Isoperimetry between exponential and Gaussian

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Abstract

We study in details the isoperimetric profile of product probability measures with tails between the exponential and the Gaussian regime. In particular we exhibit many examples where coordinate half-spaces are approximate solutions of the isoperimetric problem.

1 Introduction

This paper establishes infinite dimensional isoperimetric inequalities for a wide class of probability measures. We work in the setting of a Riemannian manifold (M, g). The geodesic distance on M is denoted by d. Furthermore M is equipped with a Borel probability measure μ which is assumed to be absolutely continuous with respect to the volume measure. For $h \geq 0$ the closed h-enlargement of a set $A \subset M$ is

$$A_h := \{x \in M;\ d(x, A) \leq h\}.$$

We may define the boundary measure, in the sense of μ, of a Borel set A by

$$\mu_s(\partial A) := \liminf_{h \to 0^+} \frac{\mu(A_h \setminus A)}{h}.$$

An isoperimetric inequality is a lower bound on the boundary measure of sets in terms of their measure. Their study is an important topic in geometry, see e.g. [33]. Finding sets of minimal boundary measure is very difficult. In many cases the only hope is to estimate the isoperimetric function of the metric measured space (M, d, μ), denoted by I_μ

$$I_\mu(a) := \inf\{\mu_s(\partial A);\ \mu(A) = a\}, \quad a \in [0, 1].$$

For $h > 0$ one may also investigate the best function R_h such that $\mu(A_h) \geq R_h(\mu(A))$ holds for all Borel sets. The two questions are related, and even equivalent in simple situations, see [16]. Since the function $\alpha(h) = 1 - R_h(1/2)$ is the so-called concentration function, the isoperimetric problem for probability measures is closely related to the concentration of measure phenomenon. We refer the reader to the book [28] for more details on this topic.

The main probabilistic example where the isoperimetric problem is completely solved is the Euclidean space $(\mathbb{R}^n, |\cdot|)$ with the standard Gaussian measure, denoted γ^n in order to emphasize its product structure

$$d\gamma^n(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}}, \quad x \in \mathbb{R}^n.$$

Sudakov-Tsirel'son [35] and Borell [18] have shown that among sets of prescribed measure, half-spaces have h-enlargements of minimal measure. Setting $G(t) = \gamma((-\infty, t])$, their result reads as follows: for $A \subset \mathbb{R}^n$ set $a = G^{-1}(\gamma^n(A))$, then

$$\gamma^n(A_h) \geq \gamma((-\infty, a+h]) = G\Big(G^{-1}(\gamma^n(A)) + h\Big),$$

and letting h go to zero

$$(\gamma^n)_s(\partial A) \geq G'(a) = G'(G^{-1}(\gamma^n(A)).$$

These inequalities are best possible, hence $I_{\gamma^n} = G' \circ G^{-1}$ is independent of the dimension n. Such dimension free properties are crucial in the study of large random systems, see e.g. [27, 37]. Asking which measures enjoy such a dimension free isoperimetric inequality is therefore a fundamental question. Let us be more specific about the products we are considering: if μ is a probability measure on (M, g), we consider the product μ^n on the product Riemannian manifold M^n where the geodesic distance is the ℓ_2 combination of the distances on the factors. Considering the ℓ_∞ combination is easier and leads to different results, see [11, 15, 6]. It can be shown that Gaussian measures on the line are the only ones for which coordinate half-spaces $\{x_1 \leq t\}$ solve the isoperimetric problem in any dimension [13, 24, 30]. Therefore it is natural to investigate measures for which one dimensional sets are approximate solutions of the isoperimetric problem for the products, that is for which there exists $c > 0$ with

$$I_\mu \geq I_{\mu^\infty} \geq c I_\mu,$$

where by definition $I_{\mu^\infty} := \inf_{n \geq 1} I_{\mu^n}$. The central limit theorem and an argument of Talagrand [36] show that measures on the real line with this property must have at least Gaussian tails and at most exponential tails. For the symmetric exponential law $d\nu(t) = e^{-|t|} dt/2$, $t \in \mathbb{R}$, Bobkov and Houdré [14] actually showed $I_{\nu^\infty} \geq I_\nu/(2\sqrt{6})$. Their argument uses a functional isoperimetric inequality with the tensorization property. In an earlier paper, Talagrand [36] proved a different dimension free isoperimetric inequality for the exponential measure, where the enlargements involve mixtures of ℓ_1 and ℓ_2 balls with different scales (this result does not provide lower bounds on the boundary measure of sets).

In a recent paper [7] we have studied in depth various types of inequalities allowing the precise description of concentration phenomenon and isoperimetric profile for probability measures, in the intermediate regime between exponential and Gaussian. Our approach of the isoperimetric inequality followed the

one of Ledoux [3]: we studied the improving properties of the underlying semigroups, but we had to replace Gross hypercontractivity by a notion of Orlicz hyperboundedness, closely related to F-Sobolev inequalities. This approach yields a dimension free description of the isoperimetric profile for the measures $d\nu_\alpha(t) = e^{-|t|^\alpha} dt$ for $1 \leq \alpha \leq 2$: there exists a universal constant K such that for all $\alpha \in [1, 2]$

$$I_{\nu_\alpha} \geq I_{\nu_\alpha^\infty} \geq \frac{1}{K} I_{\nu_\alpha}.$$

It is plain that the method in [7] allows to deal with more general measures, at the price of rather heavy technicalities.

In this paper, we wish to point out a softer approach to isoperimetric inequalities. It was recently developed by Wang and his coauthors [38, 21, 39] and relies on so called super-Poincaré inequalities. It can be combined with our techniques in order to provide dimension free isoperimetric inequalities for large classes of measures. Among them are the measures on the line with density $e^{-\Phi(|t|)}\, dt/Z$ where $\Phi(0) = 0$, Φ is convex and $\sqrt{\Phi}$ is concave. This is achieved in the first part of the paper: Sections 2–5. The dimension free inequalities are still valid for slight modifications of the above examples. Other approaches and a few examples of perturbation results are developed in the last sections of the article.

Finally, let us present the super-Poincaré inequality as introduced by Wang in order to study the essential spectrum of Markov generators (actually we have found convenient to exchange the roles of s and $\beta(s)$ in the definition below). We shall say that a probability measure μ on (M, g) satisfies a super-Poincaré inequality, if there exists a nonnegative function β defined on $[1, +\infty[$ such that for all smooth $f : M \to \mathbb{R}$ and all $s \geq 1$,

$$\int f^2 d\mu - s \left(\int |f| d\mu \right)^2 \leq \beta(s) \int |\nabla f|^2 d\mu.$$

Optimizing on s turns this family of inequalities into a single one, which belongs to Nash inequalities. But it is often easier to work with the first form. Similar inequalities appear in the literature, see [9, 20]. Wang discovered that super-Poincaré inequalities imply precise isoperimetric estimates, and are related to Beckner-type inequalities via F-Sobolev inequalities. In fact, Beckner-type inequalities, as developed by Latała-Oleszkiewicz [25] were crucial in deriving dimension-free concentration in our paper [7]. In full generality they read as follows: for all smooth f and all $p \in [1, 2)$,

$$\int f^2 d\mu - \left(\int |f|^p d\mu \right)^{\frac{2}{p}} \leq T(2-p) \int |\nabla f|^2 d\mu,$$

where $T : (0, 1] \to \mathbb{R}^+$ is a non-decreasing function. Following [8] we could characterize the measures on the line which enjoy this property, and then take advantage of the tensorization property. As the reader noticed, the super-Poincaré and Beckner-type inequalities are formally very similar. It turns out that the

tools of [8] apply to both, see for example Lemma 3 below. This remark allows us to present a rather concise proof of the dimension-free isoperimetric inequalities, since the two functional inequalities involved (Beckner type for the tensorization property, and super-Poincaré for its isoperimetric implications) can be studied in one go.

2 A measure-Capacity sufficient condition for super-Poincaré inequality

This section provides a sufficient condition for the super Poincaré inequality to hold, in terms of a comparison between capacity of sets and their measure. This point of view was put forward in [7] in order to give a natural unified presentation of the many functional inequalities appearing in the field.

