



HAL
open science

On the 3-distortion of a path.

Pierre Dehornoy

► **To cite this version:**

| Pierre Dehornoy. On the 3-distortion of a path.. 2006. hal-00017449v1

HAL Id: hal-00017449

<https://hal.science/hal-00017449v1>

Preprint submitted on 21 Jan 2006 (v1), last revised 18 May 2008 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON THE 3-DISTORTION OF A PATH

PIERRE DEHORNOY

ABSTRACT. We prove that, when a path of length n is embedded in \mathbb{R}^2 , the 3-distortion is an $\Omega(n^{1/2})$, and that, when embedded in \mathbb{R}^d , the 3-distortion is an $O(n^{\frac{1}{d-1}})$.

The general context is the study of the distortion that appears when a metric space is embedded into a Euclidean space. For (V, ρ) a metric space and f a non-expanding (*i.e.*, 1-Lipschitz) embedding of V into \mathbb{R}^d , the distortion $\Delta(f)$ of f is defined to be the supremum of the compression ratio between the distance of two points in (V, ρ) and that of their images in \mathbb{R}^d :

$$(1) \quad \Delta(f) = \sup \left\{ \frac{\rho(P, Q)}{\text{Dist}(f(P), f(Q))}; P, Q \in V \right\}.$$

By construction, $\Delta(f)$ is at least 1, and the larger it is, the bigger the deformation of distances caused by f .

In [1], U. Feige introduces a generalized distortion by considering k -tuples of points and the volume of the polytope they determine instead of pairs of points and their distance. Here we shall only consider the case $k = 3$, *i.e.*, images of triangles. In the denominator of (1), the length of the segment $[f(P), f(Q)]$ is replaced with the area of the triangle $[f(P), f(Q), f(R)]$. As for the numerator, it makes no sense in the source space (V, ρ) , but we observe that $\rho(P, Q)$ is the sup of the lengths $\text{Dist}(g(P), g(Q))$ for g a non-expanding embedding of V to \mathbb{R}^d (provided $d \geq 1$). This naturally leads to letting $\rho_3(P, Q, R)$ be the sup of $\text{Area}([g(P), g(Q), g(R)])$ for g a non-expanding embedding of V to \mathbb{R}^d (provided $d \geq 2$), and defining the 3-distortion of f to be

$$(2) \quad \Delta_3(f) = \sup \left\{ \frac{\rho_3(P, Q, R)}{\text{Area}([f(P), f(Q), f(R)])}; P, Q, R \in V \right\}.$$

We shall be interested in the minimal possible value of $\Delta_3(f)$, *i.e.*, in the configurations that minimize the distortion of triangles.

Definition. The 3-distortion $\delta_3(V, \mathbb{R}^d)$ is defined to be the infimum of $\Delta_3(f)$ over all non-expanding embeddings f of V into \mathbb{R}^d .

This notion of 3-distortion is essentially the one considered by U. Feige in [1]. The definition for k -tuples would be similar.

1991 *Mathematics Subject Classification.* 68U05.

Key words and phrases. distortion, triangle, non-expanding embedding.

This research was done while the author was visiting to Institute for Theoretical Computer Science (ITI) of the Charles University in Prague and was supported by it; the author wishes to express his gratitude to Jiří Matoušek for his warm hospitality and his help, as well as to Charles University for its support.

Graphs with length 1 edges and the geodesic distance provide natural examples of non-Euclidean metric spaces. In this note, we investigate the simplest of them, namely a path. We denote by Π_n the graph consisting of $n+1$ vertices $0, 1, \dots, n$, and n edges connecting i to $i+1$ for each i ; we make Π_n a metric space by declaring the distance between i and j to be $|j-i|$. If g is a non-expanding embedding of Π_n into \mathbb{R}^d , we have $\text{Dist}(g(i), g(j)) \leq |i-j|$ and therefore for $i < j < k$ we have $\text{Area}([g(i), g(j), g(k)]) \leq (j-i)(k-j)/2$; on the other hand, provided $d \geq 2$, we can always find g such that the latter inequality is an equality. Hence, for $0 \leq i < j < k \leq n$, we have

$$\rho_3(i, j, k) = (j-i)(k-j)/2.$$

So, for f a non-expanding embedding of Π_n into \mathbb{R}^d , (2) takes the form

$$(3) \quad \Delta_3(f) = \sup \left\{ \frac{(j-i)(k-j)/2}{\text{Area}([f(i), f(j), f(k)])}; 0 \leq i < j < k \leq n \right\}.$$

