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Christophe Chesneau

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On Wavelet Block Thresholding in Nonparametric Estimation

Chesneau Christophe

Laboratoire de Probabilités et Modèles Aléatoires,
CNRS-UMR 7599, Université Paris VI, UFR de Mathématiques,
175 rue de Chevaleret F-75013 Paris, France.
E-mail: christophe.chesneau4@wanadoo.fr
<http://www.chesneau-stat.com>

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Abstract

Starting from a general statistical model, we investigate the performance of wavelet block thresholding procedures via the maxiset approach under the \mathbb{L}^p risk ($p > 1$) for a rate of convergence of the form $n^{-\epsilon}$. We prove that such procedures can be better in the maxiset sense than the hard thresholding procedures. Moreover, we show that they can be optimal in the minimax sense over the regular zone of Besov balls.

Key Words: Maxiset, minimax, adaptive estimation, wavelet block thresholding, Besov spaces.

AMS subject classification: Primary : 62G07; Secondary : 62C20.

1 Motivation

In this Section, let us suppose that we are given data (Y_1, \dots, Y_n) defined by

$$Y_i = f(x_i) + z_i, \quad i = 1, \dots, n, \quad (1.1)$$

where $x_i = i/n$ and z_i are i.i.d normal variables with mean zero and variance one. The standard wavelet method to recover the unknown function f consists in transforming the data into empirical wavelet coefficients and in proceeding to an individual selection : we keep only those which are greater than a fixed threshold. Such local constructions include the soft and hard thresholding procedures which have been studied by many authors starting from [Donoho and Johnstone \(1994, 1995\)](#). More recently, [Kerkycharian et al. \(1996\)](#), [Juditsky \(1997\)](#), [Hall et al. \(1998\)](#) and [Cai \(1998, 1999, 2002\)](#) have developed adaptive wavelet procedures based on thresholding several empirical wavelet coefficients simultaneously rather than individually as in the local approach.

For the regression problem as given in (1.1), it was proved that such procedures can enjoy better minimax properties than the hard (and soft) thresholding procedures if we consider the Besov balls and the \mathbb{L}^2 risk. More precisely, these adaptive procedures attain the optimal rate of convergence without logarithmic factor contrary to the hard thresholding procedures. Similar results have been established for other models than (1.1). For the regression model with nonequispaced data see [Chicken \(2003\)](#), for the density estimation see [Hall et al. \(1998\)](#), [Pensky \(1999\)](#) and [Cai and Chicken \(2005\)](#), for the nonparametric regression with long memory error see [Li and Xiao \(2004\)](#) and for the nonparametric density deconvolution see [Pensky and Vidakovic \(1998\)](#).

By taking the problem under the \mathbb{L}^p risk for $p > 1$, the purpose of the present paper is to investigate the performance of the block wavelet thresholding procedures via two statistical methods. Firstly, we adopt the maxiset approach for a rate of convergence of the form $n^{-\gamma p}$ (without logarithmic factor). In our context, this point of view consists in determining the function spaces \mathcal{A}^γ which satisfy the following equivalence :

$$f \in \mathcal{A}^\gamma \iff \sup_{n>0} (n^{\gamma p} \mathbb{E}_f^n (\int_0^1 |\hat{f}(t) - f(t)|^p dt)) < \infty,$$

where \hat{f} is an estimate of f constructed from block thresholding rules and \mathbb{E}_f^n is the expectation with respect to the law \mathbb{P}_f^n of the observations. A similar maxiset study is given for the hard thresholding procedures. Let us mention that to exhibit the maxisets above, we use the powerful geometrical properties of certain wavelet bases for the \mathbb{L}^p norms which have been pointed out by [Kerkycharian and Picard \(2000\)](#) (unconditional nature and Temlyakov's property). The main conclusions of this first part are the following : The maxisets associated to the block thresholding rules can be bigger than those associated to the hard thresholding rules. This fact has been mentioned by [Autin \(2005\)](#) in the context of the white noise model when the the Besov risk is considered.

Secondly, we use the results above to investigate the minimax properties of the block thresholding procedures over Besov balls $B_{\pi,r}^s(R)$ (to be defined in Section 4). More precisely, we show that they can be optimal in the minimax sense over $B_{\pi,r}^s(R)$ under the \mathbb{L}^p risk. In particular, if we consider the model (1.1) then there exists a constant $C > 0$ such that the block thresholding procedure \hat{f} satisfies :

$$\sup_{f \in B_{\pi,r}^s(R)} \mathbb{E}_f^n \left(\int_0^1 |\hat{f}(t) - f(t)|^p dt \right) \leq C n^{-\frac{sp}{1+2s}}, \quad s \geq 1, \quad \pi \geq p, \quad r \geq 1,$$

for n large enough. It is important to notice that we provide general statistical results which can be apply for different models than (1.1), including certain inverse problems as for instance the deconvolution in white noise studied by Cavalier and Tsybakov (2002) and Johnstone et al. (2004). This study can be viewed as a \mathbb{L}^p complement of some results developed by Cai (1998), Cohen et al. (2000) and Autin (2005).

The rest of the paper is organized as follows. Section 2 describes wavelet bases on the interval, some of their geometrical properties in the \mathbb{L}^p norms, our main wavelet procedures and the maxiset point of view. By considering a general statistical model, Section 3 investigates and compares the maxiset properties of the block and the hard thresholding procedures. Some upper bounds over classical Besov balls are developed in Section 4. Section 5 investigates two applications of our general results. Section 6 contains proofs of Theorems and Propositions.

2 Methodology

Throughout this paper, we set :

$$\mathbb{L}^p([0, 1]) = \{f \text{ measurable on } [0, 1]; \|f\|_p^p = \int_0^1 |f(t)|^p dt < +\infty\}.$$

The notation $a \asymp b$ means : there exist two constants $C > 0$ and $c > 0$ such that $cb \leq a \leq Cb$. The notations $a \wedge b$ and $a \vee b$ mean respectively : $\min(a, b)$ and $\max(a, b)$.

2.1 Wavelet bases and some of their geometrical properties under the \mathbb{L}^p norms

We summarize in this subsection the basics on wavelet bases on the unit interval $[0, 1]$ and we introduce some of their powerful geometrical properties in the \mathbb{L}^p norms : the unconditional property and the Temlyakov's property.

Let us consider the wavelet basis of $[0, 1]$ described by Cohen et al. (1993) : We consider ϕ a "father" wavelet of a multiresolution analysis on \mathbb{R} and ψ the associated "mother" wavelet. Assume that $Supp(\phi) = Supp(\psi) = [1 - N, N]$ and $\int_{1-N}^N \phi(t) dt = 1$, $\int_{1-N}^N t^l \psi(t) dt = 0$ for $l = 0, \dots, N - 1$.

Let us set :

$$\phi_{j,k}(x) = 2^{\frac{j}{2}} \phi(2^j x - k) \quad \text{and} \quad \psi_{j,k}(x) = 2^{\frac{j}{2}} \psi(2^j x - k).$$

Then there exists an integer τ satisfying $2^\tau \geq 2N$ such that the collection ζ defined by :

$$\zeta = \{\phi_{\tau,k}(\cdot), k = 0, \dots, 2^\tau - 1; \psi_{j,k}(\cdot); j \geq \tau, k = 0, \dots, 2^j - 1\}$$

with an appropriate treatments at the boundaries, is an orthonormal basis of $\mathbb{L}^2([0, 1])$. Another wavelet basis on the interval $[0, 1]$ shall be introduced in Section 5.

Let $1 \leq p < \infty$. Any function f of $\mathbb{L}^p([0, 1])$ can be decomposed on ζ as :

$$f(x) = \sum_{k \in \Delta_\tau} \alpha_{\tau,k} \phi_{\tau,k}(x) + \sum_{j \geq \tau} \sum_{k \in \Delta_j} \beta_{j,k} \psi_{j,k}(x), \quad x \in [0, 1],$$

where $\alpha_{j,k} = \int_0^1 f(t) \phi_{j,k}(t) dt$, $\beta_{j,k} = \int_0^1 f(t) \psi_{j,k}(t) dt$ and $\Delta_j = \{0, \dots, 2^j - 1\}$.

Lemma 2.1 (Unconditional nature-Temlyakov's property). *Let $p > 1$. Let us set $\psi_{\tau-1,k} = \phi_{\tau,k}$. Then the basis ζ is unconditional for $\mathbb{L}^p([0, 1])$ i.e for any sequence $u = (u_{j,k})_{j,k}$ we have :*

$$\| \sum_{j \geq \tau-1} \sum_{k \in \Delta_j} u_{j,k} \psi_{j,k} \|_p^p \asymp \| (\sum_{j \geq \tau-1} \sum_{k \in \Delta_j} |u_{j,k} \psi_{j,k}|^2)^{\frac{1}{2}} \|_p^p. \quad (2.1)$$

Moreover, if for any subset Λ of \mathbb{N} there exists a constant $C > 0$ such that the positive sequence $\sigma = (\sigma_j)_j$ satisfies the following inequalities :

$$\sum_{j \in \Lambda} 2^j \sigma_j^2 \leq C \sup_{j \in \Lambda} (2^j \sigma_j^2), \quad (2.2)$$

then the weighted basis ξ_σ defined by

$$\xi_\sigma = \{\sigma_\tau \phi_{\tau,k}(\cdot), k \in \Delta_\tau; \sigma_j \psi_{j,k}(\cdot), j \geq \tau, k \in \Delta_j\}$$

satisfies the Temlyakov property for $p > 2$ i.e for any subset $A \subseteq \{\tau - 1, \dots\}$ and any subset $C \subseteq \Delta_j$ we have :

$$\|(\sum_{j \in A} \sum_{k \in C} |\sigma_j \psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p \asymp \sum_{j \in A} \sum_{k \in C} \sigma_j^p \|\psi_{j,k}\|_p^p. \quad (2.3)$$

Let us precise that for $p = 2$, the inequality (2.3) holds without any condition on σ .

For exact references concerning the previous properties, we refer the reader to [Kerkyacharian and Picard \(2000, Subsection 4.1.1\)](#) and [Johnstone et al. \(2004, Theorem 2\)](#).

The following Lemma is standard :

Lemma 2.2. *Let $p \geq 1$. For any sequence $u = (u_{j,k})_{j,k}$ and any $j \geq \tau$ there exists a constant $C > 0$ such that :*

$$\| \sum_{k \in \Delta_j} u_{j,k} \phi_{j,k} \|_p^p \leq C 2^{j(\frac{p}{2}-1)} \sum_{k \in \Delta_j} |u_{j,k}|^p.$$

This inequality holds if we exchange ϕ by ψ .

2.2 Block thresholding procedures, hard thresholding procedures and maxisets

Here and after, we assume that we observe a sequence of models E_n in which we are able to produce estimates $\hat{\beta}_{j,k}$ (resp. $\hat{\alpha}_{j,k}$) of the wavelets coefficients $\beta_{j,k}$ (resp. $\alpha_{j,k}$) of an unknown function f .

The following block thresholding procedures will be at the heart of our statistical study. They have been inspired by the work of [Efroimovich \(1985\)](#) and recently developed in the framework of wavelet analysis by [Kerkyacharian et al. \(1996\)](#), [Juditsky \(1997\)](#), [Hall et al. \(1998\)](#), [Cai \(1998\)](#) and [Picard and Tribouley \(2000\)](#).

Definition 2.1 (Block thresholding procedures). *Let $1 < p < \infty$ and $0 < \nu \leq 2$. Let j_1 be an integer satisfying :*

$$2^{j_1} \asymp \ln(n)^{\frac{p}{2} \vee 1}$$

and let σ be a known increasing positive sequence such that there exists $\nu > 0$ satisfying :

$$\sigma_{j_1} \asymp \ln(n)^\nu.$$

Let j_2 be an integer satisfying $2^{j_2} \asymp n^{\frac{\nu}{2}}$. For all j in $\{j_1, \dots, j_2 - 1\}$, let us divide Δ_j into consecutive nonoverlapping blocks $\mathcal{B}_{j,K}$ of length l_j (non decreasing in j) i.e :

$$\mathcal{B}_{j,K} = \{k \in \Delta_j : (K-1)l_j \leq k \leq Kl_j - 1\}, \quad K \in \mathcal{A}_j,$$

where the sets \mathcal{A}_j are defined by :

$$\mathcal{A}_j = \{1, \dots, 2^j l_j^{-1}\}$$

(for convenience we suppose that $2^j l_j^{-1} \in \mathbb{N}^*$ for all j in $\{j_1, \dots, j_2 - 1\}$). We define the block thresholding procedure \hat{f} by :

$$\hat{f}(x) = \sum_{j \leq j_1} \hat{\alpha}_{j_1,k} \phi_{j_1,k}(x) + \sum_{j_1 \leq j < j_2} \sum_{K \in \mathcal{A}_j} \sum_{k \in \mathcal{B}_{j,K}} \hat{\beta}_{j,k} 1_{\{\hat{b}_{j,K}(p) \geq \kappa \sigma_j n^{-\frac{1}{2}}\}} \psi_{j,k}(x), \quad x \in [0, 1], \quad (2.4)$$

where κ is a positive real number and $\hat{b}_{j,K}(p)$ is the normalized l_p -norm of estimators $(\hat{\beta}_{j,k})_{k \in \mathcal{B}_{j,K}}$ i.e :

$$\hat{b}_{j,K}(p) = (l_j^{-1} \sum_{k \in \mathcal{B}_{j,K}} |\hat{\beta}_{j,k}|^p)^{\frac{1}{p}}.$$

Starting from this general definition, we distinguish two kinds of procedures :

1. the **global thresholding procedure** (or *wild thresholding procedure*) which corresponds to the procedure \hat{f} described by (2.4) with

$$l_j = 2^j$$

(i.e $\mathcal{B}_{j,K}$ is reduced to the set Δ_j). Such procedure will be denote \hat{f}^g .

