



HAL
open science

Overpartitions, lattice paths and Rogers-Ramanujan identities

Sylvie Corteel, Olivier Mallet

► **To cite this version:**

Sylvie Corteel, Olivier Mallet. Overpartitions, lattice paths and Rogers-Ramanujan identities. 2006.
hal-00017295v1

HAL Id: hal-00017295

<https://hal.science/hal-00017295v1>

Preprint submitted on 19 Jan 2006 (v1), last revised 21 Dec 2006 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

OVERPARTITIONS, LATTICE PATHS AND ROGERS-RAMANUJAN IDENTITIES

SYLVIE CORTEEL AND OLIVIER MALLET

ABSTRACT. We define the notions of successive ranks and generalized Durfee squares for overpartitions. We show how these combinatorial statistics give extensions to overpartitions of combinatorial interpretations in terms of lattice paths of the generalizations of the Rogers-Ramanujan identities due to Burge, Andrews and Bressoud. All our proofs are combinatorial and use bijective techniques. Our result includes the Andrews-Gordon identities, the generalization of the Gordon-Göllnitz identities and Gordon's theorems for overpartitions.

1. INTRODUCTION

The starting points of this work are a result of Lovejoy of 2003

Theorem 1.1 (Gordon's theorem for overpartitions). [24] *Let $\overline{B}_k(n)$ denote the number of overpartitions of n of the form $(\lambda_1, \lambda_2, \dots, \lambda_s)$, where $\lambda_\ell - \lambda_{\ell+k-1} \geq 1$ if $\lambda_{\ell+k-1}$ is overlined and $\lambda_\ell - \lambda_{\ell+k-1} \geq 2$ otherwise. Let $\overline{A}_k(n)$ denote the number of overpartitions of n into parts not divisible by k . Then $\overline{A}_k(n) = \overline{B}_k(n)$.*

and a result of Corteel and Lovejoy of 2004

Theorem 1.2. [16] *Let $\overline{D}_k(n)$ be the number of overpartitions of n whose Frobenius representation has a top row with at most $k - 2$ successive Durfee squares in its associated partition. Then $\overline{A}_k(n) = \overline{D}_k(n)$.*

An overpartition here is a partition where the final occurrence of a part can be overlined [16]. For example there exist 8 overpartitions of 3

$$(3), (\overline{3}), (2, 1), (\overline{2}, 1), (2, \overline{1}), (\overline{2}, \overline{1}), (1, 1, 1), (1, 1, \overline{1}).$$

Overpartitions have been recently heavily studied under different names and guises. They can be called joint partitions [9], or dotted partitions [11] and they are also closely related to 2-modular diagrams [27], jagged partitions [20, 21] and superpartitions [19]. Results on (for example) combinatorics of basic hypergeometric series identities [17, 31], q -series [21, 24, 25], congruences of the overpartition function [20, 28] and supersymmetric functions [19] have been discovered.

Gordon's theorem was proved in 1961 and is the following

Theorem 1.3. [23] *Let $B_{k,i}(n)$ denote the number of partitions of n of the form $(\lambda_1, \lambda_2, \dots, \lambda_s)$, where $\lambda_\ell - \lambda_{\ell+k-1} \geq 2$ and at most $i - 1$ of the parts are equal to 1.*

2000 *Mathematics Subject Classification.* Primary 11P81; Secondary 05A17.

Key words and phrases. Partitions, overpartitions, Rogers-Ramanujan identities, lattice paths.

The authors are partially supported by the ACI Jeunes Chercheurs "Partitions d'entiers à la frontière de la combinatoire, des q -séries et de la théorie des nombres."

Let $A_{k,i}(n)$ denote the number of partitions of n into parts not congruent to $0, \pm i$ modulo $2k + 1$. Then $A_{k,i}(n) = B_{k,i}(n)$.

This theorem is an extension of the famous Rogers-Ramanujan identities proved by Rogers in 1894 [30] which correspond to the cases $k = i = 2$ and $k = 2, i = 1$. It is still a well known open problem to find a natural bijective proof of these identities, even though an impressive number of nearly combinatorial proofs have been published. A recent example is [10]. Lovejoy's result can be seen as an analog of Gordon's theorem, as the conditions on the $\overline{B}_k(n)$ reduce to the conditions on the $B_{k,k}(n)$ if the overpartition has no overlined parts and is indeed a partition.

Other combinatorial interpretations related to Gordon's theorem were given by Andrews and these became the Andrews-Gordon identities :

Theorem 1.4. [4] *Let $C_{k,i}(n)$ be the number of partitions of n whose successive ranks lie in the interval $[-i + 2, 2k - i - 1]$ and let $D_{k,i}(n)$ be the number of partitions of n with $i - 1$ successive Durfee squares followed by $k - i$ successive Durfee rectangles. Then*

$$A_{k,i}(n) = B_{k,i}(n) = C_{k,i}(n) = D_{k,i}(n).$$

Details can be found in [2, Chapter 7]. It is well understood combinatorially that $B_{k,i}(n) = C_{k,i}(n) = D_{k,i}(n)$ and that result was established by some beautiful work of Burge [14, 15] using some recursive arguments. This work was reinterpreted by Andrews and Bressoud [7] who showed that Burge's argument could be rephrased in terms of binary words and that Gordon's theorem can be established thanks to these combinatorial arguments and the Jacobi Triple product identity [22]. Finally Bressoud [12] reinterpreted these in terms of ternary words and showed some direct bijections between the objects counted by $B_{k,i}(n)$, $C_{k,i}(n)$, $D_{k,i}(n)$ and the ternary words.

The purpose of this paper is therefore to extend these works [7, 12, 14, 15] to overpartitions to try to generalize both Gordon's theorem for overpartitions and the Andrews-Gordon identities.

Our main result is the following and is proved totally combinatorially:

Theorem 1.5.

- Let $\overline{B}_{k,i}(n, j)$ be the number of overpartitions of n with j overlined parts and of the form $(\lambda_1, \lambda_2, \dots, \lambda_s)$, where $\lambda_\ell - \lambda_{\ell+k-1} \geq 1$ if $\lambda_{\ell+k-1}$ is overlined and $\lambda_\ell - \lambda_{\ell+k-1} \geq 2$ otherwise and at most $i - 1$ parts are equal to 1.
- Let $\overline{C}_{k,i}(n, j)$ be the number of overpartitions of n with j non-overlined parts in the bottom row of their Frobenius representation and whose successive ranks lie in $[-i + 2, 2k - i - 1]$.
- Let $\overline{D}_{k,i}(n, j)$ be the number of overpartitions of n with j overlined parts and $i - 1$ successive Durfee squares followed by $k - i$ successive Durfee rectangles, the first one being a generalized Durfee square/rectangle.
- Let $\overline{E}_{k,i}(n, j)$ be the number of paths that use four kinds of unitary steps with special (k, i) -conditions, major index n , and j South steps.

Then $\overline{B}_{k,i}(n, j) = \overline{C}_{k,i}(n, j) = \overline{D}_{k,i}(n, j) = \overline{E}_{k,i}(n, j)$.

We use the classical q -series notations : $(a)_\infty = (a; q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i)$, $(a)_n = (a)_\infty / (aq^n)_\infty$ and $(a_1, \dots, a_k; q)_\infty = (a_1; q)_\infty \dots (a_k; q)_\infty$. The generating function $\overline{\mathcal{E}}_{k,i}(a, q) = \sum_{n,j} \overline{E}_{k,i}(n, j) q^n a^j$ is :

Theorem 1.6.

$$(1.1) \quad \bar{\mathcal{E}}_{k,i}(a, q) = \frac{(-aq)_\infty}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n a^n q^{kn^2+(k-i+1)n} \frac{(-1/a)_n}{(-aq)_n}.$$

In some cases, we can use the Jacobi Triple Product identity [22]:

$$(-1/z, -zq, q; q)_\infty = \sum_{n=-\infty}^{\infty} z^n q^{\binom{n+1}{2}}$$

and show that this generating function has a very nice form. For example,

Corollary 1.1.

$$(1.2) \quad \bar{\mathcal{E}}_{k,i}(0, q) = \frac{(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty}$$

$$(1.3) \quad \bar{\mathcal{E}}_{k,i}(1/q, q^2) = \frac{(q^2; q^4)_\infty (q^{2i-1}, q^{4k+1-2i}, q^{4k}; q^{4k})_\infty}{(q)_\infty}$$

$$(1.4) \quad \bar{\mathcal{E}}_{k,i}(1, q) = \frac{(-q)_\infty}{(q)_\infty} \sum_{j=0}^{2(k-i)} (-1)^j (q^{i+j}, q^{2k-i-j}, q^{2k}; q^{2k})_\infty$$

$$(1.5) \quad \bar{\mathcal{E}}_{k,i}(1/q, q) = \frac{(-q)_\infty}{(q)_\infty} ((q^i, q^{2k-i}, q^{2k}; q^{2k})_\infty + (q^{i-1}, q^{2k+1-i}, q^{2k}; q^{2k})_\infty)$$

Hence our result gives a general view of different problems on partitions and overpartitions and shows how they are related.

