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Abstract The flat rank of a totally disconnected locally compact group G, denoted flat-rk(G), is an invariant of the topological group structure of G. It is defined thanks to a natural distance on the space of compact open subgroups of G. For a topological Kac-Moody group G with Weyl group W, we derive the inequalities: alg-rk$(W) \leq$ flat-rk$(G) \leq$ rk$(|W|_0)$. Here, alg-rk(W) is the maximal $\mathbb{Z}$-rank of abelian subgroups of W, and rk$(|W|_0)$ is the maximal dimension of isometrically embedded flats in the $\mathsf{CAT}(0)$-realization $|W|_0$. We can prove these inequalities under weaker assumptions. We also show that for any integer $n \geq 1$ there is a topologically simple, compactly generated, locally compact, totally disconnected group G, with flat-rk$(G) = n$ and which is not linear.

Key words totally disconnected group, flat rank, automorphism group, scale function, twin building, strong transitivity, Kac-Moody group.

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Introduction

The general structure theory of locally compact groups is a well-established topic in mathematics. One of its main achievements is the solution to Hilbert's 5th problem on characterizing Lie groups. The general structure results [MZ66] are still used in recent works. For instance, the No Small Subgroup Theorem [MZ66, 4.2] is used in Gromov's characterization of finitely generated groups of polynomial growth [Gro81]. More recently, the theory was used in rigidity problems for discrete groups, in order to attach suitable boundaries to quite general topological groups [BM02].

Simultaneously to these applications, recent years have seen substantial progress in extending known results about connected, locally compact groups to arbitrary locally compact groups. For example in the theory of random walks on groups it was shown in [JRW96] that the concentration functions for an irreducible probability measure on a non-compact group converge to 0, while [DSW04] contains the classification of ergodic $\mathbb{Z}^d$-actions on a locally compact group by automorphisms.

This progress has been due to structure theorems for totally disconnected, locally compact groups established in [Wil94], and further advanced in [Wil01] and [Wil04].

The study of particular classes of examples has played an important role in informing further developments of the structure theory of totally disconnected, locally compact groups, beginning with the study of the classes of p-adic Lie groups [Glö98,GW01] and automorphism groups of graphs [Möl02]. This paper starts the examination of the topological invariants of totally disconnected groups which are closed automorphism groups of buildings with sufficiently transitive actions.

Topological Kac-Moody groups form a subclass of the latter class of groups, which is of particular interest to us. From a combinatorial viewpoint [RR06, 1.C] topological Kac-Moody groups generalize semi-simple algebraic groups and therefore should be expected to inherit some of the properties of linear groups. For instance, their Tits system structure and the virtual pro-p-ness of their maximal compact subgroups are used to prove their topological simplicity [Rém04, 2.A.1]; these properties are well known in the algebraic case. On the other hand, it is known that some of these groups are non-linear [Rém04, 4.C.1]. This may imply that the topological invariants of topological Kac-Moody groups differ substantially from those of algebraic groups over local fields in some important aspects. In this context we mention a challenging question, which also motivates our interest in topological Kac-Moody groups, namely whether a classification of topologically simple, compactly generated, totally disconnected, locally compact groups is a reasonable goal.

In this paper we focus on the most basic topological invariant of topological Kac-Moody groups G, the **flat rank** of G, denoted flat-rk(G). This rank is defined using the space $\mathcal{B}(G)$ of compact open subgroups of G. This space is endowed with a natural distance: for $V, W \in \mathcal{B}(G)$, the numbers $d(V, W) = \log\big(|V : V \cap W| \cdot |W : V \cap W|\big)$ define a discrete metric, for which conjugations in G are isometries. A subgroup $O \leqslant G$ is called **tidy** for an element $g \in G$ if it minimizes the displacement function of g, and a subgroup H is called **flat** if all its elements have a common tidy subgroup. A flat subgroup H has a natural abelian quotient, whose rank is called its flat rank. Finally, the flat rank of G is the supremum of the flat ranks of the flat subgroups of G. For details, we refer to Subsection 1.3. The two main results of the paper provide an upper and a lower bound for the flat rank of a sufficiently transitive automorphism group G of a locally finite building.

A summary of the main results about the upper bound for flat-rk(G) is given by the following statement. The assumptions made in this theorem are satisfied by topological Kac-Moody groups (Subsection 1.2).

Theorem A. *Let $(\mathcal{C}, S)$ be a locally finite building with Weyl group W. Denote by δ and by X the W-distance function and the $\mathsf{CAT}(0)$-realization of $(\mathcal{C}, S)$, respectively. Let G be a closed subgroup of the group of automorphisms of $(\mathcal{C}, S)$. Assume that the G-action is transitive on ordered pairs of chambers at given δ-distance. Then the following statements hold.*

(i) *The map $\varphi \colon X \to \mathcal{B}(G)$ mapping a point to its stabilizer, is a quasi-isometric embedding.*
(ii) *For any point $x \in X$, the image of the orbit map $g \mapsto g.x$, restricted to a flat subgroup of flat rank n in G, is an n-dimensional quasi-flat of X.*
(iii) *We have: flat-rk$(G) \leq$ rk(X).*
(iv) *If X contains an n-dimensional flat, so does any of its apartments.*

As a consequence, we obtain: flat-rk$(G) \leq$ rk$(|W|_0)$, where rk$(|W|_0)$ is the maximal dimension of flats in the $\mathsf{CAT}(0)$-realization $|W|_0$.

The strategy for the proof of Theorem A is as follows. The inequality flat-rk$(G) \leq$ rk$(|W|_0)$ is obtained from the statements (i)–(iv), which are proven in the listed order under weaker hypotheses. Statement (i) is part of Theorem 7 — usually called the Comparison Theorem in this paper, (ii) is in fact proved in Proposition 8 under (i) as assumption, (iii) is a formal consequence of results of Kleiner's, and (iv) is Proposition 13.

The main result about the lower bound for flat-rk(G) is the second half of Theorem 17, which we reproduce here as Theorem B. The class of groups satisfying the assumptions of Theorem B is contained in the class of groups satisfying the assumptions of Theorem A and contains all topological Kac-Moody groups (Subsection 1.2).

Theorem B. *Let G be a group with a locally finite twin root datum of associated Weyl group W. We denote by $\overline{G}$ the geometric completion of G, i.e. the closure of the G-action in the full automorphism group of the positive building of G. Let A be an abelian subgroup of W. Then A lifts to a flat subgroup $\widetilde{A}$ of $\overline{G}$ such that flat-rk$(\widetilde{A}) = \text{rank}_{\mathbb{Z}}(A)$.*
As a consequence, we obtain: alg-rk$(W) \leq$ flat-rk(G), where alg-rk(W) is the maximal $\mathbb{Z}$-rank of abelian subgroups of W.

The first half of Theorem 17 (not stated here) asserts that the flat rank of the rational points of a semi-simple group over a local field k coincides with the algebraic k-rank of this group. This is enough to exhibit topologically simple, compactly generated, locally compact, totally disconnected groups of arbitrary flat rank $d \geq 1$, e.g. by taking the sequence $\left(\text{PSL}_{d+1}(\mathbb{Q}_p)\right)_{d \geq 1}$. Theorem C (Theorem 25 in the text) enables us to exhibit a sequence of *non-linear* groups with the same properties:

Theorem C. *For every integer $n \geq 1$ there is a non-linear, topologically simple, compactly generated, locally compact, totally disconnected group of flat rank n.*

These examples are provided by Kac-Moody groups. The combinatorial data leading to these groups are obtained by gluing a hyperbolic Coxeter diagram arising from a non-linear Kac-Moody group, together with an affine diagram which ensures the existence of a sufficiently large abelian group in the resulting Weyl group.

Let us finish this introduction with a conjecture. In Theorems A and B, the upper and lower bound on flat-rk(G) depend only on the associated Weyl group W. We think that the bounds are equal; if this is indeed so, the present paper computes the exact rank of any geometric completion of a finitely generated Kac-Moody group thanks to a result in Daan Krammer's PhD thesis, [Kra94, Theorem 6.8.3]; see Subsection 4.2.

Conjecture. *Let W be a finitely generated Coxeter group. Then we have:* $\mathrm{rk}(|W|_0) = \mathrm{alg\text{-}rk}(W)$, *where* $\mathrm{rk}(|W|_0)$ *is the maximal dimension of flats in the* $\mathsf{CAT}(0)$-*realization of the Coxeter complex of W and* $\mathrm{alg\text{-}rk}(W)$ *is the maximal rank of free abelian subgroups of W.*

This conjecture was checked by Frédéric Haglund in the case of right-angled Coxeter groups [Hag]. By Bieberbach theorem, to prove this conjecture it is enough to show the existence of a "periodic" flat of dimension equal to $\mathrm{rk}(|W|_0)$ in $|W|_0$; i. e. one which admits a cocompact action by a subgroup of W.

The following, more general, conjecture, seems to be the natural framework for questions of this kind: *If a group G acts cocompactly on a proper* $\mathsf{CAT}(0)$-*space X, then the rank of X equals the maximal rank of an abelian subgroup of G.* Some singular cases [BB94,BB95] of this generalization as well as the smooth analytic case [BBS85,BS91] have been proved.

ORGANIZATION OF THE PAPER. Section 1 is devoted to recalling basic facts on buildings (chamber systems), on the combinatorial approach to Kac-Moody groups (twin root data), and on the structure theory of totally disconnected locally compact groups (flat rank). We prove the upper bound inequality in Section 2, and the lower bound inequality in Section 3. In Section 4, we exhibit a family of non-linear groups of any desired positive flat rank; we also discuss applications of the flat rank to isomorphism problems.