Given $A \subset \Omega$, the *capacity* $\mathrm{Cap}_\mu(A, \Omega)$, is defined as

$$\begin{aligned} \mathrm{Cap}_\mu(A,\Omega) &= \inf\left\{ \int |\nabla f|^2 d\mu;\ f_{|A} \geq 1,\ f_{|\Omega^c} = 0 \right\} \\ &= \inf\left\{ \int |\nabla f|^2 d\mu;\ \mathbf{1}_A \leq f \leq \mathbf{1}_\Omega \right\}, \end{aligned}$$

where the infimum is over locally Lipschitz functions. The latter equality follows from an easy truncation argument, reducing to functions with values in $[0,1]$. Finally we defined in [8] the capacity of A with respect to μ when $\mu(A) < 1/2$ as

$$\mathrm{Cap}_\mu(A) := \inf\{\mathrm{Cap}(A,\Omega);\ A \subset \Omega,\ \mu(\Omega) \leq 1/2\}.$$

Theorem 1. *Assume that for every measurable $A \subset M$ with $\mu(A) < 1/2$, one has*

$$\mathrm{Cap}_\mu(A) \geq \sup_{s \geq 1} \frac{1}{\beta(s)} \left(\frac{\mu(A)}{1 + (s-1)\mu(A)} \right).$$

for some function β defined on $[1, +\infty[$.

Then, for every smooth $f : M \to \mathbb{R}$ and every $s \geq 1$ one has

$$\int f^2 d\mu - s \left(\int |f| d\mu \right)^2 \leq 4\beta(s) \int |\nabla f|^2 d\mu.$$

Proof. We use four results that we recall or prove just after this proof. Let $s \geq 1$, $f : M \to \mathbb{R}$ be locally Lipschitz and m a median of the law of f under μ. Define $F_+ = (f-m)_+$ and $F_- = (f-m)_-$. Setting

$$\mathcal{G}_s = \left\{ g : M \to [0,1);\ \int (1-g)^{-1} d\mu \leq 1 + \frac{1}{s-1} \right\},$$

it follows from Lemmas 2 and 3 (used with $A = s-1$ and $a = 1/2$) that

$$\begin{aligned}\int f^2 d\mu - s\left(\int |f| d\mu\right)^2 &\leq \int (f-m)^2 d\mu - (s-1)\left(\int |f-m|\, d\mu\right)^2 \\ &\leq \sup\left\{\int (f-m)^2 g\, d\mu; g \in \mathcal{G}_s\right\} \\ &\leq \sup\left\{\int F_+^2 g\, d\mu; g \in \mathcal{G}_s\right\} \\ &\qquad + \sup\left\{\int F_-^2 g\, d\mu; g \in \mathcal{G}_s\right\},\end{aligned}$$

where we have used the fact that the supremum of a sum is less than the sum of the suprema. We deal with the first term of the right hand side. By Theorem 5 we have

$$\sup\left\{\int F_+^2 g\, d\mu; g \in \mathcal{G}_s\right\} \leq 4B_s \int |\nabla F_+|^2 d\mu$$

where B_s is the smallest constant so that for all $A \subset M$ with $\mu(A) < 1/2$

$$B_s \mathrm{Cap}_\mu(A) \geq \sup\left\{\int \mathbb{1}_A g\, d\mu; g \in \mathcal{G}_s\right\}.$$

On the other hand Lemma 4 insures that

$$\begin{aligned}\sup\left\{\int \mathbb{1}_A g\, d\mu; g \in \mathcal{G}_s\right\} &= \mu(A)\left(1 - \left(1 + \frac{1}{(s-1)\mu(A)}\right)^{-1}\right) \\ &= \frac{\mu(A)}{1 + (s-1)\mu(A)}.\end{aligned}$$

Thus, by our assumption, $B_s \leq \beta(s)$. We proceed in the same way for F_-. Summing up, we arrive at

$$\begin{aligned}\int f^2 d\mu - s\left(\int |f| d\mu\right)^2 &\leq 4\beta(s)\left(\int |\nabla F_+|^2 d\mu + \int |\nabla F_-|^2 d\mu\right) \\ &\leq 4\beta(s)\int |\nabla f|^2 d\mu.\end{aligned}$$

In the last bound we used the fact that since f is locally Lipschitz and μ absolutely continuous, the set $\{f = m\} \cap \{\nabla f \neq 0\}$ is μ-negligible. □

Lemma 2. *Let (X, P) be a probability space. Then for any function $g \in L^2(P)$, for any $s \geq 1$,*

$$\int g^2 dP - s\left(\int |g| dP\right)^2 \leq \int (g-m)^2 dP - (s-1)\left(\int |g-m| dP\right)^2$$

where m is a median of the law of g under P.

Proof. We write

$$\int g^2 dP - s\left(\int |g| dP\right)^2 = \mathbf{Var}_P(|g|) - (s-1)\left(\int |g| dP\right)^2.$$

By the variational definition of the median and the variance respectively, we have $\mathbf{Var}_P(|g|) \leq \mathbf{Var}_P(g) \leq \int (g-m)^2 dP$ and $\int |g-m| dP \leq \int |g| dP$. The result follows. □

Lemma 3 ([8]). *Let φ be a non-negative integrable function on a probability space (X, P). Let $A \geq 0$ and $a \in (0,1)$, then*

$$\begin{aligned} & \int \varphi \, dP - A\left(\int \varphi^a dP\right)^{\frac{1}{a}} \\ = \; & \sup\left\{\int \varphi g \, dP; g : X \to (-\infty, 1) \text{ and } \int (1-g)^{\frac{a}{a-1}} dP \leq A^{\frac{a}{a-1}}\right\} \\ \leq \; & \sup\left\{\int \varphi g \, dP; g : X \to [0, 1) \text{ and } \int (1-g)^{\frac{a}{a-1}} dP \leq 1 + A^{\frac{a}{a-1}}\right\}. \end{aligned}$$

Note that in [8] it is assumed that $A > 0$. The case $A = 0$ is easy.

Lemma 4 ([8]). *Let $a \in (0,1)$. Let Q be a finite measure on a space X and let $K > Q(X)$. Let $A \subset X$ be measurable with $Q(A) > 0$. Then*

$$\begin{aligned} & \sup\left\{\int_X \mathbb{1}_A g \, dQ; g : X \to [0,1) \text{ and } \int_X (1-g)^{\frac{a}{a-1}} dQ \leq K\right\} \\ = \; & Q(A)\left(1 - \left(1 + \frac{K - Q(X)}{Q(A)}\right)^{\frac{a-1}{a}}\right). \end{aligned}$$

Theorem 5. *Let $\mathcal{G}$ be a family of non-negative Borel functions on M, $\Omega \subset M$ with $\mu(\Omega) \leq 1/2$ and for any measurable function f vanishing on Ω^c set*

$$\Phi(f) = \sup_{g \in \mathcal{G}} \int_\Omega f g \, d\mu.$$

Let B denote the smallest constant such that for all $A \subset \Omega$ with $\mu(A) < 1/2$ one has

$$B \operatorname{Cap}_\mu(A) \geq \Phi(\mathbb{1}_A).$$

Then for every smooth function $f : M \to \mathbb{R}$ vanishing on Ω^c it holds

$$\Phi(f^2) \leq 4B \int |\nabla f|^2 d\mu.$$

Proof. We start with a result of Maz'ja [29], also discussed in [7, Proposition 13]: given two absolutely continuous measures μ, ν on M, denote by B_ν the smallest constant such that for all $A \subset \Omega$ one has

$$B_\nu \operatorname{Cap}_\mu(A, \Omega) \geq \nu(A).$$

Then for every smooth function $f : M \to \mathbb{R}$ vanishing on Ω^c

$$\int f^2 d\nu \leq 4B_\nu \int |\nabla f|^2 d\mu.$$

Following an idea of Bobkov and Götze [12] we apply the previous inequality to the measures $d\nu = g d\mu$ for $g \in \mathcal{G}$. Thus for f as above

$$\Phi(f) = \sup_{g\in\mathcal{G}} \int_\Omega fg \, d\mu \leq \sup_{g\in\mathcal{G}} B_{g\,d\mu} \int |\nabla f|^2 d\mu.$$

It remains to check that the constant B is at most $\sup_{g\in\mathcal{G}} B_{g\,d\mu}$. This follows from the definition of Φ and the inequality $\mathrm{Cap}_\mu(A) \leq \mathrm{Cap}_\mu(A, \Omega)$. □

Corollary 6. *Assume that $\beta : [1, +\infty) \to \mathbb{R}^+$ is non-increasing and that $s \mapsto s\beta(s)$ is non-decreasing on $[2, +\infty)$. Then, for every $a \in (0, 1/2)$,*

$$\frac{1}{2} \frac{a}{\beta(1/a)} \leq \sup_{s\geq 1} \frac{a}{1 + (s-1)a} \frac{1}{\beta(s)} \leq 2 \frac{a}{\beta(1/a)}. \tag{1}$$

In particular, if for every measurable $A \subset M$ with $\mu(A) < 1/2$, one has

$$\mathrm{Cap}_\mu(A) \geq \frac{\mu(A)}{\beta(1/\mu(A))},$$

then, for every $f : M \to \mathbb{R}$ and every $s \geq 1$ one has

$$\int f^2 d\mu - s \left(\int |f| d\mu \right)^2 \leq 8\beta(s) \int |\nabla f|^2 d\mu.$$

Proof. The choice $s = 1/a$ gives the first inequality in (1). For the second part of (1), we consider two cases:

If $a(s-1) \leq 1/2$ then $s \leq 1 + \frac{1}{2a} \leq \frac{1}{a}$, where we have used $a < 1/2$. Hence, the monotonicity of β yields

$$\frac{a}{1 + (s-1)a} \frac{1}{\beta(s)} \leq \frac{a}{\beta(s)} \leq \frac{a}{\beta(1/a)}.$$

If $a(s-1) > 1/2$, note that $a/(1 + (s-1)a) \leq 1/s$. Thus by monotonicity of $s \mapsto s\beta(s)$ and since $s \geq 1 + 1/2a = \frac{1+2a}{2a} \geq 2$,

$$\frac{a}{1 + (s-1)a} \frac{1}{\beta(s)} \leq \frac{1}{s\beta(s)} \leq \frac{2a}{(1+2a)\beta(1 + \frac{1}{2a})} \leq \frac{2a}{\beta(1/a)}.$$

The last step uses the inequality $1 + \frac{1}{2a} \leq \frac{1}{a}$ and the monotonicity of β.

