Minor identities for quasi-determinants and quantum determinants

Daniel Krob, Bernard Leclerc

MINOR IDENTITIES FOR QUASI-DETERMINANTS AND QUANTUM DETERMINANTS

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Abstract. We present several identities involving quasi-minors of noncommutative generic matrices. These identities are specialized to quantum matrices, yielding q-analogues of various classical determinantal formulas.

1. Introduction

Defining a "good" noncommutative notion of determinant is a very old problem that can be traced back to Cayley (cf [**Ca**]). There have been several attempts at this problem since the beginning of this century. However it is only very recently that I.M. Gelfand and V.S. Retakh made a major breakthrough by introducing the concept of *quasi-determinant* which generalizes within a totally noncommutative framework the classical concept of determinant (see [**GR1**], [**GR2**]). Their main idea was to abandon the multiplicativity of commutative determinants and to focus on their properties with respect to inversion. It also happens that quasi-determinants are very closely related to the representation aspect of automata theory initiated by M.P. Schützenberger (cf [**BR**], [**Sc**] for instance). The presentation we give here is indeed highly influenced by this point of view.

Quasi-determinants are defined for generic matrices with entries in the free skew field, allowing therefore to work with them in arbitrary skew fields. Gelfand and Retakh developed their theory in the two seminal papers [**GR1**] and [**GR2**] where they obtained a lot of noncommutative versions of classical results such as Cayley-Hamilton's theorem, Capelli's identity, Gauss' decomposition of a matrix, ... They gave in particular noncommutative versions of several classical determinantal identities. They also showed that various noncommutative determinants – quantum determinants, Berezinians, Dieudonné determinants – can be expressed as products of quasi-determinants (see also Section 3.2). It is therefore of major interest to investigate general identities satisfied by quasi-minors in order to deduce from them identities for these other types of determinants.

In this article, we focus our interest on relating more strongly quasi-determinants and *quantum determinants*. The recent development of quantum group theory has indeed led to the discovery of noncommutative analogues for the main concepts of multilinear algebra (cf [**Sk**], [**RTF**], [**Ta**], [**Dr**], [**HH**], [**PW**] for instance). In this picture, quantum determinants play a major role. For example, the study of the quadratic identities satisfied by minors

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of a quantum matrix made it possible to define quantum deformations of Grassmann and flag manifolds (cf [**LR**], [**TT**]). These identities are quantum analogues of the well-known Plücker and Garnir relations and of the straightening formula of classical invariant theory. This raises the question of finding quantum analogues for other classical identities of degree > 2, like Sylvester's or Bazin's, which are of great use in many problems of commutative algebra. We present here such quantum identities as simple consequences of general noncommutative identities for quasi-determinants.

The article is organized as follows. Its first part is devoted to recall several relations between quasi-minors of the generic matrix. More precisely, we review in Sections 2.1 and 2.2 the definition and basic properties of quasi-determinants. We describe then in Section 2.3 noncommutative analogues of several classical theorems, including Cayley's law of complementaries, Muir's law of extensible minors, Sylvester's theorem, Bazin's theorem and Schweins' series. Finally, these results are applied in Section 3 to quantum determinants yielding quantum analogues of the same theorems.

2. Quasi-determinants

2.1. Definitions and notations

Let k be a field, let n be an integer and let $\mathbb{A} = \{ a_{ij},\ 1 \leq i, j \leq n \}$ be an alphabet of order n^2. Let $k \langle\!\langle \mathbb{A} \rangle\!\rangle$ be the *free skew field* constructed on k and generated by $\mathbb{A}$ (cf [**Co**] for details). The matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ is then called the *generic matrix* of order n. It is useful to associate with A the automaton $\mathcal{A}$ which is the graph whose transition matrix is A. In other words, $\mathcal{A}$ is the complete oriented graph constructed over $\{1, \ldots, n\}$ where the edge relating i to j is labelled by a_{ij} for every $i, j \in \{1, \ldots, n\}$.

Example 2.1. — For $n = 2$, the automaton $\mathcal{A}$ is given below.

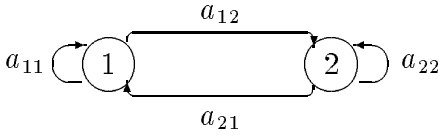

Figure 2.1

One can now define the *star* of the matrix A as follows

$$A^* = (1 - A)^{-1} = \sum_{i=0}^{+\infty} A^i \ .$$

The entries of A^* have a simple and well-known automata-theoretic interpretation (see [**BR**] or [**Ei**] for details). Indeed an easy induction on n shows that $(A^*)_{ij}$ is the sum of all words labelling the paths that relate i to j in $\mathcal{A}$. Using this property, one can obtain in particular the classical formula (see [**Co**] p. 27 for instance)

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^* = \begin{pmatrix} (a_{11} + a_{12}\, a_{22}^*\, a_{21})^* & a_{11}^*\, a_{12}\, (a_{22} + a_{21}\, a_{11}^*\, a_{12})^* \\ a_{22}^*\, a_{21}\, (a_{11} + a_{12}\, a_{22}^*\, a_{21})^* & (a_{22} + a_{21}\, a_{11}^*\, a_{12})^* \end{pmatrix} . \tag{2.1}$$

Let us prove for instance that the above equality holds for entries of order 11. Thus we must show that the entry of order 11 in the matrix of the right side of relation (2.1) represents

the set of words labelling a path from 1 to 1 in the automaton $\mathcal{A}$ of Figure 2.1. Indeed such a path can be decomposed as a sequence of paths going from 1 to 1 in $\mathcal{A}$ without using 1 as an intermediate state. But these last paths are obviously equal to $a_{11} + a_{12}\, a_{22}^*\, a_{21}$. Since the star of a set L of words is the set that consists in arbitrary sequences of words of L, this explains the entry of order 11 in formula (2.1). The same kind of interpretation also holds clearly for the other entries.

Using appropriate substitutions, formula (2.1) gives in fact a recursive definition of the star of a generic matrix of any order and shows that the entries of A^* still belong to $k\langle\!\langle \mathbb{A} \rangle\!\rangle$. Let us now connect the star and the inverse operations. According to a result of Cohn (see **[Co]** p. 89), we can define an involutive field automorphism ω of $k\langle\!\langle \mathbb{A} \rangle\!\rangle$ by setting

$$\omega(a_{ij}) = \begin{cases} 1 - a_{ii} & \text{if } \ i = j \\ -a_{ij} & \text{if } \ i \neq j \end{cases}$$

for every $1 \leq i, j \leq n$. This involution clearly maps the generic matrix A on $I_n - A$. Hence we have $\omega(A^{-1}) = A^*$ and conversely $\omega(A^*) = A^{-1}$. We can now give the following result, which shows that the generic matrix has always an inverse that can be constructed by means of recursive formulas.

PROPOSITION 2.2. — *Let P, Q and R, S be two partitions of $\{1, \ldots, n\}$ such that $|P| = |R|$ and $|Q| = |S|$. Let us decompose the generic matrix A of order n as*

$$A = \begin{array}{c} \\ P \\ Q \end{array} \begin{array}{c} \begin{array}{cc} R & S \end{array} \\ \begin{pmatrix} A_{PR} & A_{PS} \\ A_{QR} & A_{QS} \end{pmatrix} \end{array} .$$

Then the inverse of A is recursively given by

$$A^{-1} = \begin{array}{c} \\ R \\ S \end{array} \begin{array}{c} \begin{array}{cc} P & \qquad\qquad Q \end{array} \\ \begin{pmatrix} (A_{PR} - A_{PS}A_{QS}^{-1}A_{QR})^{-1} & -A_{PR}^{-1}A_{PS}(A_{QS} - A_{QR}A_{PR}^{-1}A_{PS})^{-1} \\ -A_{QS}^{-1}A_{QR}(A_{PR} - A_{PS}A_{QS}^{-1}A_{QR})^{-1} & (A_{QS} - A_{QR}A_{PR}^{-1}A_{PS})^{-1} \end{pmatrix} \end{array} .$$

Proof. — The use of appropriate permutation matrices reduces the proof to the case $P = R$ and $Q = S$. Using substitutions, we can even reduce it to the case $|P| = |Q| = 1$. Then the above formula is just the image by ω of formula (2.1). □

Let us denote by A^{pq} the matrix obtained from A by deleting the p-th row and the q-th column. Let also $\xi_{pq} = (a_{p1}, \ldots, \hat{a}_{pq}, \ldots, a_{pn})$ and $\eta_{pq} = (a_{1q}, \ldots, \hat{a}_{pq}, \ldots, a_{nq})$. Applying now Proposition 2.2 with $P = \{q\}$ and $R = \{p\}$, we get

$$(A^{-1})_{pq} = (\, a_{pq} - \xi_{pq}(A^{pq})^{-1}\eta_{pq}\,)^{-1} \tag{2.2}$$

for every $1 \leq p, q \leq n$, where ξ_{pq} is understood as a row matrix and η_{pq} as a column matrix. Formula (2.2) therefore expresses in a symmetric way the inverse of the generic matrix A. Now the following definition makes sense.