2. the **BlockShrink procedure** \hat{f}^o which corresponds to the procedure \hat{f} described by (2.4) with

$$l_j \asymp \ln(n)^{\frac{p}{2} \vee 1}$$

Such procedure will be denote \hat{f}^o .

The Definition below introduces a well-known wavelet thresholding procedure which has been intensively used in several papers. See for instance [Donoho and Johnstone \(1994, 1995\)](#).

Definition 2.2 (Hard thresholding procedures). Let $\nu > 0$ and let $\sigma = (\sigma_j)_j$ be an increasing positive sequence. We define the hard thresholding procedure \hat{f}^h by :

$$\hat{f}^h(x) = \sum_{k \in \Delta_\tau} \hat{\alpha}_{\tau,k} \phi_{\tau,k}(x) + \sum_{j < j_2} \sum_{k \in \Delta_j} \hat{\beta}_{j,k} 1_{\left\{ |\hat{\beta}_{j,k}| \geq \kappa \sigma_j \sqrt{\frac{\ln(n)}{n}} \right\}} \psi_{j,k}(x), \quad x \in [0, 1], \quad (2.5)$$

where j_2 is an integer satisfying $2^{j_2} \asymp n^{\frac{\nu}{2}}$.

Typically, the sequence σ satisfies : $\sigma_j \asymp 2^{\delta j}$ where $\delta \geq 0$ and it often appears in the literature of inverse problems (see for instance [Kerkycharian et al. \(2005\)](#), [Cavalier et al. \(2003\)](#) and [Pensky and Vidakovic \(1998\)](#)). For an example of a such inverse problem, see Section 5.

The main differences between the global thresholding procedures and the BlockShrink procedures are the length and the form of the blocks size. One depends on the level j and the other depends on the number of observations n independently of j .

Definition 2.3 (Maxisets). Let $1 \leq p < \infty$ and $n_0 \in \mathbb{N}^*$. Let \hat{f} be an estimate of f . The maxiset of \hat{f} at the rate u_n under the \mathbb{L}^p risk is the set of functions f such that there exists $R > 0$ satisfying :

$$\sup_{n \geq n_0} u_n^{-1} \mathbb{E}_f^n (\|\hat{f} - f\|_p^p) \leq R < \infty.$$

Such maxiset will be denoted $\mathcal{M}_{n_0}(\hat{f}, p, u_n)$.

Such a point of view provides a functional set which is authentically connected to the procedure and the model. The interest of a maxiset is twofold. Firstly, it allows to compare the performance of several procedures at a given rate of convergence. Secondly, it provides upper bounds over 'standard' function spaces. Several maxiset results can be found in [Cohen et al. \(2000\)](#), [Kerkycharian and Picard \(2000\)](#), [Rivoirard \(2004\)](#) and [Autin \(2005\)](#).

3 Maxiset results

3.1 Function spaces

Let us introduce the function spaces which shall appear in the expression of our maxisets. For sake of legibility, we shall adopt the following notations :

$$\sum_K = \sum_{K \in \mathcal{A}_j} \quad \text{and} \quad \sum_{(K)} = \sum_{k \in \mathcal{B}_{j,K}}.$$

Definition 3.1 (Strong Besov spaces). Let $1 \leq p < \infty$. We say that a function f of $\mathbb{L}^p([0, 1])$ belongs to $\mathcal{B}_{p,\infty}^\alpha$ if and only if there exists $R > 0$ such that :

$$\sup_{u > 0} 2^{u\alpha p} \left\| \sum_{j \geq u} \sum_{k \in \Delta_j} \beta_{j,k} \psi_{j,k} \right\|_p^p \leq R < \infty.$$

Definition 3.2 (\mathcal{W} -spaces). Let $0 < r < p < \infty$ and let $\sigma = (\sigma_j)_j$ be a positive sequence. We say that :
a function f of $\mathbb{L}^p([0, 1])$ belongs to $\mathcal{W}_\sigma(r, p)$ if and only if there exists $R > 0$ such that :

– for $p \geq 2$:

$$\sup_{u>0} u^{r-p} \left\| \sum_j \sum_{k \in \Delta_j} \beta_{j,k} \mathbf{1}_{\{b_j(p) \leq u \sigma_j\}} \psi_{j,k} \right\|_p^p \leq R < \infty,$$

– for $p \leq 2$:

$$\sup_{u>0} u^{r-p} \sum_j \sum_{k \in \Delta_j} |\beta_{j,k}|^p \mathbf{1}_{\{b_j(p) \leq u \sigma_j\}} 2^{j(\frac{p}{2}-1)} \leq R < \infty,$$

where $b_j(p)$ is the the normalized l_p -norm of wavelet coefficients $(\beta_{j,k})_{k \in \Delta_j}$ i.e :

$$b_j(p) = (2^{-j} \sum_{k \in \Delta_j} |\beta_{j,k}|^p)^{\frac{1}{p}}. \quad (3.1)$$

The previous function spaces can be viewed as a weighted \mathbb{L}^p versions of those introduced by [Autin \(2005, Subsection 6.3.1\)](#). The following $\overline{\mathcal{W}}$ -spaces will play an intermediate role to prove [Proposition 3.2](#) below.

Definition 3.3 ($\overline{\mathcal{W}}$ -spaces). Let $0 < r < p < \infty$, $\kappa \in \mathbb{R}^{+*}$, $n_0 \in \mathbb{N}^*$ and let $\sigma = (\sigma_j)_j$ be a positive sequence. Let us consider the sets $\mathcal{B}_{j,K}$ with $l_j \asymp \ln(n)^{\frac{p}{2} \vee 1}$. We say that :

1. a function f of $\mathbb{L}^p([0, 1])$ belongs to $\overline{\mathcal{W}}_{\sigma, \kappa, n_0}(r, p)$ if and only if there exists $R > 0$ such that :

– for $p \geq 2$:

$$\sup_{n>n_0} n^{\frac{p-r}{2}} \sum_{m \in \mathbb{N}} 2^{-mp} \left\| \sum_j \sum_K \sum_{(K)} \beta_{j,k} \mathbf{1}_{\{b_{j,K}(p) \leq \kappa 2^{m n^{-\frac{1}{2}}} \sigma_j\}} \psi_{j,k} \right\|_p^p \leq R < \infty,$$

– for $p \leq 2$:

$$\sup_{n>n_0} n^{\frac{p-r}{2}} \sum_{m \in \mathbb{N}} 2^{-mp} \sum_j \sum_K \sum_{(K)} |\beta_{j,k}|^p \mathbf{1}_{\{b_{j,K}(p) \leq \kappa 2^{m n^{-\frac{1}{2}}} \sigma_j\}} 2^{j(\frac{p}{2}-1)} \leq R < \infty.$$

2. a function f of $\mathbb{L}^p([0, 1])$ belongs to $\overline{\mathcal{W}}_{\sigma, \kappa, n_0}^*(r, p)$ if and only if there exists $R > 0$ such that :

$$\sup_{n>n_0} n^{\frac{p-r}{2}} \left\| \sum_j \sum_K \sum_{(K)} \beta_{j,k} \mathbf{1}_{\{b_{j,K}(p) \leq \frac{\kappa}{2} n^{-\frac{1}{2}} \sigma_j\}} \psi_{j,k} \right\|_p^p \leq R < \infty,$$

where $b_{j,K}(p)$ is the the normalized l_p -norm of wavelet coefficients $(\beta_{j,k})_{k \in \mathcal{B}_{j,K}}$ i.e :

$$b_{j,K}(p) = \left(\frac{1}{\ln(n)^{\frac{p}{2} \vee 1}} \sum_{(K)} |\beta_{j,k}|^p \right)^{\frac{1}{p}}. \quad (3.2)$$

The following spaces naturally appear when we study the maximal spaces of hard thresholding rules for a rate of convergence of the form $n^{-\gamma p}$.

Definition 3.4 ($\widehat{\mathcal{W}}$ -spaces). Let $0 < r < p < \infty$, $v \in]0, 1[$ and let $\sigma = (\sigma_j)_j$ be a positive sequence. We say that a function f of $\mathbb{L}^p([0, 1])$ belongs to $\widehat{\mathcal{W}}_\sigma(r, p)$ if and only if there exists $R > 0$ such that :

$$\sup_{u>0} \eta(u)^{r-p} \left\| \sum_j \sum_{k \in \Delta_j} \beta_{j,k} \mathbf{1}_{\{|\beta_{j,k}| \leq \sigma_j u\}} \psi_{j,k} \right\|_p^p \leq R < \infty,$$

where η is the continuous non decreasing function defined by :

$$\eta(u) = \begin{cases} u \ln((u \wedge v)^{-1})^{-\frac{1}{2}}, & u > 0, \\ 0, & u = 0, \end{cases}$$

where v is a real number such that $0 < v \leq \exp(-\frac{p-r}{2r})$.

3.2 Maxisets associated to block thresholding rules

Let us first set the assumptions we shall need to prove our Theorems. Let us recall that the $\hat{\alpha}_{j,k}$ and $\hat{\beta}_{j,k}$ are estimators of the wavelet coefficients $\alpha_{j,k}$ and $\beta_{j,k}$ and that σ_j is a known positive sequence.

1. **Assumption 1** There exists a constant $C > 0$ such that the following moments condition holds :

$$\mathbb{E}_f^n (|\hat{\alpha}_{j_1,k} - \alpha_{j_1,k}|^p) \leq C \sigma_{j_1}^p n^{-\frac{p}{2}},$$

2. **Assumption 2** There exists a constant $C > 0$ such that the following moments condition holds :

$$\mathbb{E}_f^n (|\hat{\beta}_{j,k} - \beta_{j,k}|^{2p}) \leq C \sigma_j^{2p} n^{-p}, \quad j \in \{j_1, \dots, j_2 - 1\},$$

3. **Assumption 3** There exists a constant $C > 0$ such that the following concentration condition holds :

$$\mathbb{P}_f^n \left((l_j^{-1} \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^p)^{\frac{1}{p}} \geq \lambda \sigma_j n^{-\frac{1}{2}} \right) \leq C n^{-h(\lambda)}, \quad K \in \mathcal{A}_j, \quad j \in \{j_1, \dots, j_2 - 1\},$$

where λ is a large enough real number and h is a positive function such that $\lim_{x \rightarrow \infty} h(x) = \infty$.

4. **Assumption 4** There exists a constant $C > 0$ such that the sequence σ and the integer j_2 satisfy the following weighted inequality :

$$\sum_{j < j_2} \sum_{k \in \Delta_j} \sigma_j^p \|\psi_{j,k}\|_p^p \leq C n^{\frac{p}{2}}.$$

5. **Assumption 5** There exists $C > 0$ such that the following concentration inequality holds :

$$\mathbb{P}_f^n (|\hat{\beta}_{j,k} - \beta_{j,k}| \geq \lambda \sigma_j \sqrt{\frac{\ln(n)}{n}}) \leq C n^{-h(\lambda)}, \quad j \in \{j_1, \dots, j_2 - 1\},$$

where λ is a large enough real number and h is a function of \mathbb{R}_+^* such that $\lim_{x \rightarrow \infty} h(x) = \infty$.

Theorems 3.1 and 3.2 below investigate the maxiset properties of the block thresholding procedures \hat{f}^g and \hat{f}^o measured under the \mathbb{L}^p risk for the rate of convergence $n^{-\frac{\alpha p}{2}}$.

Theorem 3.1. *Let $1 < p < \infty$, let $\sigma = (\sigma_j)_j$ be a known positive increasing sequence and let \hat{f} be the block thresholding procedure described by (2.4). Then, under the assumptions 1 – 4, for any α in $]0, 1[$, any $\kappa > \kappa_0$ with κ_0 large enough, and n_0 large enough, the maxiset associated to the global thresholding procedure satisfies :*

$$\mathcal{M}_{n_0} \left(\hat{f}^g, p, n^{-\frac{\alpha p}{2}} \right) \subseteq \mathcal{B}_{p,\infty}^{\frac{\alpha}{p}} \cap \mathcal{W}_{\sigma}((1-\alpha)p, p)$$

and the maxiset associated to the BlockShrink procedure satisfies :

$$\mathcal{M}_{n_0} \left(\hat{f}^o, p, n^{-\frac{\alpha p}{2}} \right) \subseteq \mathcal{B}_{p,\infty}^{\frac{\alpha}{p}} \cap \overline{\mathcal{W}}_{\sigma,\kappa,n_0}^*((1-\alpha)p, p).$$

Theorem 3.2. *Let $1 < p < \infty$, let $\sigma = (\sigma_j)_j$ be a known positive increasing sequence which satisfies the inequality (2.2) and let \hat{f} be the block thresholding procedure described by (2.4). Then, under the assumptions 1 – 4, for any α in $]0, 1[$, any $\kappa > \kappa_0$ with κ_0 large enough, and n_0 large enough, the maxiset associated to the global thresholding procedure satisfies :*

$$\mathcal{B}_{p,\infty}^{\frac{\alpha}{p}} \cap \mathcal{W}_{\sigma}((1-\alpha)p, p) \subseteq \mathcal{M}_{n_0} \left(\hat{f}^g, p, n^{-\frac{\alpha p}{2}} \right)$$

and the maxiset associated to the BlockShrink procedure satisfies :

$$\mathcal{B}_{p,\infty}^{\frac{\alpha}{p}} \cap \overline{\mathcal{W}}_{\sigma,\kappa,n_0}^*((1-\alpha)p, p) \subseteq \mathcal{M}_{n_0} \left(\hat{f}^o, p, n^{-\frac{\alpha p}{2}} \right).$$

It is important to mention that the assumptions of Theorem 3.1 are not very restrictive (see Section 5). Proposition 3.1 below shows that under certain conditions on the factor α , we can reduce the expression of the maxiset associated to the global thresholding procedure.