By construction, we always have $\Delta_3(f) \geq 1$, and the larger $\Delta_3(f)$ is, the bigger the deformation of triangles caused with f is. For example, when f is some isometrical embedding of Π_n in \mathbb{R}^d , all triangles of the image are flat, then the distortion $\Delta_3(f)$ is infinite.

We shall investigate the asymptotic behavior of $\delta_3(\Pi_n, \mathbb{R}^d)$ when d is fixed and n tends to ∞ . As \mathbb{R}^d isometrically embeds in \mathbb{R}^{d+1} , we always have $\delta_3(\Pi_n, \mathbb{R}^{d+1}) \leq \delta_3(\Pi_n, \mathbb{R}^d)$, hence in particular $\delta_3(\Pi_n, \mathbb{R}^d) \leq \delta_3(\Pi_n, \mathbb{R}^2)$ for $d \geq 3$. The result is clear: when we have more space, we can more easily embed with small distortion. For $d = 2$ (the planar case), hence for every d , it is known that $\delta_3(\Pi_n, \mathbb{R}^d)$ is at most linear in n . Here we compare $\delta_3(\Pi_n, \mathbb{R}^d)$ with the polynomial functions n^α , $0 < \alpha < 1$. We prove one lower bound result for $d = 2$, and one upper bound result for $d \geq 2$.

Proposition 1. *The 3-distortion $\delta_3(\Pi_n, \mathbb{R}^2)$ is an $\Omega(n^{1/2})$.*

Proposition 2. *For each fixed d , the 3-distortion $\delta_3(\Pi_n, \mathbb{R}^d)$ is an $O(n^{\frac{1}{d-1}})$.*

The results are likely not to be optimal: we conjecture that $\delta_3(\Pi_n, \mathbb{R}^2)$ might be an $\Omega(n)$, and that $\delta_3(\Pi_n, \mathbb{R}^d)$ might be lower than polynomial, typically polylogarithmic, for $d \geq 3$.

1. A LOWER BOUND IN THE PLANAR CASE

In the sequel, we shall forget about embeddings and only work inside the target space \mathbb{R}^d .

Definition. A finite sequence of points (M_0, \dots, M_n) in \mathbb{R}^d is said to be *tame* if, for each i , we have $\text{Dist}(M_i, M_{i+1}) \leq 1$. In this case, we put

$$(4) \quad \Delta_3(M_0, \dots, M_n) = \sup \left\{ \frac{(j-i)(k-j)/2}{\text{Area}([M_i, M_j, M_k])}; 0 \leq i < j < k \leq n \right\}.$$

Then the 3-distortion can be expressed in terms of tame sequences of points:

Lemma 3. *For all n, d , we have*

$$(5) \quad \delta_3(\Pi_n, \mathbb{R}^d) = \inf \{ \Delta_3(M_0, \dots, M_n); (M_0, \dots, M_n) \text{ a tame sequence in } \mathbb{R}^d \}.$$

Proof. If f is an embedding of Π_n into \mathbb{R}^d , then the sequence $(f(0), \dots, f(n))$ is tame, and, conversely, each tame sequence determines a unique embedding. Now, translating (3) gives (4) for $M_i = f(i)$. \square

In order to prove Proposition 1, we shall consider an arbitrary tame sequence in \mathbb{R}^2 , and prove that some triangle is much distorted. To this end we observe that points in convex position provide a triangle with large 3-distortion.

Say that a sequence (P_0, \dots, P_{m-1}) of points in the plane is *convex* if the border of the convex hull of $\{P_0, \dots, P_{m-1}\}$ is exactly the polygon with vertices P_0, \dots, P_{m-1} in this order.