1 Buildings and totally disconnected groups

1.1 Buildings and their automorphism groups

1.1.1 Buildings as chamber systems. In this paper a building is a chamber system, throughout denoted $\mathcal{C}$, together with a distance function with values in a Coxeter group. A chamber system is a set $\mathcal{C}$, called **the set of chambers**, together with a family S of partitions of $\mathcal{C}$, called **the adjacency relations**. Each element s of S defines an equivalence relation which we will not distinguish from s. For each s in S the equivalence classes of s should be thought of as the set of chambers sharing a fixed 'face' of 'color' s.

A finite sequence of chambers such that consecutive members are adjacent (that is, contained in some adjacency relation) is called a **gallery**. A gallery is said to **join** its first and last term. A gallery is called **non-stuttering** if consecutive members are different. For every non-stuttering gallery $(c_0, \ldots, c_n)$, any word $s_1 \cdots s_n$ in the free monoid S^* on S such that c_{j-1} and c_j are s_j-adjacent for all $1 \leq j \leq n$, is called a **type** of the gallery $(c_0, \ldots, c_n)$ (a gallery does not necessarily have a unique type). A gallery having a type contained in the submonoid of S^* generated by a subset T of S is called a **T-gallery**. A maximal subset of chambers which can be joined by a T-gallery is called a **T-residue**. We call a chamber system $(\mathcal{C}, S)$ **locally finite** if S and all $\{s\}$-residues for $s \in S$ are finite.

A permutation of the underlying set $\mathcal{C}$ of a chamber system $(\mathcal{C}, S)$ is said to be an **automorphism** of $(\mathcal{C}, S)$ if it induces a permutation of S. The group of all automorphisms of $(\mathcal{C}, S)$ will be denoted $\mathsf{Aut}(\mathcal{C}, S)$. An automorphism of $(\mathcal{C}, S)$ is said to be **type-preserving** if it induces the identity permutation of S. The group of all type-preserving automorphisms of $(\mathcal{C}, S)$ will be denoted $\mathsf{Aut}_0(\mathcal{C}, S)$.

Each Coxeter system (W, S) gives rise to a chamber system: its set of chambers is W, and for $s \in S$, we say that w and w' are s-adjacent if and only if $w' \in \{w, ws\}$. A word f in the free monoid on S is called **reduced** if it has minimal length among all such words representing their product s_f as an element of W. If (W, S) is a Coxeter system and T is a subset of S then the subgroup of W generated by T is called a **special subgroup** and is denoted W_T. A subset T of S is called **spherical** if W_T is finite.

Let (W, S) be a Coxeter system. A **building** of type (W, S) is a chamber system $\mathcal{C}$ with adjacency relations indexed by the elements of S, each consisting of sets containing at least 2 elements. We also require the existence of a **W-distance function** $\delta \colon \mathcal{C} \times \mathcal{C} \to W$ such that whenever f is a reduced word on S, then for chambers x and y we have $\delta(x, y) = s_f$ if and only if there is a (non-stuttering) gallery of type f joining x to y. In a building a non-stuttering gallery has a unique type. Other basic properties of W-distance functions can be found in [Ron89, 3.1].

1.1.2 Non-positively curved realization of a building. A chamber system can be realized as a topological space so that each chamber is homeomor-

phic to a model space X, and adjacency of chambers is represented by them sharing a preassigned subspace of X as a common 'face'. We now explain a very flexible way to do this for chamber systems which are buildings. Moussong attributes the method to Vinberg [Vin71]. We follow Davis' exposition [Dav98].

Let $(\mathcal{C}, S)$ be a building of type (W, S). We start out with a topological space X, which will be our model for a chamber, and a family of closed subspaces $(X_s)_{s\in S}$, which will be our supply of 'faces'. The pair $(X, (X_s)_{s\in S})$ will be called a **model space**. For each point x in X we define a subset $S(x)$ of S by setting $S(x) := \{s \in S : x \in X_s\}$. Further, we define an equivalence relation $\sim$ on the set $\mathcal{C} \times X$ by $(c, x) \sim (c', x')$ if and only if $x = x'$ and $\delta(c, c') \in W_{S(x)}$. The X-**realization of** $(\mathcal{C}, S)$, written $X(\mathcal{C})$, is the quotient space $(\mathcal{C} \times X)/\sim$, where $\mathcal{C}$ carries the discrete topology.

If $(\mathcal{C}, S)$ is a building, and $(X, (X_s)_{s\in S})$ is a model space, then any type-preserving automorphism of $(\mathcal{C}, S)$ induces a homeomorphism of $X(\mathcal{C})$ via the induced permutation action on $\mathcal{C} \times X$. Further, this assignment defines a homomorphism from $\mathsf{Aut}_0(\mathcal{C}, S)$ into the group of homeomorphisms of $X(\mathcal{C})$. This homomorphism is injective if $X \setminus \bigcup_{s\in S} X_s \neq \varnothing$. Automorphisms of $(\mathcal{C}, S)$ which are not type-preserving will not induce homeomorphisms of $X(\mathcal{C})$ unless the model space $(X, (X_s)_{s\in S})$ admits symmetries realizing the possible type permutations. We will spell out appropriate conditions below for specific choices of model spaces.

We now introduce the model spaces $(X, (X_s)_{s\in S})$ which define the Davis realization of a building $(\mathcal{C}, S)$ leading to a $\mathsf{CAT}(0)$-structure on $X(\mathcal{C})$. A variant of it, available for a subclass of buildings, is used by Moussong to define a $\mathsf{CAT}(-1)$-structure on $X(\mathcal{C})$ [Mou88]. *We assume from now on that $(\mathcal{C}, S)$ is a building of finite rank, i.e. S is finite.* The model spaces X in both cases are metric simplicial complexes (with a family of subcomplexes $(X_s)_{s\in S}$) with the same underlying abstract simplicial complex, namely the flag complex of the poset of spherical subsets of S ordered by inclusion. For $s \in S$ the subcomplex X_s is the union of all chains starting with the set $\{s\}$ (and all their sub-chains). This model space always supports a natural piecewise Euclidean structure [Dav98] as well as a piecewise hyperbolic structure [Mou88, section 13]

Our assumption that S is a finite set implies that X is a finite complex; in particular $X(\mathcal{C})$ has only finitely many cells up to isometry. Therefore Bridson's theorem [BH99, I.7.50] implies that $X(\mathcal{C})$ with the path metric is a complete geodesic space. Moreover X has finite diameter since only compact simplices are used for the hyperbolic structure. Suppose that all s-residues of $(\mathcal{C}, S)$ are finite for $s \in S$. Then, because of the way we defined the family of subspaces $(X_s)_{s\in S}$ encoding the faces, $X(\mathcal{C})$ is locally finite. The geometric realization of a Coxeter complex based on model spaces for the Davis realization is $\mathsf{CAT}(0)$. The Moussong realization of a Coxeter complex is $\mathsf{CAT}(-1)$ if and only if the Coxeter group is Gromov-hyperbolic. Both results are derived in [Mou88]. Using retractions onto apartments one

shows that analogous results hold for buildings whose Coxeter group is of the appropriate type [Dav98, section 11].

For both the Davis and Moussong realizations the map which assigns to a type-preserving automorphism of the building $(\mathcal{C}, S)$ the self-map of $X(\mathcal{C})$ induced by the permutation of $\mathcal{C}$ defines a homomorphism from $\mathsf{Aut}_0(\mathcal{C}, S)$ into the group of simplicial isometries of $X(\mathcal{C})$, which we will denote by $\mathsf{Isom}(X(\mathcal{C}))$. The metric structures on the corresponding model spaces are in addition invariant under all diagram automorphisms of the Coxeter diagram of the building. Hence automorphisms of $(\mathcal{C}, S)$ also induce simplicial isometries of $X(\mathcal{C})$ in both cases and the analogous result holds for automorphisms of $(\mathcal{C}, S)$ which are not necessarily type-preserving. Since the vertex $\varnothing$ of X is not contained in any of the subcomplexes X_s for $s \in S$, these homomorphisms are injective. We denote the Davis and Moussong realizations of a building $(\mathcal{C}, S)$ by $|\mathcal{C}|_0$ and $|\mathcal{C}|_{-1}$ respectively.

The natural topology on the group of automorphisms of a chamber system is the permutation topology:

Definition 1. *Suppose that a group G acts on a set M. Denote the stabilizer of a subset F of M by $G_{\{F\}}$. The **permutation topology** on G is the topology with the family $\{G_{\{F\}} \colon F$ a finite subset of $M\}$ as neighborhood base of the identity in G.*

As we already noted, automorphisms of a chamber system can be viewed as permutations of the set of chambers. The permutation topology maps to another natural topology under the injection $\mathsf{Aut}(\mathcal{C}, S) \to \mathsf{Isom}(|\mathcal{C}|_\epsilon)$ for $\epsilon \in \{0, -1\}$.