DEFINITION 2.3. — (Gelfand – Retakh; **[GR1]**) *The quasi-determinant $|A|_{pq}$ of order pq of the generic matrix A is the element of $k\langle\!\langle \mathbb{A} \rangle\!\rangle$ defined by*

$$|A|_{pq} = a_{pq} - \xi_{pq}\,(A^{pq})^{-1}\,\eta_{pq} \ = a_{pq} - \sum_{i \neq p, j \neq q} a_{pj}\,((A^{pq})^{-1})_{ji}\,a_{iq} \ .$$

NOTATION 2.4. — It is convenient to adopt the following more explicit notation

$$|A|_{pq} = \begin{vmatrix} a_{11} & \dots & a_{1q} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{p1} & \dots & \boxed{a_{pq}} & \dots & a_{pn} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nq} & \dots & a_{nn} \end{vmatrix} .$$

NOTES 2.5. — 1. Quasi-determinants are here only defined for generic matrices. However this is not a real restriction since one can clearly transport the above definition to invertible matrices with entries in an arbitrary skew field using substitutions. In fact, one can even work in a noncommutative ring when A^{pq} is an invertible matrix.

2. The involution ω is the tool that connects quasi-determinants and automata theory. The relations between these theories already appeared in the proof of Proposition 2.2.

EXAMPLE 2.6. — For $n = 2$, there are 4 quasi-determinants :

$$\begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12}\, a_{22}^{-1}\, a_{21} \ , \quad \begin{vmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{vmatrix} = a_{12} - a_{11}\, a_{21}^{-1}\, a_{22} \ ,$$

$$\begin{vmatrix} a_{11} & a_{12} \\ \boxed{a_{21}} & a_{22} \end{vmatrix} = a_{21} - a_{22}\, a_{12}^{-1}\, a_{11} \ , \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix} = a_{22} - a_{21}\, a_{11}^{-1}\, a_{12} \ .$$

Let us finally give some notations that will be used throughout this paper. Let P, Q be subsets of $\{1, \dots, n\}$ of the same cardinality. We shall denote by A^{PQ} the matrix obtained by removing from A the rows whose indices belong to P and the columns whose indices belong to Q. We also set $A_{PQ} = A^{\overline{P}\ \overline{Q}}$, where $\overline{P}$ and $\overline{Q}$ are the complements of P and Q in $\{1, 2, \dots, n\}$. Finally, if a_{ij} is an entry of some submatrix A^{PQ} or A_{PQ}, we shall denote by $|A^{PQ}|_{ij}$ or $|A_{PQ}|_{ij}$ the corresponding quasi-minor.

2.2. Fundamental properties

We shall now recall the main properties of quasi-determinants. Most of the results in this subsection are due to Gelfand and Retakh (cf **[GR1]**, **[GR2]**). We shall often use in the sequel quasi-determinants of non-generic matrices, following Note 2.5.1.

2.2.1. The inverse matrix

The following result is just a rewriting of formula (2.2) and Definition 2.3.

PROPOSITION 2.7. — (Gelfand – Retakh; **[GR1]**) *Let A be the generic matrix of order n and let $B = A^{-1} = (b_{pq})_{1 \leq p,q \leq n}$ be its inverse. Then one has $|A|_{pq} = b_{qp}^{-1}$ for every $1 \leq p, q \leq n$.*

NOTES 2.8. — 1. It follows from Proposition 2.7 that $|A|_{pq} = (-1)^{p+q} \det A / \det A^{pq}$ when k is a commutative field. Thus quasi-determinants may be regarded as noncommutative analogues of the ratio of a determinant to one of its principal minors.

2. If A is an arbitrary invertible matrix, one can show that the above relation also holds for every p, q such that $b_{qp} \neq 0$ (cf **[GR1]**, **[GR2]**).

As a consequence of Proposition 2.7, we can now give the following formula

$$|A|_{pq} = a_{pq} - \sum_{i \neq p, j \neq q} a_{pj} \, |A^{pq}|^{-1}_{ij} \, a_{iq} \tag{2.3}$$

which is also just a rewriting of Definition 2.3. However formula (2.3) is very useful since it may be considered as a recursive definition of quasi-determinants.

2.2.2. Permutation of rows and columns

PROPOSITION 2.9. — (Gelfand – Retakh; **[GR1]**) *A permutation of the rows or columns of a quasi-determinant does not change its value.*

Proof. — Let $\sigma \in \mathfrak{S}_n$ and let P_σ be the associated permutation matrix. Then we have

$$|P_\sigma \, A \, P_\sigma^{-1}|^{-1}_{pq} = ((P_\sigma \, A \, P_\sigma^{-1})^{-1})_{qp} = ((P_\sigma \, A^{-1} \, P_\sigma^{-1})_{qp} = (A^{-1})_{\sigma(q)\sigma(p)} = |A|^{-1}_{\sigma(p)\sigma(q)}$$

for every $1 \leq p, q \leq n$, according to Proposition 2.7. □

EXAMPLE 2.10. — As an application of Proposition 2.9, we have for instance

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{21} & a_{23} \\ a_{12} & a_{11} & a_{13} \\ a_{32} & \boxed{a_{31}} & a_{33} \end{vmatrix} .$$

2.2.3. Elementary operations on the rows or columns

PROPOSITION 2.11. — (Gelfand – Retakh; **[GR1]**) *If the matrix B is obtained from the matrix A by multiplying the p-th row* on the left *by λ, then*

$$|B|_{kq} = \begin{cases} \lambda \, |A|_{pq} & \text{for} \quad k = p \ , \\ |A|_{kq} & \text{for} \quad k \neq p \ . \end{cases}$$

Similarly, if the matrix C is obtained from the matrix A by multiplying the q-th column on the right *by μ, then*

$$|C|_{pl} = \begin{cases} |A|_{pq} \, \mu & \text{for} \quad l = q \ , \\ |A|_{pl} & \text{for} \quad l \neq q \ . \end{cases}$$

Finally, if the matrix D is obtained from A by adding to some row (resp. column) of A its k-th row (resp. column), then $|D|_{pq} = |A|_{pq}$ for every $p \neq k$ (resp. $q \neq k$).

Proof. — The two first properties are easily proved by induction on n using relation (2.3). Let us now show the final property for row addition for instance. Let D be obtained from A by adding its k-th row to its l-th row. Let then $M = I_n + E_{lk}$ where E_{lk} denotes the matrix whose unique non-zero entry is the lk-th entry which is equal to 1. Clearly $D = MA$. Using Proposition 2.7, we get

$$|D|^{-1}_{pq} = (D^{-1})_{qp} = (A^{-1}M^{-1})_{qp} = A^{-1}_{qp} = |A|^{-1}_{pq}$$

for every $p \neq k$, since multiplying a matrix by M on the right does only change its k-th column. □

2.2.4. Homological relations

The following proposition gives important relations between quasi-minors of a matrix that are called *homological relations*. Its proof is based on technical computations mainly using Proposition 2.2. We omit it here, rather refering to [**GR2**] where a sketch of proof may be found.