Proposition 3.1. *Let $2 \leq p < \infty$ and let \hat{f}^g be the global thresholding procedure. Then, under the assumptions 1 – 4, for any α in $]0, 1[$, any $\kappa > \kappa_0$ with κ_0 large enough, and n_0 large enough, we have :*

$$\mathcal{M}_{n_0} \left(\hat{f}^g, p, n^{-\frac{\alpha p}{2}} \right) = \begin{cases} \mathcal{B}_{p, \infty}^{\frac{\alpha}{p}} \cap \mathcal{W}_\sigma((1 - \alpha)p, p) & \text{if } \alpha \in]0, 1[, \\ \mathcal{W}_\sigma((1 - \alpha)p, p) & \text{if } \alpha \in]1 - \frac{p}{2}, 1[. \end{cases}$$

Let us precise that Proposition 3.1 above has been proved for the case $p = 2$ and $\sigma = 1$ in Cohen et al. (2000, Theorem 5).

Remark 3.1. *The exact maxiset associated to the BlockShrink procedure is difficult to express because of the presence of the number n in the length of each block. Moreover, let us notice that the maxiset comparison of the block thresholding procedures is not always possible due to the different forms of the blocks.*

3.3 Maxisets comparison between hard thresholding and BlockShrink procedures

Theorem 3.3 below exhibits the maxisets associated to the hard thresholding procedures for the rate of convergence $n^{-\frac{\alpha p}{2}}$.

Theorem 3.3 (Maxiset associated to \hat{f}^h). *Let $1 < p < \infty$, let $\sigma = (\sigma_j)_j$ be a known positive sequence which satisfies the condition (2.2) and let \hat{f}^h be the hard thresholding procedure described by (2.5). Suppose that $\hat{\alpha}_{j,k}$ and $\hat{\beta}_{j,k}$ satisfy the assumptions 1 – 2 and 4 – 5 with $j_1 = \tau$. Then for any $\alpha \in]0, 1[$, any $\kappa > \kappa_0$ with κ_0 large enough, and n_0 large enough, we have :*

$$\mathcal{M}_{n_0} \left(\hat{f}^h, p, n^{-\frac{\alpha p}{2}} \right) = \mathcal{B}_{p, \infty}^{\frac{\alpha}{p}} \cap \widehat{\mathcal{W}}_\sigma((1 - \alpha)p, p).$$

A similar maxiset Theorem has been proved in Kerkycharian and Picard (2000) for the rate of convergence $(\frac{\ln(n)}{n})^{\frac{\alpha p}{2}}$. Proposition 3.2 below does the maxiset comparison between the BlockShrink procedures and the hard thresholding procedures for $p \geq 2$.

Proposition 3.2 (Maxisets comparison). *Let $2 \leq p < \infty$, let \hat{f}^o be the BlockShrink procedure and let \hat{f}^h be the hard thresholding procedure. Under the assumptions the assumptions 1 – 5, for any α in $]0, 1[$, any $\kappa > \kappa_0$ with κ_0 large enough, and n_0 large enough, we have :*

$$\mathcal{M}_{n_0} \left(\hat{f}^h, p, n^{-\frac{\alpha p}{2}} \right) \subseteq \mathcal{M}_{n_0} \left(\hat{f}^o, p, n^{-\frac{\alpha p}{2}} \right).$$

In other words, \hat{f}^o is better in the maxiset sense than \hat{f}^h .

We conclude that the BlockShrink procedures \hat{f}^o can enjoy better maxiset properties than the hard thresholding procedures when we take the problem under the \mathbb{L}^p risk. A similar result has been pointed out by Autin (2005, Chapter 6, Proposition 6.3) for the white noise model under the Besov risk.

Let us notice that our result is proved for $p \geq 2$. So, do we have we the embedding

$$\mathcal{M}_{n_0} \left(\hat{f}^h, p, n^{-\frac{\alpha p}{2}} \right) \subseteq \mathcal{M}_{n_0} \left(\hat{f}^o, p, n^{-\frac{\alpha p}{2}} \right)$$

for the case $1 < p < 2$? This is an open question.

In practice, numerous simulations show that the BlockShrink estimator has excellent numerical performance relative to more traditional wavelet estimators (for further details on these simulations, see for instance <http://stat.wharton.upenn.edu/~tcai/paper/html/Neighblock.html>).

4 Minimax results over Besov balls

We investigate in this subsection the upper bounds of the block thresholding rules over Besov balls under the \mathbb{L}^p for general statistical models (including certain inverse problems).

Definition 4.1 (Besov balls). Let $N \in \mathbb{N}^*$, $R > 0$, $0 < s < N$, $1 \leq r \leq \infty$ and $1 \leq \pi \leq \infty$. For any function f measurable on $[0, 1]$, we denote the associated N -th order modulus of smoothness as

$$\rho^N(t, f, \pi) = \sup_{|h| \leq t} \left(\int_{J_{Nh}} \left| \sum_{k=0}^N \binom{N}{k} (-1)^k f(u + kh) \right|^\pi du \right)^{\frac{1}{\pi}}$$

where $J_{Nh} = \{x \in [0, 1] : x + Nh \in [0, 1]\}$. We say that a function f of $\mathbb{L}^\pi([0, 1])$ belongs to the Besov balls $B_{\pi, r}^s(R)$ if and only if

$$\left(\int_0^1 \left(\frac{\rho^N(t, f, \pi)}{t^s} \right)^r \frac{1}{t} dt \right)^{\frac{1}{r}} \leq R < \infty$$

with the usual modification if $r = \infty$.

The equivalence below shows the link which exists between the Besov balls and the wavelet basis ζ . See [Meyer \(1990\)](#) for a detailed proof. For $0 < s < N$ and $1 \leq \pi \leq \infty$, we have :

$$f \in B_{\pi, r}^s(R) \iff \begin{cases} (\sum_{j \geq \tau-1} (2^{j(s+\frac{1}{2}-\frac{1}{\pi})} (\sum_{k \in \Delta_j} |\beta_{j,k}|^\pi)^{\frac{1}{\pi}})^r)^{\frac{1}{r}} \leq R < \infty & \text{if } r < \infty, \\ \sup_{j \geq \tau-1} 2^{j(s+\frac{1}{2}-\frac{1}{\pi})} (\sum_{k \in \Delta_j} |\beta_{j,k}|^\pi)^{\frac{1}{\pi}} \leq R < \infty & \text{if } r = \infty. \end{cases}$$

The minimax results presented in [Theorem 4.1](#) below are direct consequences of the maxiset results developed in [Theorem 3.2](#).

Theorem 4.1 (Minimax results over Besov balls (regular zone)). Let $1 < p < \infty$, $R > 0$, $\delta \geq 0$, $\sigma = (2^{\delta j})_j$ and \hat{f} be the BlockShrink procedure taken with $\nu = \frac{2}{1+2\delta}$ (see [Definition 2.1](#)). Under the assumptions 1–4, for $s > 0$, $r \geq 1$, $\pi \geq p$ (regular zone), any $\kappa > \kappa_0$ with κ_0 large enough, there exists a constant $C > 0$ such that :

$$\sup_{f \in B_{\pi, r}^s(R)} \mathbb{E}_f^n (\|\hat{f} - f\|_p^p) \leq C n^{-\frac{sp}{1+2s+2\delta}}, \quad n \geq n_0.$$

If we compare the previous upper bound with that reached by the hard thresholding procedures under the \mathbb{L}^p risk over the regular zone of Besov balls, we remark that it is better in the sense where it is without logarithmic factor (see for instance [Kerkycharian and Picard \(2000, Theorem 6.1\)](#)).

Remark 4.1. *Theorem 4.1 can be viewed as a generalization of several minimax results obtained by [Juditsky \(1997\)](#), [Cai \(1998\)](#) and [Pensky and Vidakovic \(1998\)](#).*

In order to complete the previous results, [Theorem 4.2](#) below investigates the upper bounds for the BlockShrink procedure over the Besov balls for $p > \pi$. In particular, we show that they are similar to the upper bounds attained by the hard the hard thresholding procedure.

Theorem 4.2 (Minimax results over Besov balls for the \hat{f}^o : sparse and critical zone). Let $2 \leq p < \infty$, $R > 0$, $\delta \geq 0$, $\sigma = (2^{\delta j})_j$ and \hat{f}^o be the BlockShrink procedure taken with $\nu = \frac{2}{1+2\delta}$ (see [Definition 2.1](#)). Under the assumptions 1–4, for $s > 0$, $r \geq 1$, $p > \pi$, any $\kappa > \kappa_0$ with κ_0 large enough, and n_0 large enough, there exists a constant $C > 0$ such that :

$$\begin{cases} \sup_{f \in B_{\pi, r}^s(R)} \mathbb{E}_f^n (\|\hat{f}^o - f\|_p^p) \leq C \left(\frac{\ln(n)}{n} \right)^{\frac{sp}{2s+2\delta+1}}, & \text{if } \pi s + (\delta + \frac{1}{2})(\pi - p) > 0, \\ \sup_{f \in B_{\pi, r}^s(R)} \mathbb{E}_f^n (\|\hat{f}^o - f\|_p^p) \leq C \left(\frac{\ln(n)}{n} \right)^{\frac{(s-\frac{1}{\pi}+\frac{1}{p})p}{2(s-\frac{1}{\pi}+\delta)+1}}, & \text{if } \pi s + (\delta + \frac{1}{2})(\pi - p) < 0, \\ \sup_{f \in B_{\pi, r}^s(R)} \mathbb{E}_f^n (\|\hat{f}^o - f\|_p^p) \leq C \left(\frac{\ln(n)}{n} \right)^{\frac{(s-\frac{1}{\pi}+\frac{1}{p})p}{2(s-\frac{1}{\pi}+\delta)+1}} \ln(n)^{(p-\frac{\pi}{2})_+}, & \text{if } \pi s + (\delta + \frac{1}{2})(\pi - p) = 0, \end{cases}$$

for $n \geq n_0$.

It is important to mention that [Theorem 4.2](#) above can not be apply for the global thresholding procedure. The lengths of the blocks $l_j = \ln(n)^{\frac{\delta}{2}}$ play a determinant role in the proof.

The rate of convergence exhibit in the previous theorems can be minimax for several statistical models under the \mathbb{L}^p risk over Besov balls. In other words, for several statistical models and several values of s , π and r , the BlockShrink procedure satisfies :

$$\inf_{\tilde{f}} \sup_{f \in B_{\pi,r}^s(L)} \mathbb{E}_f^n(\|\tilde{f} - f\|_p^p) \asymp \sup_{f \in B_{\pi,r}^s(L)} \mathbb{E}_f^n(\|\hat{f}^o - f\|_p^p), \quad p > 1,$$

where $\inf_{\tilde{f}}$ is the infimum taken over all the estimators \tilde{f} . It is important to notice the assumptions 1 – 4 are satisfied with $\delta = 0$ for the majority of the 'standard' models (white noise, regression with equispaced data, for instance). The choice of $\delta > 0$ often appears in the context of inverse problems.

Section 5 below gives two applications of the results above by considering two statistical models :

1. the regression model with Gaussian errors,
2. a deconvolution in white noise.

5 Applications

Here and later, C denotes a constant (independent of n and f) which may be different from one term to the other.

5.1 Equispaced regression model with Gaussian errors

In this subsection, assume that we observe the random variables (Y_1, \dots, Y_n) governed by the equation (1.1). For minimax studies under various setting concerning this model, we refer the reader to the book of [Tsybakov \(2004\)](#). Let \hat{f} be the block thresholding procedures described by (2.4) with $\sigma = 1$, $\delta = 0$, $\nu = 2$ and the estimators :

$$\hat{\alpha}_{j,k} = \frac{1}{n} \sum_{i=1}^n Y_i \phi_{j,k}(\frac{i}{n}) \quad \text{and} \quad \hat{\beta}_{j,k} = \frac{1}{n} \sum_{i=1}^n Y_i \psi_{j,k}(\frac{i}{n}).$$

In order to apply our Theorems and Propositions, the rest of the subsection focus on the assumptions 1 – 5.

For any b in $]0, 1]$, let us defined the Hölder balls $\mathcal{H}^b(L)$ by :

$$\mathcal{H}^b(L) = \{h \text{ measurable on } [0, 1]; |h(x) - h(y)| \leq L|x - y|^b, \quad x, y \in [0, 1]\}.$$

Let us introduce an immediate consequence of [Donoho and Johnstone \(1999, Lemma 5.1\)](#) :

Lemma 5.1. *For any $j \geq \tau$ and $k, k' \in \Delta_j$, there exists a constant $C > 0$ such that*

$$n^{-1} \sum_{i=1}^n \phi_{j,k}(\frac{i}{n}) \phi_{j,k'}(\frac{i}{n}) \leq C \left(1_{\{k=k'\}} + 2^j n^{-1} 1_{\{S_{j,k} \cap S_{j,k'} \neq \emptyset\}} \right)$$

where $S_{j,k}$ denotes the support of $\psi_{j,k}$. This inequality holds if we replace ϕ by ψ .

Under the assumption that $f \in \mathcal{H}^{\frac{1}{2}}(L)$, the inequalities obtained in [Picard and Tribouley \(2000, Subsection 9.1.2\)](#) and [Lemma 5.1](#) imply that :

$$\begin{aligned} \mathbb{E}_f^n(|\hat{\beta}_{j,k} - \beta_{j,k}|^{2p}) &\leq 2^{p-1} \left(\left| \frac{1}{n} \sum_{i=1}^n (f(\frac{i}{n}) \psi_{j,k}(\frac{i}{n}) - \beta_{j,k}) \right|^{2p} + \mathbb{E}_f^n \left(\left| \frac{1}{n} \sum_{i=1}^n \psi_{j,k}(\frac{i}{n}) z_i \right|^{2p} \right) \right) \\ &\leq C \left(n^{-p} 2^{-jp} + 2^{jp} n^{-2p} + n^{-p} \left(\frac{1}{n} \sum_{i=1}^n |\psi_{j,k}(\frac{i}{n})|^2 \right)^p \right) \leq C n^{-p} \end{aligned}$$

for any $j_1 \leq j < j_2$. Thus the assumptions 1 – 2 are satisfied.