Lemma 4. *Assume that (P_0, \dots, P_{m-1}) is a convex sequence with $m \geq 3$. Then there exists i such that the 3-distortion of the triangle $P_i P_{i+1} P_{i+2}$ —where indices are taken modulo m —is at least $m/(2\pi)$.*

Proof. The sum of angles $\angle P_0 P_1 P_2 + \angle P_1 P_2 P_3 + \dots + \angle P_{m-1} P_0 P_1$ is $(m-2)2\pi$. As all angles are lower π , then, by the pigeonhole principle, one of them is at least $\frac{m-2}{m}\pi$. The 3-distortion of the corresponding triangle is then at least $m/(2\pi)$. \square

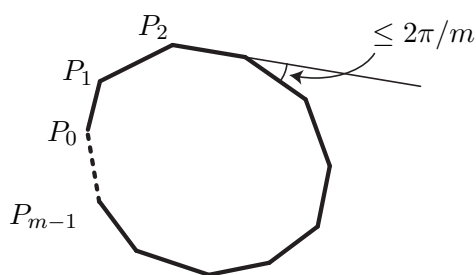


FIGURE 1. Convex sequence of points

Lemma 5. *Assume that (M_0, \dots, M_n) is a tame sequence in \mathbb{R}^2 , and that δ is an integer greater or equal to $\Delta_3(M_0, \dots, M_n)$. Then the sequence $(M_0, M_\delta, M_{2\delta}, \dots, M_{\lfloor \frac{n}{\delta} \rfloor \delta})$ is convex.*

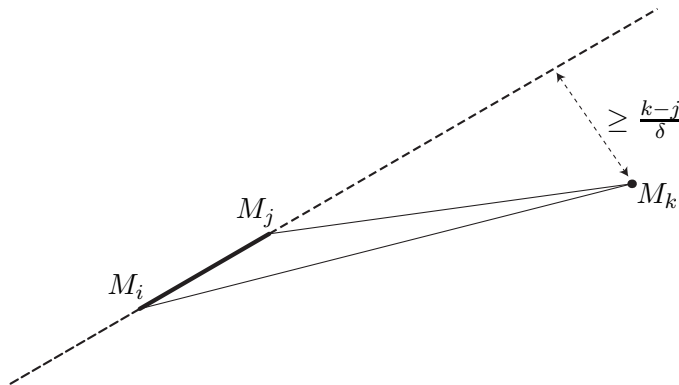


FIGURE 2. Minimal distance from the point M_k to the line $(M_i M_j)$

Proof. Let $\delta_0 := \Delta_3(M_0, \dots, M_n)$. For all $i < j$, we have $\text{Dist}(M_i, M_j) \leq |j - i|$. Since for $k > j$ the area of the triangle $[M_i, M_j, M_k]$ is at least $\frac{(k-j)(j-i)}{2\delta_0}$, hence *a fortiori* $\frac{(k-j)(j-i)}{2\delta}$, the distance between the point M_k and the line $(M_i M_j)$ is at least $\frac{k-j}{\delta}$ (Figure 1). Therefore, for $k \geq j + \delta$, the points M_k and M_{k+1} lie on the same side of the line $(M_i M_j)$: otherwise, the distance between M_k and M_{k+1} would be at least $2\frac{k-j}{\delta}$, contrary to the tameness hypothesis. Hence, for $k \geq j + \delta$, the point M_k lies on the same side of the line $(M_i M_j)$ as $M_{j+\delta}$.

For a contradiction, assume that, for some i , the sequence $(M_{i\delta}, M_{(i+1)\delta}, M_{(i+2)\delta}, M_{(i+3)\delta})$ is not convex. Then either the four points are not in convex position, or they are in convex position but they do not appear in the right order on the border of their convex hull.

In the first case, one point lies in the convex hull of the three others. But this contradicts the hypothesis that adjacent points lie on the same side of each line $(M_{j\delta} M_{(j+1)\delta})$ (Figure 3).