- Equation (1.2): The case $a \rightarrow 0$ corresponds to the Andrews-Gordon identities [4].
- Equation (1.3) : The case $q \rightarrow q^2$ and $a \rightarrow 1/q$ corresponds to Andrews's generalization of the Gordon-Göllnitz identities [5, 7].
- Equations (1.4) and (1.5) : The cases $a \rightarrow 1$ and $i = k$ and $a \rightarrow 1/q$ and $i = 1$ correspond to the two Gordon's theorems for overpartitions of Lovejoy [24].

Therefore our extension of the work on the Andrews-Gordon identities [7, 12, 14, 15] to the case of overpartitions includes these identities, but it also includes Andrews's generalization of the Gordon-Göllnitz identities and Gordon's theorems for overpartitions. We state in details the result for the Gordon-Göllnitz identities (Case $q \rightarrow q^2$ and $a \rightarrow 1/q$).

Theorem 1.7. *All the following are equal :*

- *The number of partitions of n with parts not congruent to $2 \pmod{4}$ and $0, \pm(2i-1) \pmod{4k}$;*
- *The number of partitions of n of the form $(\lambda_1, \lambda_2, \dots, \lambda_s)$ with unpeated odd parts, where $\lambda_\ell - \lambda_{\ell+k-1} \geq 3$ if $\lambda_{\ell+k-1}$ is even and 2 otherwise;*
- *The number of 2-modular diagrams of n whose successive ranks lie in $[-i+2, 2k-i-1]$;*
- *The number of 2-modular diagrams of n with $i-1$ successive Durfee squares followed by $k-i$ successive Durfee rectangles, the first one being a generalized Durfee square/rectangle.*

- *The number of paths that use four kinds of unitary steps with special (k, i) -conditions where n is the sum of twice the abscissa of each peak minus the sum the number of South steps to the left of each peak minus the number of South steps.*

Finally it also gives new partition theorems, that can be seen as generalizations of Gordon's theorem for overpartitions [24]. They are the combinatorial interpretation of Equations (1.4) and (1.5). Superpartitions [19] are overpartitions where the first occurrence of a part can be overlined and the part $\bar{0}$ can appear. Let $\bar{B}_{k,i}(n)$ be the number of overpartitions of n of the form $(\lambda_1, \lambda_2, \dots, \lambda_s)$, where $\lambda_\ell - \lambda_{\ell+k-1} \geq 1$ if $\lambda_{\ell+k-1}$ is overlined and $\lambda_\ell - \lambda_{\ell+k-1} \geq 2$ otherwise and at most $i - 1$ parts are equal to 1.

Theorem 1.8. *For $1 \leq i \leq k - 1$, the number of overpartitions counted by $\bar{B}_{k,i}(n)$ plus the number of overpartitions counted by $\bar{B}_{k,i+1}(n)$ is equal to the number of superpartitions where the non-overlined parts are not congruent to $0, \pm i$ modulo $2k$.*

Theorem 1.9. *For $2 \leq i \leq k - 1$, the number of superpartitions of n of the form $(\lambda_1, \lambda_2, \dots, \lambda_s)$, where $\lambda_\ell - \lambda_{\ell+k-1} \geq 1$ if λ_j is overlined and $\lambda_\ell - \lambda_{\ell+k-1} \geq 2$ otherwise and where the number of non-overlined ones plus the number of $\bar{0}$ is at most $i - 1$ is equal to the number of overpartitions of n where the non-overlined parts are not congruent to $0, \pm i$ modulo $2k$ plus the number of overpartitions of n where the non-overlined parts are not congruent to $0, \pm i - 1$ modulo $2k$.*

We start by some definitions in Section 2. In Section 3 we present the paths counted by $\bar{E}_{k,i}(n, j)$ and compute the generating function. In Section 4 we present a direct bijection between the paths counted by $\bar{E}_{k,i}(n, j)$ and the overpartitions counted by $\bar{C}_{k,i}(n, j)$. In Section 5 we present a recursive bijection between the paths counted by $\bar{E}_{k,i}(n, j)$ and the overpartitions counted by $\bar{B}_{k,i}(n, j)$. We also give a generating function proof. In Section 6, we present a combinatorial argument that shows that the paths counted by $\bar{E}_{k,i}(n, j)$ and the overpartitions counted by $\bar{D}_{k,i}(n, j)$ are in bijection. All these bijections are refinements of Theorem 1.5. The number of the peaks of the paths will correspond respectively to the number of columns of the Frobenius representations, the total weight of the "multiples" and the size of the generalized Durfee square. We conclude in Section 7 with open further questions.

2. DEFINITIONS ON OVERPARTITIONS

We will define all the notions in terms of overpartitions. We refer to [2] for definitions for partitions. In all of the cases the definitions coincide when the overpartition has no overlined parts.

An overpartition of n is a non-increasing sequence of natural numbers whose sum is n in which the final occurrence (equivalently, the first occurrence) of a number may be overlined. Alternatively n can be called the weight of the overpartition. Since the overlined parts form a partition into distinct parts and the non-overlined parts form an ordinary partition, the generating function of overpartitions is $\frac{(-q)_\infty}{(q)_\infty}$. The Ferrers diagram of an overpartition is a classical Ferrers diagram where the corners can be marked (see Figure 1). A 2-modular diagram is a Ferrers diagram of an overpartition where the marked corners are filled with ones and the other cells are filled with twos (see Figure 2). The weight is the sum of the entries.

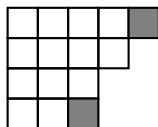
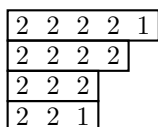
FIGURE 1. Ferrers diagram of the overpartition $\lambda = (\overline{5}, 4, 3, \overline{3})$.

FIGURE 2. Example of a 2-modular diagram.

The *multiplicity* of the part j of an overpartition, denoted by f_j , is the number of occurrences of this part. We overline the multiplicity if the part appears overlined. For example, the multiplicity of the part 4 in the overpartition $(6, 6, 5, 4, 4, \overline{4}, 3, \overline{1})$ is $f_4 = \overline{3}$. The multiplicity sequence is the sequence (f_1, f_2, \dots) . For example the previous overpartition has multiplicity sequence $(\overline{1}, 0, 1, \overline{3}, 1, 2)$.

The Frobenius representation of an overpartition [16, 26] of n is a two-rowed array

$$\begin{pmatrix} a_1 & a_2 & \dots & a_N \\ b_1 & b_2 & \dots & b_N \end{pmatrix}$$

where (a_1, \dots, a_N) is a partition into distinct nonnegative parts and (b_1, \dots, b_N) is an overpartition into nonnegative parts where the first occurrence of a part can be overlined and $N + \sum(a_i + b_i) = n$.

We now define the successive ranks.

Definition 2.1. The *successive ranks* of an overpartition can be defined from its Frobenius representation. If an overpartition has Frobenius representation

$$\begin{pmatrix} a_1 & a_2 & \dots & a_N \\ b_1 & b_2 & \dots & b_N \end{pmatrix}$$

then its i th successive rank r_i is $a_i - b_i$ minus the number of non-overlined parts in $\{b_{i+1}, \dots, b_N\}$.

This definition is an extension of Lovejoy's definition of the rank [26]. For example, the successive ranks of $\begin{pmatrix} 7 & 4 & 2 & 0 \\ \overline{3} & 3 & 1 & \overline{0} \end{pmatrix}$ are $(2, 0, 1, 0)$.

We say that the *generalized Durfee square* of an overpartition λ has side N if N is the largest integer such that the number of overlined parts plus the number of non-overlined parts greater or equal to N is greater than or equal to N (see Figure 3).

Proposition 2.1. *There exists a bijection between overpartitions whose Frobenius representation has N columns and whose bottom line has j overlined parts and overpartitions with generalized Durfee square of size N and $N - j$ overlined parts.*

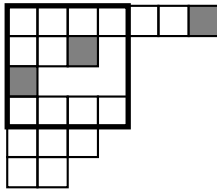


FIGURE 3. The generalized Durfee square of $\lambda = (\overline{7}, 4, 3, \overline{3}, 2, \overline{1})$ has side 4.