Since $2^{j_2} \asymp n$, it is obvious that :

$$\sum_{j < j_2} \sum_{k \in \Delta_j} \|\psi_{j,k}\|_p^p = \sum_{j < j_2} 2^{\frac{jp}{2}} \leq C 2^{\frac{j_2 p}{2}} \leq C n^{\frac{p}{2}}.$$

The assumption 4 is hence satisfied.

Let us focus on the assumption 3. Lemma 5.2 below will be intensively used in the sequel.

Lemma 5.2 (Cirelson, Ibragimov, Sudakov's inequality (Cirelson et al. (1976))). *Let $(\eta_t, t \in T)$ be a Gaussian process. Let N and W be respectively defined by $\mathbb{E}(\sup_{t \in T} \eta_t)$ and $\sup_{t \in T} \text{Var}(\eta_t)$. Then for all $c > 0$ we have :*

$$\mathbb{P}(\sup_{t \in T} \eta_t \geq c + N) \leq \exp\left(-\frac{c^2}{2W}\right). \quad (5.1)$$

• Firstly, let us consider the case where $p \geq 2$.

Let us set

$$e_{j,k,n} = \frac{1}{n} \sum_{i=1}^n \psi_{j,k}\left(\frac{i}{n}\right) z_i,$$

$\mathcal{C}_q = \left\{ a = (a_{j,k}); \sum_{(K)} |a_{j,k}|^q \leq 1 \right\}$ where q is the real number satisfying $q^{-1} + p^{-1} = 1$, and $\{\mathcal{Z}(a), a \in \mathcal{C}_q\}$ the centered Gaussian process defined by :

$$\mathcal{Z}(a) = \sum_{(K)} a_{j,k} e_{j,k,n}.$$

An argument of duality gives us :

$$\sup_{a \in \mathcal{C}_q} \mathcal{Z}(a) = \left(\sum_{(K)} |e_{j,k,n}|^p \right)^{\frac{1}{p}}.$$

This allows us to apply the inequality (5.1). Using Holder's inequality and the assumption 2 which has been previously proved, one gets :

$$N = \mathbb{E}_f^n(\sup_{a \in \mathcal{C}_q} \mathcal{Z}(a)) \leq \left(\sum_{(K)} \mathbb{E}_f^n(|e_{j,k,n}|^p) \right)^{\frac{1}{p}} \leq C n^{-\frac{1}{2}} l_j^{\frac{1}{p}}. \quad (5.2)$$

Since $\mathbb{E}_f^n(e_{j,k,n}) = 0$ and $\mathbb{E}_f^n(z_i z_{i'}) = 1$ if $i = i'$ and 0 otherwise, it follows from Lemma 5.1 and l_p -Hölder's inequality that :

$$\begin{aligned} W &= \sup_{a \in \mathcal{C}_q} \text{Var}_f^n(\mathcal{Z}(a)) = \sup_{a \in \mathcal{C}_q} \mathbb{E}_f^n\left(\sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} a_{j,k'} e_{j,k,n} e_{j,k',n} \right) \\ &= \sup_{a \in \mathcal{C}_q} \left(n^{-2} \sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} a_{j,k'} \sum_{i=1}^n \sum_{i'=1}^n \psi_{j,k}\left(\frac{i}{n}\right) \psi_{j,k'}\left(\frac{i'}{n}\right) \mathbb{E}_f^n(z_i z_{i'}) \right) \\ &= n^{-1} \sup_{a \in \mathcal{C}_q} \sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} a_{j,k'} \left(n^{-1} \sum_{i=1}^n \psi_{j,k}\left(\frac{i}{n}\right) \psi_{j,k'}\left(\frac{i}{n}\right) \right) \\ &\leq C n^{-1} \sup_{a \in \mathcal{C}_q} \left(\sum_{k \in \mathcal{B}_{j,K}} a_{j,k}^2 + \sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} a_{j,k'} 1_{\{S_{j,k} \cap S_{j,k'} \neq \emptyset\}} \right). \end{aligned} \quad (5.3)$$

Let us denote $\tilde{S}_{j,k}(x) = \{k; x \in S_{j,k}\}$. If $S_{j,k} \cap S_{j,k'} \neq \emptyset$ then there exists x of $[0, 1]$ such that $x \in S_{j,k}$ and $x \in S_{j,k'}$. Considering such x , Hölder's inequality, the fact that ψ is compactly supported and that $q \leq 2$ yield :

$$\begin{aligned} \sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} a_{j,k'} 1_{\{S_{j,k} \cap S_{j,k'} \neq \emptyset\}} &\leq \left(\sum_{k \in \mathcal{B}_{j,K}} a_{j,k} 1_{\{\tilde{S}_{j,k}(x)\}} \right)^2 \leq \sum_{k \in \mathcal{B}_{j,K}} a_{j,k}^2 \sum_{k \in \mathcal{B}_{j,K}} 1_{\{\tilde{S}_{j,k}(x)\}} \\ &\leq C \sum_{k \in \mathcal{B}_{j,K}} a_{j,k}^2 \leq \left(\sum_{k \in \mathcal{B}_{j,K}} |a_{j,k}|^q \right)^{\frac{1}{q}}. \end{aligned} \quad (5.4)$$

Putting (5.3) and (5.4) together, we observe that $W \leq C n^{-1}$. Taking $\lambda > 0$ large enough and c of the form $(\lambda - 2C) n^{-\frac{1}{2}} l_j^{\frac{1}{p}}$, the inequality (5.1) says that there exists a positive function h which satisfies

$\lim_{x \rightarrow +\infty} h(x) = \infty$ and :

$$\begin{aligned} \mathcal{V}_n &= \mathbb{P}_f^n \left((l_j^{-1} \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^p)^{\frac{1}{p}} \geq \lambda n^{-\frac{1}{2}} \right) \leq \mathbb{P}_f^n \left((l_j^{-1} \sum_{(K)} |e_{j,k,n}|^p)^{\frac{1}{p}} \geq (\lambda - C)n^{-\frac{1}{2}} \right) \\ &\leq \mathbb{P}_f^n \left(\sup_{a \in \mathcal{C}_q} \mathcal{Z}(a) \geq c + N \right) \leq \exp(-l_j^{\frac{2}{p}} h(\lambda)). \end{aligned}$$

Thus, if we work with the global thresholding procedure (i.e $l_j \asymp 2^j$ and $2^{j_1} \asymp \ln(n)^{\frac{p}{2}}$) then for $j_1 < j$ we have :

$$\mathcal{V}_n \leq \exp(-2^{\frac{2j_1}{p}} h(\lambda)) \leq n^{-h(\lambda)}.$$

If we work with the BlockShrink procedure (i.e $l_j \asymp \ln(n)^{\frac{p}{2}}$) then we have clearly :

$$\mathcal{V}_n \leq n^{-h(\lambda)}.$$

• Secondly, let us consider the case where $p \leq 2$. Thanks to the comparison between the l_p norm, one gets :

$$\begin{aligned} \mathbb{P}_f^n \left((l_j^{-1} \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^p)^{\frac{1}{p}} \geq \lambda n^{-\frac{1}{2}} \right) &\leq \mathbb{P}_f^n \left((l_j^{-1} \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^2)^{\frac{1}{2}} \geq \lambda n^{-\frac{1}{2}} \right) \\ &\leq \exp(-l_j h(\lambda)). \end{aligned}$$

Thus, if we work with the global thresholding procedure (i.e $l_j \asymp 2^j$ and $2^{j_1} \asymp \ln(n)$ for $p \leq 2$) then for $j_1 < j$ we have :

$$\mathcal{V}_n \leq \exp(-2^{j_1} h(\lambda)) \leq n^{-h(\lambda)}.$$

If we work with the BlockShrink procedure (i.e $l_j \asymp \ln(n)$) then we have clearly :

$$\mathcal{V}_n \leq n^{-h(\lambda)}.$$

In each case, we obtain the desired result. The assumption 3 holds.

By virtue of a Gaussian inequality and Lemma 5.1, one gets :

$$\begin{aligned} \mathbb{P}_f^n \left(|\hat{\beta}_{j,k} - \beta_{j,k}| \geq \lambda \sqrt{\frac{\ln(n)}{n}} \right) &\leq \mathbb{P}_f^n \left(\left| \frac{1}{n} \sum_{i=1}^n \psi_{j,k} \left(\frac{i}{n} \right) z_i \right| \geq \lambda \sqrt{\frac{\ln(n)}{n}} \right) \\ &\leq 2 \exp \left(- \frac{\lambda^2 \ln(n)}{\frac{2}{n} \sum_{i=1}^n |\psi_{j,k} \left(\frac{i}{n} \right)|^2} \right) \leq C n^{-h(\lambda)} \end{aligned}$$

for any $\lambda > 0$. So the assumption 5 holds.

Therefore, if we suppose that $f \in \mathcal{H}^{\frac{1}{2}}(L)$ then we are in position to apply Theorem 3.1, Theorem 3.2, Proposition 3.1, Proposition 3.2 and Theorem 4.1.

5.2 A deconvolution in white noise

Here, we consider an application of our maxiset results to a deconvolution problem which appears in Cavalier and Tsybakov (2002) and Johnstone et al. (2004). Suppose that we observe the random process $(Y_t)_{t \in [0,1]}$ defined by :

$$dY_t = f \star g(t) dt + n^{-\frac{1}{2}} dB_t, \quad t \in [0, 1],$$

where $(B_t)_{t \in [0,1]}$ is a standard Brownian motion on $[0, 1]$ and

$$f \star g(t) = \int_0^1 f(t-u)g(u)du.$$

We wish to recover the unknown function f via the the observations $(Y_t)_{t \in [0,1]}$. The function g is assumed to be known. Further, we assume that f and g are periodic on the unit interval and that the Fourier coefficients of g decay in a polynomial fashion i.e there exists a real number $\delta > 2^{-1}$ satisfying :

$$\mathcal{F}(g)(l) \asymp |l|^{-\delta}, \quad l \in \mathbb{Z}, \quad (5.5)$$

where \mathcal{F} is the Fourier transform defined by $\mathcal{F}(h)(l) = \int_0^1 h(x)e^{-2i\pi lx} dx$ for $h \in \mathbb{L}^1([0, 1])$.

Instead of working with the basis ζ as before, we consider a basis constructed from Meyer-type wavelet adapted to the unit interval by periodization. We denote this family by :

$$\zeta^M = \{\phi_{\tau,k}^M(\cdot), k = 0, \dots, 2^\tau - 1; \psi_{j,k}^M(\cdot); j \geq \tau, k = 0, \dots, 2^j - 1\}.$$

The main advantage of this choice is that the Fourier transforms of ψ^M and ϕ^M are compactly supported. Moreover, the Temlyakov property and the unconditional property described in Lemma 2.1 hold for ζ^M and ζ_σ^M (see Johnstone et al. (2004)).

Let \hat{f} be the block thresholding procedure defined by (2.4) with $\sigma = (2^{\delta j})_j$, $\nu = \frac{2}{1+2\delta}$ and the following unbiased estimators of $\alpha_{j,k}$ and $\beta_{j,k}$:

$$\hat{\alpha}_{j,k} = \sum_{l \in C_j} \frac{\mathcal{F}^*(Y)(l)}{\mathcal{F}(g)(l)} \mathcal{F}(\phi_{j,k}^M)(l) \quad \text{and} \quad \hat{\beta}_{j,k} = \sum_{l \in C_j} \frac{\mathcal{F}^*(Y)(l)}{\mathcal{F}(g)(l)} \mathcal{F}(\psi_{j,k}^M)(l)$$

where :

$$- C_j = \{l; \mathcal{F}(\psi_{j,k}^M)(l) \neq 0\} = \{l; |l| \in [2\pi 3^{-1} 2^j, 8\pi 3^{-1} 2^j]\},$$

- starting from a process $(R_t)_{t \in [0,1]}$ which satisfies $\mathbb{E}(|R_t|) < \infty$, the operator $\mathcal{F}^*(R)$ is defined by

$$\mathcal{F}^*(R)(l) = \int_0^1 e^{-2i\pi lx} dR_x.$$

The assumptions 1 – 2 and 5 have been shown in Johnstone et al. (2004, Section 5).

Since $2^{j_2} \asymp n^{\frac{1}{1+2\delta}}$, it is obvious that :

$$\sum_{j < j_2} \sum_{k \in \Delta_j} \sigma_j^p \|\psi_{j,k}^M\|_p^p = \sum_{j < j_2} 2^{jp(\delta + \frac{1}{2})} \leq C 2^{j_2 p(\delta + \frac{1}{2})} \leq C n^{\frac{p}{2}}.$$

Hence the assumption 4 holds.

Let us prove that the assumption 3 is satisfied.