In the second case, the points are in convex position, but the segment $[M_{(i+1)\delta}, M_{(i+2)\delta}]$ crosses the line $(M_{i\delta} M_{(i+3)\delta})$. Then there exists j with $(i+1)\delta \leq j \leq (i+2)\delta$ such that the distance from M_j to $(M_{i\delta} M_{(i+3)\delta})$ is at most $1/2$. The area of the triangle $[M_{i\delta}, M_j, M_{(i+3)\delta}]$ is therefore at most $3\delta/4$. On the other side, by definition of δ_0 , this area is at least $((i+3)\delta - j)(j - i\delta)/2\delta_0$, hence *a fortiori* $((i+3)\delta - j)(j - i\delta)/2\delta$. Since $(i+1)\delta \leq j \leq (i+2)\delta$, the latter quantity is at least δ , a contradiction (Figure 4). \square

Proof of Proposition 1. Let (M_0, \dots, M_n) be a tame sequence in \mathbb{R}^2 , and let $\delta = \lceil \Delta_3(M_0, \dots, M_n) \rceil$. If we have $\lfloor \frac{n}{\delta} \rfloor < 2$, then we have $\delta \geq n/2$, hence $\delta \in \Omega(n^{1/2})$ *a fortiori*. Assume now $\lfloor \frac{n}{\delta} \rfloor \geq 2$. Then by Lemma 5, the sequence $(M_0, M_\delta, \dots, M_{\lfloor \frac{n}{\delta} \rfloor \delta})$ is convex, and by Lemma 4 there is a triangle whose distortion is least $\lfloor \frac{n}{\delta} \rfloor / 2\pi$. By definition, this quantity is at most δ , hence we have $\delta \in \Omega(\frac{n}{\delta})$. So in any case, $\delta_3(\Pi_n, \mathbb{R}^2)$ lies in $\Omega(n^{1/2})$. \square

Remark. The proof of Lemma 5 gives many constraints for the sequence (M_0, \dots, M_n) . Here we use these constraints to construct a convex subsequence of size \sqrt{n} , but it is

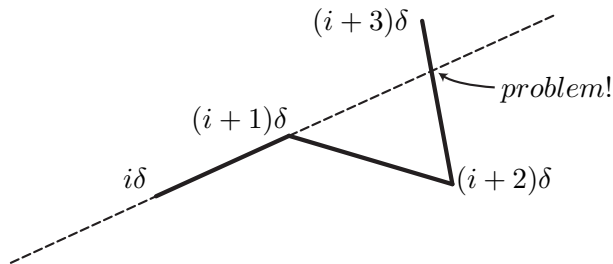


FIGURE 3. Four points not in convex position: a problem arises between $(i+2)\delta$ and $(i+3)\delta$

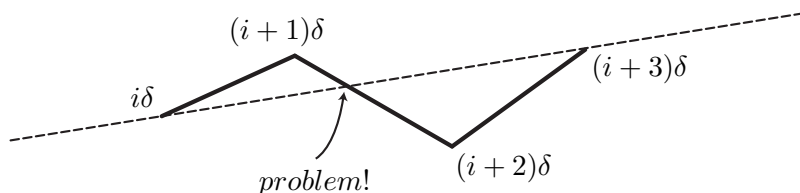


FIGURE 4. Four points not in ordered convex position: a problem arises between $(i+1)\delta$ and $(i+2)\delta$

likely that larger subsequences with properties slightly weaker than convexity could be constructed as well. So we think that the result of Proposition 1 is not optimal.

2. CONSTRUCTION OF A d -DIMENSIONAL EMBEDDING

Now we turn to dimension d and we wish to establish the lower bound result stated as Proposition 2. Our aim is to construct for each n a tame sequence of length n in \mathbb{R}^d with a small 3-distortion, *i.e.*, such that all extracted triangles are not too much distorted.