PROPOSITION 2.12. — (Gelfand – Retakh; [**GR1**]) *The quasi-minors of the generic matrix A are related by the following relations :*

$$|A|_{ij}\,(|A^{il}|_{kj})^{-1} = -|A|_{il}\,(|A^{ij}|_{kl})^{-1}\ ,$$
$$(|A^{kj}|_{il})^{-1}\,|A|_{ij} = -(|A^{ij}|_{kl})^{-1}\,|A|_{kj}\ .$$

EXAMPLE 2.13. — The following example illustrates Proposition 2.12.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & \boxed{a_{32}} \end{vmatrix}^{-1} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & \boxed{a_{33}} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix}^{-1} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & \boxed{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\ .$$

2.2.5. Expansion by a row or column

PROPOSITION 2.14. — *For quasi-determinants, there holds the following analogue of the classical expansion of a determinant by one of its rows or columns :*

$$|A|_{pq} = a_{pq} - \sum_{j\neq q} a_{pj}\,(|A^{pq}|_{kj})^{-1}\,|A^{pj}|_{kq}\ ,$$
$$|A|_{pq} = a_{pq} - \sum_{i\neq p} |A^{iq}|_{pl}\,(|A^{pq}|_{il})^{-1}\,a_{iq}\ ,$$

for every $k \neq p$ and $l \neq q$.

Proof. — Let us prove for instance the first relation. By Proposition 2.7, we have

$$1 = \sum_{j=1}^{n} a_{p,j}\,|A|_{pj}^{-1}\ .$$

One now gets the desired row expansion by multiplying on the right this last expression by $|A|_{pq}$ and using the first homological relations given by Proposition 2.12. □

EXAMPLE 2.15. — Let $n = p = q = 4$. Then Proposition 2.14 gives us

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & \boxed{a_{44}} \end{vmatrix} = a_{44} - a_{43} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & \boxed{a_{33}} \end{vmatrix}^{-1} \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & \boxed{a_{34}} \end{vmatrix}$$

$$-a_{42}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & \boxed{a_{32}} & a_{33} \end{vmatrix}^{-1}\begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & \boxed{a_{34}} \end{vmatrix} - a_{41}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \boxed{a_{31}} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}^{-1}\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & \boxed{a_{34}} \end{vmatrix} .$$

2.3. Minors identities for quasi-determinants

In this section, we give the noncommutative analogues of several classical theorems. The reader is referred to [**Le**] for a review of these theorems in the commutative case.

2.3.1. Jacobi's ratio theorem

In the commutative case, Jacobi's ratio theorem (cf [**Ja**], [**Tu**] or [**Bo**] Exercise III. 11. 9) states that each minor of the inverse matrix A^{-1} is equal, up to a sign factor, to the ratio of the corresponding complementary minor of the transpose of A to $det\, A$. In the noncommutative case, we have the following analogue of this theorem.

THEOREM 2.16. — (Gelfand – Retakh; [**GR1**]) *Let A be the generic matrix of order n, let B be its inverse and let $(\{i\}, L, P)$ and $(\{j\}, M, Q)$ be two partitions of $\{1, 2, \ldots, n\}$ such that $|L| = |M|$ and $|P| = |Q|$. Then there holds :*

$$|B_{M\cup\{j\},L\cup\{i\}}|_{ji} = |A_{P\cup\{i\},Q\cup\{j\}}|_{ij}^{-1} .$$

Proof. — Using appropriate permutation matrices allows to reduce the proof to the case $i = j$, $L = M$ and $P = Q$. The image by ω of the relation to be proved is then equal to

$$|(A^*)_{L\cup\{i\},L\cup\{i\}}|_{ii} = |(I - A)_{P\cup\{i\},P\cup\{i\}}|_{ii}^{-1} . \tag{2.4}$$

According to Definition 2.3, we have

$$|(A^*)_{L\cup\{i\},L\cup\{i\}}|_{ii} = (A^*)_{ii} - (A^*)_{iL}((A^*)_{LL})^{-1}(A^*)_{Li} .$$

Using now Proposition 2.7, we get

$$|(I - A)_{P\cup\{i\},P\cup\{i\}}|_{ii}^{-1} = ((I - A_{P\cup\{i\},P\cup\{i\}})^{-1})_{ii} = ((A_{i\cup P,i\cup P})^*)_{ii} .$$

Therefore we have to prove that

$$(A^*)_{ii} - (A^*)_{iL}((A^*)_{LL})^{-1}(A^*)_{Li} = ((A_{i\cup P,i\cup P})^*)_{ii} .$$

Applying Proposition 2.2 to the matrix $I - A$, we find that

$$((A^*)_{LL})^{-1} = I - A_{LL} - A_{L,i\cup P}(A_{i\cup P,i\cup P})^* A_{i\cup P,L} .$$

Thus the identity to be checked may also be written

$$(A^*)_{ii} + (A^*)_{iL}(A_{LL} + A_{L,i\cup P}(A_{i\cup P,i\cup P})^* A_{i\cup P,L})(A^*)_{Li} = ((A_{i\cup P,i\cup P})^*)_{ii} + (A^*)_{iL}(A^*)_{Li} . \tag{2.5}$$

Let us now notice that one has

$$(A^*)_{Li} = (A_{LL} + A_{L,i\cup P}\,(A_{i\cup P,i\cup P})^*\, A_{i\cup P,L})\,(A^*)_{Li} + (A_{Li} + A_{L,i\cup P}\,(A_{i\cup P,i\cup P})^*\, A_{i\cup P,i}) .$$

Indeed this relation just expresses that the set of the paths going from L to i in the automaton $\mathcal{A}$ whose transition matrix is A, can be decomposed in the two disjoint sets

consisting respectively in the paths using an intermediate state belonging to L and in the paths using no such a state. Using now this last relation, we can rewrite relation (2.5) in the following equivalent way

$$(A^*)_{ii} = (A^*)_{iL}\,(A_{Li} + A_{L,i\cup P}\,(A_{i\cup P,i\cup P})^*\,A_{i\cup P,i}) + ((A_{i\cup P,i\cup P})^*)_{ii}\ ,$$

which expresses again an obvious decomposition of the set of paths going from i to i in $\mathcal{A}$. This ends therefore our proof. □

EXAMPLE 2.17. — Take $n = 5$, $i = 3$, $j = 4$, $L = \{1,2\}$, $M = \{1,3\}$, $P = \{4,5\}$ and $Q = \{2,5\}$. Theorem 2.16 shows that

$$\begin{vmatrix} a_{32} & \boxed{a_{34}} & a_{35} \\ a_{42} & a_{44} & a_{45} \\ a_{52} & a_{54} & a_{55} \end{vmatrix} = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & \boxed{b_{43}} \end{vmatrix}^{-1} .$$

2.3.2. Cayley's law of complementaries

In the commutative case, Cayley's law of complementaries assumes the following form. Let I be an identity between minors of the generic matrix A. If every minor is replaced by its complement in A (multiplied by a suitable power of $det\,A$), a new identity I^C is obtained, which is said to result from I by application of the law of complementaries (cf **[Mu]** and also **[Bo]** Exercise III. 11. 10). In the noncommutative case, we have the following analogue of this law.

THEOREM 2.18. — *Let I be an identity between quasi-minors of the generic matrix A of order n. If every quasi-minor $|A_{L,M}|_{ij}$ involved in I is replaced by $|A_{\overline{M}\cup\{j\},\overline{L}\cup\{i\}}|_{ji}^{-1}$, where $\overline{L} = \{1,2,\ldots,n\} - L$ and $\overline{M} = \{1,2,\ldots,n\} - M$, there results a new identity I^C.*

Proof. — Applying identity I to A^{-1} gives identity I^C by means of Theorem 2.16. □

EXAMPLE 2.19. — Let $n = 3$ and let I be the identity

$$a_{13}^{-1}\begin{vmatrix} \boxed{a_{12}} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{13}^{-1}a_{12} - a_{33}^{-1}a_{32} \qquad (I)\ .$$

By means of the law of complementaries, one can deduce from I the new identity I^C :

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix}\begin{vmatrix} a_{11} & a_{12} \\ \boxed{a_{21}} & a_{22} \end{vmatrix}^{-1} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \boxed{a_{21}} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}^{-1}$$

$$- \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & \boxed{a_{33}} \end{vmatrix}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & \boxed{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}^{-1} \qquad (I^C)\ .$$

2.3.3. Muir's law of extensible minors

Let us first recall Muir's law of extensible minors in the commutative case (cf **[Mu]** and also **[Bo]** Exercise III. 11. 11). Let B be a square matrix of order $n + p$, let

$A = B_{P,Q}$, $C = B^{P,Q}$ where P, Q are two subsets of $\{1,\dots,n+p\}$ of cardinality n and let I be an identity between minors of A. When every minor $|A_{L,M}|$ involved in I is replaced by its *extension* $|B_{L\cup\overline{P},M\cup\overline{Q}}|$ (multiplied by a suitable power of the *pivot* $|C|$ if the obtained identity is not homogeneous), a new identity I^E is obtained, which is called an *extensional* of I. A similar rule holds in the noncommutative case.