Lemma 2. *Let $(\mathcal{C}, S)$ be a building with S and all s-residues finite for $s \in S$. Then the permutation topology on $\mathsf{Aut}(\mathcal{C}, S)$ maps to the compact-open topology under the map $\mathsf{Aut}(\mathcal{C}, S) \to \mathsf{Isom}(|\mathcal{C}|_\epsilon)$ for $\epsilon \in \{0, -1\}$.* $\qquad\square$

1.2 Topological automorphism groups of buildings

The examples of topological automorphism groups of buildings we are most interested in are Kac-Moody groups over finite fields. We will not define them and refer the reader to [Tit87, Subsection 3.6] and [Rém02b, Section 9] for details instead. A Kac-Moody group over a finite field is an example of a group G with twin root datum $\big((U_a)_{a \in \Phi}, H\big)$ (of type (W, S); compare [Rém02b, 1.5.1] for the definition) such that all the root groups are finite. We will call a group which admits a twin root datum consisting of finite groups **a group with a locally finite twin root datum**.

Any group G with locally finite twin root datum of type (W, S) admits an action on a twin building (compare [Rém02b, 2.5.1] for the definition) of type (W, S) having finite s-residues for all $s \in S$. Let $(\mathcal{C}, S)$ be the positive twin (which is isomorphic to the negative twin). Its geometric realizations $|\mathcal{C}|_0$ and $|\mathcal{C}|_{-1}$ are locally finite. The action of G on $(\mathcal{C}, S)$ is strongly transitive

in the sense of [Ron89, p. 56] (called strongly transitive with respect to an apartment in [Rém02b, 2.6.1]). The group H is the fixator of an apartment $\mathcal{A}$ of $(\mathcal{C}, S)$ with respect to this action of G. Hence we have a short exact sequence $1 \to H \to N \to W \to 1$, where N is the stabilizer of $\mathcal{A}$ (note that both BN-pairs are saturated).

If G is a group with a locally finite twin root datum $\big((U_a)_{a \in \Phi}, H\big)$, then the G-actions on each building have a common kernel K. Moreover the root groups embed in G/K and $\big((U_a)_{a \in \Phi}, H/K\big)$ is a twin root datum for G/K with the same associated Coxeter system and twin buildings [RR06, Lemma 1]. When G is a Kac-Moody group over a finite field, we have $K = Z(G)$. The **topological group associated to** G, denoted $\overline{G}$, is the closure of G/K with respect to the topology on $\mathsf{Aut}(\mathcal{C}, S)$ defined in the previous subsection.

1.3 Structure of totally disconnected, locally compact groups

The structure theory of totally disconnected, locally compact groups is based on the notions of tidy subgroup for an automorphism and the scale function. These notions were defined in [Wil94] in terms of the topological dynamics of automorphisms and the definitions were reformulated in [Wil01]. We take the geometric approach to the theory as outlined in [BW06] and further elaborated on in [BMW04].

Let G be a totally disconnected, locally compact group and let $\mathsf{Aut}(G)$ be the group of bicontinuous automorphisms of G. We want to analyze the action of subgroups of $\mathsf{Aut}(G)$ on G; we will be primarily interested in groups of inner automorphisms of G. To that end we consider the induced action of $\mathsf{Aut}(G)$ on the set

$$\mathcal{B}(G) := \{V : V \text{ is a compact, open subgroup of } G\}.$$

The function

$$d(V, W) := \log(|V : V \cap W| \cdot |W : W \cap V|)$$

defines a metric on $\mathcal{B}(G)$ and $\mathsf{Aut}(G)$ acts by isometries on the discrete metric space $(\mathcal{B}(G), d)$. Let α be an automorphism of G. An element O of $\mathcal{B}(G)$ is called **tidy for** α if the **displacement function** of α, denoted by $d_\alpha : \mathcal{B}(G) \to \mathbb{R}$ and defined by $d_\alpha(V) = d(\alpha(V), V)$, attains its minimum at O. Since the set of values of the metric d on $\mathcal{B}(G)$ is a well-ordered discrete subset of $\mathbb{R}$, every $\alpha \in \mathsf{Aut}(G)$ has a subgroup tidy for α. Suppose that O is tidy for α. The integer

$$s_G(\alpha) := |\alpha(O) : \alpha(O) \cap O|,$$

which is also equal to $\min\big\{|\alpha(V) : \alpha(V) \cap V| : V \in \mathcal{B}(G)\big\}$, is called **the scale of** α. An element of $\mathcal{B}(G)$ is called **tidy for a subset** $\mathcal{M}$ of $\mathsf{Aut}(G)$ if and only if it is tidy for every element of $\mathcal{M}$. A subgroup $\mathcal{H}$ of $\mathsf{Aut}(G)$

is called **flat** if and only if there is an element of $\mathcal{B}(G)$ which is tidy for $\mathcal{H}$. We will call a subgroup H of G flat if and only if the group of inner automorphisms induced by H is flat.

Later we will uncover implications of the flatness condition for groups acting in a nice way on $\mathsf{CAT}(0)$-spaces. They are based on the following properties of flat groups which hold for automorphism groups of general totally disconnected, locally compact groups. Suppose that $\mathcal{H}$ is a flat group of automorphisms. The set

$$\mathcal{H}(1) := \{\alpha \in \mathcal{H}: s_G(\alpha) = 1 = s_G(\alpha^{-1})\}$$

is a normal subgroup of $\mathcal{H}$ and $\mathcal{H}/\mathcal{H}(1)$ is free abelian. The **flat rank** of $\mathcal{H}$, denoted flat-rk($\mathcal{H}$), is the $\mathbb{Z}$-rank of $\mathcal{H}/\mathcal{H}(1)$. If $\mathcal{A}$ is a group of automorphisms of the totally disconnected, locally compact group G then its flat rank is defined to be the supremum of the flat ranks of all flat subgroups of $\mathcal{A}$. The flat rank of the group G itself is the flat rank of the group of inner automorphisms of G.

If $\mathcal{H}$ is a flat group of automorphisms with O tidy for $\mathcal{H}$ then by setting $\|\alpha\mathcal{H}(1)\|_{\mathcal{H}} := d(\alpha(O), O)$, one defines a norm on $\mathcal{H}/\mathcal{H}(1)$ [BMW04, Lemma 11]. That is, $\|\cdot\|_{\mathcal{H}}$ satisfies the axioms of a norm on a vector space with the exception that we restrict scalar multiplication to integers. This norm can be given explicitly in terms of a set of epimorphisms $\Phi(\mathcal{H}, G) \subseteq \operatorname{Hom}(\mathcal{H}, \mathbb{Z})$ of **root functions** and a set of **scaling factors** s_{ρ}; $\rho \in \Phi(\mathcal{H}, G)$ associated to $\mathcal{H}$. In terms of these invariants of $\mathcal{H}$ the norm may be expressed as $\|\alpha\mathcal{H}(1)\|_{\mathcal{H}} = \sum_{\rho \in \Phi} \log(s_{\rho}) |\rho(\alpha)|$ [BMW04, Remark 13]. In particular the function $\|\cdot\|_{\mathcal{H}}$ extends to a norm in the ordinary sense on the vector space $\mathbb{R} \otimes \mathcal{H}/\mathcal{H}(1)$. For further information on flat groups see [Wil04].

2 Geometric rank as an upper bound

In this section, we show that the flat rank of a locally compact, strongly transitive group G of automorphisms of a locally finite building, is bounded above by the geometric rank of the Weyl group. This inequality actually holds under the weaker assumption that the G-action is δ-2-transitive:

Definition 3. *Suppose the group G acts on a building $(\mathcal{C}, S)$ with Weyl group W and W-valued distance function δ. Then the action of G on $(\mathcal{C}, S)$ is said to be $\boldsymbol{\delta}$-**2-transitive** if whenever (c_1, c_2), and (c_1', c_2') are ordered pairs of chambers of $(\mathcal{C}, S)$ with $\delta(c_1, c_2) = \delta(c_1', c_2')$, then the diagonal action of some element of G maps (c_1, c_2) to (c_1', c_2').*

If the action of G on a building is δ-2-transitive, then so is its restriction to the finite index subgroup of type-preserving automorphisms in G.

2.1 Consequences of δ-2-transitivity

In this subsection our main result is Theorem 7, which compares the Davis-realization of a locally finite building with the space of compact open subgroups of a closed subgroup of its automorphism group acting δ-2-transitively.

The following proposition allows us to compute the distance between stabilizers of chambers for such groups.

Proposition 4. *Suppose that the action of a group G on a locally finite building $(\mathcal{C}, S)$ with Weyl-distance δ is δ-2-transitive and type-preserving. For each type s let $q_s + 1$ be the common cardinality of s-residues in $(\mathcal{C}, S)$. Let (c, c') be a pair of chambers of $(\mathcal{C}, S)$ and let $s_1 \cdots s_l$ be the type of some minimal gallery connecting c to c'. Then $|G_c : G_c \cap G_{c'}| = \prod_{j=1}^{l} q_{s_j}$.*

Proof.   Set $w = \delta(c, c')$. The index $|G_c : G_c \cap G_{c'}|$ is the length of the orbit of the chamber c' under the action of G_c. Since the action of G is type-preserving, this orbit is contained in the set $c_w = \{d : \delta(c, d) = \delta(c, c')\}$. Since the action of G is δ-2-transitive as well, the orbit is equal to this set. It remains to show that the cardinality of c_w is equal to $\prod_{j=1}^{l} q_{s_j}$.

Pick $d \in c_w$. By definition of buildings in terms of W-distance and by [Ron89, (3.1)v], the chamber c is connected to d by a unique minimal gallery of type $s_1 \cdots s_l$. On the other hand, any endpoint of a gallery of type $s_1 \cdots s_l$ starting at c will belong to c_w. Therefore the cardinality of c_w, hence the index $|G_c : G_c \cap G_{c'}|$, equals the number of galleries of type $s_1 \cdots s_l$ starting at c. That latter number is equal to $\prod_{j=1}^{l} q_{s_j}$, establishing the claim.   □

Before we continue with the preparation of Theorem 7, we derive the following corollary.