The second part of the Corollary is a direct consequence of Theorem 1 and (1) (replacing β by $\beta/2$). □

3 Beckner type versus super Poincaré inequality

In this section we use the previous measure-capacity sufficient condition to derive super Poincaré inequality from Beckner type inequality.

The following criterion was established in [7, Theorem 18 and Lemma 19] in the particular case of $M = \mathbb{R}^n$. As mentioned in the introduction of [7] the extension to Riemannian manifolds is straightforward.

Theorem 7 ([7]). *Let $T : [0,1] \to \mathbb{R}^+$ be non-decreasing and such that $x \mapsto T(x)/x$ is non-increasing. Let C be the optimal constant such that for every smooth $f : M \to \mathbb{R}$ one has (Beckner type inequality)*

$$\sup_{p\in(1,2)} \frac{\int f^2 d\mu - \left(\int |f|^p d\mu\right)^{\frac{2}{p}}}{T(2-p)} \leq C \int |\nabla f|^2 d\mu.$$

Then $\frac{1}{6}B(T) \leq C \leq 20B(T)$, where $B(T)$ is the smallest constant so that every $A \subset M$ with $\mu(A) < 1/2$ satisfies

$$B(T)\mathrm{Cap}_\mu(A) \geq \frac{\mu(A)}{T\left(1/\log\left(1+\frac{1}{\mu(A)}\right)\right)}.$$

If $M = \mathbb{R}$, m is a median of μ and ρ_μ is its density, we have more explicitly

$$\frac{1}{6}\max(B_-(T), B_+(T)) \leq C \leq 20\max(B_-(T), B_+(T))$$

where

$$\begin{aligned} B_+(T) &= \sup_{x>m} \mu([x,+\infty)) \frac{1}{T\left(1/\log\left(1+\frac{1}{\mu([x,+\infty))}\right)\right)} \int_m^x \frac{1}{\rho_\mu} \\ B_-(T) &= \sup_{x<m} \mu((-\infty,x]) \frac{1}{T\left(1/\log\left(1+\frac{1}{\mu((-\infty,x])}\right)\right)} \int_x^m \frac{1}{\rho_\mu}. \end{aligned}$$

The relations between Beckner-type and super-Poincaré inequalities have been explained by Wang, via F-Sobolev inequalities. Here we give an explicit connection under a natural condition on the rate function T.

Corollary 8 (From Beckner to Super Poincaré). *Let $T : [0,1] \to \mathbb{R}^+$ be non-decreasing and such that $x \mapsto T(x)/x$ is non-increasing. Assume that there exists a constant C such that for every smooth $f : M \to \mathbb{R}$ one has*

$$\sup_{p\in(1,2)} \frac{\int f^2 d\mu - \left(\int |f|^p d\mu\right)^{\frac{2}{p}}}{T(2-p)} \leq C \int |\nabla f|^2 d\mu. \tag{2}$$

Define $\beta(s) = T\left(1/\log(1+s)\right)$ for $s \geq e-1$ and $\beta(s) = T(1)$ for $s \in [1, e-1]$.

Then, every smooth $f : M \to \mathbb{R}$ satisfies for every $s \geq 1$,

$$\int f^2 d\mu - s\left(\int |f| d\mu\right)^2 \leq 48C\beta(s) \int |\nabla f|^2 d\mu.$$

Proof. By Theorem 7, Inequality (2) implies that every $A \subset M$ with $\mu(A) < 1/2$ satisfies

$$6\, C\, \mathrm{Cap}_\mu(A) \geq \frac{\mu(A)}{T\left(1/\log\left(1+\frac{1}{\mu(A)}\right)\right)} = \frac{\mu(A)}{\beta\left(\frac{1}{\mu(A)}\right)}.$$

Since T is non-decreasing, β is non-increasing on $[1,\infty)$. On the other hand, for $s \geq e-1$, we have

$$s\beta(s) = sT\left(1/\log(1+s)\right) = \log(1+s)T\left(1/\log(1+s)\right)\frac{s}{\log(1+s)}.$$

The map $x \mapsto T(x)/x$ is non-increasing and $s \mapsto \frac{s}{\log(1+s)}$ is non-decreasing. It follows that $s \mapsto s\beta(s)$ is non-decreasing. Corollary 6 therefore applies and yields the claimed inequality. □

A remarkable feature of Beckner type inequalities (2) is the tensorization property: if μ_1 and μ_2 both satisfy (2) with constant C, then so does $\mu_1 \otimes \mu_2$ [25]. For this reason inequalities for measures on the real line are inherited by their infinite products. In dimension 1 the criterion given in Theorem 7 allows us to deal with probability measures $d\mu_\Phi(x) = Z_\Phi^{-1} e^{-\Phi(|x|)} dx$ with quite general potentials Φ:

Proposition 9. *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing convex function with $\Phi(0) = 0$ and consider the probability measure $d\mu_\Phi(x) = Z_\Phi^{-1} e^{-\Phi(|x|)} dx$. Assume that Φ is $\mathcal{C}^2$ on $[\Phi^{-1}(1), +\infty)$ and that $\sqrt{\Phi}$ is concave. Define $T(x) = [1/\Phi' \circ \Phi^{-1}(1/x)]^2$ for $x > 0$ and $\beta(s) = [1/\Phi' \circ \Phi^{-1}(\log(1+s))]^2$ for $s \geq e-1$ and $\beta(s) = [1/\Phi' \circ \Phi^{-1}(1)]^2$ for $s \in [1, e-1]$. Then there exists a constant $C > 0$ such that for any $n \geq 1$, every smooth function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies*

$$\sup_{p \in (1,2)} \frac{\int f^2 d\mu_\Phi^n - \left(\int |f|^p d\mu_\Phi^n\right)^{\frac{2}{p}}}{T(2-p)} \leq C \int |\nabla f|^2 d\mu_\Phi^n.$$

In turn, for any $n \geq 1$, every smooth function $f : \mathbb{R}^n \to \mathbb{R}$ and every $s \geq 1$,

$$\int f^2 d\mu_\phi^n - s\left(\int |f| d\mu_\Phi^n\right)^2 \leq 48 C \beta(s) \int |\nabla f|^2 d\mu_\Phi^n.$$

Proof. The proof of the Beckner type inequality comes from [7, proof of Corollary 32]: the hypotheses on Φ allow to compute an equivalent of $\mu_\Phi([x,+\infty))$ when x tends to infinity (namely $e^{-\phi}/\phi'$) and thus to bound from above the quantities $B_+(T)$ and $B_-(T)$ of Theorem 7. This yields the Beckner type inequality in dimension 1. Next we use the tensorization property.

The second part follows from Corollary 8 (the hypotheses on Φ ensure that T is non-decreasing and $T(x)/x$ is non-increasing). □

Example 1. A first family of examples is given by the measures $d\mu_p(x) = e^{-|x|^p} dx/(2\Gamma(1+1/p))$, $p \in [1,2]$. The potential $x \mapsto |x|^p$ fulfills the hypotheses of Proposition 9 with $T_p(x) = \frac{1}{p^2} x^{2(1-\frac{1}{p})}$. Thus, by Proposition 9, for any $n \geq 1$,

μ_p^n satisfies a super Poincaré inequality with function $\beta(s) = c_p/\log(1+s)^{2(1-\frac{1}{p})}$ where c_p depends only on p and not on the dimension n.

Note that the corresponding Beckner type inequality

$$\sup_{q\in(1,2)} \frac{\int f^2 d\mu_p^n - \left(\int |f|^q d\mu_p^n\right)^{\frac{2}{q}}}{(2-q)^{2(1-\frac{1}{p})}} \leq \tilde{c}_p \int |\nabla f|^2 d\mu_p^n,$$

goes back to Latała and Oleszkiewicz [25] with a different proof, see also [8].