THEOREM 2.20. — *Let B be the generic matrix of order $n+p$, let $A = B_{P,Q}$ where P, Q are subsets of $\{1,\dots,n+p\}$ of cardinality n and let I be an identity between quasi-minors of A. If every quasi-minor $|A_{L,M}|_{ij}$ involved in I is replaced by its extension $|B_{L\cup\overline{P},M\cup\overline{Q}}|_{ij}$, a new identity I^E is obtained which is called an extensional of I. The submatrix $B_{\overline{P},\overline{Q}}$ is called the pivot of the extension.*

Proof. — As shown by Muir, Theorem 2.20 results from two successive applications of Theorem 2.18. Indeed, a first application of the law of complementaries to identity I transforms it into an other identity I^C between quasi-minors of A. But quasi-minors of A may be seen as quasi-minors of B and identity I^C may be seen as an identity between quasi-minors of B. A new application to I^C of the law of complementaries, but taking now the complements relatively to B, yields identity I^E. □

EXAMPLE 2.21. — The following identity results clearly from proposition 2.9 :

$$a_{11}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \boxed{a_{21}} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}^{-1} + a_{12}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \boxed{a_{22}} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}^{-1} + a_{13}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & \boxed{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}^{-1} = 0 \qquad (I)\ .$$

An extensional of identity I is, for example, the following identity that illustrates Theorem 1.3 of **[GR2]** :

$$\begin{vmatrix} \boxed{a_{11}} & a_{14} & a_{15} \\ a_{41} & a_{44} & a_{45} \\ a_{51} & a_{54} & a_{55} \end{vmatrix}\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ \boxed{a_{21}} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix}^{-1}$$

$$+\begin{vmatrix} \boxed{a_{12}} & a_{14} & a_{15} \\ a_{42} & a_{44} & a_{45} \\ a_{52} & a_{54} & a_{55} \end{vmatrix}\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & \boxed{a_{22}} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix}^{-1}$$

$$+\begin{vmatrix} \boxed{a_{13}} & a_{14} & a_{15} \\ a_{43} & a_{44} & a_{45} \\ a_{53} & a_{54} & a_{55} \end{vmatrix}\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & \boxed{a_{23}} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix}^{-1} = 0 \qquad (I^E)\ .$$

2.3.4. Sylvester's theorem

Another important application of Muir's law of extensible minors is the noncommutative version of Sylvester's theorem. As in the commutative case, it is obtained by applying Theorem 2.20 to the complete expansion of a quasi-determinant.

THEOREM 2.22. — (Gelfand – Retakh; **[GR1]**) *Let A be the generic matrix of order n and let P, Q be two subsets of $\{1, \ldots, n\}$ of cardinality k. For $i \in \overline{P}$ and $j \in \overline{Q}$, we set $b_{ij} = |A_{P\cup\{i\},Q\cup\{j\}}|_{ij}$ and form the matrix $B = (b_{ij})_{i\in\overline{P},j\in\overline{Q}}$ of order $n-k$. Then one has*

$$|A|_{lm} = |B|_{lm}$$

for every $l \in \overline{P}$ and $m \in \overline{Q}$.

EXAMPLE 2.23. — Let us take $n = 3$, $P = Q = \{3\}$ and $l = m = 1$. Applying Muir's law to the expansion of $|A_{\{1,2\},\{1,2\}}|_{11}$:

$$\begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12}\, a_{22}^{-1}\, a_{21} \ ,$$

we get the identity

$$\begin{vmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} \boxed{a_{11}} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - \begin{vmatrix} \boxed{a_{12}} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} \boxed{a_{22}} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{-1} \begin{vmatrix} \boxed{a_{21}} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} \boxed{\begin{vmatrix} \boxed{a_{11}} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}} & \begin{vmatrix} \boxed{a_{12}} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} \boxed{a_{22}} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} \boxed{a_{21}} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix}$$

which is the simplest instance of Sylvester's theorem for quasi-determinants.

Note also that Sylvester's theorem furnishes a recursive method for evaluating quasi-determinants since it allows for instance to reduce the computation of a quasi-determinant of order n to the computation of a quasi-determinant of order $n-1$ composed of $(n-1)^2$ quasi-determinants of order 2. As one can easily check, this clearly leads to a cubic algorithm for computing quasi-determinants.

NOTE 2.24. — We presented here Sylvester's theorem as a simple consequence of Theorem 2.20. This was not the method originally followed by Gelfand and Retakh in **[GR1]**. In fact, Sylvester's theorem for quasi-determinants is just a rewriting of Proposition 2.2 and can therefore be directly obtained from it as the reader will easily check.

2.3.5. Bazin's theorem

Sylvester's theorem expresses a relation between (quasi-)minors of a square matrix. Bazin's theorem deals with maximal (quasi-)minors of a rectangular matrix. In fact, in both commutative and noncommutative cases, these theorems are equivalent and each one may be deduced from the other by specialization to a suitable matrix.

We introduce some notations. Let A be a matrix of order $n \times 2n$. Then, for every subset P of cardinality n of $\{1, \ldots, 2n\}$, we denote by A_P the square submatrix of A whose columns are indexed by P. In the sequel, we also index all quasi-minors of order $P \times Q$ of a matrix by their relative indices in $P \times Q$. We can now state Bazin's theorem for quasi-determinants.

THEOREME 2.25. — *Let A be the generic matrix of order $n \times 2n$ and let m be an integer in $\{1, \ldots, n\}$. For $1 \leq i, j \leq n$, we set $b_{ij} = |A_{\{j,n+1,\ldots,n+i-1,n+i+1,\ldots,2n\}}|_{mj}$ and form the matrix $B = (b_{ij})_{1 \leq i,j \leq n}$. Then we have*

$$|B|_{kl} = |A_{\{n+1,\ldots,2n\}}|_{m,n+k} \ |A_{\{1,\ldots,l-1,l+1,\ldots,n,n+k\}}|^{-1}_{m,n+k} \ |A_{\{1,\ldots,n\}}|_{ml}$$

for every integers k, l in $\{1, \ldots, n\}$.

Proof. — Let us consider the $2n \times 2n$ matrix C defined by

$$C = \begin{pmatrix} A_{\{1,\ldots,n\}} & A_{\{n+1,\ldots,2n\}} \\ 0_n & I_n \end{pmatrix} ,$$

where I_n and 0_n denote respectively the unit and zero matrix of order n. Applying Sylvester's theorem to this matrix with $C_{\{1,\ldots,n\},\{n+1,\ldots,2n\}}$ as pivot, we now get

$$|C|_{n+k,l} = \left| \left(\begin{array}{cc} A_j & A_{\{n+1,\ldots,2n\}} \\ \boxed{0} & u_{n+i} \end{array} \right)_{1 \leq i,j \leq n} \right|_{kl}$$

where u_i denotes for every integer i the row vector whose only non-zero entry is the i-th entry equal to 1. Developping on the last row every quasi-determinant involved in the above identity and using then Proposition 2.11, we easily obtain

$$|C|_{n+k,l} = - \, |A_{\{n+1,\ldots,2n\}}|^{-1}_{m,n+k} \ |B|_{kl} \qquad (2.6) \ .$$