Corollary 5. *A closed subgroup G of the automorphism group of a locally finite building satisfying the conditions of Proposition 4, is unimodular. In particular, the scale function of G assumes the same value at a group element and its inverse.*

Proof.   Let α be a bicontinuous automorphism of a totally disconnected, locally compact group G. Then the module of α equals $|\alpha(V) : \alpha(V) \cap V| \cdot |V : \alpha(V) \cap V|^{-1}$, where V is an arbitrary compact, open subgroup of G.

If we apply this observation in the situation of Proposition 4 with $V = G_c$ and α being inner conjugation by $x \in G$, we conclude that the modular function of G takes the value $|G_{x.c} : G_{x.c} \cap G_c| \cdot |G_c : G_{x.c} \cap G_c|^{-1}$ at x. Since traversing a minimal gallery joining $x.c$ to c in the opposite order gives a minimal gallery joining c to $x.c$, the formula for the index derived in Proposition 4 shows that x has module 1. Since x was arbitrary, G is unimodular.

Our second claim follows from the first one, because the value of the modular function of G at an automorphism α is $s_G(\alpha)/s_G(\alpha^{-1})$. We recall the proof. Choose V to be tidy for the automorphism α of G. Then

$|\alpha(V) : \alpha(V) \cap V|$ equals the scale of α while $|V : \alpha(V) \cap V|$ equals the scale of α^{-1}. By the definition of the modular function Δ_G of G, we have $\Delta_G(\alpha) = s_G(\alpha)/s_G(\alpha^{-1})$ as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following concept will be used to make the comparison in Theorem 7.

Definition 6 (adjacency graph of a chamber system). *Let $(\mathcal{C}, S)$ be a chamber system and let $d : s \mapsto d_s$ be a map from S to the positive real numbers. The **adjacency graph of** $(\mathcal{C}, S)$ **with respect to** $(d_s)_{s \in S}$ is defined as follows. It is the labeled graph $\Gamma(\mathcal{C}, S, (d_s)_{s \in S})$ having $\mathcal{C}$ as set of vertices; two vertices $c, c' \in \mathcal{C}$ are connected by an edge of label s if and only if c and c' are s-adjacent for some $s \in S$; each edge of label s is defined to have length d_s.*

We are ready to state and prove our comparison theorem.

Theorem 7 (Comparison Theorem). *Suppose that $(\mathcal{C}, S)$ is a locally finite building with Weyl group distance function δ and let $\epsilon \in \{0, -1\}$.*

1. *Suppose that $(d_s)_{s \in S}$ is a set of positive real numbers indexed by S. Then any map $\psi : |\mathcal{C}|_\epsilon \to \Gamma(\mathcal{C}, S, (d_s)_{s \in S})$ which sends a point x to some chamber $c \in \mathcal{C}$ such that $x \in |c|$ is a quasi-isometry.*
2. *Let G be a closed subgroup of the group of automorphisms of $(\mathcal{C}, S)$ such that the action of G is δ-2-transitive. Then the map $\varphi : |\mathcal{C}|_\epsilon \to \mathcal{B}(G)$ mapping a point to its stabilizer is a quasi-isometric embedding.*

Proof. We begin by proving the first claim. Both $m := \min\{d_s : s \in S\}$ and $M := \max\{d_s : s \in S\}$ are finite and positive, because S is a finite set and $\{d_s : s \in S\}$ is a set of positive numbers. The image of ψ is obviously $M/2$-quasi-dense in $\Gamma(\mathcal{C}, S, (d_s)_{s \in S})$ and we need to prove that ψ is a quasi-isometric embedding as well.

To see this, we compare distances between points in the space $|\mathcal{C}|_\epsilon$ and the graph $\Gamma(\mathcal{C}, S, (d_s)_{s \in S})$ to the gallery-distance between chambers corresponding to these points. To that end, denote by $d_\mathcal{C}$, d_Γ and d_ϵ the gallery-distance on the set of chambers, the distance in the graph $\Gamma(\mathcal{C}, S, (d_s)_{s \in S})$ and the distance on $|\mathcal{C}|_\epsilon$ respectively. Factor ψ as the composite of a map $\psi' : (|\mathcal{C}|_\epsilon, d_\epsilon) \to (\mathcal{C}, d_\mathcal{C})$, sending each point to some chamber containing it and the map $\iota : (\mathcal{C}, d_\mathcal{C}) \to (\Gamma(\mathcal{C}, S, (d_s)_{s \in S}), d_\Gamma)$ induced by the identity on $\mathcal{C}$. For any pair of chambers, c and c' we have

$$m \, d_\mathcal{C}(c, c') \leq d_\Gamma(c, c') \leq M \, d_\mathcal{C}(c, c') \,,$$

which shows that ι is a quasi-isometric embedding.

Furthermore, given two points, x and y say, in $|\mathcal{C}_\epsilon|$, there is a minimal gallery such that the geometric realizations of the chambers in that gallery do cover the geodesic joining x to y. Denoting by D the diameter of the geometric realization $|c|$ of a chamber in $|\mathcal{C}|_\epsilon$ and by r the maximal gallery-distance between chambers in the same spherical residue we conclude that

$$d_\epsilon(x, y) - 2rD \leq D \, d_\mathcal{C}(\psi'(x), \psi'(y)) \leq d_\epsilon(x, y) + 2rD \,.$$

It follows that ψ' is a quasi-isometric embedding, and so ψ is as well, finishing the proof of the first claim.

The second claim will be derived from the first and Proposition 4, once we have shown that we can reduce to the case where G acts by type-preserving automorphisms. Let G° be the subgroup of type-preserving automorphisms in G. It is a closed subgroup of finite index, say n, in G. Therefore it is open in G, and $d(O, G^\circ \cap O) \le \log(|G : G^\circ|) = \log n$ for each open subgroup O of G. It follows that the map $\mathcal{B}(G^\circ) \to \mathcal{B}(G)$ induced by the inclusion $G^\circ \hookrightarrow G$ is a quasi-isometry. Since G° is δ-2-transitive as well, we may — and shall — assume that $G = G^\circ$.

Choose some map ψ satisfying the conditions on the map with the same name in part 1. As before denote the flag consisting of the single vertex $\varnothing$ in X by $(\varnothing)$. It defines a vertex of the simplicial complex underlying the model space X of the Davis- and Moussong realizations. Denote the equivalence class with respect to the relation $\sim$ containing the pair $(c, x) \in \mathcal{C} \times X$ by $[c, x]$. Let $\mathcal{C}_\varnothing := \{[c, |\varnothing|] : c \in \mathcal{C}\}$, a set of vertices of the simplicial complex underlying $|\mathcal{C}|_\epsilon$. Let q_s be the common cardinality of s-residues in $(\mathcal{C}, S)$. Proposition 4 implies that for the choice $d_s = 2 \log q_s$ for $s \in S$ the map $\nu : \mathrm{im}(\varphi) \to \Gamma(\mathcal{C}, S, (d_s)_{s \in S})$ defined by $\nu(G_{[c, |\varnothing|]}) := \psi([c, |\varnothing|])$ is an isometric embedding. Since the composite of the restriction of φ to $\mathcal{C}_\varnothing$ with ν equals the restriction of ψ to $\mathcal{C}_\varnothing$ it follows that the restriction of φ to $\mathcal{C}_\varnothing$ is a quasi-isometric embedding, because ψ is by part 1, which we already proved.

It follows that φ is a quasi-isometric embedding as well because $\mathcal{C}_\varnothing$ is quasi-dense in $|\mathcal{C}|_\epsilon$ and the distance between the stabilizer of a point in the geometric realization $|c|$ of a chamber c and the stabilizer of the point $[c, |\varnothing|] \in \mathcal{C}_\varnothing \cap |c|$ is bounded above by a constant M independent of c since $|c|$ is a finite complex and that G acts transitively on chambers. $\qquad\square$

2.2 Consequences of the Comparison Theorem

The most important consequence of the Comparison Theorem, Theorem 7, is the inequality between the flat rank of a δ-2-transitive automorphism group and the rank of the building the group acts on. This result follows from the following

Proposition 8. *Let G be a totally disconnected, locally compact group. Suppose that G acts on a metric space X in such a way that G-stabilizers of points are compact, open subgroups of G. Assume that the map $X \to \mathcal{B}(G)$, which assigns a point its stabilizer, is a quasi-isometric embedding. Let H be a flat subgroup of G of finite flat rank n. Then, for any point x in X the inclusion of the H-orbit of x defines an n-quasi-flat in X.*

Proof. Let x be any point in X. Since the map $x \mapsto G_x$ is an equivariant quasi-isometric embedding, the orbit of x under H is quasi-isometric to the orbit of its stabilizer G_x under H acting by conjugation. The latter orbit

is quasi-isometric to the H-orbit of a tidy subgroup, say O, for H. But $H.O$ is isometric to the subset $\mathbb{Z}^n$ of $\mathbb{R}^n$ with the norm $\|\cdot\|_H$ introduced in Subsection 1.3. The subset $\mathbb{Z}^n$ is quasi-dense in $\mathbb{R}^n$ equipped with that norm and therefore, we obtain a quasi-isometric embedding of $\mathbb{R}^n$ equipped with $\|\cdot\|_H$ into X. But the identity map between $\mathbb{R}^n$ equipped with $\|\cdot\|_H$ and $\mathbb{R}^n$ with the Euclidean norm is bi-Lipschitz. Composing with this map, we obtain an n-quasi-flat in X as claimed. $\square$

Before deriving the rank inequality, we prove the following characterization for existence of fixed points.