Proof. An overpartition with generalized Durfee square of size N can be decomposed into a partition α into at most N parts and an overpartition γ into N parts whose non-overlined parts are $\geq N$. An overpartition whose Frobenius representation has N columns can be decomposed into a partition β into N distinct parts (β is obtained by adding 1 to each part of the top line), a partition δ into distinct parts which lie between 0 and $N - 1$ and a partition ε into at most N parts. δ and ε are obtained from the bottom line as follows : we first initialize ε to the bottom line, then if the i th part of the bottom line is overlined, we take off its overline, we decrease the first $(i - 1)$ parts of ε by 1 and add a part $i - 1$ to δ .

Now there exists a bijection between ordered pairs (β, δ) and overpartitions γ defined previously. This bijection is defined as follows : we overlined all the parts of β , then for each part i in δ , we add i to β_{i+1} and we remove the overlining. We then reorder the parts, which gives us γ . This is similar to the Algorithm Z [8].

The decompositions of the first paragraph and the bijection of the second paragraph show that there is indeed a bijection between overpartitions whose Frobenius representation has N columns and overpartition with generalized Durfee square of size N . If there are j overlined parts in the bottom line of the Frobenius symbol, there are j parts in δ and by the bijection, there are $N - j$ overlined parts in γ . The proposition is thus established. \square

The generating function of overpartitions with generalized Durfee square of size N where the exponent of q counts the weight and the exponent of a the number of overlined parts is

$$\frac{a^N q^{\binom{N+1}{2}} (-1/a)_N}{(q)_N (q)_N}.$$

Definition 2.2. The *successive Durfee squares* of an overpartition are its generalized Durfee square and the successive Durfee squares of the partition below the generalized Durfee square, if we represent the partition as in Figure 3, with the overlined parts above the non-overlined ones. We can also define similarly the *successive Durfee rectangles* by dissecting the overpartition with $d \times (d + 1)$ -rectangles instead of squares. In this case, we also impose the condition that the partition on the right of a $d \times (d + 1)$ -rectangle cannot have more than d parts.

These definitions imply that

$$(2.1) \quad \sum_{n_1 \geq \dots \geq n_{k-1} \geq 0} \frac{q^{\binom{n_1+1}{2} + n_i + \dots + n_{k-1}} (-1/a)_{n_1} a^{n_1}}{(q)_{n_1}} \times \left(q^{n_2^2} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \right) \left(q^{n_3^2} \begin{bmatrix} n_2 \\ n_3 \end{bmatrix}_q \right) \dots \left(q^{n_{k-1}^2} \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q \right)$$

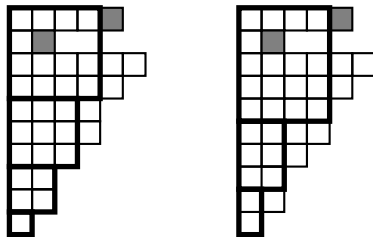


FIGURE 4. Successive Durfee squares and successive Durfee rectangles of $(6, 5, \bar{5}, 4, 4, 3, 2, 2, \bar{2}, 1)$.



FIGURE 5. This path has four peaks : two NES peaks (located at $(2, 2)$ and $(6, 1)$) and two NESE peaks (located at $(4, 1)$ and $(7, 1)$). Its major index is $2 + 4 + 6 + 7 = 19$.

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q)_n}{(q)_k (q)_{n-k}}$$

is the generating function of overpartitions with $i - 1$ successive Durfee squares followed by $k - i$ successive Durfee rectangles, the first one being a generalized Durfee square/rectangle.

3. PATHS AND GENERATING FUNCTION

This part is an extension of papers of Andrews and Bressoud [7, 12] based on ideas of Burge [14, 15]. We study paths in the first quadrant that use four kinds of unitary steps :

- North-East $NE : (x, y) \rightarrow (x + 1, y + 1)$,
- South-East $SE : (x, y) \rightarrow (x + 1, y - 1)$,
- South $S : (x, y) \rightarrow (x, y - 1)$,
- East $E : (x, 0) \rightarrow (x + 1, 0)$.

The *height* corresponds to the y -coordinate. A South step can only appear after a North-East step and an East step can only appear at height 0. The paths must end with a North-East or South step. A *peak* is a vertex preceded by a North-East step and followed by a South step (in which case it will be called a *NES peak*) or by a South-East step (in which case it will be called a *NESE peak*). If the path ends with a North-East step, its last vertex is also a NESE peak. The *major index* of a path is the sum of the x -coordinates of its peaks (see Figure 5 for an example). When the paths have no South steps, this is the definition of the paths in [12].

We say that a path satisfies the *special (k, i) -conditions* if it starts at height $k - i$ and its height is less than k . Let $\bar{E}_{k,i}(n, j, N)$ be the number of such paths of major index n with N peaks, j South steps which satisfy the special (k, i) -conditions. Let $\bar{\mathcal{E}}_{k,i}(N)$ be the generating function of those paths, that is $\bar{\mathcal{E}}_{k,i}(N) = \bar{\mathcal{E}}_{k,i}(N, a, q) = \sum_{n,j} \bar{E}_{k,i}(n, j, N) a^j q^n$.

Then

Proposition 3.1.

$$\begin{aligned}
\bar{\mathcal{E}}_{k,i}(N) &= q^N \bar{\mathcal{E}}_{k,i+1}(N) + q^N \bar{\Gamma}_{k,i-1}; & i < k \\
\bar{\Gamma}_{k,i}(N) &= q^N \bar{\Gamma}_{k,i-1}(N) + (a + q^{N-1}) \bar{\mathcal{E}}_{k,i+1}(N-1); & i > 0 \\
\bar{\mathcal{E}}_{k,k}(N) &= \frac{q^N}{1 - q^N} \bar{\Gamma}_{k,k-1}(N) \\
\bar{\mathcal{E}}_{k,i}(0) &= 1 & \bar{\Gamma}_{k,0}(N) = 0
\end{aligned}$$

Proof. We prove that by induction on the length of the path. If the path is empty, then its major index is 0 and $N = 0$. Moreover if $N = 0$ the only path counted in $\bar{\mathcal{E}}_{k,i}(0)$ is the empty path. If the path is not empty, then we take off its first step. If $i < k$, then a path counted in $\bar{\mathcal{E}}_{k,i}(N)$ starts with a North-East (defined by $q^N \bar{\Gamma}_{k,i-1}(N)$) or a South-East step ($q^N \bar{\mathcal{E}}_{k,i+1}(N)$). If $i > 0$, $\bar{\Gamma}_{k,i}(N)$ is the generating function of paths counted in $\bar{\mathcal{E}}_{k,i+1}(N)$ where the first North-East step was deleted. These paths can start with a North-East step ($q^N \bar{\Gamma}_{k,i-1}(N)$), a South step ($a \bar{\mathcal{E}}_{k,i+1}(N-1)$) or a South-East step ($q^{N-1} \bar{\mathcal{E}}_{k,i+1}(N-1)$). If $i = k$ then a path counted in $\bar{\mathcal{E}}_{k,k}(N)$ starts with a North-East ($q^N \bar{\Gamma}_{k,k-1}(N)$) or an East step ($q^N \bar{\mathcal{E}}_{k,k}(N)$). The height of the paths is less than k , therefore no path which starts at height $k-1$ can start with a North-East step and $\bar{\Gamma}_{k,0}(N) = 0$. \square

These recurrences uniquely define the series $\bar{\mathcal{E}}_{k,i}(N)$ and $\bar{\Gamma}_{k,i}(N)$. We get that :

Theorem 3.1.

$$\begin{aligned}
\bar{\mathcal{E}}_{k,i}(N) &= a^N q^{\binom{N+1}{2}} (-1/a)_N \sum_{n=-N}^N (-1)^n \frac{q^{kn^2+n(k-i)-\binom{n}{2}}}{(q)_{N-n} (q)_{N+n}} \\
\bar{\Gamma}_{k,i}(N) &= a^N q^{\binom{N}{2}} (-1/a)_N \sum_{n=-N}^{N-1} (-1)^n \frac{q^{kn^2+n(k-i)-\binom{n+1}{2}}}{(q)_{N-n-1} (q)_{N+n}}
\end{aligned}$$

The proof is omitted. It uses simple algebraic manipulation to prove that these generating functions satisfy the recurrence relations of Proposition 3.1.

We just need a small proposition to prove Theorem 1.6.

Proposition 3.2. *For any $n \in \mathbb{Z}$*

$$\sum_{N \geq |n|} \frac{(-azq)_n (-q^n/a)_{N-n} q^{\binom{N+1}{2}-\binom{n+1}{2}} z^{N-n} a^{N-n}}{(zq)_{N+n} (q)_{N-n}} = \frac{(-azq)_\infty}{(zq)_\infty}.$$

Proof. We present an analytical proof and a combinatorial one.

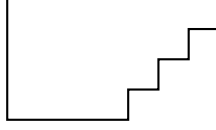


FIGURE 6. “Trapezoid” of bases n and N and height $N - n$ with $N = 7$ and $n = 3$.