Example 2. Consider now the larger family of examples given by $d\mu_{p,\alpha}(x) = Z_{p,\alpha}^{-1} e^{-|x|^p \log(\gamma+|x|)^\alpha} dx$, $p \in [1,2]$, $\alpha \geq 0$ and $\gamma = e^{\alpha/(2-p)}$. One can see that $\mu_{p,\alpha}^n$ satisfies a super Poincaré inequality with function

$$\beta(s) = \frac{c_{p,\alpha}}{(\log(1+s))^{2(1-\frac{1}{p})}(\log\log(e+s))^{2\alpha/p}}, \qquad s \geq 1.$$

4 Isoperimetric inequalities

In this section we collect results which relate super-Poincaré inequalities with isoperimetry. They follow Ledoux approach of Buser's inequality [26]. This method was developed by Bakry-Ledoux [3] and Wang [32, 38], see also [7].

The following result, a particular case of [3, Inequality (4.3)], allows to derive isoperimetric estimates from semi-group bounds.

Theorem 10 ([3]). *Let μ be a probability measure on (M, g) with density e^{-V} with respect to the volume measure. Assume that V is $\mathcal{C}^2$ and such that* $\mathrm{Ricci} + \nabla\nabla V \geq -Rg$ *for some $R \geq 0$. Let $(\mathbf{P}_t)_{t\geq 0}$ be the corresponding semi-group with generator $\Delta - \nabla V \cdot \nabla$. Then, for $t > 0$, every measurable set $A \subset M$ satisfies*

$$2\sqrt{t}\left(\frac{4tR}{1-e^{-4tR}}\right)^{\frac{1}{2}} \mu_s(\partial A) \geq \mu(A) - \int (\mathbf{P}_t \mathbb{1}_A)^2 d\mu = \mu(A^c) - \int (\mathbf{P}_t \mathbb{1}_{A^c})^2 d\mu.$$

In order to exploit this result we need the following proposition due to Wang [38]. We sketch the proof for completeness.

Proposition 11 ([38]). *Let μ be a probability measure on M with density e^{-V} with respect to the volume measure. Assume that V is $\mathcal{C}^2$. Let $(\mathbf{P}_t)_{t\geq 0}$ be the corresponding semi-group with generator $L := \Delta - \nabla V \cdot \nabla$. Then the following are equivalent*
(i) μ satisfies a Super Poincaré inequality: every smooth $f : M \to \mathbb{R}$ satisfies for every $s \geq 1$

$$\int f^2 d\mu - s\left(\int |f| d\mu\right)^2 \leq \beta(s) \int |\nabla f|^2 d\mu.$$

(ii) For every $t \geq 0$, every smooth $f : M \to \mathbb{R}$, and all $s \geq 1$

$$\int (\mathbf{P}_t f)^2 d\mu \leq e^{-\frac{2t}{\beta(s)}} \int f^2 d\mu + s(1 - e^{-\frac{2t}{\beta(s)}}) \left(\int |f| d\mu\right)^2.$$

Proof. (i) follows from (ii) by differentiation at $t = 0$.

On the other hand, if $u(t) = \int (\mathbf{P}_t f)^2 d\mu$, (i) implies that

$$u'(t) = 2 \int \mathbf{P}_t f L \mathbf{P}_t f \, d\mu = -2 \int |\nabla \mathbf{P}_t f|^2 d\mu \leq -\frac{2}{\beta(s)} \left[u(t) - s \left(\int |f| d\mu \right)^2 \right]$$

since $\int |\mathbf{P}_t f| d\mu \leq \int |f| d\mu$. The result follows by integration. □

Theorem 12 ([38]). *Let μ be a probability measure on (M, g) with density e^{-V} with respect to the volume measure. Assume that V is $\mathcal{C}^2$ and such that* $\mathrm{Ricci} + \nabla\nabla V \geq -Rg$ *for some $R \geq 0$. Let $(\mathbf{P}_t)_{t \geq 0}$ be the corresponding semi-group with generator $\Delta - \nabla V \cdot \nabla$. Assume that every smooth $f : M \to \mathbb{R}$ satisfies for every $s \geq 1$*

$$\int f^2 d\mu - s \left(\int |f| d\mu \right)^2 \leq \beta(s) \int |\nabla f|^2 d\mu,$$

with β decreasing. Then there exists a positive number $C(R, \beta(1))$ such that every measurable set $A \subset M$ satisfies

$$\mu_s(\partial A) \geq C(R, \beta(1)) \mu(A)(1 - \mu(A)).$$

If $\lim_{+\infty} \beta = 0$, any measurable set $A \subset M$ with $p := \min(\mu(A), \mu(A^c)) \leq \min(1/2, 1/(2\beta^{-1}(1/R)))$ satisfies

$$\mu_s(\partial A) \geq \frac{1}{7} \frac{p}{\sqrt{\beta\left(\frac{1}{2p}\right)}}.$$

Proof. From the super-Poincaré inequality and Proposition 11 we have for any smooth $f : M \to \mathbb{R}$ and all $s \geq 1$

$$\int (\mathbf{P}_t f)^2 d\mu \leq e^{-\frac{2t}{\beta(s)}} \int f^2 d\mu + s(1 - e^{-\frac{2t}{\beta(s)}}) \left(\int |f| d\mu \right)^2.$$

Applying this to approximations of characteristic functions we get for any measurable set $A \subset M$,

$$\int (\mathbf{P}_t \mathbb{1}_A)^2 d\mu \leq e^{-\frac{2t}{\beta(s)}} \mu(A) + s(1 - e^{-\frac{2t}{\beta(s)}}) \mu(A)^2 \qquad \forall s \geq 1.$$

Hence by Theorem 10, we have for all $t > 0, s \geq 1$,

$$\mu_s(\partial A) \geq \mu(A)(1 - s\mu(A)) \frac{1 - e^{-\frac{2t}{\beta(s)}}}{2\sqrt{t}} \left(\frac{1 - e^{-4tR}}{4tR} \right)^{\frac{1}{2}}. \tag{3}$$

The first isoperimetric inequality is obtained when choosing $s = 1, t = \beta(1)$. In fact this is almost exactly the method used by Ledoux to derive Cheeger's

inequality from Poincaré inequality when the curvature is bounded from below [26].

For a set A of measure at most $1/2$, we take $s = 1/(2\mu(A))$ and $t = \beta(s)/2 = \frac{1}{2}\beta\left(\frac{1}{2\mu(A)}\right)$. If $\mu(A)$ is small enough to ensure that $R\beta(1/(2\mu(A))) \leq 1$, and since $x \mapsto (1-e^{-x})/x$ is decreasing, the last factor in the latter inequality is easily bounded from below. We obtain

$$\mu_s(\partial A) \geq \frac{1}{4}(1-e^{-1})\sqrt{1-e^{-2}}\frac{\mu(A)}{\sqrt{\beta\left(\frac{1}{2\mu(A)}\right)}} \geq \frac{1}{7}\frac{\mu(A)}{\sqrt{\beta\left(\frac{1}{2\mu(A)}\right)}}.$$

For sets with $\mu(A) > 1/2$ we work instead with the expression involving A^c in Theorem 10. □

Combining Theorem 7, the tensorization property of Beckner type-inequalities, Corollary 8 and Theorem 12 allows to derive dimension-free isoperimetric inequalities for the products of large classes of probability measures in the real line. In the next section we focus on log-concave densities.

5 Isoperimetric profile for log-concave measures

Here we apply the previous results to infinite product of the measures: $\mu_\Phi(dx) = Z_\Phi^{-1}\exp\{-\Phi(|x|)\}dx = \varphi(x)dx$, $x \in \mathbb{R}$, with Φ convex and $\sqrt{\Phi}$ concave. The isoperimetric profile of a symmetric log-concave density on the line (with the usual metric) was calculated by Bobkov [10]. He showed that half-lines have minimal boundary among sets of given measure. Since the boundary measure of $(-\infty, x]$ is given by the density of the measure at x, the isoperimetric profile is $I_\Phi(t) = \varphi(H^{-1}(\min(t, 1-t)) = \varphi(H^{-1}(t))$, $t \in [0,1]$ where H is the distribution function of μ_Φ. It compares to the function

$$L_\Phi(t) = \min(t, 1-t)\Phi' \circ \Phi^{-1}\left(\log\frac{1}{\min(t,1-t)}\right),$$

where Φ' is the right derivative. More precisely,

Proposition 13. *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing convex function. Assume that in a neighborhood of $+\infty$, the function Φ is $\mathcal{C}^2$ and $\sqrt{\Phi}$ is concave.*

Let $d\mu_\Phi(x) = Z_\Phi^{-1}e^{-\Phi(|x|)}dx$ be a probability measure with density φ. Let H be the distribution function of μ and $I_\Phi(t) = \varphi(H^{-1}(t))$, $t \in [0,1]$. Then,

$$\lim_{t\to 0}\frac{I_\Phi(t)}{t\Phi' \circ \Phi^{-1}(\log\frac{1}{t})} = 1.$$

Consequently, if $\Phi(0) < \log 2$, L_Φ is defined on $[0,1]$ and there exists constants $k_1, k_2 > 0$ such that for all $t \in [0,1]$,

$$k_1 L_\Phi(t) \leq I_\Phi(t) \leq k_2 L_\Phi(t).$$

This result appears in [4, 17] in the particular case $\Phi(x) = |x|^p$.