Corollary 9. *Let G be a totally disconnected, locally compact group. Suppose that G acts isometrically on a complete $\mathsf{CAT}(0)$–space X with compact, open point stabilizers. Assume that the map $X \to \mathcal{B}(G)$, which assigns a point its stabilizer, is a quasi-isometric embedding. Then an element g of G has a fixed point in X if, and only if, $s_G(g) = 1 = s_G(g^{-1})$.*

Proof.   Let g be an element of G and let x be a point of X. The subgroup $\langle g \rangle$ is a flat subgroup of G of flat rank 0 or 1 and we have flat-rk$(\langle g \rangle) = 0$ if and only if $s_G(g) = 1 = s_G(g^{-1})$. Hence to prove our claim, we need to show that g has a fixed point in X if and only if flat-rk$(\langle g \rangle) = 0$.

Proposition 8 is applicable with H equal to $\langle g \rangle$. Therefore the set $\langle g \rangle.x$ is quasi-isometric to a point or the real line in the cases flat-rk$(\langle g \rangle) = 0$ and flat-rk$(\langle g \rangle) = 1$ respectively. We conclude that the set $\langle g \rangle.x$ is bounded if and only if flat-rk$(\langle g \rangle) = 0$.

Since $\langle g \rangle$ acts by isometries on the complete $\mathsf{CAT}(0)$-space X, if it has a bounded orbit, it admits a fixed point by [BH99, II.2, Corollary 2.8(1)]. The converse of the latter statement is trivial. We conclude that g has a fixed point in X if and only if flat-rk$(\langle g \rangle) = 0$ as had to be shown. $\square$

We adopt the following definition for the rank of a complete $\mathsf{CAT}(0)$-space. For alternative definitions see [Gro93, pp. 127-133].

Definition 10. *The **rank** of a complete $\mathsf{CAT}(0)$-space X, denoted $\mathrm{rk}(X)$, is the maximal dimension of an isometrically embedded Euclidean space in X.*

Recall that a metric space X is called **cocompact** if and only if the isometry group of X acts cocompactly on X [BH99, p. 202].

Theorem 11. *Let G be a totally disconnected, locally compact group. Suppose that G acts isometrically on a complete, locally compact, cocompact $\mathsf{CAT}(0)$-space X with compact, open point stabilizers. Assume that the map $X \to \mathcal{B}(G)$, which assigns a point its stabilizer, is a quasi-isometric embedding. Then flat-rk$(G) \le \mathrm{rk}(X)$; in particular the flat rank of G is finite. We have flat-rk$(G) = 0$ if, and only if, every element of G fixes a point in X.*

Proof.   The hypotheses on X guarantee that the rank of X is finite and equals the maximal dimension of quasi-flats in X by [Kle99, Theorem C]. The hypotheses on the action of G on X enable us to apply Proposition 8, which, together with the first observation of this proof, implies that flat-$\mathrm{rk}(G) \leq \mathrm{rk}(X) < \infty$. The last statement follows from Corollary 9.   □

The rank of a Gromov-hyperbolic $\mathsf{CAT}(0)$-space is 1. This leads to the following special case of Theorem 11.

Corollary 12. *Let G, X and $X \to \mathcal{B}(G)$ be as above. Assume further that X is Gromov-hyperbolic. Then* flat-$\mathrm{rk}(G) = 1$, *unless every element of G has a fixed point in X, in which case* flat-$\mathrm{rk}(G) = 0$.   □

2.3 Equality of the rank of a building and the rank of an apartment

The purpose of this subsection is to prove that the Davis-realization of a building has the same rank as any of its apartments. We do not claim that any flat of the building is contained in an apartment, though this is probably true as well. This stronger statement is known to be true for Euclidean buildings by Theorem 1 in [Bro89, Chapter VI, Section 7]. We believe, we can show it to be true also if the building is Moufang and locally finite.

Proposition 13. *Let $(\mathcal{C}, S)$ be a building with S finite and Weyl group W. If $|\mathcal{C}|_0$ contains a d-flat, then so does $|W|_0$. Hence* $\mathrm{rk}(|\mathcal{C}|_0) = \mathrm{rk}(|W|_0)$.

Proof.   Let F be a d-flat in $|\mathcal{C}|_0$. Applying Lemma 9.34 in chapter II of [BH99] with Y equal to $\mathbb{R}^d$ and X equal to $|W|_0$ and using the isomorphism of the geometric realization of any apartment with $|W|_0$, we see that it suffices to show that for each n in $\mathbb{N}$ there is an apartment A_n of $(\mathcal{C}, S)$ which contains an isometric copy of the ball of radius n around $\mathbf{0}$ in $\mathbb{R}^d$.

This isometric copy, B_n, of the ball of radius n around $\mathbf{0}$ in $\mathbb{R}^d$ will be taken to lie inside F. Let n be a natural number, o some point in F and B_n the ball of radius n around o in F. To show that there is an apartment A_n containing B_n, we will prove that there are two chambers c_n and c'_n such that the minimal galleries connecting c_n and c'_n cover B_n. Then by combinatorial convexity any apartment A_n containing c_n and c'_n contains B_n and by our introductory remark the proposition follows, because n was arbitrary.

To determine how we should choose the chambers c_n and c'_n, we first take a look at the way walls in $|\mathcal{C}|_0$ intersect the flat F. Since any geodesic joining two points of a wall lies entirely inside that wall, the intersection of a wall with F is an affine subspace of F. (It can be shown that the affine subspaces of F arising in this way are either empty or of codimension at most 1 in F, but we will not make use of this additional information.) Note further that the family of affine subspaces arising as intersections of walls in $|\mathcal{C}|_0$ with F is locally finite.

If M is a wall in $|\mathcal{C}|_0$ and two points p and p' in F are separated by $M \cap F$ in F, then p and p' are separated by M in $|\mathcal{C}|_0$. Therefore, if we choose chambers c_n and c'_n to contain points p_n and p'_n in F such that no intersection of a wall with F separates any point in B_n from both p_n and p'_n, then the minimal galleries connecting c_n and c'_n cover B_n. The following lemma demonstrates that such a choice of p_n and p'_n is always possible. This concludes the proof modulo Lemma 14. $\qquad \square$

The following Lemma completes the proof of Proposition 13.

Lemma 14. *Let B be a bounded, convex subset of $\mathbb{R}^d$ with non-empty interior and let $\mathcal{M}$ be a locally finite collection of hyperplanes in $\mathbb{R}^d$. Then there exist two points p and p' in the complement of $\bigcup \mathcal{M}$ such that no element of $\mathcal{M}$ separates any point in B from both p and p'.*

Proof. Let $\mathcal{M}_\cap$ be the subfamily of $\mathcal{M}$ consisting of those members of $\mathcal{M}$ which intersect the interior of B. Since $\mathcal{M}$ is locally finite and B is bounded, $\mathcal{M}_\cap$ is finite.

We will choose the points p and p' from the complement of $\bigcup \mathcal{M}$ to have the following two properties.

1. The line segment $[p, p']$ intersects the interior of B.
2. Each element of $\mathcal{M}_\cap$ separates p from p'.

The first property ensures that for every element M of $\mathcal{M} \setminus \mathcal{M}_\cap$ at least one of the points p and p' lies on the same side of M as B, while the second property ensures that for every element M of $\mathcal{M}_\cap$ any point of B lies in the same closed halfspace with respect to M as either p or p'. Therefore the two conditions ensure that no element of $\mathcal{M}$ separates any point in B from both p and p', and it will suffice to conform with the conditions 1 and 2 to conclude the proof.

Since $\mathcal{M}_\cap$ is finite, there is at least one unbounded component, say C, of the complement of $\bigcup \mathcal{M}_\cap$ which contains interior points of B. This can be shown by induction on the number of elements in $\mathcal{M}_\cap$. Let p be any interior point of B inside C. By the definition of $\mathcal{M}_\cap$, the point p does not lie in any element of $\mathcal{M}$. Note that whenever the point p' is subsequently chosen, the pair (p, p') will satisfy property 1 by our choice of p.

Let H be the collection of all open halfspaces with respect to $\mathcal{M}_\cap$, which do not contain p. Since the component C of the complement of $\bigcup \mathcal{M}_\cap$ which contains p is unbounded, $\bigcap H$ is not empty. The set $\bigcap H$ is also open in $\mathbb{R}^d$. Since $\mathcal{M}$ is locally finite, we can therefore choose a point p' in $\bigcap H$ not contained in $\bigcup \mathcal{M}$. By construction, the pair (p, p') satisfies property 2 and we have already noted that it satisfies property 1 as well. As seen above, this proves that p and p' have the property sought in the statement of the Lemma. $\qquad \square$

3 Algebraic rank as a lower bound

In this section, we show that when G is either a semi-simple algebraic group over a local field or a topological Kac-Moody group over a finite field, the

algebraic rank of the Weyl group W (Definition 16) is a lower bound for the flat rank of G.