On the other hand, it easily follows from Propositions 2.2 and 2.7 that

$$|C|^{-1}_{n+k,l} = -(A^{-1}_{\{1,\ldots,n\}} \, A_{\{1,\ldots,n\},\{n+1,\ldots,2n\}})_{l,n+k} = - \sum_{j=1}^{n} \ |A_{\{1,\ldots,n\}}|^{-1}_{j,l} \, a_{j,n+k} \ .$$

Using Definition 2.3, this last relation can be rewritten in an equivalent way as

$$|C|^{-1}_{n+k,l} = \left| \begin{array}{cc} A_{\{1,\ldots,n\}} & A_{n+k} \\ u_l & \boxed{0} \end{array} \right| \ .$$

Expanding now this quasi-determinant on the last row, we get the identity

$$|C|^{-1}_{n+k,l} = - \, |A_{\{1,\ldots,n\}}|^{-1}_{ml} \ |A_{\{1,\ldots,l-1,l+1,\ldots,n,n+k\}}|_{m,n+k}$$

from which, comparing it to relation (2.6), we can immediately conclude. □

EXAMPLE 2.26. — Let $n = 3$ and $k = l = m = 1$. Let us adopt more appropriate notations, writing for example $|2\boxed{4}5|$ instead of $|M_{\{2,4,5\}}|_{14}$. Bazin's identity says then that

$$\left| \begin{array}{ccc} |\boxed{1}56| & |\boxed{2}56| & |\boxed{3}56| \\ |\boxed{1}46| & |\boxed{2}46| & |\boxed{3}46| \\ |\boxed{1}45| & |\boxed{2}45| & |\boxed{3}45| \end{array} \right| = |\boxed{4}56| \ |23\boxed{4}|^{-1} \ |\boxed{1}23| \ .$$

2.3.6. Schweins' series

"Schweins found an important series, in 1825, for the quotient of two n-rowed determinants which differ only in one column. This series is of great use in many branches of algebra and analysis, and many interesting cases arise by treating one column as a column of the unit matrix" (cf [**Tu**]). Here is an example of Schweins' commutative series.

$$\frac{(abcd)_{1234}}{(abce)_{1234}} = \frac{(abc)_{123}(abed)_{1234}}{(abe)_{123}(abce)_{1234}} + \frac{(ab)_{12}(aed)_{123}}{(ae)_{12}(abe)_{123}} + \frac{a_1(ed)_{12}}{e_1(ae)_{12}} + \frac{d_1}{e_1} \ ,$$

where for instance $(aed)_{123}$ denotes the determinant $\begin{vmatrix} a_1 & e_1 & d_1 \\ a_2 & e_2 & d_2 \\ a_3 & e_3 & d_3 \end{vmatrix}$.

Schweins' series is still valid in the noncommutative case. Keeping the notations of 2.3.5, let us first note that, according to the homological relations, one has for instance in the case of the generic matrix A of order 3×6 :

$$|A_{123}|_{13}^{-1}\,|A_{124}|_{14} = |A_{123}|_{23}^{-1}\,|A_{124}|_{24} = |A_{123}|_{33}^{-1}\,|A_{124}|_{34} \ .$$

This common value will be denoted for short by $|12\boxed{3}|^{-1}|12\boxed{4}|$. We can now state Schweins' series for quasi-determinants. For convenience, we limit ourselves to the case of quasi-determinants of order 3 and 4, the general case being easily induced from these.

THEOREM 2.27. — *The maximal quasi-minors of a* 3×6 *generic matrix satisfy the relation*

$$|12\boxed{3}|^{-1}|12\boxed{4}| = |12\boxed{3}|^{-1}|12\boxed{5}|\,|23\boxed{5}|^{-1}|23\boxed{4}| +$$
$$|25\boxed{3}|^{-1}|25\boxed{6}|\,|35\boxed{6}|^{-1}|35\boxed{4}| + |56\boxed{3}|^{-1}|56\boxed{4}| \ .$$

The maximal quasi-minors of a 4×8 *generic matrix satisfy the relation*

$$|123\boxed{4}|^{-1}|123\boxed{5}| = |123\boxed{4}|^{-1}|123\boxed{6}||234\boxed{6}|^{-1}|234\boxed{5}|$$
$$+|236\boxed{4}|^{-1}|236\boxed{7}||346\boxed{7}|^{-1}|346\boxed{5}| + |367\boxed{4}|^{-1}|367\boxed{8}||467\boxed{8}|^{-1}|467\boxed{5}| + |678\boxed{4}|^{-1}|678\boxed{5}| \ .$$

Proof. — Let us take again the notations of Example 2.26. Applying Bazin's theorem for $n = 2$ to the matrix (4513), we get :

$$|\boxed{4}1| - |\boxed{5}1|\,|\boxed{5}3|^{-1}\,|\boxed{4}3| = \begin{vmatrix} |\boxed{4}3| & |\boxed{5}3| \\ |\boxed{4}1| & |\boxed{5}1| \end{vmatrix} = |1\boxed{3}|\,|5\boxed{3}|^{-1}\,|\boxed{4}5| \ .$$

Multiplying the above relation on the left by $|1\boxed{3}|^{-1}$, using then Muir's law by extending the obtained identity on the second column, one now easily obtains with the help of Proposition 2.9 the relation

$$|12\boxed{3}|^{-1}|12\boxed{4}| = |12\boxed{3}|^{-1}|12\boxed{5}||23\boxed{5}|^{-1}|23\boxed{4}| + |25\boxed{3}|^{-1}|25\boxed{4}|$$

for quasi-determinants of order 3. Schweins' series for order 3 results from two applications of this lemma. The general case is similar. □

As noted by Turnbull, interesting corollaries are obtained by specialization to a particular matrix some columns of which are columns of the unit matrix. Let us mention the following, which for convenience is stated for order 3 and 4 only.

THEOREM 2.28. — *The quasi-minors of a 3×4 generic matrix satisfy the relation*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & \boxed{a_{33}} \end{vmatrix}^{-1} \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & \boxed{a_{34}} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & \boxed{a_{33}} \end{vmatrix}^{-1} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & \boxed{a_{34}} \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & \boxed{a_{23}} \end{vmatrix}^{-1} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & \boxed{a_{24}} \end{vmatrix} + a_{13}^{-1} a_{14} \ .$$

The quasi-minors of a 4×5 generic matrix satisfy the relation

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & \boxed{a_{44}} \end{vmatrix}^{-1} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{35} \\ a_{41} & a_{42} & a_{43} & \boxed{a_{45}} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & \boxed{a_{44}} \end{vmatrix}^{-1} \begin{vmatrix} a_{11} & a_{12} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{44} & \boxed{a_{45}} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & \boxed{a_{34}} \end{vmatrix}^{-1} \begin{vmatrix} a_{11} & a_{14} & a_{15} \\ a_{21} & a_{24} & a_{25} \\ a_{31} & a_{34} & \boxed{a_{35}} \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & \boxed{a_{24}} \end{vmatrix}^{-1} \begin{vmatrix} a_{14} & a_{15} \\ a_{24} & \boxed{a_{25}} \end{vmatrix} + a_{14}^{-1} a_{15} \ .$$

Proof. — Let us prove for instance the first relation, the general case beeing similar. We consider the 3×6 matrix

$$M = \begin{pmatrix} 0 & 0 & a_{11} & a_{12} & a_{13} & a_{14} \\ 1 & 0 & a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 1 & a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \ .$$

Using Proposition 2.12, we easily get the following relation between quasi-minors of M :

$$|12\boxed{3}|^{-1} \, |12\boxed{5}| \, |23\boxed{5}|^{-1} \, |23\boxed{4}| = a_{13}^{-1} \, a_{11} \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & \boxed{a_{21}} \end{vmatrix}^{-1} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & \boxed{a_{24}} \end{vmatrix}$$

$$= - \begin{vmatrix} a_{13} & a_{13} \\ a_{21} & \boxed{a_{23}} \end{vmatrix} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & \boxed{a_{24}} \end{vmatrix} \ .$$

Arguing in the same way, one can also obtain the relation

$$|25\boxed{3}|^{-1} |25\boxed{6}| \, |35\boxed{6}|^{-1} |35\boxed{4}| = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & \boxed{a_{33}} \end{vmatrix}^{-1} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & \boxed{a_{34}} \end{vmatrix}$$

between quasi-minors of M. The desired relation now becomes an obvious consequence of the first identity of Proposition 2.27 as easily checked. □

The last result is exactly the analogue for quasi-determinants of the commutative example given above. However, it looks more natural this way.