- Analytical proof :

$$\begin{aligned}
& \sum_{N=n}^{\infty} \frac{(-azq)_n (-q^n/a)_{N-n} q^{\binom{N+1}{2} - \binom{n+1}{2}} z^{N-n} a^{N-n}}{(zq)_{N+n} (q)_{N-n}} \\
&= \sum_{N=0}^{\infty} \frac{(-azq)_n (-q^n/a)_N q^{\binom{N+n+1}{2} - \binom{n+1}{2}} z^N a^N}{(q)_N (zq)_{N+2n}} \\
&= \frac{(-azq)_n}{(zq)_{2n}} \sum_{N=0}^{\infty} \frac{q^{Nn + \binom{N+1}{2}} (-q^n/a)_N z^N a^N}{(q)_N (zq^{2n+1})_N} \\
&= \frac{(-azq)_n}{(zq)_{2n}} \frac{(-azq^{n+1})_{\infty}}{(zq^{2n+1})_{\infty}}
\end{aligned}$$

by Corollary 2.4 of [2] with $a \rightarrow zq$, $b \rightarrow a$ and $c \rightarrow 0$

$$= \frac{(-azq)_{\infty}}{(zq)_{\infty}}$$

- Combinatorial proof : the factor $(-azq)_n$ corresponds to the overlined parts $\leq n$ and the factor $\frac{1}{(zq)_{N+n}}$ corresponds to the non-overlined parts $\leq N + n$.

The remaining factor corresponds to an overpartition into $N - n$ parts whose overlined parts are $> n$ and whose non-overlined parts are $\geq N + n$. To prove this, let us show that there exists a bijection between such overpartitions and triples $(\varepsilon, \zeta, \eta)$ where ε is a “trapezoid” (see Figure 6) of bases n and N and height $N - n$ (factor $q^{\binom{N+1}{2} - \binom{n+1}{2}} z^{N-n}$), ζ is a partition into distinct parts which lie between n and $N - 1$ (factor $(-q^n/a)_{N-n}$) and η is a partition into at most $N - n$ parts (factor $\frac{1}{(q)_{N-n}}$). This bijection (similar to the Algorithm Z [8]) is defined as follows : we add to ε the parts in η and we overline all the parts of the partition θ we obtain (which gives the factor a^{N-n}), then for all part $n + i$ in ζ , we add $n + i$ to θ_{i+1} and we remove the overlining. We then reorder the parts.

Let us show that the decomposition used in this proof is unique. Let n be a fixed integer. N is the greatest integer such that the sum of the number of overlined parts $> n$ and of the number of non-overlined parts $\geq N + n$ is $\geq N - n$. By definition of N , if it exists, it is unique. But $N = n$ satisfies the condition : the sum of the number of overlined parts $> n$ and of the number of non-overlined parts $\geq 2n$ is indeed ≥ 0 . Therefore, N exists and it is unique.

Note that if $n = 0$, we get the generalized Durfee square.

□

Summing on N using the previous proposition we get

$$\sum_{N \geq 0} \bar{\mathcal{E}}_{k,i}(N) = \frac{(-aq)_\infty}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n a^n q^{kn^2+(k-i+1)n} \frac{(-1/a)_n}{(-aq)_n}.$$

This is equation (1.1) of Theorem 1.6.

4. PATHS AND SUCCESSIVE RANKS

This section is a generalization of Bressoud's correspondence for partitions presented in [12] and based on ideas of Burge [14, 15]. The aim of this section is the following:

Proposition 4.1. *There exists a one-to-one correspondence between the paths of major index n with j south steps counted by $\bar{E}_{k,i}(n, j)$ and the overpartitions of n with j non-overlined parts in the bottom line of their Frobenius representation and whose successive ranks lie in $[-i + 2, 2k - i - 1]$ counted by $\bar{C}_{k,i}(n, j)$. This correspondence is such that the paths have N peaks if and only if the Frobenius representation of the overpartition has N columns.*

The proof of this proposition can be done with a recursive argument. We can indeed show that $\bar{C}_{k,i}(N)$ the generating function of overpartitions whose Frobenius representation has N columns and whose successive ranks lie in $[-i + 2, 2k - i - 1]$ follows the same recurrences as $\bar{E}_{k,i}(N)$. This is done in [29]. We propose here a direct mapping that is a generalisation of [12].

Given a lattice path which starts at $(0, a)$ and a peak (x, y) , we map this peak to the pair (s, t) where

$$\begin{aligned} s &= (x + a - y + u)/2 \\ t &= (x - a + y - 2 - u)/2 \end{aligned}$$

if there are an even number of East steps to the left of the peak (we then say that the peak is of *type 0*), and

$$\begin{aligned} s &= (x + a + y - 1 + u)/2 \\ t &= (x - a - y - 1 - u)/2 \end{aligned}$$

if there are an odd number of East steps to the left of the peak (we then say that the peak is of *type 1*). Moreover, we overline t if the peak is a NESE peak. The parameter u is the number of South steps to the left of the peak. In both cases, s and t are integers and we have $s + t + 1 = x$. In the case of partitions treated in [12], u is always 0.

Let N be the number of peaks in the path and j the number of South steps of the paths. Let (x_i, y_i) be the coordinates of the i th peak from the right and (s_i, t_i) be the corresponding pair.

Proposition 4.2. *The sequence (s_1, s_2, \dots, s_N) is a partition into distinct non-negative parts and the sequence (t_1, t_2, \dots, t_N) is an overpartition into nonnegative parts with j non-overlined parts.*

Proof. We need to prove the following results :

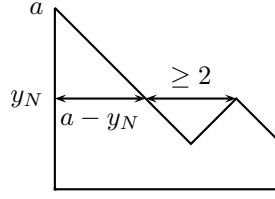


FIGURE 7. We show that $x_N \geq 2 + a - y_N$ in the case $y_N \leq a$.

- *The s_i are nonnegative.* We consider a peak of type 0. It is sufficient to prove that $x - y \geq -a$ since $u \geq 0$. It is obvious that any vertex has a greater (or equal) value of $x - y$ than the previous vertex in the path. Since the path begins at $(0, a)$, we have $x - y = -a$ at the beginning of the path and thus we have $x - y \geq -a$ for all vertices. Now we consider a peak of type 1. For all i , we have $x_i + y_i \geq x_{i+1} + y_{i+1}$ and $u_i \geq u_{i+1}$, so $x_i + y_i + u_i > x_{i+1} + y_{i+1} + u_{i+1}$. Let p be the leftmost peak of type 1 in the path. We have $x_p \geq a + 1 + y_p + u_p$ (see Figure 8), $y_p \geq 1$, $u_p \geq 0$ and thus $s_p \geq 0$. Hence all the s_i are nonnegative for peaks of type 1.
- *The t_i are nonnegative.* We consider a peak of type 0. We remark that $x_i + y_i - u_i \geq x_{i+1} + y_{i+1} - u_{i+1}$ (remember that the peaks are counted from the right). For $i = N$, we have $u_N = 0$, $x_N \geq 2 + a - y_N$ (if $a - y_N < 0$, this is clear since $x_N \geq 1$; for the case $y_N \leq a$, see Figure 7) and thus $x_N + y_N - u_N \geq a + 2$. Hence t_N is nonnegative and so are all the t_i for peaks of type 0. Now we consider a peak of type 1. We have $x_i - y_i - u_i \geq x_{i+1} - y_{i+1} - u_{i+1}$ for all i . Since we have $x_p \geq a + 1 + y_p + u_p$, we get that $t_p \geq 0$.
- *The sequence s is a partition into distinct parts.* We need to prove that for all i , $s_i - s_{i+1} > 0$. If the peaks i and $i + 1$ are both of type 0, it is clearly true since $x_i - y_i > x_{i+1} - y_{i+1}$ (two peaks cannot have the same value of $x - y$), and $u_i \geq u_{i+1}$ (remember that u_i is the number of NES peaks to the left of the i th peak). If i and $i + 1$ are both of type 1, the result comes from the fact that $x_i + y_i + u_i > x_{i+1} + y_{i+1} + u_{i+1}$. If i is of type 0 and $i + 1$ is of type 1, we have $s_i - s_{i+1} = \frac{1}{2}(x_i - x_{i+1} - y_i - y_{i+1} + u_i - u_{i+1} + 1)$. Since $u_i \geq u_{i+1}$, it is sufficient to prove that $x_i - x_{i+1} - y_i - y_{i+1} \geq 0$. This comes from the fact that there is an East step between i and $i + 1$ (see Figure 9). If i is of type 1 and $i + 1$ is of type 0, the proof is similar.
- *The sequence t is an overpartition (where the first occurrence of a part can be overlined).* We need to prove that for all i , $t_i - t_{i+1} > 0$ if t_{i+1} is overlined and $t_i - t_{i+1} \geq 0$ otherwise. The fact that we always have $t_i - t_{i+1} \geq 0$ is proved in the same way as with the s_i . If $i + 1$ is overlined, then it corresponds to a NESE peak, so we have $x_i - x_{i+1} \geq 2$ and $u_{i+1} = u_i$. By considering the expression of $t_i - t_{i+1}$ in the four cases (i and $i + 1$ are of type 0, i and $i + 1$ are of type 1, i is of type 0 and $i + 1$ of type 1, i is of type 1 and $i + 1$ of type 0), the result is easily shown.