One strategy to get a lower bound of flat-rk(G) of geometric nature, i.e. coming from the building X, is to use the stabilizer map $\varphi : x \mapsto G_x$. Indeed, according to Theorem 7, φ maps an n-quasi-flat in $|\mathcal{C}|_\epsilon$ to an n-quasi-flat in $\mathcal{B}(G)$, which can even be assumed to consist of vertex stabilizers. Still, in order to make this strategy work, one needs a connection between flats in the space $\mathcal{B}(G)$ and flat subgroups in G itself. One connection is stated by the following

Conjecture 15. Let $\mathcal{H}$ be a group of automorphisms of a totally disconnected, locally compact group G. Suppose that some (equivalently every) orbit of $\mathcal{H}$ in $\mathcal{B}(G)$ is quasi-isometric to $\mathbb{R}^n$. Then $\mathcal{H}$ is flat, of flat rank n.

This conjecture has been verified for $n = 0$ [BW06, Proposition 5]. Going back to the announced algebraic lower bound, we now introduce the algebraic rank of a group.

Definition 16. *Let H be a group. The algebraic rank of H, denoted* alg-rk(H), *is the supremum of the ranks of the free abelian subgroups of H.*

In order to achieve the aim of this section, we will show that given a free abelian subgroup A of the Weyl group W of the building X, we can lift A to a flat subgroup of the isometry group G. This holds in both classes of groups considered.

Theorem 17.

1. *Let $\mathbf{G}$ be an algebraic semi-simple group $\mathbf{G}$ over a local field k, and let $G = \mathbf{G}(k)$ be its rational points. Let $\mathbf{S}$ be a maximal k-split torus in $\mathbf{G}$, and let $W = W_{\mathrm{aff}}(\mathbf{S}, \mathbf{G})$ be the affine Weyl group of $\mathbf{G}$ with respect to $\mathbf{S}$. Then $\mathbf{S}(k)$ is a flat subgroup of G, of flat rank* alg-rk$(W) = k$-rk$(\mathbf{G})$. *In particular, we have:* alg-rk$(W) \leq$ flat-rk(G).

2. *Let G be a group with a locally finite twin root datum of associated Weyl group W. Let A be an abelian subgroup of W and let $\widetilde{A}$ be the inverse image of A under the natural map $N \to W$. Then $\widetilde{A}$ is a flat subgroup of $\overline{G}$ and* flat-rk$(\widetilde{A}) = \mathrm{rank}_{\mathbb{Z}}(A)$. *In particular, we have:* alg-rk$(W) \leq$ flat-rk(G).

Before beginning the proof of the above theorem, we note the following corollary, which was obtained by different means in unpublished work of the first and third authors.

Corollary 18. *Let $\mathbf{G}$ be an algebraic semi-simple group over a local field k, with affine Weyl group W. Then* flat-rk$(\mathbf{G}(k)) = k$-rk$(\mathbf{G}) =$ alg-rk(W).

This result and its proof are valid for any closed subgroup lying between $\mathbf{G}(k)$ and its closed subgroup $\mathbf{G}(k)^+$ generated by unipotent radicals of

parabolic k-subgroups — see e.g. [Mar89, I.2.3] for a summary about $\mathbf{G}(k)^+$. It suffices to replace $\mathbf{S}(k)$ by $\mathbf{S} \cap \mathbf{G}(k)^+$.

Proof.   Let $\mathbf{S}$ be a maximal k-split torus of $\mathbf{G}$ and let W be the affine Weyl group of $\mathbf{G}$ with respect to $\mathbf{S}$. Part 1 of Theorem 17 shows that $k\text{-rk}(\mathbf{G}) = \text{alg-rk}(W) \leq \text{flat-rk}(\mathbf{G}(k))$.

To show the inequality $\text{alg-rk}(W) \geq \text{flat-rk}(\mathbf{G}(k))$, note that the action of $\mathbf{G}(k)$ on its affine building X is given by a BN-pair in $\mathbf{G}(k)$, which implies that this action is strongly transitive. In particular, the action of $\mathbf{G}(k)$ on X is δ-2-transitive for the canonical W-metric δ. Part 2 of Theorem 7 shows that the map assigning a point of X its stabilizer in $\mathbf{G}(k)$ is a quasi-isometric embedding. Hence Theorem 11 is applicable and yields $\text{flat-rk}(\mathbf{G}(k)) \leq \text{rk}(X)$. In the case at hand, we have $\text{rk}(X) = \text{alg-rk}(W)$ because maximal flats in X are apartments in X [Bro89, VI.7], and W is virtually free abelian.

We finally conclude that $k\text{-rk}(\mathbf{G}) = \text{alg-rk}(W) = \text{flat-rk}(\mathbf{G}(k))$. $\square$

Proof of Theorem 17.   We treat separately cases 1 and 2.

In case 1, the group $\mathbf{S}(k)$ is flat by Theorem 5.9 of [Wil04], being topologically isomorphic to a power of the multiplicative group of k, which itself is generated by the group of units and a uniformizer of k.

It remains to show that $\text{flat-rk}(\mathbf{S}(k)) = \text{rank}_{\mathbb{Z}}(\mathbf{S}(k)/(\mathbf{S}(k))(1))$ equals $\text{alg-rk}(W) = k\text{-rk}(\mathbf{G})$. As seen in the proof of Corollary 18, the map assigning to a point of the affine building X its stabilizer in $\mathbf{G}(k)$ is a quasi-isometric embedding. Hence Corollary 9 can be applied to the action of $G = \mathbf{G}(k)$ on X. Therefore $(\mathbf{S}(k))(1)$ is the subgroup of $\mathbf{S}(k)$ of elements admitting fixed points in X. Denote by $A_{\mathbf{S}}$ the affine apartment of X associated to $\mathbf{S}/$ The group $\mathbf{S}(k)$ leaves $A_{\mathbf{S}}$ invariant. Since $A_{\mathbf{S}}$ is a convex subspace of the $\mathsf{CAT}(0)$-space X, every element of $\mathbf{S}(k)$ admitting a fixed point in X fixes a point of $A_{\mathbf{S}}$ (the projection of the fixed point onto $A_{\mathbf{S}}$). Therefore $\mathbf{S}(k)/(\mathbf{S}(k))(1)$ identifies with the group of translations of $A_{\mathbf{S}}$ induced by $\mathbf{S}(k)$, which is a subgroup of finite index in the translation lattice of W, which is itself an abelian subgroup of W of maximal rank. Therefore $\text{flat-rk}(\mathbf{S}(k)) = \text{alg-rk}(W)$, which coincides with $k\text{-rk}(\mathbf{G})$. This settles case 1.

We now reduce case 2 to Proposition 19, which we will prove later. Let $(H, (U_a)_{a \in \Phi})$ be the locally finite root datum in G, $(\mathcal{C}, S)$ the positive building of $(H, (U_a)_{a \in \Phi})$ and $\mathcal{A}$ the standard apartment in $(\mathcal{C}, S)$. Let δ be the W-distance on $(\mathcal{C}, S)$ and let $\epsilon \in \{0, -1\}$.

Since the group H is finite, $\widetilde{A}$ is finitely generated. Since A is abelian, the commutator of two elements of $\widetilde{A}$ is contained in H, hence is of finite order. It follows that $\widetilde{A}$ is a flat subgroup of $\overline{G}$ by Proposition 19.

The G-action on $(\mathcal{C}, S)$ is δ-2-transitive, hence so is the $\overline{G}$-action. Hence the map sending a point in $|\mathcal{C}|_\epsilon$ to its stabilizer in $\overline{G}$ is a quasi-isometric embedding by Theorem 7. Corollary 9 then implies that the subgroup $\widetilde{A}(1)$ is the set of elements in $\widetilde{A}$ which fix some point in $|\mathcal{C}|_\epsilon$.

The group N leaves $\mathcal{A}$ invariant and the induced action of N on $|\mathcal{A}|_\epsilon$ is equivariant to the action of W when $\mathcal{A}$ is identified with the Coxeter complex of W. Since the kernel of the action of N on $|\mathcal{A}|_\epsilon$ is the finite group H, the action of N on $|\mathcal{A}|_\epsilon$ is proper. Hence an element of infinite order in N does not have a fixed point in $|\mathcal{A}|_\epsilon$ and therefore has no fixed point in $|\mathcal{C}|_\epsilon$ either, because $|\mathcal{A}|_\epsilon$ is an N-stable convex subspace of the $\mathsf{CAT}(0)$-space $|\mathcal{C}|_\epsilon$. Therefore $\widetilde{A}(1)$ is the set of elements of $\widetilde{A}$ of finite order. This implies that $\mathrm{rank}_\mathbb{Z}(\widetilde{A}/\widetilde{A}(1)) = \mathrm{rank}_\mathbb{Z}(A)$, as claimed. $\qquad\square$

The remainder of this section is devoted to the proof of the following

Proposition 19. *Let G be a totally disconnected locally compact group and let A be a subgroup of G which is a finite extension of a finitely generated abelian group. Then A is a flat subgroup of G.*

We split the proof into several subclaims.

Lemma 20. *If C is a compact subgroup of G, then there is a base of neighborhoods of e consisting of C-stable, compact, open subgroups. In particular, any compact subgroup is flat.*

Proof. If V is a compact, open subgroup of G, then $O = \bigcap_{c \in C} cVc^{-1}$ is a compact, open subgroup of G which is contained in V and is normalized by C. Since G has a base of neighborhoods consisting of compact, open subgroups, this proves the first claim. The second claim follows from it. $\square$

We remind the reader on the tidying procedure defined in [Wil04] which for any automorphism α of G outputs a subgroup tidy for α. It will be used in the proofs of Lemmas 22 to 24.