3. Quantum determinants

In this section we derive from the previous theorems on quasi-determinants the corresponding identities for quantum determinants.

3.1. Definitions and notations

We first review the definitions of quantum matrices and determinants. The reader is referred to **[RTF]**, **[Ta]** or **[Ma]** for further details. Consider the coordinate ring of the manifold of quantum rectangular $m \times n$ matrices. It is the polynomial $\mathbb{C}[q, q^{-1}]$-algebra generated by mn symbols a_{ij} subject to the following relations

$$a_{ik}a_{il} = q^{-1}a_{il}a_{ik} \quad \text{for } k < l, \quad a_{ik}a_{jk} = q^{-1}a_{jk}a_{ik} \quad \text{for } i < j,$$
$$a_{il}a_{jk} = a_{jk}a_{il} \quad \text{for } i < j,\ k < l,$$
$$a_{ik}a_{jl} - a_{jl}a_{ik} = (q^{-1} - q)\, a_{il}a_{jk} \quad \text{for } i < j,\ k < l\ .$$

In such a situation, we say that $A = (a_{ij})$ is a *quantum $m \times n$ matrix*. Note that the transpose of a quantum $m \times n$ matrix is a quantum $n \times m$ matrix. Let $P = \{i_1, \ldots, i_k\}$ and $Q = \{j_1, \ldots, j_k\}$ be two subsets of $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$ of same cardinality k. The *quantum minor* of A indexed by P and Q is defined by

$$det_q\, A_{PQ} = \sum_{\sigma \in S_k} (-q)^{-\ell(\sigma)}\, a_{i_1 j_{\sigma(1)}} \ \ldots\ a_{i_k j_{\sigma(k)}}\ ,$$

where S_k is the symmetric group on $\{1, \ldots, k\}$ and $\ell(\sigma)$ denotes the length of the permutation σ. In particular, for $m = n = k$, one obtains the *quantum determinant* $det_q\, A$ of the quantum square matrix $A = (a_{ij})$. The following theorem summarizes some basic and well-known properties of quantum matrices and quantum determinants.

THEOREM 3.1. — *(i) Let $A = (a_{ij})_{1 \le i,j \le n}$ be a square $n \times n$ quantum matrix. Let us then set $S(A) = (\alpha_{ij})_{1 \le i,j \le n}$ where*

$$\alpha_{ij} = (-q)^{j-i}\, det_q\, (A^{ji})$$

denotes for every $1 \le i, j \le n$ the quantum cofactor of a_{ij}. Then, one has

$$A\ S(A) = S(A)\ A = det_q\, A\, .\, I_n \qquad (3.1)\ .$$

Moreover the matrix $S(A)$ is also a quantum matrix for the parameter q^{-1}.

(ii) Let $A = (a_{ij})_{1 \le i,j \le n}$ be a square $n \times n$ quantum matrix. The quantum determinant $det_q\, A$ commutes with each generator a_{ij}.

We refer the reader to **[Me]** for a q-analogue of Binet-Cauchy formula, which generalizes the multiplication formula for quantum determinants. Note that (i) amounts to the expansion of $det_q\, A$ by one of its rows or columns. More generally there exists also a q-analogue of Laplace expansion for quantum determinants. Identity (3.1) also shows that by adjoining to the symbols a_{ij} the new symbol t submitted to the relations

$$t\ det_q\, A = det_q\, A\ t = 1\ ,$$

the matrix A becomes invertible in the $\mathbb{C}[q, q^{-1}]$-algebra generated by t and the a_{ij}. This algebra is therefore called the ring of coordinates of the quantum linear group. The matrix

$A^{-1} = tS(A)$ is called the inverse of the quantum matrix A. It is also a quantum matrix for the parameter q^{-1}.

3.2. Quantum determinants and quasi-determinants

We shall now consider the connection between quantum determinants and quasi-determinants. As recalled in section 2.1, quasi-determinants are noncommutative analogues of the ratio of a determinant to one of its principal minors. Thus if the entries a_{ij} of a matrix A belong to a commutative field, one has the following expression of $det\, A$ in terms of quasi-determinants

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} \boxed{a_{11}} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \begin{vmatrix} \boxed{a_{22}} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \dots & a_{nn} \end{vmatrix} \dots a_{nn} \ .$$

The following theorem provides an analogue of this formula for quantum determinants.

THEOREM 3.2. — (Gelfand – Retakh; **[GR1]**) *Let $A = (a_{ij})_{1 \le i,j \le n}$ be a quantum $n \times n$ matrix. In the skew field generated by the a_{ij}, one has*

$$det_q\, A = |A|_{11}\, |A^{11}|_{22} \dots a_{nn}$$

and the quasi-minors in the right-hand side commute all together. More generally, let $\sigma = i_1 \dots i_n$ and $\tau = j_1 \dots j_n$ be two permutations of S_n. There holds

$$det_q\, A = (-q)^{\ell(\sigma)-\ell(\tau)}\, |A|_{i_1 j_1}\, |A^{i_1 j_1}|_{i_2 j_2} \dots a_{i_n j_n} \tag{3.2}$$

and the quasi-minors in the right-hand side commute all together.

Proof. — By Proposition 2.7, the quasi-determinants of the quantum $n \times n$ matrix A are the inverses of the entries of $A^{-1} = (det_q\, A)^{-1}\ S(A)$. Hence we have

$$det_q\, A = (-q)^{i-j}\, |A|_{ij}\, det_q\, A^{ij} = (-q)^{i-j}\, det_q\, A^{ij}\, |A|_{ij} \tag{3.3}$$.

Using an induction on n, relation (3.2) immediately follows. Let us now prove that the quasi-determinants involved in (3.2) commute all together. For simplicity, we only argue here in the case $i_1 = j_1 = 1, \dots, i_n = j_n = n$. Using an induction on n, we can reduce the problem to prove that $|A|_{11}$ commutes with $|A^{11}|_{22}, \dots, a_{nn}$. Using relation (3.3), it suffices to show that $|A|_{11}$ commutes for every $1 \le i \le n-1$ with $det_q\, A_{\{1,\dots,i\},\{1,\dots,i\}}$. But relation (3.3) and Theorem 3.1 (ii) show that this is equivalent to ask that $det_q\, A^{11}$ commutes with $det_q\, A_{\{1,\dots,i\},\{1,\dots,i\}}$ for every $1 \le i \le n$ and this last property is true according to Theorem 3.1 (ii) applied to A^{11}. □

EXAMPLE 3.3. — For $n = 2$, we have

$$\begin{aligned} det_q\, A &= (a_{11} - a_{12} a_{22}^{-1} a_{21})\, a_{22} = (-q)^{-1}\, (a_{12} - a_{11} a_{21}^{-1} a_{22})\, a_{21} \\ &= (-q)\, (a_{21} - a_{22} a_{12}^{-1} a_{11})\, a_{12} = (a_{22} - a_{21} a_{11}^{-1} a_{12})\, a_{11} \ . \end{aligned}$$

Note that the parameter q no longer appears in the first and fourth expression.

3.3. Minor identities for quantum determinants

3.3.1. Jacobi's ratio theorem

Recall that Jacobi's theorem states that each minor of the inverse matrix A^{-1} is equal, up to a sign factor, to the ratio of the corresponding complementary minor of the transpose of A to $det\, A$. For quantum determinants, we have the following quantum analogue.