□

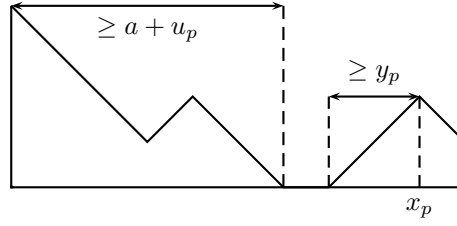


FIGURE 8. If p is the leftmost peak of type 1, $x_p \geq a + 1 + y_p + u_p$.

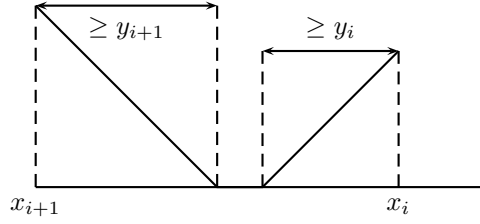


FIGURE 9. If i is a peak of type 0 and $i + 1$ is a peak of type 1, we have $x_i - x_{i+1} \geq y_i + y_{i+1}$.

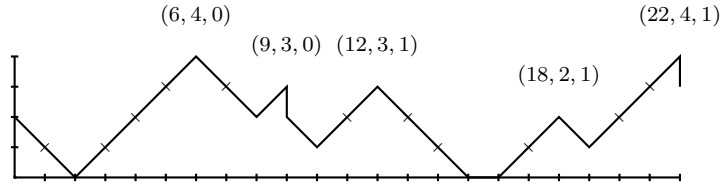


FIGURE 10. Illustration of the correspondence between paths and successive ranks. The values of x , y and u are given for each peak.

Therefore $\begin{pmatrix} s_1 & s_2 & \cdots & s_N \\ t_1 & t_2 & \cdots & t_N \end{pmatrix}$ is the Frobenius representation of an overpartition whose weight is

$$\sum_{i=1}^N (s_i + t_i + 1) = \sum_{i=1}^N x_i$$

i.e. the major index of the corresponding path.

As an example, the path in Figure 10 corresponds to the overpartition

$$\begin{pmatrix} 14 & 11 & 6 & 4 & 2 \\ 7 & \bar{6} & \bar{5} & 4 & \bar{3} \end{pmatrix}.$$

The peaks all have height at least one, thus for a peak (x, y) which is preceded by an even number of East steps, we have :

$$\begin{aligned} 1 &\leq y = a + 1 + t - s + u \leq k - 1 \\ \Leftrightarrow a - k + 2 &\leq s - t - u \leq a \\ \Leftrightarrow \text{the corresponding successive rank is } &\geq a - k + 2 \text{ and } \leq a \end{aligned}$$

and if the peak is preceded by an odd number of East steps, we have :

$$\begin{aligned} 1 \leq y = s - t - u - a &\leq k - 1 \\ \Leftrightarrow a + 1 \leq s - t - u &\leq k + a - 1 \\ \Leftrightarrow \text{the corresponding successive rank is } &\geq a + 1 \text{ and } \leq k + a - 1 \end{aligned}$$

Thus, given a Frobenius representation of an overpartition and a nonnegative integer a , there is a unique corresponding path which starts at $(0, a)$.

In our paths, $a = k - i$, therefore in the first case the successive rank $r \in [-i + 2, k - i]$ and in the second case $r \in [k - i + 1, 2k - i - 1]$.

The map is easily reversible. This proves Proposition 4.1.

5. PATHS AND MULTIPLICITIES

Recall that $\overline{B}_{k,i}(n, j)$ is the number of overpartitions λ of n with j overlined parts such that for all ℓ ,

$$\left\{ \begin{array}{l} \lambda_\ell - \lambda_{\ell+k-1} < \begin{cases} 1 & \text{if } \lambda_{\ell+k-1} \text{ is overlined} \\ 2 & \text{otherwise} \end{cases} \\ f_1 < i \end{array} \right.$$

or equivalently

$$\left\{ \begin{array}{l} \forall \ell, f_\ell + f_{\ell+1} \leq \begin{cases} k + 1 & \text{if a part } \ell \text{ is overlined} \\ k & \text{otherwise} \end{cases} \\ f_1 < i \end{array} \right.$$

The aim of this section is the following:

Proposition 5.1. *There exists a one-to-one correspondence between the paths counted by $\overline{E}_{k,i}(n, j)$ and the overpartitions counted by $\overline{B}_{k,i}(n, j)$. This correspondence is such that the paths have N peaks if and only if the total weight of the “multiples” of the overpartition is N .*

We will first give a generating function proof of that proposition (without the refinement). Then we will give a combinatorial proof which is a generalization of Burge’s correspondence for partitions presented in [14].

5.1. A generating function proof. Let $\overline{B}_{k,i}(a, q) = \sum_{n \geq 0} \overline{B}_{k,i}(n, j) a^j q^n$. We prove that

Proposition 5.2.

$$\overline{B}_{k,i}(a, q) = \overline{E}_{k,i}(a, q)$$

Proof. We generalize Lovejoy’s proof of Theorem 1.1 of [24]. Let

$$\begin{aligned} J_{k,i}(a, x, q) &= H_{k,i}(a, xq, q) - axqH_{k,i-1}(a, xq, q) \\ H_{k,i}(a, x, q) &= \sum_{n=0}^{\infty} \frac{x^{kn} q^{kn^2+n-in} a^n (1-x^i q^{2ni})(axq^{n+1})_{\infty} (1/a)_n}{(q)_n (xq^n)_{\infty}}. \end{aligned}$$

Andrews showed in [2, p. 106-107] that for $2 \leq i \leq k$,

$$\begin{aligned} J_{k,i}(a, x, q) - J_{k,i-1}(a, x, q) &= (xq)^{i-1} (J_{k,k-i+1}(a, xq, q) - aJ_{k,k-i+2}(a, xq, q)) \\ J_{k,1}(a, x, q) &= J_{k,k}(a, x, q). \end{aligned}$$

This functional equation of $J_{k,i}(a, x, q)$ implies that

$$\bar{\mathcal{B}}_{k,i}(a, q) = J_{k,i}(-a, 1, q).$$

Hence

$$\begin{aligned} \bar{\mathcal{B}}_{k,i}(a, q) &= \frac{(-aq)_\infty}{(q)_\infty} \sum_{n=0}^{\infty} (-1)^n a^n \frac{q^{kn^2+n(k-i+1)} (-1/a)_n (1 - q^{(2n+1)i})}{(-aq)_{n+1}} \\ &\quad + aq \frac{(-aq)_\infty}{(q)_\infty} \sum_{n=0}^{\infty} (-1)^n a^n \frac{q^{kn^2+n(k-i+2)} (-1/a)_n (1 - q^{(2n+1)(i-1)})}{(-aq)_{n+1}} \\ &= \frac{(-aq)_\infty}{(q)_\infty} \left(\sum_{n=0}^{\infty} (-1)^n a^n \frac{q^{kn^2+n(k+1)} (-1/a)_n (q^{-in} + aq^{1-(i-1)n})}{(-aq)_{n+1}} \right. \\ &\quad \left. - \sum_{n=0}^{\infty} (-1)^n a^n \frac{q^{kn^2+n(k+1)} (-1/a)_n (q^{(n+1)i} + aq^{(n+1)(i-1)+1})}{(-aq)_{n+1}} \right) \\ &= \frac{(-aq)_\infty}{(q)_\infty} \left(\sum_{n=0}^{\infty} (-1)^n a^n \frac{q^{kn^2+n(k+1-i)} (-1/a)_n}{(-aq)_n} \right. \\ &\quad \left. - \sum_{n=0}^{\infty} (-1)^n a^{n+1} \frac{q^{kn^2+n(k+i)+i} (-1/a)_{n+1}}{(-aq)_{n+1}} \right) \\ &= \frac{(-aq)_\infty}{(q)_\infty} \left(\sum_{n=0}^{\infty} (-1)^n a^n \frac{q^{kn^2+n(k+1-i)} (-1/a)_n}{(-aq)_n} \right. \\ &\quad \left. + \sum_{n=-\infty}^{-1} (-1)^n a^{-n} \frac{q^{kn^2+n(k-i)} (-1/a)_{-n}}{(-aq)_{-n}} \right) \\ &= \frac{(-aq)_\infty}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n a^n \frac{q^{kn^2+n(k+1-i)} (-1/a)_n}{(-aq)_n} \\ &= \bar{\mathcal{E}}_{k,i}(a, q) \end{aligned}$$

□

5.2. A combinatorial proof. This part is a generalization of [14, Section 3]. Like Burge, we define operations on overpartitions represented by their multiplicity sequence.