A natural idea would be to construct the n th sequence $(M_{0,n}, \dots, M_{n,n})$ by taking more and more points on a single curve Γ of length 1, and rescaling. But then a small 3-distortion would require a complicated curve Γ . Indeed, assume that Γ is of class C^2 . As Γ is compact, the infimum r_Γ of the radii of the osculating circles of Γ is reached at some point, and therefore it is non-zero. For any n , there exists i such that the curvilinear distance between $M_{i,n}$ and $M_{i+2,n}$ is lower than $2/n$ before rescaling. Then the distances between $M_{i,n}$ and $M_{i+1,n}$, and between $M_{i+1,n}$ and $M_{i+2,n}$ are lower than $2/n$ too. Therefore the sinus of the angle between the lines $(M_{i,n}M_{i+1,n})$ and $(M_{i+1,n}M_{i+2,n})$ is at most r_Γ/n , and the distortion of the triangle $M_{i,n}M_{i+1,n}M_{i+2,n}$ is at least n/r_Γ . This leads to a 3-distortion in $\Omega(n)$ for $(M_{0,n}, \dots, M_{n,n})$. So, in order to construct sequences of points with small 3-distortion, we have either to use curves depending on n , or to use a non- C^2 curve (typically a fractal curve). In the following construction we choose the first option.

Proof of Proposition 2. For simplicity, we assume $n = m^d$ for some m . We recursively construct a family of curves $\Gamma_{m,d}$ in \mathbb{R}^d , and, on each of them, we mark $m^d + 1$ points $P_{m,d,0}, \dots, P_{m,d,m^d}$ in such a way that $\Delta_3(P_{m,d,0}, \dots, P_{m,d,m^d})$ lies in $O(m)$ for each fixed d .

When $m + 1$ points lie at mutual distance 1 on an arc of circle, the distortion is $\Theta(m)$. The idea of our construction is to use this fact and to recursively put circles one above the others.

Let Γ_0 be the sixth of a circle whose radius r will be chosen later. On Γ_0 we put points P_0, \dots, P_m with regular angular distance $\frac{\pi}{3m}$. Then we replace the arc between P_i and P_{i+1} with a coplanar arc of radius $2r$ lying between the original arc and the chord connecting P_i to P_{i+1} . We rescale the figure so that the curvilinear coordinate of P_i becomes i for each i . We let $\Gamma_{m,2}$ be the resulting curve (oriented from P_0 to P_m) and $P_{m,2,0}, \dots, P_{m,2,m}$ be the marked points on $\Gamma_{m,2}$.