THEOREM 3.4. — *Let A be a quantum $n \times n$ matrix, let $P = \{i_1 < \ldots < i_k\}$, $Q = \{j_1 < \ldots < j_k\}$, $\overline{P} = \{i_{k+1} < \ldots < i_n\}$ and $\overline{Q} = \{j_{k+1} < \ldots < j_n\}$, let $\sigma = i_1 \ldots i_n$ and $\tau = j_1 \ldots j_n$ and let $det_{q^{-1}} A^{-1}_{P,Q}$ be one of the quantum minors of its inverse A^{-1}. We have*

$$det_{q^{-1}}\, A^{-1}_{P,Q} = (-q)^{\ell(\tau)-\ell(\sigma)}\, det_q\, A^{Q,P}\ (det_q\, A)^{-1}\ .$$

Proof. — It suffices to express $det_{q^{-1}} A^{-1}_{P,Q}$ as a product of quasi-determinants by means of Theorem 3.2 and to apply to each quasi-determinant Jacobi's Theorem 2.16. Then, using Theorem 3.2 a second time, we get our result. □

EXAMPLE 3.5. — Let $A^{-1} = (b_{ij})_{1 \leq i,j \leq 5}$ and take $n = 5$, $P = \{1, 3, 4\}$ and $Q = \{1, 2, 3\}$. Then Theorem 3.4 gives us the following relation

$$\begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{vmatrix}_{q^{-1}} = (-q)^{-2} \begin{vmatrix} a_{42} & a_{45} \\ a_{52} & a_{55} \end{vmatrix}_q (det_q\, A)^{-1}\ .$$

3.3.2. Cayley's law of complementaries

For quantum determinants, we obtain the following analogue of Cayley's law of complementaries (see also Section 2.3.2).

THEOREM 3.6. — *Let I be a polynomial identity with coefficients in $\mathbb{C}[q, q^{-1}]$ between quantum minors of the quantum $n \times n$ matrix A. If every minor $det_q\, A_{P,Q}$ involved in I is replaced by its complement $det_q\, A^{P,Q}$ multiplied by $(det_q\, A)^{-1}$ and if, in addition, the substitution $q \to q^{-1}$ is made in the coefficients of I, there results a new identity I^C.*

Proof. — Let us apply identity I to the transpose of the matrix A^{-1}, which is a q^{-1}-quantum matrix. Identity I^C is then obtained by means of Theorem 3.4. □

EXAMPLE 3.7. — Take $n = 4$ and consider the following identity (see Proposition 3.13)

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_q a_{23} = q^{-1} a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_q \qquad (I)\ .$$

Applying Cayley's law for quantum determinants, we get

$$\begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}_q \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}_q = q \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}_q \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}_q \qquad (I^C)\ .$$

3.3.3. Muir's law of extensible minors

From Cayley's law for quantum determinants is deduced the following quantum analogue of Muir's law as in the case of quasi-determinants.

THEOREM 3.8. — *Let A be a $(n+p)\times(n+p)$ quantum matrix, let B be the $n\times n$ quantum matrix $A_{P,Q}$ where P, Q are subsets of $\{1,\dots,n+p\}$ of cardinality n and let I be a polynomial identity with coefficients in $\mathbb{C}[q,q^{-1}]$ between quantum minors of B. When every quantum minor $det_q\, B_{L,M}$ involved in I is replaced by its extension $det_q\, A_{L\cup\overline{P},M\cup\overline{Q}}$ (multiplied by a suitable power of the pivot $det_q\, A_{\overline{P},\overline{Q}}$ if the identity is not homogeneous), a new identity I^E is obtained, which is called an extensional of I.*

EXAMPLE 3.9. — Take $n=2,\ p=2,\ P=\{2,4\},\ Q=\{2,3\}$ and consider the identity

$$a_{22}a_{23} = q^{-1}a_{23}a_{22} \qquad (I)$$

which may be regarded as an identity between minors of $A_{24,23}$. Applying Muir's law, we get the following identity I^E

$$\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix}_q \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \end{vmatrix}_q = q^{-1} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \end{vmatrix}_q \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix}_q \qquad (I^E)\ .$$

3.3.4. Sylvester's theorem

An important consequence of Muir's law is the quantum analogue of Sylvester's theorem.

THEOREM 3.10. — *Let A be a quantum $n\times n$ matrix and choose a quantum $k\times k$ minor $det_q\, A_{P,Q}$ (the pivot of the extension). For $i\in\overline{P}$ and $j\in\overline{Q}$, set $b_{ij} = det_q\, A_{P\cup\{i\},Q\cup\{j\}}$ and form the matrix $B=(b_{ij})_{i\in\overline{P},j\in\overline{Q}}$. Then B is a quantum $(n-k)\times(n-k)$ matrix, and there holds*

$$det_q\, B = det_q\, A\ (det_q\, A_{P,Q})^{n-k-1}\ .$$

Proof. — B is a quantum matrix in view of Muir's law. Indeed the commutation relations for the b_{ij} are nothing but the commutation relations for the a_{ij} extended by the pivot $det_q\, A_{P,Q}$ (see Example 3.9). Now, applying Muir's law to the complete expansion of the quantum minor $A^{P,Q}$ yields Sylvester's theorem, the term $(det_q\, A_{P,Q})^{n-k-1}$ in the right-hand side being an homogeneity factor. □

EXAMPLE 3.11. — Let $n=3,\ k=1,\ P=Q=\{3\}$. Sylvester's theorem gives then

$$\begin{vmatrix} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}_q & \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}_q \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}_q & \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}_q \end{vmatrix}_q = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_q a_{33}\ .$$

NOTE 3.12. — Sylvester's quantum theorem can also be directly obtained from the noncommutative corresponding Theorem 2.22, using the same method than in the proof of Theorem 3.4.

3.3.5. Bazin's theorem

In the commutative and noncommutative cases, Bazin's theorem may be obtained by applying Sylvester's theorem to a special matrix some rows of which are rows of the unit matrix. However this method can no longer be used for quantum determinants, since the unit matrix is not a quantum matrix. We shall therefore go back to quasi-determinants and deduce the quantum analogue of Bazin's theorem from its noncommutative version (Theorem 2.25). The proof requires several lemmas of independent interest, describing commutation relations obeyed by certain minors of a quantum matrix.

Let $A = (b_{ij})_{1 \leq i,j \leq n}$ be a non-necessarily quantum square matrix of order n with entries in $\mathbb{C}[q, q^{-1}](a_{ij})$. We then denote

$$|A|_q = \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} \, b_{1\sigma(1)} \, \ldots \, b_{n\sigma(n)} \, .$$

If A a quantum matrix, this expression is just the quantum determinant of A. Note also that there exists an expansion of $|A|_q$ along the last row, but not in general along the other rows, given as in the usual quantum case by

$$|A|_q = \sum_{j=1}^{n} (-q)^{j-n} \, |A^{n,j}| \, b_{nj} \, .$$

We may now give two lemmas, required in the proof of Bazin's quantum theorem.

LEMMA 3.13. — *Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be a $n \times n$ quantum matrix and let $A(k)$ denote the matrix obtained from A by replacing by 0 all entries a_{nj} with $j \geq k$. We have then the following expansion of $|A(k)|_q$*

$$|A(k)|_q = \sum_{j<k} (-q)^{j-n+1} \, |A^{n-1,j}(k)|_q \, a_{n-1,j} \; + \; \sum_{j \geq k} (-q)^{j-n-1} \, |A^{n-1,j}|_q \, a_{n-1,j} \, .$$

Proof. — Expand first $|A(k)|_q$ along the last row. Expand again along the last row all the quantum determinants that appear in this expansion. Using quantum relations, grouping terms according to powers of q and applying Theorem 3.1 (i) allows then easily to get our lemma. □

LEMMA 3.14. — *Let $A = (a_{ij})_{1 \leq i,j \leq n+1}$ be a $(n+1) \times (n+1)$ quantum matrix. Then, using the notations of Lemma 3.13, one has*

$$det_q \, A^{n+1,j} \, a_{n+1,j} - a_{n+1,j} \, det_q \, A^{n+1,j} = (q^{-1} - q) \, (-q)^{n+2-j} \, |A(j)|_q \, .$$

Proof. — Expand $det_q \, A^{n+1,j}$ along the last row. Applying the basic quantum relations to this expansion, we can then obtain the desired relation by means of an induction and of Lemma 3.13. □

From now on, $A = (a_{ij})$ denotes a quantum $n \times m$ matrix with $n < m$. The quantum maximal minor formed on the columns $j_1 < \ldots < j_n$ is written for short $[j_1, \ldots, j_n]_q$.