• For the the case $p \geq 2$. As in the previous subsection, the aim is to apply the concentration inequality (5.1). Let us set

$$\tilde{e}_{j,k,n} = \hat{\beta}_{j,k} - \beta_{j,k} = n^{-\frac{1}{2}} \sum_l \frac{\mathcal{F}^*(B)(l)}{\mathcal{F}(g)(l)} \mathcal{F}(\psi_{j,k}^M)(l)$$

and let us observe that $\mathcal{F}^*(B)(l) \sim \mathcal{N}(0, 1)$ and $\mathbb{E}_f^n(\mathcal{F}^*(B)(l)\mathcal{F}^*(B)(l')) = \int_0^1 e^{-2i\pi(l-l')t} dt = 1$ if $l = l'$ and 0 otherwise. Proceeding in the same way as in (5.2), let us consider the centered Gaussian process $\{\tilde{\mathcal{Z}}(a), a \in C_q\}$ defined by :

$$\tilde{\mathcal{Z}}(a) = \sum_{(K)} a_{j,k} \tilde{e}_{j,k,n}.$$

By virtue of the Holder inequality and the assumption 2 which has been previously proved, we have :

$$N = \mathbb{E}_f^n(\sup_{a \in C_q} \tilde{\mathcal{Z}}(a)) \leq \left(\sum_{(K)} \mathbb{E}_f^n(|\hat{\beta}_{j,k} - \beta_{j,k}|^p) \right)^{\frac{1}{p}} \leq C n^{-\frac{1}{2}} l_j^{\frac{1}{p}} 2^{\delta j}.$$

Moreover, under the assumption (5.5), let us remark that :

$$\sup_{l \in C_j} |\mathcal{F}(g)(l)|^{-2} \leq C \sup_{l \in C_j} |l|^{2\delta} \leq C 2^{2\delta j} (= C \sigma_j^2).$$

Taking in account the previous remarks and applying Plancherel's inequality, we can dominate W by :

$$\begin{aligned}
W &= \sup_{a \in \mathcal{C}_q} \text{Var}_f^n(\tilde{Z}(a)) = \sup_{a \in \mathcal{C}_q} \mathbb{E}_f^n \left(\sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} a_{j,k'} \tilde{e}_{j,k,n} \tilde{e}_{j,k',n} \right) \\
&= n^{-1} \sup_{a \in \mathcal{C}_q} \left(\sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} \overline{a_{j,k'}} \sum_l \sum_{l'} \mathcal{F}(g)(l)^{-1} \mathcal{F}(\psi_{j,k}^M)(l) \dots \right. \\
&\quad \left. \dots \overline{\mathcal{F}(g)(l')}^{-1} \mathcal{F}(\psi_{j,k'}^M)(l') \mathbb{E}_f^n(\mathcal{F}^*(B)(l) \mathcal{F}^*(B)(l')) \right) \\
&= n^{-1} \sup_{a \in \mathcal{C}_q} \left(\sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} \overline{a_{j,k'}} \sum_l |\mathcal{F}(g)(l)|^{-2} \mathcal{F}(\psi_{j,k}^M)(l) \overline{\mathcal{F}(\psi_{j,k'}^M)(l)} \right) \\
&\leq n^{-1} \sup_{a \in \mathcal{C}_q} \left(\sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} \overline{a_{j,k'}} \sup_{l \in \mathcal{C}_j} |\mathcal{F}(g)(l)|^{-2} \sum_l \mathcal{F}(\psi_{j,k}^M)(l) \overline{\mathcal{F}(\psi_{j,k'}^M)(l)} \right) \\
&= n^{-1} \sup_{l \in \mathcal{C}_j} |\mathcal{F}(g)(l)|^{-2} \sup_{a \in \mathcal{C}_q} \left(\sum_{k \in \mathcal{B}_{j,K}} \sum_{k' \in \mathcal{B}_{j,K}} a_{j,k} \overline{a_{j,k'}} \int \psi_{j,k}^M(x) \overline{\psi_{j,k'}^M(x)} dx \right) \\
&= n^{-1} \sup_{l \in \mathcal{C}_j} |\mathcal{F}(g)(l)|^{-2} \sup_{a \in \mathcal{C}_q} \left(\sum_{k \in \mathcal{B}_{j,K}} |a_{j,k}|^2 \right) \leq C 2^{2\delta j} n^{-1}.
\end{aligned}$$

Taking λ large enough and c of the form $(\lambda - C)n^{-\frac{1}{2}}l_j^{\frac{1}{p}}2^{\delta j}$, the inequality (5.1) says that there exists a positive function h which satisfies $\lim_{x \rightarrow +\infty} h(x) = \infty$ and :

$$\mathcal{V}_n = \mathbb{P}_f^n \left((l_j^{-1} \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^p)^{\frac{1}{p}} \geq \sigma_j \lambda n^{-\frac{1}{2}} \right) \leq \mathbb{P}_f^n \left(\sup_{a \in \mathcal{C}_q} \tilde{Z}(a) \geq c + N \right) \leq \exp(-l_j^{\frac{2}{p}} h(\lambda)).$$

we conclude by proceeding exactly as in the previous subsection.

- For $p \leq 2$. We have clearly :

$$\mathcal{V}_n = \mathbb{P}_f^n \left((l_j^{-1} \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^p)^{\frac{1}{p}} \geq \sigma_j \lambda n^{-\frac{1}{2}} \right) \leq \mathbb{P}_f^n \left((l_j^{-1} \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^2)^{\frac{1}{2}} \geq \sigma_j \lambda n^{-\frac{1}{2}} \right) \leq \exp(-l_j h(\lambda))$$

Using again similar arguments than the previous subsection, we deduce that the assumption 3 holds.

Therefore, if we suppose that g satisfies the condition (5.5) then we are in position to apply Theorem 3.1, Theorem (3.2), Proposition 3.1, Proposition 3.2 and Theorem 4.1.

Remark 5.1. *In the two previous statistical models, we have proved that the corresponding BlockShrink procedure*

1. *is better in the maxiset sense than the corresponding hard thresholding procedure under the \mathbb{L}^p risk and for the rate of convergence $n^{-\frac{ap}{2}}$,*
2. *is optimal in the minimax sense over Besov balls $B_{\pi,r}^s(R)$ under the \mathbb{L}^p risk if $\pi \geq p > 1$, $r \geq 1$ and $s > 0$ contrary to the corresponding hard thresholding procedure.*

6 Appendix : proofs of Theorems and Propositions

Before proving our Theorems and Propositions, let us set several important remarks which will be determinant in the sequel.

Preliminary remarks and Lemmas : For any sequence $u = (u_{j,k})_{j,k}$, $v = (v_{j,K})_{j,K}$ and any $j \geq \tau$, we can write :

$$\sum_K \sum_{(K)} u_{j,k} v_{j,K} \psi_{j,k}(x) = \sum_{k \in \Delta_j} u_{j,k}^* \psi_{j,k}(x),$$

where the coefficients $(u_{j,0}^*, u_{j,1}^*, \dots, u_{j,2^j}^*)$ are respectively :

$$\underbrace{(u_{j,0}^* v_{j,1}, u_{j,1}^* v_{j,1}, \dots, u_{j,l_j-1}^* v_{j,1})}_{\text{length } l_j} \underbrace{(u_{j,l_j}^* v_{j,2}, \dots, \dots, u_{j,2^j}^* v_{j,2^{j-1}})}_{\text{length } l_j}.$$

It follows from the unconditional nature of ζ (see Lemma 2.1) implies that :

$$\begin{aligned} \left\| \sum_{j \geq \tau-1} \sum_K \sum_{(K)} u_{j,k} v_{j,K} \psi_{j,k} \right\|_p^p &\asymp \left\| \left(\sum_{j \geq \tau-1} \sum_{k \in \Delta_j} |u_{j,k}^* \psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \\ &= \left\| \left(\sum_{j \geq \tau-1} \sum_K \sum_{(K)} |u_{j,k} v_{j,K} \psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p. \end{aligned} \quad (6.1)$$

Moreover, Lemma 2.2 says that for any sequence $u = (u_{j,k})_{j,k}$, $v = (v_{j,K})_{j,K}$ and any $j \geq \tau$ we have :

$$\left\| \sum_K \sum_{(K)} u_{j,k} v_{j,K} \psi_{j,k} \right\|_p^p \leq C 2^{j(\frac{p}{2}-1)} \sum_{k \in \Delta_j} |u_{j,k}^*|^p = C 2^{j(\frac{p}{2}-1)} \sum_K \sum_{(K)} |u_{j,k} v_{j,K}|^p. \quad (6.2)$$

For sake of simplicity in exposition of the proofs, we shall refer to the inequalities above when it is necessary.

We now recall two elementary inequalities for the l_p -norms.

Lemma 6.1. *Let $a = (a_i)_i \in \mathbb{R}^m$, $m \in \mathbb{N}^*$ and $0 < u \leq v \leq \infty$. Then the following inequalities hold :*

$$\left(\sum_{i=1}^m |a_i|^v \right)^{\frac{1}{v}} \leq \left(\sum_{i=1}^m |a_i|^u \right)^{\frac{1}{u}} \leq m^{\frac{1}{u} - \frac{1}{v}} \left(\sum_{i=1}^m |a_i|^v \right)^{\frac{1}{v}}. \quad (6.3)$$

We can now proceed to the proofs of the theorems and propositions. The proofs of Theorem 3.1 and Theorem 3.2 below are based on mathematical arguments similar to Kerkycharian and Picard (2000, Proofs of Theorems 5.1 and 5.2).

Proof of Theorem 3.1. Let $p > 1$. Let us show that for any constant $C > 0$ we have :

$$\begin{cases} \mathbb{E}_f^n (\|\hat{f}^g - f\|_p^p) \leq C n^{-\frac{\alpha p}{2}} \Rightarrow f \in \mathcal{B}_{p,\infty}^{\frac{\alpha}{p}} \cap \mathcal{W}_\sigma((1-\alpha)p, p), & n > n_0, \\ \mathbb{E}_f^n (\|\hat{f}^o - f\|_p^p) \leq C n^{-\frac{\alpha p}{2}} \Rightarrow f \in \mathcal{B}_{p,\infty}^{\frac{\alpha}{p}} \cap \overline{\mathcal{W}}_{\sigma,\kappa,n_0}^*((1-\alpha)p, p), & n > n_0, \end{cases}$$

for $\kappa > \kappa_0$ with κ_0 large enough, and n_0 large enough.

Let \hat{f} be the block thresholding procedure defined by (2.4) (including \hat{f}^g and \hat{f}^o).

For any constant $C > 0$ such that $\mathbb{E}_f^n (\|\hat{f} - f\|_p^p) \leq C n^{-\frac{\alpha p}{2}}$, the inequality (6.1) yields :

$$\begin{aligned} n^{-\frac{\alpha p}{2}} C &\geq \mathbb{E}_f^n (\|\hat{f} - f\|_p^p) \\ &\geq C \mathbb{E}_f^n \left(\left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\hat{\beta}_{j,k} 1_{\{\hat{b}_{j,K}(p) \geq \sigma_j \kappa n^{-\frac{1}{2}}\}} - \beta_{j,k}|^2 |\psi_{j,k}|^2 + \sum_{j \geq j_2} \sum_{k \in \Delta_j} |\beta_{j,k} \psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \right) \\ &\geq C \max(T_1, T_2) \end{aligned}$$

where we have set :

$$T_1 = \mathbb{E}_f^n \left(\left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\beta_{j,k}|^2 1_{\{\hat{b}_{j,K}(p) < \sigma_j \kappa n^{-\frac{1}{2}}\}} |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \right)$$

and

$$T_2 = \left\| \left(\sum_{j \geq j_2} \sum_{k \in \Delta_j} |\beta_{j,k} \psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p.$$

Thus we have :

$$\max(T_1, T_2) \leq C n^{-\frac{\alpha p}{2}}. \quad (6.4)$$

• Let us show that $f \in \mathcal{B}_{p,\infty}^{\frac{\alpha}{p}}$. Using the inequalities (2.1) and (6.4), one gets :

$$\left\| \sum_{j \geq j_2} \sum_{k \in \Delta_j} \beta_{j,k} \psi_{j,k} \right\|_p^p \leq C \left\| \left(\sum_{j \geq j_2} \sum_{k \in \Delta_j} |\beta_{j,k} \psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p = C T_2 \leq C n^{-\frac{\alpha p}{2}}. \quad (6.5)$$

So, if $u \leq c$ where c denotes a small constant then the inequality (2.1) gives us :

$$\left\| \sum_{j \geq u} \sum_{k \in \Delta_j} \beta_{j,k} \psi_{j,k} \right\|_p^p \leq C \|f\|_p^p \leq C 2^{-u \frac{\alpha p}{\nu}}.$$

For $u > c$, let us remark that there exists an integer n satisfying $j_2 < u < j_2 + 1$. Using the inequalities (2.1) and (6.5), one gets :

$$\left\| \sum_{j \geq u} \sum_{k \in \Delta_j} \beta_{j,k} \psi_{j,k} \right\|_p^p \leq C \left\| \sum_{j \geq j_2} \sum_{k \in \Delta_j} \beta_{j,k} \psi_{j,k} \right\|_p^p \leq C n^{-\frac{\alpha p}{2}} \leq C 2^{-j_2 \frac{\alpha p}{\nu}} \leq C 2^{-u \frac{\alpha p}{\nu}}.$$

We conclude that $f \in \mathcal{B}_{p,\infty}^{\frac{\alpha}{\nu}}$.

• Let us show that $f \in \mathcal{W}_\sigma((1-\alpha)p, p)$ if $\hat{f} = \hat{f}^g$ and $f \in \overline{\mathcal{W}}_{\sigma,\kappa,n_0}^*((1-\alpha)p, p)$ if $\hat{f} = \hat{f}^o$. Applying Minkowski's inequality with the inequality (6.3), one gets :

$$\left\| \sum_j \sum_K \sum_{(K)} \beta_{j,k} 1_{\left\{ b_{j,\kappa}(p) \leq \sigma_j \frac{\kappa n - \frac{1}{2}}{2} \right\}} \psi_{j,k} \right\|_p^p \leq 3^{p-1} (W_1 + W_2 + W_3) \quad (6.6)$$

where

$$W_1 = \left\| \sum_{j < j_1} \sum_K \sum_{(K)} \beta_{j,k} 1_{\left\{ b_{j,\kappa}(p) \leq \sigma_j \frac{\kappa n - \frac{1}{2}}{2} \right\}} \psi_{j,k} \right\|_p^p,$$

$$W_2 = \left\| \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \beta_{j,k} 1_{\left\{ b_{j,\kappa}(p) \leq \sigma_j \frac{\kappa n - \frac{1}{2}}{2} \right\}} \psi_{j,k} \right\|_p^p$$

and

$$W_3 = \left\| \sum_{j \geq j_2} \sum_K \sum_{(K)} \beta_{j,k} 1_{\left\{ b_{j,\kappa}(p) \leq \sigma_j \frac{\kappa n - \frac{1}{2}}{2} \right\}} \psi_{j,k} \right\|_p^p.$$

Let us analyze each term W_i , $i = 1, 2, 3$ in turn.