The operation α is defined as follows. We divide the overpartition into tuples (which we call *multuples* i.e. “tuples of multiplicities”) of the form $(f_m, \dots, f_{m+\ell})$ with $\ell \geq 1$ starting at the smallest part. When we find a multiplicity $f_m > 0$, we open a parenthesis to its left. If f_m is not overlined then we close the parenthesis to the right of f_{m+1} . Otherwise, we look for the next non-overlined multiplicity, say f_p . If $f_p = 0$ then we close the parenthesis to its right, otherwise we close the parenthesis to the right of f_{p+1} . Then we look for the next positive multiplicity, and so on. Finally, for each $(\ell + 1)$ -tuple $(f_m, \dots, f_{m+\ell})$, we do :

- $f_m \leftarrow f_m - 1$
- $f_{m+\ell} \leftarrow f_{m+\ell} + 1$
- if f_m is overlined, we remove its overlining and we overline the smallest non-overlined multiplicity in the $(\ell + 1)$ -tuple.

The operation β (resp. δ) consists in setting $f_0 = 1$ (resp. $f_0 = \bar{1}$) and applying α .

The inverse operation α^{-1} is performed by first dividing the overpartition into $(\ell + 1)$ -tuples of the form $(f_m, \dots, f_{m+\ell})$, with $\ell \geq 1$ starting at the largest part, such that :

- $f_{m+\ell} > 0$
- f_m is not overlined
- f_{m+p} is overlined for $1 \leq p \leq \ell - 1$

and then doing for each $(\ell + 1)$ -tuple :

- if $f_{m+\ell} = \bar{1}$:
 - remove the overlining of $f_{m+\ell}$
 - underline f_m
- else if $\ell > 1$:
 - remove the overlining of $f_{m+\ell-1}$
 - underline f_m
- $f_{m+\ell} \leftarrow f_{m+\ell} - 1$
- $f_m \leftarrow f_m + 1$

If there is an $(\ell + 1)$ -tuple (f_0, \dots, f_ℓ) , the operation α^{-1} will produce a zero part, which may be overlined or not. The operation β^{-1} (resp. δ^{-1}) consists in applying α^{-1} and removing the non-overlined (resp. overlined) zero part.

The inverse operations allow us to define a reduction process for overpartitions which is similar to Burge's reduction for partitions [14] :

- if f_0 does not belong to a multiple then we apply α^{-1}
- if there is a pair (f_0, f_1) where $f_1 \neq \bar{1}$, we apply β^{-1}
- otherwise we apply δ^{-1}

We do this recursively until we get the empty partition. An example is shown on Table 1.

Let $\bar{B}_{k,i}(n, j, N)$ be the number of partitions counted by $\bar{B}_{k,i}(n, j)$ such that $N = \sum_{(\ell+1)\text{-tuples}} \ell$ (we say that N is the total weight of the multiples, if we define the weight of an $(\ell + 1)$ -tuple to be ℓ). Note that for partitions, we always have $\ell = 1$ and N is the number of pairs of multiplicities [14]. Let $\bar{B}_{k,i}(N) = \sum_{n,j} \bar{B}_{k,i}(n, j, N) q^n a^j$. Starting with an overpartition counted in $\bar{B}_{k,i}(N)$, when we apply the reduction the weight will decrease by N . We can apply β^{-1} or δ^{-1} only if $i > 1$.

Proposition 5.3. *When we apply α^{-1} (resp. β^{-1} , (resp. δ^{-1})), N stays the same (resp. stays the same or decreases by 1 [in which case the next reduction is an α^{-1}], (resp. decreases by 1)) and i increases by 1 (resp. decreases by 1, (resp. stays the same)).*

Proof.

- When we apply α^{-1} :
 - N stays the same. To show it, we consider a block of consecutive multiples (i.e. multiples not separated by zeros). We restrict ourselves to the case of two multiples. Moreover, we only treat the case where both pairs have their last multiplicity $\neq \bar{1}$ (the other cases are similar). As we can see in Table 2, if we have two multiples of length $\ell_1 + 1$ and $\ell_2 + 1$ (total contribution to N : $\ell_1 + \ell_2$), after applying α^{-1} , we have

Operation	N	i	0	1	2	3	4	5
α^{-1}	3	1	0	(0	$\bar{1}$)	(1	$\bar{1}$	$\bar{3}$)
δ^{-1}	3	2	(0	$\bar{1}$)	(0	$\bar{2}$)	(1	$\bar{2}$)
α^{-1}	2	2	0	0	(1	$\bar{1}$)	(2	$\bar{1}$)
α^{-1}	2	3	0	(0	$\bar{2}$)	(0	$\bar{3}$)	
α^{-1}	2	4	0	(1	$\bar{1}$)	(1	$\bar{2}$)	
β^{-1}	2	4	(0	$\bar{2}$)	0	(2	$\bar{1}$)	
δ^{-1}	2	3	(0	$\bar{1}$)	(0	$\bar{3}$)		
α^{-1}	1	3	0	0	(1	$\bar{2}$)		
α^{-1}	1	4	0	0	(2	$\bar{1}$)		
α^{-1}	1	4	0	(0	$\bar{3}$)			
α^{-1}	1	4	0	(1	$\bar{2}$)			
α^{-1}	1	4	0	(2	$\bar{1}$)			
β^{-1}	1	4	(0	$\bar{3}$)				
β^{-1}	1	3	(0	$\bar{2}$)				
δ^{-1}	1	2	(0	$\bar{1}$)				
	0	2	0					

TABLE 1. Reduction of the overpartition $(5, 5, \bar{5}, \bar{4}, 3, \bar{2})$.

$$\begin{array}{cccccccc} 0 & (0 & * & * & * & * & * & \bar{2}) & (1 & * & * & * & * & \bar{2}) \\ (0 & \bar{1} & * & * & * & *) & (* & \bar{1} & \bar{2} & * & * & *) & (* & \bar{1}) \end{array}$$

TABLE 2. The contribution to N of this block of multiples is not affected by the operation α^{-1} .

three multiples of length ℓ_1 , $\ell_2 + 1$ and 2. Their total contribution to N is $\ell_1 - 1 + \ell_2 + 1 = \ell_1 + \ell_2$, so the contribution of the block to N was not affected by the operation. Since two multiples belonging to different blocks cannot interact and thus cannot modify N , we get that N stays the same.

- if there is a multiple which begins by f_1 , the operation increases f_1 by 1 ; otherwise f_1 remains the same. Thus i increases by 1.
- When we apply β^{-1} :
 - if $\ell > 1$ or $\ell = 1$ and $f_1 = 1$, N decreases by 1 because the multiple (f_0, \dots, f_ℓ) is transformed into $(f_0, \dots, f_{\ell-2})(f_{\ell-1}, f_\ell - 1)$. if $\ell = 1$ and $f_1 \neq 1$, the pair (f_0, f_1) remains and N is unchanged.
 - β^{-1} is applied if there is a pair (f_0, f_1) with $f_1 \neq \bar{1}$: the operation decreases f_1 by 1. Thus i decreases by 1.
- When we apply δ^{-1} :
 - N decreases by 1 (same reasoning as for β^{-1})
 - if there is a pair (f_0, f_1) (in this case $f_1 = \bar{1}$), the operation decreases f_1 by 1 ; otherwise f_1 stays the same. Thus i stays the same.

□

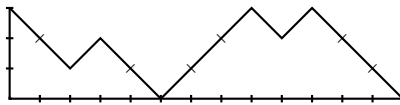


FIGURE 11. Example of a path.