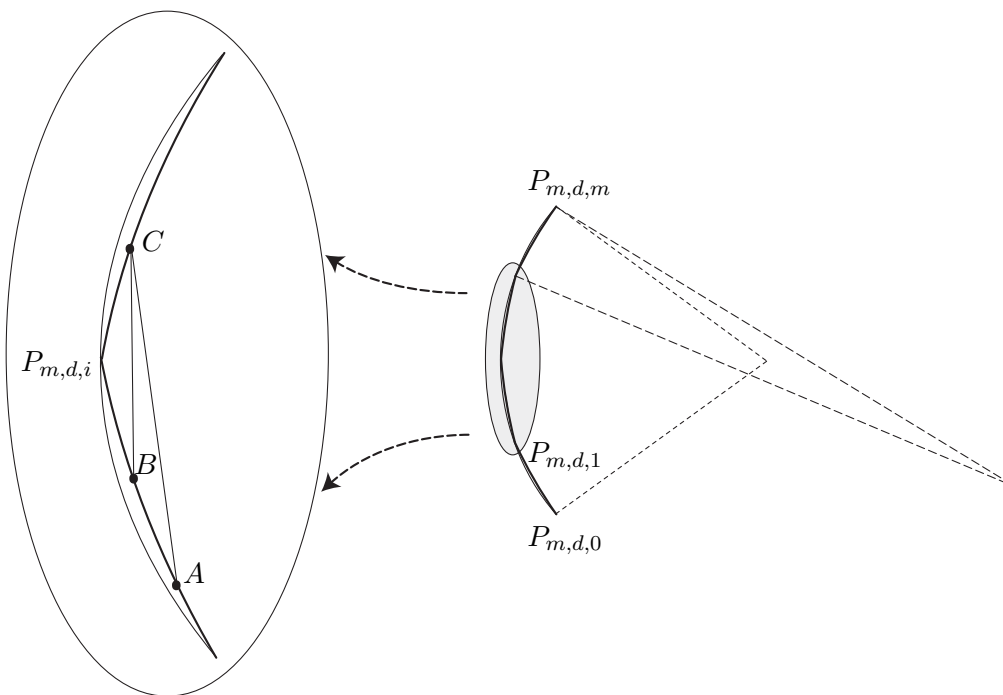


FIGURE 5. On the right: the curve Γ_0 and the points $P_{m,d,0}, \dots, P_{m,d,m}$. On the left: three points A, B, C with at least one $P_{m,d,i}$ between them yield an angle $\angle ABC \leq \pi(1 - \frac{1}{6m})$

The main remark for the proof is that, for all triples A, B, C taken in increasing order on $\Gamma_{m,2}$ (not necessarily some $P_{m,2,i}$'s) and not all lying on some arc $(P_{m,2,i}, P_{m,2,i+1})$, we have $\angle ABC \leq \pi(1 - \frac{1}{6m})$. As the Euclidean distance between two points of $\Gamma_{m,2}$ is at least $3/\pi$ times their curvilinear distance, it follows that the 3-distortion of the triangle ABC is in $O(m)$.

The idea for the induction is to add a copy of $\Gamma_{m,2}$ between $P_{m,d-1,i}$ and $P_{m,d-1,i+1}$, orthogonally to the hyperplane in which $\Gamma_{m,d-1}$ lies. More precisely, we construct $\Gamma_{m,d}$ from $\Gamma_{m,d-1}$ so that the following induction hypothesis is preserved:

(i) $\Gamma_{m,d}$ is a curve of length m^d in \mathbb{R}^d such that two points at curvilinear distance ℓ lie at euclidian distance at least $(2/\sqrt{3})^{-d+2}\pi/3 \times \ell$;

(ii) Let A, B, C be three points that do not all lie on some arc $(P_{m,d,i}P_{m,d,i+1})$ for any i . Then the 3-distortion of the triangle $[A, B, C]$ is at most $c_d m$, where $c_d = (2/\sqrt{3})^{-d+2} \times 6/\pi$.

Notice that the induction hypothesis holds for $d = 2$.

The construction of $\Gamma_{m,d}$ is as follows. We identify \mathbb{R}^d with $\mathbb{R}^{d-1} \times \mathbb{R}$, where \mathbb{R}^{d-1} is the space containing $\Gamma_{m,d-1}$. Next we work in the cylinder $\Gamma_{m,d-1} \times \mathbb{R}_+$ with the induced metric. For each i between 0 and $m^{d-1} - 1$, we insert a copy of $\Gamma_{m,2}$ from $P_{m,d-1,i}$ to $P_{m,d-1,i+1}$. In this way, we obtain $m^d + 1$ points denoted $P_{m,d,0}, \dots, P_{m,d,m^d}$. We then rescale the figure so that the curvilinear distance between consecutive points $P_{m,d,i}$'s is 1. We call $\Gamma_{m,d}$ the resulting curve.

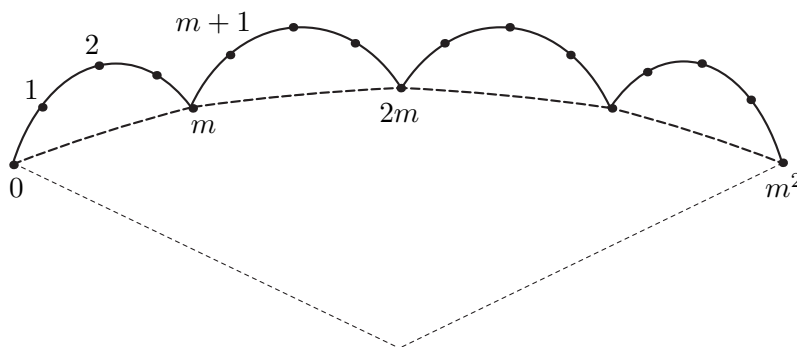


FIGURE 6. The curve $\Gamma_{m,2}$ in the space

It remains to show that the induction hypothesis is preserved.