– The upper bound for the term W_1 . By virtue of Minkowski's inequality, the inequality (6.2), the fact that $|\mathcal{A}_j| = 2^j l_j^{-1}$ and that $\alpha \in]0, 1[$, one gets :

$$\begin{aligned} W_1 &\leq \left(\sum_{j < j_1} \left\| \sum_K \sum_{(K)} \beta_{j,k} \psi_{j,k} 1_{\left\{ b_{j,\kappa}(p) \leq \sigma_j \frac{\kappa n - \frac{1}{2}}{2} \right\}} \right\|_p \right)^p \\ &\leq C \left(\sum_{j < j_1} 2^{j(\frac{1}{2} - \frac{1}{p})} \left(\sum_K \sum_{(K)} |\beta_{j,k}|^p 1_{\left\{ b_{j,\kappa}(p) \leq \sigma_j \frac{\kappa n - \frac{1}{2}}{2} \right\}} \right)^{\frac{1}{p}} \right)^p \\ &= C \left(\sum_{j < j_1} 2^{j(\frac{1}{2} - \frac{1}{p})} l_j^{\frac{1}{p}} \left(\sum_K b_{j,\kappa}(p)^p 1_{\left\{ b_{j,\kappa}(p) \leq \sigma_j \frac{\kappa n - \frac{1}{2}}{2} \right\}} \right)^{\frac{1}{p}} \right)^p \\ &\leq C n^{-\frac{p}{2}} \left(\sum_{j < j_1} 2^{\frac{j}{2}} \sigma_j \right)^p \leq C n^{-\frac{p}{2}} 2^{\frac{j_1 p}{2}} \sigma_{j_1}^p \leq C n^{-\frac{\alpha p}{2}} \end{aligned} \quad (6.7)$$

for n large enough.

– The upper bound for the term W_2 . Minkowski's inequality and the inequality (6.3) yield $W_2 \leq 2^{p-1} (U_1 + U_2)$ where :

$$U_1 = \mathbb{E}_f^n \left(\left\| \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \beta_{j,k} 1_{\left\{ b_{j,\kappa}(p) \leq \sigma_j \frac{\kappa n - \frac{1}{2}}{2} \right\}} 1_{\left\{ \hat{b}_{j,\kappa}(p) < \sigma_j \kappa n^{-\frac{1}{2}} \right\}} \psi_{j,k} \right\|_p^p \right)$$

and

$$U_2 = \mathbb{E}_f^n \left(\left\| \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \beta_{j,k} 1_{\left\{ b_{j,\kappa}(p) \leq \sigma_j \frac{\kappa n - \frac{1}{2}}{2} \right\}} 1_{\left\{ \hat{b}_{j,\kappa}(p) \geq \sigma_j \kappa n^{-\frac{1}{2}} \right\}} \psi_{j,k} \right\|_p^p \right).$$

Using the inequalities (6.1) and (6.4), one obtains :

$$U_1 \leq C \mathbb{E}_f^n \left(\left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\beta_{j,k}|^2 1_{\left\{ \hat{b}_{j,\kappa}(p) < \sigma_j \kappa n^{-\frac{1}{2}} \right\}} |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \right) = C T_1 \leq C n^{-\frac{\alpha p}{2}}.$$

In order to bound the term U_2 , let us remark that l_p -Minkowski's inequality gives us :

$$\begin{aligned} \mathbb{1}_{\left\{b_{j,\kappa}(p) \leq \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2}\right\}} \mathbb{1}_{\left\{\hat{b}_{j,\kappa}(p) \geq \sigma_j \kappa n^{-\frac{1}{2}}\right\}} &\leq \mathbb{1}_{\left\{|\hat{b}_{j,\kappa}(p) - b_{j,\kappa}(p)| \geq \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2}\right\}} \\ &\leq \mathbb{1}_{\left\{(l_j^{-1} \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^p)^{\frac{1}{p}} \geq \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2}\right\}}. \end{aligned} \quad (6.8)$$

Now, let us investigate separately the case where $p \geq 2$ and the case where $1 < p \leq 2$.

For $p \geq 2$: Using the inequality (6.1), the generalized Minkowski inequality, the inequality (6.8), the assumption 3 and the inequality (2.1), it comes :

$$\begin{aligned} U_2 &\leq C \mathbb{E}_f^n \left(\left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\beta_{j,k}|^2 \mathbb{1}_{\left\{b_{j,\kappa}(p) \leq \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2}\right\}} \mathbb{1}_{\left\{\hat{b}_{j,\kappa}(p) \geq \sigma_j \kappa n^{-\frac{1}{2}}\right\}} |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \right) \\ &\leq C \left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\beta_{j,k}|^2 \mathbb{E}_f^n \left(\mathbb{1}_{\left\{b_{j,\kappa}(p) \leq \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2}\right\}} \mathbb{1}_{\left\{\hat{b}_{j,\kappa}(p) \geq \sigma_j \kappa n^{-\frac{1}{2}}\right\}} \right) \right)^{\frac{2}{p}} |\psi_{j,k}|^2 \right\|_p^p \\ &\leq C \left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\beta_{j,k}|^2 \mathbb{P}_f^n \left((l_j^{-1} \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^p)^{\frac{1}{p}} \geq \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2} \right) \right)^{\frac{2}{p}} |\psi_{j,k}|^2 \right\|_p^p \\ &\leq C n^{-h(\frac{\alpha}{2})} \left\| \left(\sum_{j < j_2} \sum_{k \in \Delta_j} |\beta_{j,k}|^2 |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \leq C \|f\|_p^p n^{-h(\frac{\alpha}{2})} \leq C n^{-\frac{\alpha p}{2}} \end{aligned}$$

for κ large enough.

For $1 < p \leq 2$: Using the inequality (6.1), the Jensen inequality, the inequalities (6.8), the assumption 3 and the inequality (2.1), one gets :

$$\begin{aligned} U_2 &\leq C \mathbb{E}_f^n \left(\left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\beta_{j,k}|^2 \mathbb{1}_{\left\{b_{j,\kappa}(p) \leq \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2}\right\}} \mathbb{1}_{\left\{\hat{b}_{j,\kappa}(p) \geq \sigma_j \kappa n^{-\frac{1}{2}}\right\}} |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \right) \\ &\leq C \left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\beta_{j,k}|^2 \mathbb{E}_f^n \left(\mathbb{1}_{\left\{b_{j,\kappa}(p) \leq \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2}\right\}} \mathbb{1}_{\left\{\hat{b}_{j,\kappa}(p) \geq \sigma_j \kappa n^{-\frac{1}{2}}\right\}} \right) \right)^{\frac{1}{2}} |\psi_{j,k}|^2 \right\|_p^p \\ &\leq C \left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\beta_{j,k}|^2 \mathbb{P}_f^n \left((l_j^{-1} \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^p)^{\frac{1}{p}} \geq \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2} \right) \right)^{\frac{1}{2}} |\psi_{j,k}|^2 \right\|_p^p \\ &\leq C n^{-h(\frac{\alpha}{2})} \left\| \left(\sum_{j < j_2} \sum_{k \in \Delta_j} |\beta_{j,k}|^2 |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \leq C \|f\|_p^p n^{-h(\frac{\alpha}{2})} \leq C n^{-\frac{\alpha p}{2}} \end{aligned}$$

for κ large enough.

Combining the upper bounds of U_1 and U_2 we deduce that for $p > 1$:

$$W_2 \leq C n^{-\frac{\alpha p}{2}}. \quad (6.9)$$

– The upper bound for the term W_3 . By virtue of the inequality (6.1), one gets :

$$W_3 \leq C \left\| \sum_{j \geq j_2} \sum_{k \in \Delta_j} \beta_{j,k} \psi_{j,k} \right\|_p^p \leq C T_2 \leq C n^{-\frac{\alpha p}{2}}. \quad (6.10)$$

Putting (6.6), (6.7), (6.9) and (6.10) together, we deduce that :

$$\left\| \sum_j \sum_K \sum_{(K)} \beta_{j,k} \mathbb{1}_{\left\{b_{j,\kappa}(p) \leq \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2}\right\}} \psi_{j,k} \right\|_p^p \leq C n^{-\frac{\alpha p}{2}} \quad (6.11)$$

for $p > 1$. It follows that $f \in \overline{\mathcal{W}}_{\sigma, \kappa, n_0}^*((1-\alpha)p, p)$ if $\hat{f} = \hat{f}^o$ (i.e $l_j = \ln(n)^{\frac{p}{2} \vee 1}$).

In order to show that $f \in \mathcal{W}_\sigma((1-\alpha)p, p)$ if $\hat{f} = \hat{f}^g$ (i.e $l_j = 2^j$), let us distinguish two cases :

In the case $u > c$ where c denotes a small constant, the inequality (6.1) gives us :

$$\left\| \sum_j \sum_K \sum_{(K)} \beta_{j,k} \mathbb{1}_{\{b_{j,\kappa}(p) \leq \sigma_j u\}} \psi_{j,k} \right\|_p^p \leq C \|f\|_p^p \leq C u^{\alpha p}.$$

For $u < c$, let us remark that there exists an integer n such that $\frac{\kappa(n+1)^{-\frac{1}{2}}}{2} < u \leq \frac{\kappa n^{-\frac{1}{2}}}{2}$. Using the inequalities (6.1) and (6.11), one gets :

$$\begin{aligned} \left\| \sum_j \sum_K \sum_{(K)} \beta_{j,k} 1_{\{b_{j,\kappa}(p) \leq \sigma_j u\}} \psi_{j,k} \right\|_p^p &\leq \left\| \sum_j \sum_K \sum_{(K)} \beta_{j,k} 1_{\{b_{j,\kappa}(p) \leq \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2}\}} \psi_{j,k} \right\|_p^p \leq C n^{-\frac{\alpha p}{2}} \\ &\leq C u^{\alpha p} \left(\frac{n}{n+1}\right)^{-\frac{\alpha p}{2}} \leq C u^{\alpha p}. \end{aligned}$$

This ends the proof of the theorem. \square

Proof of Theorem 3.2. Let $p > 1$. Let us prove that there exists a constant $C > 0$ satisfying :

$$\begin{cases} f \in \mathcal{B}_{p,\infty}^{\frac{\alpha}{v}} \cap \mathcal{W}_\sigma((1-\alpha)p, p) \Rightarrow \mathbb{E}_f^n(\|\hat{f}^g - f\|_p^p) \leq C n^{-\frac{\alpha p}{2}}, & n > n_0, \\ f \in \mathcal{B}_{p,\infty}^{\frac{\alpha}{v}} \cap \overline{\mathcal{W}}_{\sigma,\kappa,n_0}((1-\alpha)p, p) \Rightarrow \mathbb{E}_f^n(\|\hat{f}^o - f\|_p^p) \leq C n^{-\frac{\alpha p}{2}}, & n > n_0, \end{cases}$$

for $\kappa > \kappa_0$ with κ_0 large enough, and n_0 large enough.

Let \hat{f} be the block thresholding procedure defined by (2.4) (we shall distinguish the case where $\hat{f} = \hat{f}^g$ and the case where $\hat{f} = \hat{f}^o$ when it is necessary).

Combining Minkowski's inequality with (6.3), the L^p risk of \hat{f} can be bounded by a sum of four components :

$$\mathbb{E}_f^n(\|\hat{f} - f\|_p^p) \leq 4^{p-1}(G_1 + G_2 + G_3 + G_4) \quad (6.12)$$

where

$$\begin{aligned} G_1 &= \mathbb{E}_f^n \left(\left\| \sum_{k \in \Delta_{j_1}} (\hat{\alpha}_{j_1,k} - \alpha_{j_1,k}) \phi_{j_1,k} \right\|_p^p \right), \\ G_2 &= \mathbb{E}_f^n \left(\left\| \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \beta_{j,k} 1_{\{\hat{b}_{j,\kappa}(p) < \sigma_j \kappa n^{-\frac{1}{2}}\}} \psi_{j,k} \right\|_p^p \right), \\ G_3 &= \mathbb{E}_f^n \left(\left\| \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} (\hat{\beta}_{j,k} - \beta_{j,k}) 1_{\{\hat{b}_{j,\kappa}(p) \geq \sigma_j \kappa n^{-\frac{1}{2}}\}} \psi_{j,k} \right\|_p^p \right) \end{aligned}$$

and

$$G_4 = \left\| \sum_{j \geq j_2} \sum_{k \in \Delta_j} \beta_{j,k} \psi_{j,k} \right\|_p^p.$$

Let us analyze each term G_i , $i = 1, 2, 3, 4$ in turn.

- *The upper bound for the term G_1 .* It follows from the assumption 1 and Lemma 2.2 that :

$$G_1 \leq C 2^{j_1(\frac{p}{2}-1)} \sum_{k \in \Delta_{j_1}} \mathbb{E}_f^n(|\hat{\alpha}_{j_1,k} - \alpha_{j_1,k}|^p) \leq n^{-\frac{p}{2}} 2^{\frac{j_1 p}{2}} \sigma_{j_1}^p.$$

Since j_1 is chosen such that $2^{j_1} \asymp \ln(n)^{\frac{p}{2}v}$, the sequence σ satisfies $\sigma_{j_1} \asymp \ln(n)^v$ with $v > 0$ and that $\alpha \in [0, 1]$, we see that :

$$G_1 \leq C n^{-\frac{\alpha p}{2}} \quad (6.13)$$

for n large enough.