Let $\bar{\beta}_{k,i}(N)$ (resp. $\bar{\delta}_{k,i}(N)$) be the generating functions of the overpartitions obtained by applying β^{-1} to an overpartition counted in $\bar{\mathcal{B}}_{k,i+1}(N)$ (resp. by applying δ^{-1} to an overpartition counted in $\bar{\mathcal{B}}_{k,i}(N)$). Then the previous proposition implies that

$$\begin{aligned}\bar{\mathcal{B}}_{k,i}(N) &= q^N(\bar{\mathcal{B}}_{k,i+1}(N) + \bar{\beta}_{k,i-1}(N) + a\bar{\delta}_{k,i}(N)) \text{ if } i < k \\ \bar{\mathcal{B}}_{k,k}(N) &= q^N(\bar{\mathcal{B}}_{k,k}(N) + \bar{\beta}_{k,k-1}(N) + a\bar{\delta}_{k,k}(N)) \\ \bar{\beta}_{k,i}(N) &= q^{N-1}\bar{\mathcal{B}}_{k,i+1}(N-1) + q^N\bar{\beta}_{k,i-1}(N) + q^Na\bar{\delta}_{k,i}(N) \\ \bar{\mathcal{B}}_{k,i}(0) &= 1 \\ \bar{\beta}_{k,0}(N) &= 0 \\ \delta_{k,i}(N) &= \bar{\mathcal{B}}_{k,i}(N-1) \text{ if } i > 1 \\ \delta_{k,1}(N) &= 0\end{aligned}$$

Let $\bar{\Gamma}_{k,i}(N) = \bar{\beta}_{k,i}(N) + a\bar{\delta}_{k,i+1}(N)$, then we get the recurrences defined in Proposition 3.1. Therefore $\bar{\mathcal{B}}_{k,i}(N) = \bar{\mathcal{E}}_{k,i}(N)$. This proves Proposition 5.1.

6. PATHS AND SUCCESSIVE DURFEE SQUARES

We will prove here that

Proposition 6.1.

$$\sum_{n_1 \geq \dots \geq n_{k-1} \geq 0} \frac{q^{(n_1+1)+n_2^2+\dots+n_{k-1}^2+n_i+\dots+n_{k-1}}(-1/a)_{n_1}a^{n_1}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-2}-n_{k-1}}(q)_{n_{k-1}}}$$

is the generating function of the paths counted by major index and number of South steps starting at height $k-i$, whose height is less than k and having n_j peaks of relative height $\geq j$ for $1 \leq j \leq k-1$ and is therefore equal to $\bar{\mathcal{E}}_{k,i}(a, q)$.

The relative height of a peak was defined by Bressoud in [12] when he proved that

Lemma 6.1 (Bressoud).

$$\frac{q^{n_1^2+n_2^2+\dots+n_{k-1}^2+n_i+\dots+n_{k-1}}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-2}-n_{k-1}}(q)_{n_{k-1}}}$$

is the generating function of the paths with no South steps starting at height $k-i$, whose height is less than k and having n_j peaks of relative height $\geq j$ for $1 \leq j \leq k-1$.

An example of such a path, taken from [12], is shown on Figure 11.

A mountain in a path is a portion of the path that starts at the beginning of the path or at height 0 stays above the x -axis and ends at height 0. We recall Bressoud's definition of the relative height of a peak [12]. We first map each peak of the path to a pair (y, y') where y is the height of the peak and y' is defined as

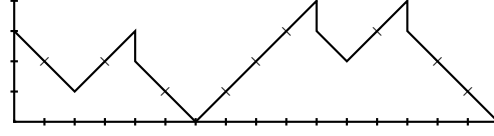
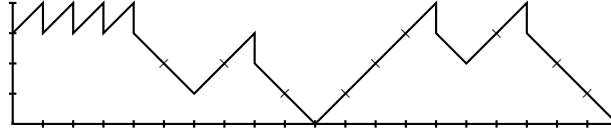


FIGURE 12. Effect of the volcanic uplift.

FIGURE 13. After adding the $n_1 - n_2 = 4$ NES peaks of relative height one.

follows. In each mountain, we choose the leftmost peak of maximal height relative to that mountain. For this peak, y' is the minimal height over all vertices to its left. Then, if there are any unchosen peaks left, we cut all the mountains off at height one. This may divide some mountains into several mountains relative to height one. For each mountain relative to height one in which no peaks have been chosen, we choose the leftmost peak of maximal height relative to that mountain ; for this peak, y' is the greater of one and the minimal height over all vertices to its left. We continue cutting the mountains off at height 2, 3, etc. until all peaks have been chosen.

Definition 6.2. [12] The *relative height of a peak* is then defined by $y - y'$.

This definition extends naturally to overpartitions. We can now move on to the proof of Proposition 6.1.

Proof. We prove the proposition using Bressoud's result. We consider a path counted by

$$(6.1) \quad \frac{q^{n_2^2 + \dots + n_{k-1}^2 + n_i + \dots + n_{k-1}}}{(q)_{n_2 - n_3} \cdots (q)_{n_{k-2} - n_{k-1}} (q)_{n_{k-1}}}$$

where $2 \leq i \leq k$. Thanks to Lemma 6.1, we know that this path starts at height $k - i$, its height is less than $k - 1$ and it has n_j peaks of relative height $\geq j - 1$ for $2 \leq j \leq k - 1$. We first insert a NES peak at each peak (see Figure 12). This "volcanic uplift" operation increases the weight of the path by

$$1 + 2 + \dots + n_2 = \binom{n_2 + 1}{2}$$

and the relative height of each peak by one.

We then insert $n_1 - n_2$ NES peaks at the beginning of the path (see Figure 13). These new peaks have total weight $\binom{n_1 - n_2 + 1}{2}$ and they increase the weight of each of the old peaks by $n_1 - n_2$. Altogether, the two operations introduce a factor

$$q^{\binom{n_2 + 1}{2} + \binom{n_1 - n_2 + 1}{2} + n_2(n_1 - n_2)} = q^{\binom{n_1 + 1}{2}}.$$

If $i = 2$, so that the path starts at $(0, k - 2)$, we have the option to introduce an extra step at the beginning of the path, from $(0, k - 1)$ to $(1, k - 2)$. This introduces the factor q^{n_1} .

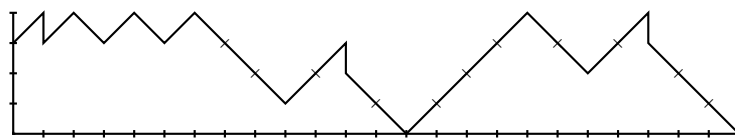


FIGURE 14. Effect of transforming some NES peaks into NESE peaks. The partition into distinct parts is (5, 4, 3, 1).

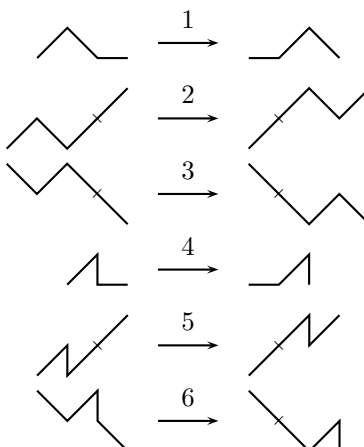


FIGURE 15. The rules for moving peaks.

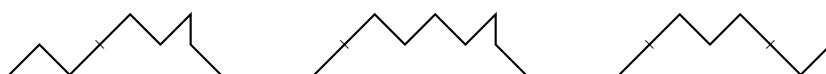


FIGURE 16. We want to move the leftmost peak to the right twice, but after the first move, we come up against a sequence of adjacent peaks. We then move the rightmost peak in this sequence.

The factor $(-1)_{n_1}$ corresponds to a partition into distinct parts which lie in $[0, n_1 - 1]$. If this partition contains a part $j - 1$ ($1 \leq j \leq n_1$), we transform the j th NES peak from the right into a NESE peak (see Figure 14). This operation increases the weight of the path by $j - 1$.

The factor $\frac{1}{(q)_{n_1 - n_2}}$ corresponds to a partition $(b_1, b_2, \dots, b_{n_1 - n_2})$ where $b_1 \geq b_2 \geq \dots \geq b_{n_1 - n_2} \geq 0$. For $1 \leq j \leq n_1 - n_2$, we move the j th peak of relative height one from the right b_j times according to the rules illustrated in Figure 15.

When we move a peak, it can meet the next peak to the right. We say that a peak (x, y) meets a peak (x', y') if

$$x' - x = \begin{cases} 2 & \text{if } (x, y) \text{ is a NESE peak} \\ 1 & \text{if } (x, y) \text{ is a NES peak} \end{cases} .$$

If this happens, we abandon the peak we have been moving and move the next one. If we come up against a sequence of adjacent peaks, we move the rightmost peak in the sequence (see Figure 16).



FIGURE 17. Case where the relative heights of two peaks are inverted. This does not modify the number of peaks with a given relative height.