For (i), we observe that the angle between any chord of $\Gamma_{m,d}$ and the hyperplan containing $\Gamma_{m,d-1}$ is lower than $\pi/6$. Therefore, when going from $\Gamma_{m,d-1}$ to $\Gamma_{m,d}$, no distance is decreased by more than a factor $2/\sqrt{3}$.

For (ii), let A, B, C be three points on $\Gamma_{m,d}$ and i such that A is before $P_{m,d,i}$ and C lies after it following the curvilinear order.

First case: There exists j such that A, B, C lie between $P_{m,d,jm}$ and $P_{m,d,(j+1)m}$. This means that A, B, C lie on some copy of $\Gamma_{m,2}$. In the case of $\Gamma_{m,2}$, we know that the distortion is at most $c_2 m$. Here there is an additive distortion due to the fact that the copy was made on a cylinder. The projection of $\Gamma_{m,d}$ on \mathbb{R}^{d-1} is $\Gamma_{m,d-1}$, and not a line as in the $d = 2$ case. By induction hypothesis, the distances on $\Gamma_{m,d-1}$ (compared with the Euclidean distances) are not contracted by more than $(2/\sqrt{3})^{-d+1}\pi/3$, hence the distortion of the triangle $[A, B, C]$ is bounded by $(2/\sqrt{3})^{-d+1}\pi/3 \times c_2 m \leq c_d m$.

Second case: There exists j such that A is before $P_{m,d,jm}$ and C after. Then, when A, B, C are projected from $\Gamma_{m,d}$ on $\Gamma_{m,d-1}$, the area of the triangle $[A, B, C]$ is decreased

by a factor at most $\sqrt{3}/2$. Since the projection has 3-distortion at most $c_{d-1}m$, the triangle $[A, B, C]$ has 3-distortion at most $c_d m$. \square

Remarks. (i) The choice of the curve $\Gamma_{m,2}$ may sound strange, in particular the choice of an arc of radius $2r$ between $P_{m,i}$ and $P_{m,i+1}$ rather than an arc of radius r or a chord. The reason is that, in both cases, the key property, namely that the triangle $[A, B, C]$ has distortion $O(m)$ if A, B, C do not all lie on some arc $(P_{d,m,i}P_{d,m,i+1})$, does not hold. With arcs of radius r , if we take A, B, C close to some $P_{d,m,i}$, then the 3-distortion of $[A, B, C]$ can be arbitrary large. With chords, if we take A, B strictly between $P_{d,m,i}$ and $P_{d,m,i+1}$ and C just after $P_{d,m,i+1}$, then the 3-distortion is not bounded either.

(ii) Our construction uses $d - 1$ pairwise orthogonal directions to draw the curves $\Gamma_{m,d}$ one above the other. We could use other fixed directions as well, the point being that the projections preserve the convexity of the specific patterns we consider. Alternatively we could replace cylinders by cones, as central projection also preserves the needed convexity. But it seems difficult to use more than one cylinder, and therefore more than one curve, for each new dimension, because no projection preserves the needed convexity for several sufficiently distinct directions simultaneously.

REFERENCES

1. U. FEIGE, Approximating the bandwidth via volume respecting embeddings, In *J. Comput. Sci.*, 60:510-539, (2000).
2. J. MATOUŠEK, Lectures on Discrete Geometry, Springer GTM Series, vol. 212, (2001).
3. S. RAO, Small distortion and volume respecting embeddings for planar and Euclidean metrics, In *Proc. 15th Annual ACM Symposium on Comput. Geometry*, pages 300–306, (1999).
4. S. VEMPALA, Random projection: a new approach to VLSI layout, In *Proc. 39th IEEE Symposium on Foundations of Computer Science*, pages 389–395, (1998).

DÉPARTEMENT DE MATHÉMATIQUES ET APPLICATIONS, ÉCOLE NORMALE SUPÉRIEURE, 45 RUE D'ULM, 75005 PARIS, FRANCE

E-mail address: `dehornoy@clipper.ens.fr`