Proof. Since Φ is convex and (strictly) increasing, note that Φ' may vanish only at 0. Under our assumptions on Φ we have $H(y) = \int_{-\infty}^{y} Z_\Phi^{-1} e^{-\Phi(|x|)} dx \sim Z_\Phi^{-1} e^{-\Phi(|y|)}/\Phi'(|y|)$ when y tends to $-\infty$ (see *e.g.* [7, Corollary 2 of section 6]). Thus using the change of variable $y = H^{-1}(t)$, we get

$$\begin{aligned}\lim_{t\to 0} \frac{I_\Phi(t)}{t\Phi' \circ \Phi^{-1}(\log \frac{1}{t})} &= \lim_{y\to-\infty} \frac{e^{-\Phi(|y|)}}{Z_\Phi H(y) \Phi' \circ \Phi^{-1}(\log \frac{1}{H(y)})} \\ &= \lim_{y\to-\infty} \frac{\Phi'(|y|)}{\Phi' \circ \Phi^{-1}(\log \frac{1}{H(y)})}.\end{aligned}$$

A Taylor expansion of $\Phi' \circ \Phi^{-1}$ between $\log \frac{1}{H(y)}$ and $\Phi(|y|)$ gives

$$\frac{\Phi' \circ \Phi^{-1}(\log \frac{1}{H(y)})}{\Phi'(|y|)} = 1 + \frac{1}{\Phi'(|y|)} \left(\log \frac{1}{H(y)} - \Phi(|y|) \right) \frac{\Phi'' \circ \Phi^{-1}(c_y)}{\Phi' \circ \Phi^{-1}(c_y)}$$

for some $c_y \in [\min(\Phi(|y|), \log \frac{1}{H(y)}), \infty)$.

Since for $y \ll -1$

$$\frac{1}{2} \frac{e^{-\Phi(|y|)}}{Z_\Phi \Phi'(|y|)} \leq H(y) \leq \frac{e^{-\Phi(|y|)}}{Z_\Phi \Phi'(|y|)}$$

we have

$$\log(Z_\Phi \Phi'(|y|)) \leq \log \frac{1}{H(y)} - \Phi(|y|) \leq \log 2 + \log(Z_\Phi \Phi'(|y|)). \tag{4}$$

On the other hand, when $\sqrt{\Phi}$ is concave and $\mathcal{C}^2$, $(\sqrt{\Phi})''$ is non positive when it is defined. This leads to $\frac{\Phi''}{\Phi'} \leq \frac{\Phi'}{2\Phi}$. Furthermore $(\sqrt{\Phi})'$ decreasing implies that $\Phi'(x) \leq c\sqrt{\Phi(x)}$ for x large enough and for some constant $c > 0$. Finally we get $\frac{\Phi''(x)}{\Phi'(x)} \leq \frac{c}{\sqrt{\Phi(x)}}$ for x large enough.

All these computations together give

$$\left| \frac{1}{\Phi'(|y|)} \left(\log \frac{1}{H(y)} - \Phi(|y|) \right) \frac{\Phi'' \circ \Phi^{-1}(c_y)}{\Phi' \circ \Phi^{-1}(c_y)} \right| \leq \frac{\log 2 + |\log(Z_\Phi \Phi'(|y|))|}{|\Phi'(|y|)|} \frac{c}{\sqrt{c_y}}$$

which goes to 0 as y goes to $-\infty$. This ends the proof. □

The following comparison result will allow us to modify measures without loosing much on their isoperimetric profile. It also shows that even log-concave measures on the real line play a central role.

Theorem 14 ([5, 33]). *Let m be a probability measure on $(\mathbb{R}, |.|)$ with even log-concave density. Let μ be a probability measure on (M, g) such that $I_\mu \geq cI_m$. Then for all $n \geq 1$, $I_{\mu^n} \geq cI_{m^n}$.*

Now we show the following infinite dimensional isoperimetric inequality.

Theorem 15. *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing convex function with $\Phi(0) = 0$ and consider the probability measure $d\mu_\Phi(x) = Z_\Phi^{-1} e^{-\Phi(|x|)} dx$. Assume that Φ is $\mathcal{C}^2$ on $[\Phi^{-1}(1), +\infty)$ and that $\sqrt{\Phi}$ is concave.*

Then there exists a constant $K > 0$ such that for all $t \in [0, 1]$ one has

$$I_{\mu_\Phi^\infty}(t) \geq K L_\Phi(t).$$

Since $I_{\mu_\Phi^\infty}(t) \leq I_{\mu_\Phi} \leq k_2 L_\Phi(t)$, we have, up to constants, the value of the isoperimetric profile of the infinite product.

Proof. For technical reasons we cannot work directly with μ_Φ since its potential Φ is not $\mathcal{C}^2$. We introduce another even convex function Ψ which is $\mathcal{C}^2$, increases on $[0, +\infty)$ and coincides with Φ outside a symmetric interval $[-a, a]$. We also consider the probability measure μ_Ψ. In the large its density differs from the one of μ_Φ exactly by the multiplicative factor Z_Φ / Z_Ψ. The first statement of Proposition 13 shows that the isoperimetric profiles of μ_Φ and μ_Ψ are equivalent when t tends to 0 or 1. Since they are continuous, there exists constants $c_1, c_2 > 0$ such that

$$c_1 I_{\mu_\Phi} \geq I_{\mu_\Psi} \geq c_2 I_{\mu_\Phi}. \tag{5}$$

The second inequality in the above formula implies that the monotone map $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = H_\Psi^{-1} \circ H_\Phi$ is Lipschitz (just compute its derivative). Here H_Φ is the distribution function of μ_Φ. Moreover, by construction the image measure of μ_Φ by T is μ_Ψ. This easily implies that any Sobolev type inequality satisfied by μ_Φ can be transported to μ_Ψ with a change in the constant, see e.g. [27, 4] for more on these methods. Applying Proposition 9 to the measure μ_Φ provides a Beckner-type inequality, with rate function T expressed in terms of Φ. For the above reasons it is inherited by μ_Ψ (we could also have used the perturbation results recalled in the last section of the paper). By tensorization the powers of this measure enjoy the same property, which implies a super-Poincaré inequality by Corollary 8. Hence there exists a constant C independent of the dimension n such that for every smooth $f : \mathbb{R}^n \to \mathbb{R}$ one has

$$\int f^2 d\mu_\Psi^n - s \left(\int |f| d\mu_\Psi^n \right)^2 \leq C\beta(s) \int |\nabla f|^2 d\mu_\Psi^n \qquad \forall s \geq 1,$$

where $\beta(s) = [1/\Phi' \circ \Phi^{-1}(\log(1+s))]^2$ for $s \geq e-1$ and $\beta(s) = [1/\Phi' \circ \Phi^{-1}(1)]^2$ for $s \in [1, e-1]$.

Next we apply Theorem 12 to the measure μ_Ψ^n. Consider first the case $\lim_{x\to\infty} \Phi'(x) = \alpha < +\infty$. The first inequality in Theorem 12 yields

$$I_{\mu_\Phi^n}(t) \geq K_1 \Phi' \circ \Phi^{-1}(1) \min(t, 1-t) \geq K_2 L_\Phi(t),$$

where the constants $K_1, K_2 > 0$ are independent of n and t.