PROPOSITION 3.15. — *Consider an increasing sequence of integers* $1 \leq j_1 < \ldots < j_n < k \leq m$. *For every* i, *we have the following commutation relation*

$$[j_1 \ldots j_n]_q \, a_{ik} = q^{-1} \, a_{ik} \, [j_1 \ldots j_n]_q \ .$$

Proof. — For simplicity, we may suppose that $j_1 = 1, \ldots, j_n = n$ and $k = n+1$. Set then $B = [1, \ldots, n]$. Expand $det_q B$ along the last row and apply the quantum relations to the expansion of the product $(det_q B)\, a_{n+1,j}$ obtained in this way. Using an induction, Lemma 3.14 and Theorem 3.1, one can prove that

$$[1 \ldots n]_q \, a_{n+1,j} = q^{-1} \, a_{n+1,j} \, [1 \ldots n]_q \ .$$

Using the fact that the transpose of a quantum matrix is still a quantum matrix with the same quantum determinant, it is now easy to conclude. □

An immediate corollary of Proposition 3.15 is the following result.

PROPOSITION 3.16. — *Consider two increasing sequences of integers* $1 \leq j_1 < \ldots < j_n \leq m$ *and* $1 \leq k_1 < \ldots < k_n \leq m$ *and suppose that for some* $s \in \{0, \ldots, n\}$, *one has* $k_s < j_1 < j_n < k_{s+1}$. *Then we have*

$$[j_1 \ldots j_n]_q \, [k_1 \ldots k_n]_q = q^{2s-n} \, [k_1 \ldots k_n]_q \, [j_1 \ldots j_n]_q \ .$$

EXAMPLE 3.17. — Let A be a 2×4 quantum matrix. We have

$$[12]_q \, [34]_q = q^{-2} \, [34]_q \, [12]_q \ , \quad [14]_q \, [23]_q = [23]_q \, [14]_q \ .$$

We can now state Bazin's theorem for quantum determinants.

THEOREM 3.18. — *Let* $J = \{j_1 < \ldots < j_n\}$ *and* $K = \{k_1 < \ldots < k_n\}$ *be two subsets of* $\{1, \ldots m\}$ *such that* $j_n < k_1$. *Then the matrix* $B_n = (b_{st})_{1 \leq s,t \leq n}$ *defined by*

$$b_{st} = [j_t, \, (K \setminus \, k_s)]_q \quad \textit{for} \ \ 1 \leq s, t \leq n \ ,$$

is a quantum $n \times n$ *matrix and we have*

$$det_q B_n = q^{\binom{n}{2}} \, [j_1 \ldots j_n]_q \, [k_1 \ldots k_n]_q^{n-1} \ .$$

Proof. — The proof is by induction on $n \geq 2$. For $n = 2$, one can check by means of Plücker relations for quantum determinants (described for example in **[TT]**) that

$$B_2 = \begin{pmatrix} [j_1 k_2]_q & [j_2 k_2]_q \\ [j_1 k_1]_q & [j_2 k_1]_q \end{pmatrix}$$

is a quantum matrix, with quantum determinant $det_q B_2 = q \, [j_1 j_2]_q \, [k_1 k_2]_q$. Using Theorem 3.8, it follows that every 2×2 submatrix of B_n is a quantum matrix, and therefore that B_n is itself a quantum $n \times n$ matrix for every $n \geq 2$. Assume now that

$$det_q B_{n-1} = q^{\binom{n-1}{2}} \, [j_1 \ldots j_{n-1}]_q \, [k_1 \ldots k_{n-1}]_q^{n-2}$$

for all sequences J and K of cardinality $n-1$ satisfying the hypothesis of Theorem 3.18. From Theorem 3.2, it results that

$$det_q\, B_n = |B_n|_{nn}\ det_q\, B_n^{nn}\ .$$

Now Muir's law and the induction hypothesis show that

$$det_q\, B_n^{nn} = q^{\binom{n-1}{2}}\ [j_1 j_2 \dots j_{n-1} k_n]_q\ [k_1 k_2 \dots k_n]_q^{n-2}\ .$$

On the other hand, expanding all entries of $|B_n|_{nn}$ according to Theorem 3.2, using then Proposition 2.6 and applying finally Bazin's theorem for quasi-determinants, one obtains

$$|B_n|_{nn} = [k_1 \dots k_n]_q\ [j_1 j_2 \dots j_{n-1} k_n]_q^{-1}\ [j_1 \dots j_n]_q\ .$$

The required result follows now from Muir's law, which shows that

$$[j_1 \dots j_n]_q\ [j_1 j_2 \dots j_{n-1} k_n]_q = q^{-1}\ [j_1 j_2 \dots j_{n-1} k_n]_q\ [j_1 \dots j_n]_q\ ,$$

and from Proposition 3.13. □

EXAMPLE 3.19. — Take $n = 3$, $J = \{1,2,3\}$ and $K = \{4,5,6\}$. In this case, Bazin's theorem reads as follows

$$\begin{vmatrix} [145]_q & [245]_q & [345]_q \\ [146]_q & [246]_q & [346]_q \\ [156]_q & [256]_q & [356]_q \end{vmatrix}_q = q^3\ [123]_q\ [456]_q^2\ .$$

3.3.6. Schweins' series

Using Theorem 3.2, Schweins' series for quasi-determinants, as given by Theorems 2.20 and 2.21, is readily turned into the following quantum analogues. Here again identities are stated for quantum determinants of order 3 and 4 only, the general case being easily understood from these.

THEOREM 3.17. — *The maximal minors of a 3×6 quantum matrix satisfy the relation*

$$[123]_q^{-1}[124]_q = [123]_q^{-1}[125]_q[235]_q^{-1}[234]_q + [253]_q^{-1}[256]_q[356]_q^{-1}[354]_q + [563]_q^{-1}[564]_q\ .$$

The maximal minors of a 4×8 quantum matrix satisfy the relation

$$[1234]_q^{-1}[1235]_q = [1234]_q^{-1}[1236]_q[2346]_q^{-1}[2345]_q$$
$$+[2364]_q^{-1}[2367]_q[3467]_q^{-1}[3465]_q + [3674]_q^{-1}[3678]_q[4678]_q^{-1}[4675]_q + [6784]_q^{-1}[6785]_q\ .$$

THEOREM 3.18. — *The minors of a 3×4 quantum matrix satisfy the relation*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_q^{-1} \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix}_q$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_q^{-1} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_q \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \end{vmatrix}_q \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}_q^{-1}$$

$$+ \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}_q^{-1} a_{11} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}_q a_{13}^{-1}$$

$$+ a_{13}^{-1} a_{14} \ .$$

The minors of a 4×5 *quantum matrix satisfy the relation*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}_q^{-1} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{45} \end{vmatrix}_q$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}_q^{-1} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_q \begin{vmatrix} a_{11} & a_{12} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{44} & a_{45} \end{vmatrix}_q \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix}_q^{-1}$$

$$+ \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix}_q^{-1} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_q \begin{vmatrix} a_{11} & a_{14} & a_{15} \\ a_{21} & a_{24} & a_{25} \\ a_{31} & a_{34} & a_{35} \end{vmatrix}_q \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix}_q^{-1}$$

$$+ \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix}_q^{-1} a_{11} \begin{vmatrix} a_{14} & a_{15} \\ a_{24} & a_{25} \end{vmatrix}_q a_{14}^{-1}$$

$$+ a_{14}^{-1} a_{15} \ .$$

4. Conclusion

We have investigated some noncommutative analogues of classical determinantal identities. Our strategy was to study first the most general case, that is, the case of the generic matrix over the free skew field. In this situation the quasi-determinants of Gel'fand and Retakh satisfy many formulas similar to the commutative case : Sylvester's theorem, Jacobi's ratio theorem [**GR1**], Cayley's law of complementaries, Muir's law of extensible minors, Bazin's theorem, Schweins' series.

Then, considering the case of the quantum group $A_q(GL(n))$ and using the expression of its quantum determinant in terms of quasi-determinants [**GR1**], we derived some quantum analogues of the same theorems.

We mention that the same approach applies also to the quantum determinants of the Yangians $Y(gl_n)$ and $Y_q(gl_n)$. This will be developped in a forthcoming paper.

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