Algorithm 21 (α-tidying procedure).
[0] Choose a compact open subgroup $O \leqslant G$.
[1] Let $^kO := \bigcap_{i=0}^k \alpha^i(O)$. For some $n \in \mathbb{N}$ (hence for all $n' \geq n$) we have
$^nO = \left(\bigcap_{i \geq 0} \alpha^i(^nO)\right) \cdot \left(\bigcap_{i \geq 0} \alpha^{-i}(^nO)\right)$. Set $O' = {}^nO$.
[2] For each compact, open subgroup V let
$\mathcal{K}_{\alpha,V} := \left\{ k \in G : \{\alpha^n(k)\}_{n \in \mathbb{Z}} \text{ is bounded and } \alpha^n(k) \in V \text{ for all large } n \right\}$.
Put $K_{\alpha,V} = \overline{\mathcal{K}}_{\alpha,V}$ and $K_\alpha = \bigcap\{K_{\alpha,V} : V \text{ is a compact, open subgroup}\}$.
[3] Form $O^* := \{x \in O' : kxk^{-1} \in O'K_\alpha \; \forall k \in K_\alpha\}$ and define $O'' := O^*K_\alpha$.
The group O'' is tidy for α and we output O''.

Lemma 22. *Let α be an automorphism of G and C an α-stable compact subgroup of G. Choose a C-stable compact, open subgroup O of G as in Lemma 20. Then $\alpha(O)$ and K_α are C-stable. Hence the output derived from applying the α-tidying procedure in Algorithm 21 to O is C-stable.*

Proof. To show that $\alpha(O)$ is C-stable, let c be an element of C. Then

$$c\alpha(O)c^{-1} = \alpha\big(\alpha^{-1}(c)O\alpha^{-1}(c)^{-1}\big) = \alpha(O),$$

which shows that $\alpha(O)$ is C-stable.

Towards proving that K_α is C-stable, we first prove that if C is α-stable and V is a C-stable compact, open subgroup of G then $\overline{\mathcal{K}}_{\alpha,V}$ is C-stable. Let c be an element of C and k an element of $\mathcal{K}_{\alpha,V}$. Then, for all $n \in \mathbb{Z}$,

$$\alpha^n(ckc^{-1}) = \alpha^n(c)\,\alpha^n(k)\,\alpha^n(c^{-1}) \subseteq C\,\{\alpha^n(k)\colon n \in \mathbb{Z}\}\,C\,,$$

and hence the set $\{\alpha^n(ckc^{-1})\colon n \in \mathbb{Z}\}$ is bounded. Further, for all n such that $\alpha^n(k) \in V$ we have

$$\alpha^n(ckc^{-1}) \in \alpha^n(c)V\alpha^n(c)^{-1} = V$$

proving that indeed $\overline{\mathcal{K}}_{\alpha,V}$ is C-stable if C is α-stable and V is a C-stable.

To derive that K_α is C stable, note that whenever $V' \leqslant V$ are compact, open subgroups of G then $\overline{\mathcal{K}}_{\alpha,V'} \leqslant \overline{\mathcal{K}}_{\alpha,V}$. Hence if $\mathcal{O}$ is a base of neighborhoods of e, consisting of compact, open subgroups of G then $K_\alpha = \bigcap\{\overline{\mathcal{K}}_{\alpha,V}\colon V \in \mathcal{O}\}$. Since C is compact, there is a base of neighborhoods $\mathcal{O}$ of e consisting of C-stable, compact, open subgroups by Lemma 20. We conclude that K_α is the intersection of a family of C-stable sets, hence is itself C-stable.

Finally, we establish that the output of Algorithm 21 is C-stable. The group O' is C-stable since it is the intersection of C-stable subgroups. Therefore O^* is C-stable as well. Since K_α is C-stable, it follows that the output O'' is also C-stable as claimed. $\qquad\square$

Lemma 23. *Let $\mathcal{A}$ be a set of automorphisms of G, C a compact subgroup of G stable under each element of $\mathcal{A}$ and let γ be an automorphism of G stabilizing C such that $[\gamma,\alpha]$ is an inner automorphism in C for each α in $\mathcal{A}$. Suppose there is a common tidy subgroup for $\mathcal{A}$ which is C-stable. Then there is a common tidy subgroup for $\mathcal{A} \cup \{\gamma\}$ which is C-stable.*

Proof. Let O be a common tidy subgroup for $\mathcal{A}$ which is C-stable. We will show that the output O'' of the γ-tidying procedure, Algorithm 21, on the input O produces a common tidy subgroup for $\mathcal{A} \cup \{\gamma\}$ which is C-stable. Lemma 22 shows that O'' is C-stable and we have to prove it is tidy for each element α in $\mathcal{A}$.

Since $[\gamma,\alpha] \in C$, for all C-stable, compact, open subgroups V, we have

$$|\alpha\gamma(V)\colon \alpha\gamma(V) \cap \gamma(V)| = |\gamma(\alpha(V))\colon \gamma(\alpha(V) \cap V)| = |\alpha(V)\colon \alpha(V) \cap V|\,.$$

Hence, if V is α-tidy and C-stable, then $\gamma(V)$ is α-tidy and it is C-stable by Lemma 22 as well. Using this observation, induction on i shows that $\gamma^i(O)$ is α-tidy for each $i \in \mathbb{N}$. Since any finite intersection of α-tidy subgroups is α-tidy by Lemma 10 in [Wil94], the output O' of step 1 of Algorithm 21 will be α-tidy.

We will show next that $\alpha(K_\gamma) = K_\gamma$. Theorem 3.3 in [Wil04] then implies that O'' is tidy for α, finishing the proof.

Towards proving our remaining claim $\alpha(K_\gamma) = K_\gamma$, we now show that if V is a C-stable, compact, open subgroup of G then $\alpha(\mathcal{K}_{\gamma,V}) = \mathcal{K}_{\gamma,\alpha(V)}$. Using our assumptions $[\gamma, \alpha] \subseteq C$ and $\gamma(C) = C$, the equation

$$[\gamma^n, \alpha] = [\gamma, \alpha]^{\gamma^{n-1}} \cdot [\gamma, \alpha]^{\gamma^{n-2}} \cdots [\gamma, \alpha]^{\gamma^{n-1}} \cdot [\gamma, \alpha]$$

shows that for all n in $\mathbb{Z}$ there are c_n in C such that $\gamma^n \alpha = \kappa(c_n) \alpha \gamma^n$, where $\kappa(g)$ denotes conjugation by g. Therefore, if $k \in \mathcal{K}_{\gamma,V}$ then

$$\gamma^n(\alpha(k)) = c_n(\alpha\gamma^n(k))c_n^{-1} \subseteq C\alpha(\gamma^n(k))C$$

is bounded. Since $\alpha(C) = C$ we have $\gamma^n \alpha = \kappa(c_n) \alpha \gamma^n = \alpha \kappa(\alpha^{-1}(c_n)) \gamma^n$ for all n. Put $c'_n = \alpha^{-1}(c_n)$ for $n \in \mathbb{Z}$. Since V is C-stable, for sufficiently large n in $\mathbb{N}$ we have

$$\gamma^n(\alpha(k)) = \alpha(c'_n \gamma^n(k) c'^{-1}_n) \in \alpha(c'_n V c'^{-1}_n) = \alpha(V) \,.$$

This shows that $\alpha(\mathcal{K}_{\gamma,V}) = \mathcal{K}_{\gamma,\alpha(V)}$, hence $\alpha(\overline{\mathcal{K}}_{\gamma,V}) = \overline{\mathcal{K}}_{\gamma,\alpha(V)}$ for all C-stable, compact, open subgroups V of G.

Now, if V runs through a neighborhood base of e consisting of C-stable compact, open subgroups (which exists by Lemma 20) then $\alpha(V)$ does as well. Since K_γ can be defined as the intersection of the family of all $\overline{\mathcal{K}}_{\gamma,W}$, where W runs through a neighborhood base of e consisting of compact, open subgroups as already observed in the proof of Lemma 22 we obtain $\alpha(K_\gamma) = K_\gamma$. □

As a corollary of Lemma 23 we obtain the following result.

Lemma 24. *Let $\mathcal{H}$ be a group of automorphisms of G, C a compact subgroup of G such that $[\mathcal{H}, \mathcal{H}]$ consists of inner automorphisms in C. Then $\mathcal{H}$ has local tidy subgroups, that is, for every finite subset $\mathbf{f}$ of $\mathcal{H}$ there is a compact, open subgroup O of G such that for any $\gamma \in \langle \mathbf{f} \rangle$ the group $\gamma(O)$ is tidy for each $\alpha \in \mathbf{f}$. Moreover, O can be chosen C-stable.*

Proof.   First, we use induction on the cardinality $f \geq 0$ of the finite set $\mathbf{f}$ to derive the existence of a common tidy subgroup for $\mathbf{f}$ which is C-stable. Lemma 20 proves the induction hypothesis in the case $f = 0$ and provides a basis for the induction. Assume the induction hypothesis is already proved for sets of cardinality $f - 1 \geq 0$, and assume that $\mathbf{f}$ is a finite set of cardinality f. Choose an element γ in $\mathbf{f}$ and put $\mathcal{A} = \mathbf{f} \setminus \{\gamma\}$. Then the induction hypothesis implies that there is a common C-stable tidy subgroup for $\mathcal{A}$ and Lemma 23 implies that the same holds for $\mathbf{f}$.