If Φ' tends to infinity, the second part of Theorem 12 allows to conclude that for $t \in [0,1]$ (note that $\Psi'' \geq 0$ and thus we may take $R = 0$)

$$I_{\mu_\Psi^n}(t) \geq K_3 \min(t, 1-t) \Phi' \circ \Phi^{-1}(\log(1 + \frac{1}{2\min(t, 1-t)})).$$

Next we use elementary inequalities to bound from below $\Phi' \circ \Phi^{-1}(\log(1 + \frac{1}{2\min(t,1-t)}))$ by $\Phi' \circ \Phi^{-1}(\log(\frac{1}{\min(t,1-t)}))$. Their proof is postponed to the next lemma. Using the bound $1 + \frac{1}{2x} \geq (\frac{1}{x})^{\frac{1}{2}}$ for $0 < x \leq 1/2$ we have $\Phi'[\Phi^{-1}(\log(1 + \frac{1}{2x}))] \geq \Phi'[\Phi^{-1}(\log(\frac{1}{x})/2)]$. Then, (i) and (iii) of Lemma 16 ensure that $\Phi'[\Phi^{-1}(\log(1 + \frac{1}{2x}))] \geq \frac{1}{2}\Phi'[\Phi^{-1}(\log(\frac{1}{x}))]$. Thus there exists a constant $K_4 > 0$ such that for any n

$$I_{\mu_\Psi^n}(t) \geq K_4 L_\Phi(t) \qquad \forall t \in [0,1].$$

This achieves the proof since the first inequality in (5) and Theorem 14 imply that $I_{\mu_{\Phi^n}} \geq \frac{1}{c_1} I_{\mu_{\Psi^n}}$. □

Remark 3. The above theorem can be extended in many ways. The regularity assumption and the concavity of $\sqrt{\Phi}$ need only be satisfied in the large. Proving this requires in particular to modify the function T in Proposition 9.

Example 4. The previous theorem applies to the family of measures $d\nu_p(x) = e^{-|x|^p} dx/(2\Gamma(1 + 1/p))$, $p \in [1,2]$. This recovers results in [14, 7].

Example 5. More generally, for $d\mu_{p,\alpha}(x) = Z_{p,\alpha}^{-1} e^{-|x|^p \log(\gamma + |x|)^\alpha} dx$, $p \in [1,2]$, $\alpha \geq 0$ and $\gamma = e^{2\alpha/(2-p)}$ we get the following isoperimetric inequality: there exists a constant $c_{p,\alpha}$ such that for any dimension n and any Borel set A with $\mu_{p,\alpha}^n(A) \leq 1/2$,

$$(\mu_{p,\alpha}^n)_s(\partial A) \geq c_{p,\alpha} \left(\log(\frac{1}{\mu_{p,\alpha}^n(A)}) \right)^{1-\frac{1}{p}} \left(\log\log\big(e + \frac{1}{\mu_{p,\alpha}^n(A)}\big) \right)^{\alpha/p}.$$

Lemma 16. *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing convex function with $\Phi(0) = 0$. Assume that $\sqrt{\Phi}$ is concave. Then,*
(i) for every $x \geq 0$: $\Phi^{-1}\left(\frac{1}{2}x\right) \geq \frac{1}{2}\Phi^{-1}(x)$;
(ii) for every $x \geq 0$: $\Phi(2x) \leq 4\Phi(x)$;
(iii) for every $x \geq 0$: $\Phi'\left(\frac{1}{2}x\right) \geq \frac{1}{2}\Phi'(x)$.

Proof. Since Φ is convex, the slope function $(\Phi(x) - \Phi(0))/x = \Phi(x)/x$ is non-decreasing. Comparing the values at x and $2x$ shows that $2\Phi(x) \leq \Phi(2x)$. The claim of (i) follows.

Assertion (ii) is proved along the same line. Since $\sqrt{\Phi}$ is concave and vanishes at 0, the ratio $\sqrt{\Phi(x)}/x$ is non-increasing. Comparing its values at x and $2x$ yields the inequality.

Point (iii) is a direct consequence of (ii). Indeed, since $\sqrt{\Phi}$ is concave, $\Phi'/(2\sqrt{\Phi})$ is non-increasing. Comparing the values at x and $2x$ and using (ii) ensures that

$$\Phi'(2x) \leq \sqrt{\frac{\Phi(2x)}{\Phi(x)}}\Phi'(x) \leq 2\Phi'(x).$$

This completes the proof. □

6 F-Sobolev versus super-Poincaré inequality

We have explained in Section 3 how to get a dimension free super-Poincaré inequality, using the (tensorizable) Beckner inequality and Theorem 1. Another family of tensorizable inequalities is discussed in [7], namely additive ϕ-Sobolev inequalities.

We shall say that μ satisfies a homogeneous F-Sobolev inequality if for all smooth f,

$$\int f^2 F\left(\frac{f^2}{\int f^2 d\mu}\right) d\mu \leq C_F \int |\nabla f|^2 d\mu. \tag{6}$$

When $F = \log$ this is the usual tight logarithmic Sobolev inequality. In this case $F(a/b) = F(a) - F(b)$ so that the previous homogeneous inequality can be rewritten in an additive form. In general however this is not the case, so that we have to introduce the additive ϕ-Sobolev inequality, i.e.

$$\int \phi(f^2) d\mu - \phi\left(\int f^2 d\mu\right) \leq C_\phi \int |\nabla f|^2 d\mu, \tag{7}$$

with for example $\phi(x) = xF(x)$. In general, Inequalities (6) and (7) have different features. Note that (6) is an equality for constant f if $F(1) = 0$. We shall say that the inequality is tight in this case, and is defective if $F(1) < 0$. Besides, Inequality (7) is tight by nature. The main advantage of additive inequalities is that they enjoy the tensorization property, see [7] Lemma 12. Both kinds of Sobolev inequalities can be related to measure-capacity inequalities. We shall below complete the picture in [7]. The next Lemma shows how to tight a defective homogeneous inequality, in a much more simple way than the extension of Rothaus lemma discussed in [7] Lemma 9 and Theorem 10.

Lemma 17. *Let $F : (0, +\infty) \to \mathbb{R}$ be a non-decreasing continuous function such that $F(x)$ tends to $+\infty$ when x goes to $+\infty$ and $xF_-(x)$ is bounded.*

Assume that μ satisfies the homogeneous F-Sobolev inequality with constant C_F and a Poincaré inequality with constant C_P. Then one can find $a \geq 0$ only depending on F and C_+ depending on F, C_F and C_P such that for all smooth f

$$\int f^2 (F - a)_+ \left(\frac{f^2}{\int f^2 d\mu}\right) d\mu \leq C_+ \int |\nabla f|^2 d\mu\,.$$

Taking a larger a if necessary, we may thus assume that $(F(1) - a)_+ = 0$.

Proof. Since F goes to ∞ at ∞, we may find some $\rho > 1$ such that $F(2\rho) \geq 0$. Define $\tilde{F}(u) = F(u) - F(2\rho)$ which is thus non-positive on $[0, 2\rho]$ and non-negative on $[2\rho, +\infty[$ since F is non-decreasing. μ is obviously still satisfying the $\tilde{F}$-Sobolev inequality. If $M = \sup_{0 \leq u \leq 2\rho}\{-u\tilde{F}(u)\}$, $M < +\infty$ thanks to our hypotheses, so that for a non-negative f such that $\int f^2 d\mu = 1$,

$$\int f^2 \tilde{F}_+(f^2) d\mu \leq C_F \int |\nabla f|^2 d\mu + M. \tag{8}$$

Let ψ defined on $\mathbb{R}^+$ as follows : $\psi(u) = 0$ if $u \leq \sqrt{2}$, $\psi(u) = u$ if $u \geq \sqrt{2\rho}$ and $\psi(u) = \sqrt{2\rho}(u - \sqrt{2})/(\sqrt{2\rho} - \sqrt{2})$ if $\sqrt{2} \leq u \leq \sqrt{2\rho}$. Since $\psi(f) \leq f$, $\int \psi^2(f) d\mu \leq 1$ so that

$$\begin{aligned}
\int f^2 \tilde{F}_+(f^2) d\mu &= \int \psi^2(f) \tilde{F}_+(\psi^2(f)) d\mu \\
&\leq \int \psi^2(f) \tilde{F}_+ \left(\frac{\psi^2(f)}{\int \psi^2(f) d\mu} \right) d\mu \\
&\leq AC_F \int |\nabla f|^2 d\mu + M \int \psi^2(f) d\mu \\
&\leq AC_F \int |\nabla f|^2 d\mu + M \int_{f^2 \geq 2} f^2 d\mu
\end{aligned}$$

where $A = \left(2/(\sqrt{2\rho} - \sqrt{2})\right)^2$. But as shown in [7],

$$\int_{f^2 \geq 2} f^2 d\mu \leq 12 C_P \int |\nabla f|^2 d\mu$$

(recall that $\int f^2 d\mu = 1$), so that we finally obtain the desired result. □

The previous Lemma is a key for the result below we shall use in the sequel.

Theorem 18. *Let $d\mu = e^{-V} dx$ a probability measure on $\mathbb{R}^d$, with V a locally bounded potential. Let $F : (0, +\infty) \to \mathbb{R}$ be a non decreasing, concave, $\mathcal{C}^1$ function, with $F(8) > 0$, satisfying for some γ and M*

(1) $F(x)$ tends to $+\infty$ when x goes to $+\infty$,

(2) $xF'(x) \leq \gamma$ for all $x > 0$,

(3) $F(xy) \leq M + F(x) + F(y)$ for all $x, y > 0$,

(4) $xF(x)$ is bounded from below.