To conclude the proof, we must show that the distribution of relative heights is not modified by the operations of Figure 15 (Proposition 6.2) and that the construction procedure is uniquely reversible (Proposition 6.3). \square

Proposition 6.2. *The operations of Figure 15 preserve the number of peaks of relative height $\geq j$ for all j .*

Proof. Let us show it for each operation :

- for operations 1 and 4, the peak forms a mountain on its own. Its relative height remains 1 after the move and the other mountains are not affected ; the relative heights thus remain unchanged.
- for operations 2 and 5 :
 - if the peak forms a mountain on its own, its relative height clearly remains 1 after the move because it then forms a mountain relative to height 1 (except in the case illustrated by Figure 17 where its relative height will become 2 whereas that of the peak on its right — which will be moved at the next step — will become 1). The relative heights of the other peaks in the mountain are not modified since those peaks have not moved and thus the mountains relative to the different heights remain unchanged.
 - if the peak is the first one in a mountain, the proof is similar ;
 - otherwise, its relative height will remain 1 ; indeed, before as well as after the move, it forms a mountain relative to height 1 on its own (except if the peak to its right has relative height 2, in which case the two peaks will switch their relative heights).
- for operations 3 and 6 :
 - if the peak is the last one in a mountain :
 - * if its height is 2, it will form a mountain on its own after the move. Its relative height will thus remain 1. The other peaks are not affected since we take into account the leftmost peak of maximal height when computing the relative heights. Naturally, the other mountains are not affected either.
 - * if its height is > 2 , the peaks will remain the last one in its mountain and will still have relative height 1 for the same reason as before the move.
 - otherwise, the relative height of the peak will remain 1. Indeed, its relative height could only change if our peak became the leftmost peak of maximal height of a relative mountain (other than the one which only includes our peak). It is impossible by construction, so our peak still has relative height 1 and the other peaks keep their relative heights.

\square

Proposition 6.3. *The construction used in the proof of Proposition 6.1 is uniquely reversible.*

Proof. We start with a path beginning at $(0, k - i)$, whose height is less than k , and with n_j peaks of relative height j for $1 \leq j \leq k$. We first have to move the peaks of relative height 1 to the left. We begin with the leftmost of these peaks and we move it to the left end of the path. If, during this move, our peak becomes adjacent to another peak, we abandon the peak we have been moving and we move the peak to its left (we do the same if we come up against a sequence of continuous peaks). The number of moves we had to perform to bring the peak to the beginning of the path gives us b_1 . We proceed similarly for the other peaks of relative height 1, which gives us a partition b into $n_1 - n_2$ parts ≥ 0 . At this step, among the n_1 peaks, some are NESE peaks. We transform these NESE peaks into NES peaks : if the i th peak from the right was transformed, it gives a part $i - 1$ in $(-1)_{n_1}$. Then, we remove the $n_1 - n_2$ NES peaks at the beginning of the path. Finally, we decrease by 1 the height of the remaining peaks by removing the NES peaks. \square

The multiple series in Proposition 6.1 can be re-expressed as (2.1), which is the generating function of overpartitions with $i - 1$ successive Durfee squares followed by $k - i$ successive Durfee rectangles, the first one being a generalized Durfee square/rectangle. For $i = k$, this is Theorem 1.2.

7. CONCLUSION

We showed in this work how the combinatorial interpretation of the Andrews-Gordon identities can be generalized to the case of overpartitions, when the combinatorial statistics (successive ranks, generalized Durfee square, multiples) are defined properly. There exist other generalizations of the Rogers-Ramanujan identities, see for example [13]. It was shown that the combinatorial interpretation in terms of lattice paths can also be done for these identities [1, 12, 14, 15]. Our work can also be extended in that direction and the results are presented in [18]. Finally there exists an extension of the concept of successive ranks for partitions due to Andrews, Baxter, Bressoud, Burge, Forrester and Viennot [6] and our goal now is to extend that notion to overpartitions.

REFERENCES

- [1] A. K. Agarwal and D. M. Bressoud, Lattice paths and multiple basic hypergeometric series. *Pacific J. Math.* **135** (1989) 209–228.
- [2] G. E. Andrews, The theory of partitions. Cambridge University Press, Cambridge, 1998.
- [3] G. E. Andrews, An analytic generalization of the Rogers-Ramanujan identities for odd moduli. *Proc. Nat. Acad. Sci. USA* **71** (1974) 4082–4085.
- [4] G. E. Andrews, Sieves in the theory of partitions. *Amer. J. Math.* **94** (1972) 1214–1230.
- [5] G. E. Andrews, A generalization of the Göllnitz-Gordon partition theorems. *Proc. Amer. Math. Soc.* **8** (1967) 945–952.
- [6] G. E. Andrews, R. J. Baxter, D. M. Bressoud, W. H. Burge, P. J. Forrester and G. Viennot, Partitions with prescribed hook differences, *Europ. J. Combinatorics* (1987) **8** 341–350.
- [7] G. E. Andrews and D. Bressoud, On The Burge correspondence between partitions and binary words. Number Theory (Winnipeg, Man., 1983). *Rocky Mountain J. Math.* **15** (1985), no. 2, 225–233.
- [8] G. E. Andrews and D. M. Bressoud, Identities in Combinatorics III : further aspects of ordered set sorting. *Discrete Math* **49** (1984) 222–236.
- [9] C. Bessenrodt and I. Pak, Partition congruences by involutions, *European J. Comb.* **25** (2004) 1139–1149.

- [10] C. Boulet and I. Pak, A combinatorial proof of the Rogers-Ramanujan and Schur identities. *J. of Combin. Theory Ser. A*, to appear.
- [11] F. Brenti, Determinants of Super-Schur Functions, Lattice Paths, and Dotted Plane Partitions *Advances in Math.*, **98** (1993), 27-64.
- [12] D. Bressoud, Lattice paths and the Rogers-Ramanujan identities. Number Theory, Madras 1987, 140–172, *Lecture Notes in Math.* **1395**, Springer, Berlin, 1989.
- [13] D. Bressoud, A generalization of the Rogers-Ramanujan identities for all moduli. *J. Comb. Theory* **27** (1979) 64–68.
- [14] W. H. Burge, A correspondence between partitions related to generalizations of the Rogers-Ramanujan identities. *Discrete Math.* **34** (1981), no. 1, 9–15.
- [15] W. H. Burge, A three-way correspondence between partitions. *European J. Combin.* **3** (1982), no. 3, 195–213.
- [16] S. Corteel, J. Lovejoy, Overpartitions. *Trans. Amer. Math. Soc.* **356** (2004), no. 4, 1623–1635 (Presented at FPSAC 2003).
- [17] S. Corteel, J. Lovejoy, Frobenius partitions and the combinatorics of Ramanujan’s ${}_1\psi_1$ summation, *J. Comb. Theory Ser. A* **97** (2002), 177–183.
- [18] S. Corteel, J. Lovejoy and O. Mallet, Overpartitions and Rogers-Ramanujan identities for even moduli, in preparation.
- [19] P. Desrosiers, L. Lapointe, and P. Mathieu, A symmetric function theory in superspace, Presented at FPSAC 2005.
- [20] J.-F. Fortin, P. Jacob, and P. Mathieu, Generating function for K -restricted jagged partitions. *Electron. J. Comb.* **12**, No.1, Research paper R12, 17 p., (2005).
- [21] J.-F. Fortin, P. Jacob, and P. Mathieu, Jagged partitions, *Ramanujan Jal* (2005), to appear.
- [22] G. Gaspar and M. Rahman, Basic hypergeometric series, Cambridge University Press (1990).
- [23] B. Gordon, A combinatorial generalization of the Rogers-Ramanujan identities, *Amer. J. Math.* **83** (1961), 393–399 .
- [24] J. Lovejoy, Gordon’s theorem for overpartitions. *J. Combin. Theory Ser. A* **103** (2003), no. 2, 393–401.
- [25] J. Lovejoy, Overpartition theorems of the Rogers-Ramanujan type *J. London Math. Soc.* **69** (2004), 562–574.
- [26] J. Lovejoy, Rank and conjugation for the Frobenius representation of an overpartition. *Ann. Combin.* **9** (2005) 321–334.
- [27] P. A. MacMahon, Combinatory Analysis, vol. 2, Cambridge Univ. Press, Cambridge, 1916.
- [28] K. Mahlburg, The overpartition function modulo small powers of 2, *Discrete Mathematics* **286** (2004), no. 3, 263–267.
- [29] O. Mallet, Rangs successifs, chemins et identités de type Rogers-Ramanujan, *Master thesis, Paris 6* (2005).
- [30] L. J. Rogers, Second memoir on the expansion of certain infinite products, *Proc. London Math. Soc.* **25** (1894), 318–343.
- [31] A. J. Yee, Combinatorial proofs of Ramanujan’s ${}_1\psi_1$ summation and the q -Gauss summation, *J. Combin. Theory Ser. A*, **105** (2004), 63–77.

CNRS PRISM, UNIVERSITÉ DE VERSAILLES, 45 AVENUE DES ETATS-UNIS, 78035 VERSAILLES CEDEX FRANCE

E-mail address: syl@prism.uvsq.fr

LIAFA, UNIVERSITÉ DENIS DIDEROT, 2 PLACE JUSSIEU, CASE 7014, F-75251 PARIS CEDEX 05

E-mail address: mallet@liafa.jussieu.fr