If O is a common C-stable tidy subgroup for $\mathbf{f}$, then the first step in the proof of Lemma 23 shows that $\gamma(O)$ is α-tidy for each α in $\mathbf{f}$ since $\gamma \in \mathcal{H}$. □

Proof of Proposition 19.   Finally, an application of Theorem 5.5 in [Wil04] enables us to derive Proposition 19 from Lemma 24. □

4 Topologically simple, non-linear groups of arbitrary flat rank

We think that the algebraic and the geometric rank of a Coxeter group of finite rank are equal. More generally, Corollary 7 in [GO] in conjunction with [Kle99, Theorem C] leads us to believe that the rank of a proper $\mathsf{CAT}(0)$-space on which a group G acts properly discontinuously and cocompactly, will turn out to be a quasi-isometry invariant and will be equal to the algebraic rank of G.

4.1 A family of non-linear groups with equality of ranks

Looking for examples with alg-rk$(W) = $ rk$(|W|_0)$ led us to the existence of non-linear, topologically simple groups of arbitrary flat rank.

Theorem 25. *For every natural number $n \geq 1$ there is a non-linear, topologically simple, compactly generated, locally compact, totally disconnected group of flat rank n.*

Proof. The idea of the proof is to show the existence of a sequence of connected Coxeter diagrams $(D_n)_{n \in \mathbb{N}}$ such that for each $n \in \mathbb{N}$, we have:

1. if W_n denotes the Coxeter group with diagram D_n, then alg-rk$(W_n) = $ rk$(|W_n|_0) = \dim(|W_n|_0)$;
2. there is a Kac-Moody root datum of associated Coxeter diagram D_n, and a finite field k such that the corresponding Kac-Moody group G_n over k is center-free and not linear over any field.

The required non-linear, topologically simple, compactly generated, locally compact, totally disconnected group of flat rank n may then be taken to be the completion $\overline{G}_n$.

We now specify a sequence of Coxeter diagrams satisfying the two conditions above. Fix a field k of cardinality at least 5. Let D_1 be a cycle of length $r = 5$ all of whose edges are labeled ∞. Then W_1 is Gromov-hyperbolic and therefore satisfies the first condition above. There is a Kac-Moody root datum whose Coxeter diagram is D_1 and whose group of k-points satisfies the conditions of [Rém04, Theorem 4.C.1] (take the remark succeeding that theorem into account). This shows that the second condition above is satisfied as well. This settles the case $n = 1$.

If $n > 1$ let D_n be the diagram obtained by joining every vertex of D_1 to every vertex of a diagram of type $\widetilde{A}_n$ by an edge labeled ∞. The translation subgroup of the special subgroup corresponding to the $\widetilde{A}_n$-subdiagram is an abelian subgroup of rank n in W_n. Furthermore, the dimension of the $\mathsf{CAT}(0)$-realization of W_n can be seen to be n as well since the size of the maximal spherical subsets of S does not grow. As the dimension of $|W_n|_0$ is an upper bound for rk$(|W_n|_0)$, the first condition above is satisfied for D_n for any $n > 1$. For every $n > 1$ the Kac-Moody data chosen in the

case $n = 1$ can be extended to a Kac-Moody data such that the associated Coxeter group has diagram D_n. (This amounts to extend the combinatorial data — that is, the finite index set of cardinality n, the generalized $n \times n$ Cartan matrix, the choice of n of vectors each in a free $\mathbb{Z}$-module of rank n and its dual, such that the matrix of the pairing between these two sets of vectors is the given generalized Cartan matrix — defining the Kac-Moody group functor. The most restrictive of these tasks is to arrange for the coefficients of a general Cartan matrix to yield the given Weyl group. This is possible as soon as the edge labels of the Coxeter diagram of the Weyl group belong to the set $\{2, 3, 4, 6, \infty\}$.) By our choice then G_n contains G_1, and the second condition is also satisfied for any $n > 1$. The group G_n is center-free whenever the above described Kac-Moody root datum is chosen to be the adjoint datum for the given generalized Cartan matrix.   □

4.2 Algebraic rank, after Krammer

We next observe that thanks to a theorem of Krammer's, the algebraic rank of a Coxeter group of finite rank can be computed from its Coxeter diagram. In order to state that theorem, we first introduce the notion of standard abelian subgroup of a Coxeter group.

Definition 26. *Let (W, S) be a Coxeter system. Let $I_1, \ldots, I_n \subseteq S$ be irreducible, non-spherical and pairwise perpendicular. For any i, let H_i be a subgroup of W_{I_i} defined as follows. If I_i is affine, H_i is the translation subgroup; otherwise, H_i is any infinite cyclic subgroup of W_{I_i}. The group $\prod_i H_i$ is called a* **standard free abelian subgroup**.

The algebraic rank of a standard abelian subgroup $\prod_i H_i$ as above equals $\sum_{I_i \text{ affine}}(\#I_i - 1) + \sum_{I_i \text{ not affine}} 1$. Moreover all possible choices of the subsets $I_1, \ldots, I_n \subseteq S$ can be enumerated, so the maximal algebraic rank of a Coxeter group is achieved by some standard free abelian subgroup because of the following theorem [Kra94, 6.8.3].

Theorem 27. *Let W be an arbitrary Coxeter group of finite rank. Then any free abelian subgroup of W has a finite index subgroup which is conjugate to a subgroup of some standard free abelian subgroup.*   □

In particular, the algebraic rank of a Coxeter group of finite rank can be computed from its Coxeter diagram.

4.3 Application to isomorphism problems

We finish by mentioning an application of the notion of flat rank to isomorphisms of discrete groups, via the use of super-rigidity.

Proposition 28. *Let Λ be an almost split Kac-Moody group over a finite field $\mathbb{F}_q$ of characteristic p. We assume that $q \geq 4$ and $W(\frac{1}{q}) < \infty$, where $W(t)$ is the growth series of the Weyl group W. Let $\mathbf{H}$ be a simple algebraic group defined and isotropic over a local field k, archimedean or not. Assume there exists a surjective group homomorphism from Λ to a lattice of $\mathbf{H}(k)$. Then, we have:* $\mathrm{char}(k) = p$ *and* $\mathrm{flat\text{-}rk}(\overline{\Lambda}) = k\text{-}\mathrm{rk}(\mathbf{H})$.

Proof.   Let Γ be a lattice in $\mathbf{H}(k)$. Let $\eta\colon \Lambda \to \Gamma$ be a surjective group homomorphism. Since $\mathbf{H}$ is k-isotropic, $\mathbf{H}(k)$ is non compact and Γ is infinite. By surjectivity of η, we have $\eta\big(Z(\Lambda)\big) \leqslant Z(\Gamma)$, so by factoring out modulo the centers we obtain a surjective group homomorphism $\bar\eta\colon \Lambda/Z(\Lambda) \to \Gamma/Z(\Gamma)$. By Borel density [Mar89, II.4.4] $Z(\Gamma)$ lies in the finite center $Z(\mathbf{H})$, so $\Gamma/Z(\Gamma)$ is infinite. By the normal subgroup property for Λ [BS06, Rém05], any proper quotient of $\Lambda/Z(\Lambda)$ is finite. This implies that $\bar\eta$ is an isomorphism, so that the group $\Lambda/Z(\Lambda)$ is linear in characteristic p. By [Rém02a, Proposition 4.3], this implies $\mathrm{char}(k) = p$. In particular, k is a non-archimedean local field, and $\mathbf{H}(k)$ is totally disconnected.

Denoting by $\widetilde{\mathbf{H}}$ the adjoint quotient of $\mathbf{H}$ and applying the embedding theorem of [Rém04, 3.A], we obtain a closed embedding of topological groups $\mu\colon \overline{\Lambda} \hookrightarrow \widetilde{\mathbf{H}}(k)$. This is where we use that $q \geq 4$ and $W(\frac{1}{q}) < \infty$. The closed subgroup $\mu(\overline{\Lambda})$ of $\widetilde{\mathbf{H}}(k)$ contains the lattice Γ. All groups in the chain $\Gamma \leqslant \mu(\overline{\Lambda}) \leqslant \widetilde{\mathbf{H}}(k)$ being unimodular, $\mu(\overline{\Lambda})$ is of finite covolume in $\widetilde{\mathbf{H}}(k)$. Therefore by [Pra77, Main Theorem], $\mu(\overline{\Lambda})$ contains $\widetilde{\mathbf{H}}(k)^+$. In view of the remark following Corollary 18, this implies $\mathrm{flat\text{-}rk}(\overline{\Lambda}) = k\text{-}\mathrm{rk}(\mathbf{H})$. $\square$

Replacing the target group $\mathbf{H}(k)$ by a closed isometry group of a suitable CAT(0)-space, this result might be generalized to provide restrictions on the existence of discrete actions on CAT(0)-cell complexes. For this, one shall replace the commensurator super-rigidity by super-rigidity of irreducible lattices, as proved in [Mon05].

Assume now that the target lattice Γ is replaced by an infinite Kac-Moody lattice (and that both Kac-Moody groups are split). Using again the normal subgroup property [BS06, Rém05], we are led to an isomorphism $\bar\eta : \Lambda/Z(\Lambda) \simeq \Gamma/Z(\Gamma)$. In this situation, a much more precise answer than ours was given by Caprace and Mühlherr [CM]. Their proof is of geometric and combinatorial nature (Bruhat-Tits fixed point lemma and twin root datum arguments).

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