If μ satisfies the homogeneous F-Sobolev inequality (6) with constant C_F, then μ satisfies an additive ϕ-Sobolev inequality with some constant C_ϕ and $\phi(x) = xF(x)$. Moreover there exists a constant D such that, for all n, the product measure μ^n satisfies a measure-capacity inequality

$$\mu^n(A) F\left(\frac{1}{\mu^n(A)}\right) \leq D \mathrm{Cap}_{\mu^n}(A), \tag{9}$$

for all A such that $\mu^n(A) \leq 1/2$.

Proof. Since μ has a locally bounded potential V, it follows from the remark after Theorem 3.1 in [32] that it satisfies a weak Poincaré inequality. It also satisfies a F-Sobolev inequality with F growing to infinity, so [1, Theorem 2.11] ensures that is verifies a Poincaré inequality (actually we also need to check that the function $xF(x)$ is bounded from below; this is a consequence of (2)).

Hence we may apply Lemma 17 and replace F by $\tilde{F} = (F-a)_+$ which is non-negative. Remark that $\tilde{F}$ satisfies (1),(2) with the same γ and (3) replacing M by $M+2a$. Also remark that if μ satisfies the measure-capacity inequality (9) with $\tilde{F}$ in place of F, it holds

$$\mu^n(A)F\left(\frac{1}{\mu^n(A)}\right) \leq D\mathrm{Cap}_{\mu^n}(A) + a\mu^n(A)$$

and the second term in the right hand side is bounded by $aC\mathrm{Cap}_{\mu^n}(A)$ thanks to Poincaré inequality which is dimension free. Hence the measure-capacity inequality is still true with F and a different constant D.

So we may and will assume in the remaining of the proof that F is non-negative and vanishes on $[0,1]$.

According to [7] Theorem 22 and Remark 23, μ will satisfy a measure-capacity inequality as soon as we can find some $x_0 > 2$ such that

(i) $x \mapsto F(x)/x$ is non-increasing on $(x_0, +\infty)$,

(ii) there exists some $\lambda > 4$ such that $4F(\lambda x) \leq \lambda F(x)$ for $x \geq x_0$.

The derivative of $F(x)/x$ has the sign of $xF'(x) - F(x)$. This is non-positive for $x \geq F^{-1}(\gamma)$ thanks to (2) and (1). So Property (i) is valid when $x_0 \geq F^{-1}(\gamma)$. For (ii) just remark that $F(8x) \leq M + F(8) + F(x) \leq 2F(x)$ if $x \geq F^{-1}(M + F(8))$, thanks to (3). We may choose x_0 as the maximum of the two previous values. As explained in [7] Remark 23, we then have $\mu(A)F(1/\mu(A)) \leq K\mathrm{Cap}_\mu(A)$ if $\mu(A) \leq 1/x_0$. It follows that

$$\mu(A)F\left(\frac{2}{\mu(A)}\right) \leq \mu(A)F\left(\frac{8}{\mu(A)}\right) \leq 2\mu(A)F\left(\frac{1}{\mu(A)}\right) \leq 2K\mathrm{Cap}_\mu(A).$$

for any A with $\mu(A) \leq 1/x_0$. If $1/x_0 \leq \mu(A) \leq 1/2$, since μ satisfies Poincaré inequality, it is known (see e.g. [7] Theorem 14) that $\mu(A) \leq C\mathrm{Cap}_\mu(A)$, for some constant $C = C(x_0, \mu)$ (independent on the dimension n). Hence, in this case, $\mu(A)F(2/\mu(A)) \leq CF(2x_0)\mathrm{Cap}_\mu(A)$. Thus,

$$\mu(A)F\left(\frac{2}{\mu(A)}\right) \leq \max(2K, CF(2x_0))\mathrm{Cap}_\mu(A),$$

for all A with $\mu(A) \leq 1/2$. Theorem 26 in [7] furnishes the additive ϕ-Sobolev inequality.

We have already mentioned that additive ϕ-Sobolev inequalities tensorize. Consequently μ^n satisfies a homogeneous F-Sobolev inequality and a constant

C_ϕ independent of n (recall that we assumed $F(1) = 0$). We therefore deduce that

$$\mu^n(A)F\left(\frac{2}{\mu^n(A)}\right) \leq D_\phi \mathrm{Cap}_{\mu^n}(A),$$

for some constant D_ϕ (independent on n) and all A with $\mu^n(A) \leq 1/2$. This achieves the proof. □

Remark 6. Part of the previous Theorem is proved in a slightly different form in [31].

Remark 7. The condition $F(8) > 0$ can be relaxed just changing $F(x)$ in $F(\rho x)$ for a large enough ρ and arguing as before it will only modify the constant C_ϕ.

Remark 8. In the capacity-measure inequality (9) we may replace F by F_+ and then by $1 + F_+$ according to Poincaré inequality, changing the constant D if necessary. The latter is of course the interesting one.

As a consequence of Corollary 6, Theorem 18 and the latter Remark we thus have

Corollary 19. *Let μ and F as in Theorem 18. Then there exists a constant K such that for all n for all $f : (\mathbb{R}^d)^n \to \mathbb{R}$ and every $s \geq 1$ one has*

$$\int f^2 d\mu^n - s\left(\int |f| d\mu^n\right)^2 \leq K\beta(s)\int |\nabla f|^2 d\mu^n,$$

with $\beta(s) = 1/(1 + F_+)(s)$.

As the reader readily sees, the previous corollary is not as esthetic as the Beckner type approach for two reasons: first F has to fulfill some hypotheses, second the constant K is not explicit (the main difficulty is to get an estimate on the Poincaré constant from the weak spectral gap property). Nonetheless combined with the results in Section 4, it allows to obtain isoperimetric inequalities for Boltzmann's measures that do not enter the framework of Section 5.

In addition it is clear that we may subtract a positive constant to F without changing the result except for the constant K (just using Poincaré as before). This remark reflects the fact that Ledoux's method uses hyperboundedness and not hypercontractivity.

Finally the results extend to Riemannian manifolds since any probability measure with a locally bounded potential satisfies a local Poincaré inequality, see [32].

7 Isoperimetric inequalities and Witten Laplacian

As remarked by many authors there is a natural isometry between the Fokker-Planck operator $\Delta - \nabla V.\nabla$ and the Schrödinger operator $\Delta - U$ with $U = |\nabla V|^2 - \Delta V$. The latter was called the Witten Laplacian by B. Helffer [22],

and this correspondence is crucial in the spectral analysis of the Fokker-Planck operator and of its analogue for particle systems. This correspondence can be explained via Ito's stochastic calculus and the Girsanov transform, see e.g. [19]. The following result is a consequence of Theorem 3 and Example 14 in [7]. We recall that a Young function Φ satifies the Δ_2 condition if $\Phi(2x) \leq K\Phi(x)$ for some $K > 1$ and $x \geq x_0 \geq 0$; and the ∇_2 condition if $2\ell\Phi(x) \leq \Phi(\ell x)$ for some $\ell > 1$ and $x \geq \tilde{x}_0 \geq 0$.

Proposition 20 ([7]). *Let $d\mu = e^{-V}dx$ be a Boltzmann probability measure on $\mathbb{R}^d$. Assume that*

(i) V is C^2 bounded from below by a possibly negative constant v.

(ii) There exist $c \in \mathbb{R}$, $u_0 > 0$ and a function $G : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying Δ_2 and ∇_2 such that $G(u) \to +\infty$ as $u \to +\infty$ and $G(u)/(u+1)$ is bounded for $u \geq u_0$, and such that for all $x \in \mathbb{R}^d$,

$$|\nabla V|^2(x) - \Delta V(x) \geq G(|V(x)|) - c.$$

(iii) There exists $q > 0$ such that $\int e^{-qG(|V|)}dx < +\infty$.

Then μ satisfies a homogeneous F-Sobolev inequality for a function F satisfying $F(u) = G(\log(u))$ for $u \geq u_0$.

Corollary 21. *Let $d\mu = e^{-V}dx$ be a Boltzmann probability measure on $\mathbb{R}^d$ with V a C^2 potential such that $V \geq -R$, $\nabla\nabla V \geq -R\,\mathrm{Id}$ for some nonnegative R. Let G be a nondecreasing, C^1 function defined on $[0, +\infty[$, such that G goes to infinity at infinity. Assume in addition that $G''(u) \leq G'(u) \leq \alpha$ for all $u > 0$, and that $G(u+v) \leq M + G(u) + G(v)$ for all $u, v > 0$. If the following two conditions are satisfied*

- *$|\nabla V|^2(x) - \Delta V(x) \geq G(|V(x)|) - c$ for some c and all x,*
- *$\int e^{-qG(|V|)}dx < +\infty$ for some $q > 0$,*

then there exists $\theta > 0$ such that for all n and all measurable sets $A \subset (\mathbb{R}^d)^n$ with $\mu^n(A) \leq 1/2$ the following isoperimetric inequality holds:

$$\mu_s^n(\partial A) \geq \theta\mu^n(A)\sqrt{G\left(\log\left(\frac{1}{2\mu^n(A)}\right)\right)}.$$

The proof is immediate using Proposition 20, Corollary 19 and Theorem 12. We show in the next section that the integrability assumption can be omitted.

