{\lambda,\nu}$ (see 2.1.2), $X \to Y$ is a torsor under $\mathcal{K}_{\lambda,\nu}$ for $\lambda = \pi$ and $\nu = 1$. Thus, the effective model is $G_R \to \mathcal{G}$ with $\mathcal{G} = \mathcal{K}_{\pi,1}$.

Example 2.2.2 Assume $m_1 = -p^2 n_1 < 0$ and $m_2 = 0$. Put $\widetilde{m}_1 = n_1(p(p-1)+1)$. Then after the change of variables $Z_1 = \pi^{pn_1} T_1$ and $Z_2 = \pi^{\widetilde{m}_1} T_2$ the map f_K extends to a cover $X \to Y$ with equations

$$\begin{cases} Z_1^p - \pi^{(p-1)pn_1} Z_1 &= w \\ Z_2^p - \pi^{(p-1)\widetilde{m}_1} Z_2 &= \pi^{p\widetilde{m}_1} w - \sum_{k=1}^{p-1} \tbinom{p}{k} \pi^{pn_1(p-1)(p-1-k)} (Z_1)^{pk} (-Z_1)^{p-k} \end{cases}$$

The action of $\mathbb{Z}/p^2\mathbb{Z}$ extends to this model as follows : for (u_1,u_2) a point of $G_R = (\mathbb{Z}/p^2\mathbb{Z})_R$,

$$(u_1,u_2).(Z_1,Z_2) = \left(Z_1 + \pi^{pn_1} u_1 \, , \, Z_2 + \pi^{\widetilde{m}_1} u_2 + \sum \tbinom{p}{k} \pi^{n_1(p(p-1)+1-pk)} (Z_1)^k (u_1)^{p-k} \right)$$

In order to find out the model $G_R \to \mathcal{G}$ we look at the subalgebra of RG generated by $v_1 = \pi^{n_1} u_1$ and $v_2 = \pi^{\widetilde{m}_1} u_2$:

$$R\mathcal{G} := R[v_1, v_2] \subset RG$$

One computes that $R\mathcal{G}$ inherits a comultiplication from RG :

$$(v_1, v_2) + (w_1, w_2) = \left(v_1 + w_1 \ , \ v_2 + w_2 + \sum \tbinom{p}{k} \pi^{n_1(p-1)^2} v_1^k w_1^{p-k} \right)$$

Thus if $p > 2$ we recognize $\mathcal{G} \simeq \mathcal{K}_{\lambda,\nu}$ for $\lambda = \pi^{n_1(p-1)^2}$ and $\nu = \pi^{n_1(p-1)}$. The action of G on X extends to an action of $\mathcal{G}$ as

$$(v_1, v_2).(Z_1, Z_2) = \left(Z_1 + \pi^{(p-1)n_1} v_1 \ , \ Z_2 + v_2 + \sum \tbinom{p}{k} \pi^{n_1(p-1)(p-1-k)} Z_1^k v_1^{p-k} \right)$$

Here $X \to Y$ is not a torsor under $\mathcal{G}$. Indeed, on the special fibre we have $\mathcal{G}_k = (\alpha_p)^2$ and the action on X_k is

$$(v_1, v_2).(Z_1, Z_2) = \left(Z_1 \ , \ Z_2 + v_2 + v_1 Z_1^{p-1} \right)$$

This action is faithful as required, but any point $(z_1, z_2) \in X_k$ has a stabilizer of order p which is the subgroup of $\mathcal{G}_k$ defined by the equation $v_2 + v_1 z_1^{p-1} = 0$.

References

[Ab] D. Abramovich, *Raynaud's group-scheme and reduction of coverings*, with an appendix by Jonathan Lubin, available at arxiv.org/math.AG/0304352.

[An] S. Anantharaman, *Schémas en groupes, espaces homogènes et espaces algébriques sur une base de dimension 1*, Bull. Soc. Math. Fr., Suppl., Mém. **33**, 5-79 (1973).

[DG] M. Demazure, P. Gabriel, *Groupes algébriques*, North-Holland Publ. Co. (1970).

[He] Y. Henrio, *Arbres de Hurwitz et automorphismes d'ordre p des disques et des couronnes p-adiques formels*, available at arxiv.org/math.AG/0011098.

[Ma] S. Maugeais, *Relèvement des revêtements p-cycliques des courbes rationnelles semi-stables*, Math. Ann. **327**, No.2, 365-393 (2003).

[OSS] F. Oort, T. Sekiguchi, N. Suwa, *On the deformation of Artin-Schreier to Kummer*, Ann. Sci. Éc. Norm. Sup. (4ème série) **22**, No.3, 345-375 (1989).

[Ra1] M. Raynaud, *Spécialisation du foncteur de Picard*, Publ. Math. IHÉS **38** (1970).

[Ra2] M. Raynaud, *Schémas en groupes de type* $(p, ..., p)$, Bull. SMF **102**, 241-280 (1974).

[Ra3] M. Raynaud, *Spécialisation des revêtements en caractéristique* $p > 0$, Ann. Sci. Éc. Norm. Sup. (4ème série) **32**, 87-126 (1999).

[Sa1] M. Saïdi, *On the existence of a torsor structure for Galois covers*, available at arxiv.org/math.AG/0403389.

[Sa2] M. Saïdi, *Cyclic p-groups and semi-stable reduction of curves in equal characteristic* $p > 0$, available at arxiv.org/math.AG/0405529.

Institut de Mathématiques
Analyse Algébrique
Université Pierre et Marie Curie
Case 82
4, place Jussieu
F-75252 Paris Cedex 05
romagny@math.jussieu.fr