- *The upper bound for the term G_2 .* By virtue of Minkowski's inequality and the inequality (6.3), we have $G_2 \leq 2^{p-1}(G_{2,1} + G_{2,2})$ where :

$$G_{2,1} = \mathbb{E}_f^n \left(\left\| \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \beta_{j,k} 1_{\{\hat{b}_{j,\kappa}(p) < \sigma_j \kappa n^{-\frac{1}{2}}\}} 1_{\{b_{j,\kappa}(p) \leq 2\sigma_j \kappa n^{-\frac{1}{2}}\}} \psi_{j,k} \right\|_p^p \right)$$

and

$$G_{2,2} = \mathbb{E}_f^n \left(\left\| \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \beta_{j,k} 1_{\{\hat{b}_{j,\kappa}(p) < \sigma_j \kappa n^{-\frac{1}{2}}\}} 1_{\{b_{j,\kappa}(p) > 2\sigma_j \kappa n^{-\frac{1}{2}}\}} \psi_{j,k} \right\|_p^p \right).$$

– The upper bound for the term $G_{2,1}$. Let us investigate separately the case where $p \geq 2$ and the case where $1 < p \leq 2$.

For $p \geq 2$: Using the inequality (6.1) and the fact that $f \in \mathcal{W}_\sigma((1-\alpha)p, p)$ or $f \in \overline{\mathcal{W}}_{\sigma, \kappa, n_0}((1-\alpha)p, p)$ (according to the form of l_j), the term $G_{2,1}$ can be bounded by :

$$G_{2,1} \leq C \left\| \sum_j \sum_K \sum_{(K)} \beta_{j,k} 1_{\{b_{j,K}(p) \leq 2\sigma_j \kappa n^{-\frac{1}{2}}\}} \psi_{j,k} \right\|_p^p \leq C n^{-\frac{\alpha p}{2}}.$$

For $1 < p \leq 2$: By virtue of the inequalities (6.1) and (6.3), and the fact that $f \in \mathcal{W}_\sigma((1-\alpha)p, p)$ or $f \in \overline{\mathcal{W}}_{\sigma, \kappa, n_0}((1-\alpha)p, p)$, we observe that :

$$\begin{aligned} G_{2,1} &\leq C \left\| \sum_j \sum_K \sum_{(K)} \beta_{j,k} 1_{\{b_{j,K}(p) \leq 2\sigma_j \kappa n^{-\frac{1}{2}}\}} \psi_{j,k} \right\|_p^p \\ &\leq C \left\| \left(\sum_j \sum_K \sum_{(K)} |\beta_{j,k}|^2 1_{\{b_{j,K}(p) \leq 2\sigma_j \kappa n^{-\frac{1}{2}}\}} |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \\ &\leq C \sum_j \sum_K \sum_{(K)} |\beta_{j,k}|^p 1_{\{b_{j,K}(p) \leq 2\sigma_j \kappa n^{-\frac{1}{2}}\}} 2^{j(\frac{p}{2}-1)} \leq C n^{-\frac{\alpha p}{2}}. \end{aligned}$$

– The upper bound for the term $G_{2,2}$. Let us remark that the l_p -Minkowski inequality gives us :

$$\begin{aligned} 1_{\{b_{j,K}(p) > 2\sigma_j \kappa n^{-\frac{1}{2}}\}} 1_{\{\hat{b}_{j,K}(p) < \sigma_j \kappa n^{-\frac{1}{2}}\}} &\leq 1_{\{|\hat{b}_{j,K}(p) - b_{j,K}(p)| \geq \sigma_j \kappa n^{-\frac{1}{2}}\}} \\ &\leq 1_{\{(l_j^{-1} \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^p)^{\frac{1}{p}} \geq \sigma_j \kappa n^{-\frac{1}{2}}\}}. \end{aligned} \quad (6.14)$$

Let us distinguish the case where $p \geq 2$ and the case where $1 < p \leq 2$.

For $p \geq 2$: Using the inequality (6.1), the generalized Minkowski inequality, the inequality (6.14), the assumption 3 and the inequality (2.1), we have :

$$\begin{aligned} G_{2,2} &\leq C \mathbb{E}_f^n \left(\left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\beta_{j,k}|^2 1_{\{b_{j,K}(p) > 2\sigma_j \kappa n^{-\frac{1}{2}}\}} 1_{\{\hat{b}_{j,K}(p) < \sigma_j \kappa n^{-\frac{1}{2}}\}} |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \right) \\ &\leq C \left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\beta_{j,k}|^2 \mathbb{E}_f^n \left(1_{\{b_{j,K}(p) > 2\sigma_j \kappa n^{-\frac{1}{2}}\}} 1_{\{\hat{b}_{j,K}(p) < \sigma_j \kappa n^{-\frac{1}{2}}\}} \right)^{\frac{2}{p}} |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \\ &\leq C \left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\beta_{j,k}|^2 \mathbb{P}_f^n \left((l_j^{-1} \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^p)^{\frac{1}{p}} \geq \sigma_j \kappa n^{-\frac{1}{2}} \right)^{\frac{2}{p}} |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \\ &\leq C n^{-h(\kappa)} \left\| \left(\sum_{j < j_2} \sum_{k \in \Delta_j} |\beta_{j,k}|^2 |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \leq C \|f\|_p^p n^{-\frac{\alpha p}{2}} \leq C n^{-\frac{\alpha p}{2}} \end{aligned}$$

for κ large enough.

For $1 < p \leq 2$: Using the inequality (6.1), the Jensen inequality, the inequalities (6.14) and (6.3), the assumption 3 and the inequality (2.1), we have :

$$\begin{aligned} G_{2,2} &\leq C \mathbb{E}_f^n \left(\left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\beta_{j,k}|^2 1_{\{b_{j,K}(p) > 2\sigma_j \kappa n^{-\frac{1}{2}}\}} 1_{\{\hat{b}_{j,K}(p) < \sigma_j \kappa n^{-\frac{1}{2}}\}} |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \right) \\ &\leq C \left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\beta_{j,k}|^2 \mathbb{E}_f^n \left(1_{\{b_{j,K}(p) > 2\sigma_j \kappa n^{-\frac{1}{2}}\}} 1_{\{\hat{b}_{j,K}(p) < \sigma_j \kappa n^{-\frac{1}{2}}\}} \right) |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \\ &\leq C \left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\beta_{j,k}|^2 \mathbb{P}_f^n \left((l_j^{-1} \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^p)^{\frac{1}{p}} \geq \sigma_j \kappa n^{-\frac{1}{2}} \right) |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \\ &\leq C n^{-h(\kappa)} \left\| \left(\sum_{j < j_2} \sum_{k \in \Delta_j} |\beta_{j,k}|^2 |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \leq C \|f\|_p^p n^{-\frac{\alpha p}{2}} \leq C n^{-\frac{\alpha p}{2}} \end{aligned}$$

for κ large enough.

– It follows from the upper bounds of $G_{2,1}$ and $G_{2,2}$ that :

$$G_2 \leq C n^{-\frac{\alpha p}{2}} \quad (6.15)$$

for $p > 1$ and κ large enough.

• *The upper bound for the term G_3 .* Combining Minkowski's inequality with (6.3), one gets $G_3 \leq 2^{p-1}(G_{3,1} + G_{3,2})$ where :

$$G_{3,1} = \mathbb{E}_f^n \left(\left\| \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} (\hat{\beta}_{j,k} - \beta_{j,k}) \mathbf{1}_{\{\hat{b}_{j,\kappa}(p) \geq \sigma_j \kappa n^{-\frac{1}{2}}\}} \mathbf{1}_{\{b_{j,\kappa}(p) < \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2}\}} \psi_{j,k} \right\|_p^p \right)$$

and

$$G_{3,2} = \mathbb{E}_f^n \left(\left\| \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} (\hat{\beta}_{j,k} - \beta_{j,k}) \mathbf{1}_{\{\hat{b}_{j,\kappa}(p) \geq \sigma_j \kappa n^{-\frac{1}{2}}\}} \mathbf{1}_{\{b_{j,\kappa}(p) \geq \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2}\}} \psi_{j,k} \right\|_p^p \right).$$

– *The upper bound for the term $G_{3,1}$.* Let us distinguish the case where $p \geq 2$ and the case where $1 < p \leq 2$.

For $p \geq 2$: It follows from the inequality (6.1), the generalized Minkowski inequality, the inequality (6.8), the Cauchy-Schwartz inequality, the assumptions 2 – 3, the fact that ζ_σ satisfies the Temlyakov property (see Lemma 2.1, inequality (2.1)) and the assumption 4 that :

$$\begin{aligned} G_{3,1} &\leq C \mathbb{E}_f^n \left(\left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^2 \mathbf{1}_{\{\hat{b}_{j,\kappa}(p) \geq \sigma_j \kappa n^{-\frac{1}{2}}\}} \mathbf{1}_{\{b_{j,\kappa}(p) < \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2}\}} |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \right) \\ &\leq C \left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \mathbb{E}_f^n (|\hat{\beta}_{j,k} - \beta_{j,k}|^p \mathbf{1}_{\{\hat{b}_{j,\kappa}(p) \geq \sigma_j \kappa n^{-\frac{1}{2}}\}} \mathbf{1}_{\{b_{j,\kappa}(p) < \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2}\}}) \right)^{\frac{2}{p}} |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \\ &\leq C \left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \mathbb{E}_f^n (|\hat{\beta}_{j,k} - \beta_{j,k}|^{2p})^{\frac{1}{p}} \mathbb{P}_f^n \left(\left(l_j^{-1} \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^p \right)^{\frac{1}{p}} \geq \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2} \right) \right)^{\frac{1}{2}} |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \\ &\leq C n^{-\frac{p}{2}} n^{-h(\frac{\kappa}{2})} \left\| \left(\sum_{j < j_2} \sum_{k \in \Delta_j} \sigma_j^2 |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \leq C n^{-\frac{p}{2}} n^{-h(\frac{\kappa}{2})} \sum_{j < j_2} \sum_{k \in \Delta_j} \sigma_j^p \|\psi_{j,k}\|_p^p \leq C n^{-\frac{\alpha p}{2}} \end{aligned}$$

for κ large enough.

For $1 < p \leq 2$: Using the inequalities (6.1) and (6.3), the Cauchy-Schwartz inequality and the assumptions 2 – 4, we find :

$$\begin{aligned} G_{3,1} &\leq C \mathbb{E}_f^n \left(\left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^2 \mathbf{1}_{\{\hat{b}_{j,\kappa}(p) \geq \sigma_j \kappa n^{-\frac{1}{2}}\}} \mathbf{1}_{\{b_{j,\kappa}(p) < \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2}\}} |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \right) \\ &\leq C \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \mathbb{E}_f^n (|\hat{\beta}_{j,k} - \beta_{j,k}|^p \mathbf{1}_{\{\hat{b}_{j,\kappa}(p) \geq \sigma_j \kappa n^{-\frac{1}{2}}\}} \mathbf{1}_{\{b_{j,\kappa}(p) < \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2}\}}) \|\psi_{j,k}\|_p^p \\ &\leq C \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \mathbb{E}_f^n (|\hat{\beta}_{j,k} - \beta_{j,k}|^{2p})^{\frac{1}{2}} \mathbb{P}_f^n \left(\left(l_j^{-1} \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^p \right)^{\frac{1}{p}} \geq \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2} \right)^{\frac{1}{2}} \|\psi_{j,k}\|_p^p \\ &\leq C n^{-\frac{p}{2}} n^{-\frac{1}{2}h(\frac{\kappa}{2})} \sum_{j < j_2} \sum_{k \in \Delta_j} \sigma_j^p \|\psi_{j,k}\|_p^p \leq C n^{-\frac{\alpha p}{2}} \end{aligned}$$

for κ large enough.

– *The upper bound for the term $G_{3,2}$.* Let us study separately the case where $p \geq 2$ and the case where $1 < p \leq 2$.