Remark 9. If $G(0) = 0$ and G is nondecreasing, concave and goes to infinity at infinity, then it fulfills the conditions in the latter corollary. Thus we get a more tractable sufficient condition than the one provided by Theorem 6, Proposition 7 and Theorem 43 in [7]. Since we may modify the constant c, the previous assumptions only need to be checked near infinity.

Remark 10. As for the logarithmic Sobolev inequality in [19], the previous result allows us to look at d dimensional spaces from the beginning. Nevertheless, if $d = 1$ it can be compared with the tractable condition one can get for the Beckner type inequality in Section 4.

Indeed assume that V' does not vanish near ∞ and that $V''/|V'|^2$ goes to 0 at ∞. Then the Laplace method, see e.g. [2, Corollaire 6.4.2], yields a sufficient condition for $B_+(T)$ and $B_-(T)$ in Theorem 7 to be finite, namely :

$$|V'|^2 T\left(\frac{1}{V+\log(|V'|)}\right) \geq C > 0, \tag{10}$$

near ∞. If $\log(|V'|) \ll V$ near ∞, (10) becomes $|V'|^2 \geq C/T(\frac{1}{V})$ i.e. we have the formal relation $1/T(1/u) = G(u)$.

Remark 11. We can also compare the results above with the one in Section 5 (corresponding to $2VV'' \leq (V')^2$) . Indeed if V is even and convex, Corollary 21 holds with $G(u) = \left(V' \circ V^{-1}(u)\right)^2$ for u large enough as soon as V'' is non increasing at infinity (in which case G is concave near infinity). This yields a new sufficient condition for a complete description of the isoperimetric profile of log-concave measures. The examples at the end of Section 3 enter this case.

However, there is more flexibility in Corollary 21 as the following example shows :

Example 12. Let $1 < \alpha < 2$. Let $V : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^2$ function with $V(x) = |x|^\alpha + \log(1+|x|\sin^2 x)$ when $|x| \geq \varepsilon > 0$. This potential is an unbounded perturbation of $|x|^\alpha$ and is not convex. Corollary 21 applies to V for $G(u) = (1-\varepsilon)u^{2(1-\frac{1}{\alpha})}$ for all $\varepsilon > 0$. Up to a constant this is the same G as for $|x|^\alpha$.

8 Perturbation results

We complete the picture by studying how perturbations affect our inequalities. In this section μ is non-negative measure, and $\nu = e^{-2V}\mu$ is a probability measure.

Recall a well known result by Holley and Stroock [23]: if a probability measure μ satisfies a logarithmic Sobolev inequality with constant C then ν satisfies a logarithmic Sobolev inequality with constant at most $Ce^{Osc(2V)}$, where $Osc(V) = \sup V - \min V$. The same holds true for the usual Poincaré inequality. In [39, Proposition 2.5], Wang proves a similar result for the generalized Beckner inequality in Theorem 7. According to Corollary 8 this allows to prove super-Poincaré inequalities for more measures.

When considering unbounded perturbations, bounds on the derivatives seem to be needed. Here is a general result in this direction, extending [7, Section 7.2.].

Theorem 22. *Assume that μ on (M, g) satisfies a homogeneous L-Sobolev inequality with constant C. Let F be a $\mathcal{C}^1$ function defined on $(0, +\infty)$, satisfying*

(i) $F(x)$ *tends to* $+\infty$ *when* x *goes to* $+\infty$,

(ii) There exists $A \in \mathbb{R}$ *such that* $F(xy) \leq A + F(x) + F(y)$ *for all* $x, y > 0$,

(iii) there exists $K \in \mathbb{R}$ *such that* $xF(x) \leq xL(x) + K/\mu(M)$ *for all* x. *If* $\mu(M) = +\infty$ *we decide that* $K = 0$.

If $F(e^{2V}) + C(\Delta_\mu V - |\nabla V|^2)$ *is bounded from above, then there exists a constant* B *such that* ν *satisfies a homogeneous* $(F - B)$*-Sobolev inequality. Here* Δ_μ *is an analogue of the Laplace operator for* μ *with the integration by part property*

$$\int f \, \Delta_\mu g \, d\mu = -\int \nabla f \cdot \nabla g \, d\mu.$$

If in addition $x \mapsto xF(x)$ *is convex, one can relax the previous assumption into the following one: there exists* $\varepsilon \in (0, 1)$ *such that*

$$\int H\left(\frac{1}{\varepsilon}\left(F(e^{2V}) + C(\Delta_\mu V - |\nabla V|^2)\right)\right) d\nu < +\infty,$$

where H *is the convex conjugate of* $x \mapsto xF(x)$.

Proof. First of all thanks to (ii),

$$F(g^2) = F(g^2 e^{-2V} e^{2V}) \leq A + F(e^{2V}) + F(g^2 e^{-2V}).$$

Hence if $\int g^2 d\nu = 1$ and $f = g\, e^{-V}$ (so that $\int f^2 d\mu = 1$),

$$\begin{aligned}
\int g^2 F(g^2) d\nu &\leq A + \int g^2 F(e^{2V}) d\nu + \int f^2 F(f^2) d\mu \\
&\leq (A + K) + \int g^2 F(e^{2V}) d\nu + \int f^2 L(f^2) d\mu \\
&\leq (A + K) + \int g^2 F(e^{2V}) d\nu + C \int |\nabla f|^2 d\mu \\
&\leq (A + K) + \int g^2 \left(F(e^{2V}) + C(\Delta V - |\nabla V|^2)\right) d\nu \\
&\qquad + C \int |\nabla g|^2 d\nu
\end{aligned}$$

using (iii), the L-Sobolev inequality for μ and an immediate integration by parts. Hence we get the first part of the theorem. If in addition $x \mapsto xF(x)$ is convex, we apply Young's inequality $xy \leq \varepsilon x F(x) + H(y/\varepsilon)$ and get the second part. □

Remark 13. Theorem 22 gives another proof of Corollary 21 without the final integrability condition, just taking $d\mu = dx$ and $L = \log_+$ and using e.g. the version of the Euclidean logarithmic Sobolev inequality given in [20, Theorem 2.2.4]. If F also fulfills the hypotheses of Theorem 18, we get a dimension-free super-Poincaré inequality, and the corresponding isoperimetric inequality.

Remark 14. The previous scheme of proof yields other perturbation results. Here is an example which we did not see in the literature: if U is locally bounded and $\mu = e^{-U} dx$ satisfies a logarithmic Sobolev inequality, then for $\varepsilon > 0$ small enough, the perturbed measure $\nu_\varepsilon = Z_\varepsilon^{-1} e^{2\varepsilon |x|^2} \mu$ also satisfies a logarithmic Sobolev inequality.

To see this, first recall that the log-Sobolev inequality implies Gaussian concentration of the measure. Hence for ε small enough, Z_ε is finite and ν_ε is a well defined probability measure.

Arguing as before, with $F = L = \log$, $V(x) = \frac{\log(Z_\varepsilon)}{2} - \varepsilon |x|^2$ and $\int g^2 d\nu_\varepsilon = 1$,

$$\begin{aligned}
\int g^2 \log g^2 d\nu_\varepsilon &= \int g^2 e^{-2V} \log(g^2 e^{-2V}) \, d\mu + 2 \int g^2 V \, d\nu_\varepsilon \\
&\leq C \int \left| \nabla (g e^{-V}) \right|^2 d\mu + 2 \int g^2 V \, d\nu_\varepsilon \\
&\leq 2C \int \left(|\nabla g|^2 + g^2 |\nabla V|^2 \right) \, d\nu_\varepsilon + 2 \int g^2 V \, d\nu_\varepsilon \\
&= 2C \int |\nabla g|^2 d\nu_\varepsilon + \int g^2 \Big(8C\varepsilon^2 - 2\varepsilon \Big) |x|^2 d\nu_\varepsilon + \log(Z_\varepsilon).
\end{aligned}$$

When $4C\varepsilon < 1$ the middle term is negative and we get a defective logarithmic Sobolev inequality. Since ν_ε has a locally bounded potential, it satisfies a weak Poincaré inequality, see [32]. By Aida's result [1], ν_ε has a spectral gap. This is enough to transform the defective log-Sobolev inequality into a tight one.

Similar results can be shown for more general F-Sobolev inequalities.

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