For $p \geq 2$: By virtue of the inequality (6.1), the generalized Minkowski inequality, the assumption 2, the fact that ζ_σ satisfies the Temlyakov property, and the Markov inequality, one gets :

$$\begin{aligned} G_{3,2} &\leq C \mathbb{E}_f^n \left(\left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^2 \mathbf{1}_{\{b_{j,\kappa}(p) \geq \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2}\}} |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \right) \\ &\leq C \left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \mathbb{E}_f^n (|\hat{\beta}_{j,k} - \beta_{j,k}|^p)^{\frac{2}{p}} \mathbf{1}_{\{b_{j,\kappa}(p) \geq \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2}\}} |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \\ &\leq C n^{-\frac{p}{2}} \left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \mathbf{1}_{\{b_{j,\kappa}(p) \geq \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2}\}} \sigma_j^2 |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \\ &\leq C n^{-\frac{p}{2}} \sum_j \sum_K \sum_{(K)} \mathbf{1}_{\{b_{j,\kappa}(p) \geq \sigma_j \frac{\kappa n^{-\frac{1}{2}}}{2}\}} \sigma_j^p \|\psi_{j,k}\|_p^p \end{aligned}$$

$$\begin{aligned}
&= Cn^{-\frac{p}{2}} \sum_{m \in \mathbb{N}} \sum_j \sum_K 1_{\left\{ \frac{\kappa n - \frac{1}{2}}{2} \sigma_j 2^m \leq b_{j,K}(p) < \sigma_j \frac{\kappa n - \frac{1}{2}}{2} 2^{m+1} \right\}} \sigma_j^p l_j 2^{j(\frac{p}{2}-1)} \\
&\leq C \sum_{m \in \mathbb{N}} 2^{-mp} \sum_j \sum_K (b_{j,K}(p))^{p-1} 1_{\left\{ b_{j,K}(p) < \sigma_j \frac{\kappa n - \frac{1}{2}}{2} 2^{m+1} \right\}} l_j 2^{j(\frac{p}{2}-1)} \\
&= C \sum_{m \in \mathbb{N}} 2^{-mp} \sum_j \sum_K \sum_{(K)} |\beta_{j,k}|^p 1_{\left\{ b_{j,K}(p) \leq \frac{\kappa n - \frac{1}{2}}{2} \sigma_j 2^{m+1} \right\}} 2^{j(\frac{p}{2}-1)} = J
\end{aligned}$$

Applying the Markov inequality and the inequalities (6.3) and (2.1), J can be majorized by :

$$\begin{aligned}
J &= C \int_0^1 \sum_{m \in \mathbb{N}} 2^{-mp} \sum_j \sum_K \sum_{(K)} |\beta_{j,k}|^p 1_{\left\{ b_{j,K}(p) \leq \frac{\kappa n - \frac{1}{2}}{2} \sigma_j 2^{m+1} \right\}} |\psi_{j,k}|^p \\
&\leq C \sum_{m \in \mathbb{N}} 2^{-mp} \int_0^1 \left(\sum_j \sum_K \sum_{(K)} |\beta_{j,k}|^2 1_{\left\{ b_{j,K}(p) \leq \frac{\kappa n - \frac{1}{2}}{2} \sigma_j 2^{m+1} \right\}} \right)^{\frac{p}{2}} |\psi_{j,k}|^2 \\
&\leq C \sum_{m \in \mathbb{N}} 2^{-mp} \left\| \sum_j \sum_K \sum_{(K)} \beta_{j,k} 1_{\left\{ b_{j,K}(p) \leq \frac{\kappa n - \frac{1}{2}}{2} \sigma_j 2^{m+1} \right\}} \right\|_p^p. \tag{6.16}
\end{aligned}$$

Thus, if $l_j \asymp 2^j$ and $f \in \mathcal{W}_\sigma((1-\alpha)p, p)$ with $\alpha \in]0, 1[$ then we have :

$$G_{3,2} \leq Cn^{-\frac{\alpha p}{2}} \sum_{m \in \mathbb{N}} 2^{(\alpha-1)mp} \leq Cn^{-\frac{\alpha p}{2}}.$$

Moreover, if $l_j \asymp \ln(n)^{\frac{p}{2}}$ and $f \in \overline{\mathcal{W}}_{\sigma, \kappa, n_0}((1-\alpha)p, p)$ then we have immediately :

$$G_{3,2} \leq Cn^{-\frac{\alpha p}{2}}.$$

For $1 < p \leq 2$: Using the inequalities (6.1) and (6.3), the assumption 2 and the Markov inequality in the same way that for the case $p \geq 2$, we find :

$$\begin{aligned}
G_{3,2} &\leq C \mathbb{E}_f^n \left(\left\| \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^2 1_{\left\{ b_{j,K}(p) \geq \sigma_j \frac{\kappa n - \frac{1}{2}}{2} \right\}} |\psi_{j,k}|^2 \right\|_p^{\frac{1}{2}} \right) \\
&\leq C \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \mathbb{E}_f^n \left(|\hat{\beta}_{j,k} - \beta_{j,k}|^p 1_{\left\{ b_{j,K}(p) \geq \sigma_j \frac{\kappa n - \frac{1}{2}}{2} \right\}} \right) \|\psi_{j,k}\|_p^p \\
&\leq Cn^{-\frac{p}{2}} \sum_j \sum_K \sum_{(K)} 1_{\left\{ b_{j,K}(p) \geq \sigma_j \frac{\kappa n - \frac{1}{2}}{2} \right\}} \sigma_j^p \|\psi_{j,k}\|_p^p \\
&\leq C \sum_{m \in \mathbb{N}} 2^{-mp} \sum_j \sum_K \sum_{(K)} |\beta_{j,k}|^p 1_{\left\{ b_{j,K}(p) \leq \frac{\kappa n - \frac{1}{2}}{2} \sigma_j 2^{m+1} \right\}} 2^{j(\frac{p}{2}-1)}.
\end{aligned}$$

Starting from the expression of the last term, it is easy to see that if $l_j \asymp 2^j$ and $f \in \mathcal{W}_\sigma((1-\alpha)p, p)$ then :

$$G_{3,2} \leq Cn^{-\frac{\alpha p}{2}} \sum_{m \in \mathbb{N}} 2^{(\alpha-1)mp} \leq Cn^{-\frac{\alpha p}{2}}.$$

Moreover, if $l_j \asymp \ln(n)$ and $f \in \overline{\mathcal{W}}_{\sigma, \kappa, n_0}((1-\alpha)p, p)$ then :

$$G_{3,2} \leq Cn^{-\frac{\alpha p}{2}}.$$

– Putting the upper bounds of $G_{3,1}$ and $G_{3,2}$ together we deduce that for $p > 1$, κ and n are large enough we have in each case :

$$G_3 \leq Cn^{-\frac{\alpha p}{2}}. \tag{6.17}$$

- The upper bound for the term G_4 . Since $f \in \mathcal{B}_{p, \infty}^{\frac{\alpha}{2}}$, we have clearly :

$$G_4 \leq Cn^{-\frac{\alpha p}{2}}. \tag{6.18}$$

Combining (6.12), (6.13), (6.15), (6.17) and (6.18), the proof of Theorem 3.2 ends. \square

Proof of Proposition 3.1. If $p \geq 2$, the following identity :

$$\mathcal{M}_{n_0} \left(\hat{f}^g, p, n^{-\frac{\alpha p}{2}} \right) = \mathcal{B}_{p,\infty}^{\frac{\alpha}{\nu}} \cap \mathcal{W}_\sigma((1-\alpha)p, p)$$

is provided by Theorem 3.1 and Theorem 3.2. Thus, we only need to show that for any $\nu > 0$, $\alpha \in]1 - \frac{\nu}{2}, 1[$, $\kappa > \kappa_0$ with κ_0 large enough, and n_0 large enough, we have the inclusion $\mathcal{W}_\sigma((1-\alpha)p, p) \subseteq \mathcal{B}_{p,\infty}^{\frac{\alpha}{\nu}}$.

If $f \in \mathcal{W}_\sigma((1-\alpha)p, p)$ and $\alpha > 1 - \frac{\nu}{2}$ then there exists a constant $C > 0$ such that the integer j_2 described in Definition 2.1 satisfies :

$$\begin{aligned} 2^{\frac{j_2 p}{2}} \sum_{j \geq j_2} \mathbb{1}_{\{b_j(p) \geq C n^{-\frac{1}{2}} \sigma_j\}} &\leq \sum_{j \geq j_2} \mathbb{1}_{\{b_j(p) \geq C n^{-\frac{1}{2}} \sigma_j\}} 2^{\frac{j p}{2}} \leq \sum_j \sum_{k \in \Delta_j} \mathbb{1}_{\{b_j(p) \geq C n^{-\frac{1}{2}} \sigma_j\}} \sigma_j^p \|\psi_{j,k}\|_p^p \\ &\leq n^{(1-\alpha)\frac{p}{2}} < n^{\frac{\nu p}{4}} \end{aligned}$$

and $2^{j_2} \geq n^{\frac{\nu}{2}}$. We used an inequality similar to (6.16) and the fact that $\inf_j \sigma_j \geq 1$ (without loss of generality). We deduce that if $j \geq j_2$ then we have necessarily $b_j(p) < C n^{-\frac{1}{2}} \sigma_j$.

Consequently, for any f of $\mathcal{W}_\sigma((1-\alpha)p, p)$, the inequality (2.1) implies that :

$$\left\| \sum_{j \geq j_2} \sum_{k \in \Delta_j} \beta_{j,k} \psi_{j,k} \right\|_p^p \leq \left\| \sum_{j \geq j_2} \sum_{k \in \Delta_j} \beta_{j,k} \mathbb{1}_{\{b_j(p) < C n^{-\frac{1}{2}} \sigma_j\}} \psi_{j,k} \right\|_p^p \leq C n^{-\frac{\alpha p}{2}}.$$

We conclude that $f \in \mathcal{B}_{p,\infty}^{\frac{\alpha}{\nu}}$. Hence the inclusion $\mathcal{W}_\sigma((1-\alpha)p, p) \subseteq \mathcal{B}_{p,\infty}^{\frac{\alpha}{\nu}}$ holds. \square

Proof of Theorem 3.3. The proof of this theorem is identical to :

1. the proof of Theorem 3.1 for the inclusion :

$$\mathcal{M}_{n_0} \left(\hat{f}^h, p, n^{-\frac{\alpha p}{2}} \right) \subseteq \mathcal{B}_{p,\infty}^{\frac{\alpha}{\nu}} \cap \widehat{\mathcal{W}}_\sigma((1-\alpha)p, p)$$

2. the proof of Theorem 3.2 for the inclusion :

$$\mathcal{B}_{p,\infty}^{\frac{\alpha}{\nu}} \cap \widehat{\mathcal{W}}_\sigma((1-\alpha)p, p) \subseteq \mathcal{M}_{n_0} \left(\hat{f}^h, p, n^{-\frac{\alpha p}{2}} \right).$$

It suffices to replace j_1 by τ , the threshold $n^{-\frac{1}{2}}$ by $\sqrt{\frac{\ln(n)}{n}}$, $\sum_K \sum_{(K)}$ by $\sum_{k \in \Delta_j}$, $\hat{b}_{j,K}(p)$ by $|\hat{\beta}_{j,k}|$, $b_{j,K}(p)$ by $|\beta_{j,k}|$ and to remark that :

$$\eta \left(\sqrt{\frac{\ln(n)}{n}} \right) \asymp n^{-\frac{1}{2}}.$$

Let us just mention that if $f \in \widehat{\mathcal{W}}_\sigma((1-\alpha)p, p)$ then the term analog to $G_{3,2}$ can be dominated by :

$$\begin{aligned} G_{3,2} &= \mathbb{E}_f^n \left(\left\| \sum_{j_1 \leq j < j_2} \sum_{k \in \Delta_j} (\hat{\beta}_{j,k} - \beta_{j,k}) \mathbb{1}_{\{|\hat{\beta}_{j,k}| \geq \sigma_j \kappa \sqrt{\frac{\ln(n)}{n}}\}} \mathbb{1}_{\{|\beta_{j,k}| \geq \sigma_j \frac{\kappa}{2} \sqrt{\frac{\ln(n)}{n}}\}} \psi_{j,k} \right\|_p^p \right) \\ &\leq \left\| \left(\sum_{j_1 \leq j < j_2} \sum_{k \in \Delta_j} \mathbb{E}_f^n (|\hat{\beta}_{j,k} - \beta_{j,k}|^{2\nu p})^{\frac{2}{\nu}} \wedge 1 \mathbb{1}_{\{|\beta_{j,k}| \geq \sigma_j \frac{\kappa}{2} \sqrt{\frac{\ln(n)}{n}}\}} |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \\ &\leq C n^{-\frac{p}{2}} \left\| \left(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \sigma_j^2 \mathbb{1}_{\{|\beta_{j,k}| \geq \sigma_j \frac{\kappa}{2} \sqrt{\frac{\ln(n)}{n}}\}} |\psi_{j,k}|^2 \right)^{\frac{1}{2}} \right\|_p^p \\ &\leq C n^{-\frac{p}{2}} \left\| \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \sigma_j \mathbb{1}_{\{|\beta_{j,k}| \geq \sigma_j \frac{\kappa}{2} \sqrt{\frac{\ln(n)}{n}}\}} \psi_{j,k} \right\|_p^p \\ &\leq C n^{-\frac{p}{2}} \left(\sum_{m \in \mathbb{N}} \left\| \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \sigma_j \mathbb{1}_{\left\{ \sigma_j 2^{m+1} \frac{\kappa}{2} \sqrt{\frac{\ln(n)}{n}} > |\beta_{j,k}| \geq \sigma_j 2^m \frac{\kappa}{2} \sqrt{\frac{\ln(n)}{n}} \right\}} \psi_{j,k} \right\|_p^p \right) \\ &\leq C \ln(n)^{-\frac{p}{2}} \left(\sum_{m \in \mathbb{N}} 2^{-m} \left\| \sum_j \sum_{k \in \Delta_j} \beta_{j,k} \mathbb{1}_{\left\{ |\beta_{j,k}| \leq \sigma_j \frac{\kappa}{2} \sqrt{\frac{\ln(n)}{n}} 2^{m+1} \right\}} \psi_{j,k} \right\|_p^p \right) \\ &\leq C \ln(n)^{-\frac{p}{2}} \left(\sum_{m \in \mathbb{N}} 2^{-m} \eta \left(\frac{\kappa}{2} \sqrt{\frac{\ln(n)}{n}} 2^{m+1} \right)^\alpha \right)^p \end{aligned}$$

$$\begin{aligned}
&\leq C n^{-\frac{p}{2}} \left(\sum_{m \in \mathbb{N}} 2^{(\alpha-1)m} \left(\frac{\ln(n)}{n} \right)^{\frac{(\alpha-1)}{2}} \left(\ln \left((2^{m+1} \frac{\kappa}{2} \sqrt{\frac{\ln(n)}{n}} \wedge v)^{-1} \right) \right)^{-\frac{\alpha}{2}} \right)^p \\
&\leq C n^{-\frac{p}{2}} \left(\frac{\ln(n)}{n} \right)^{\frac{(\alpha-1)p}{2}} \left(\ln \left(\left(\frac{\kappa}{2} \sqrt{\frac{\ln(n)}{n}} \wedge v \right)^{-1} \right) \right)^{-\frac{\alpha p}{2}} \leq C n^{-\frac{\alpha p}{2}}
\end{aligned} \tag{6.19}$$

for n large enough. We used the inequality (6.1), the generalized Minkowski inequality in the case where $p \geq 2$, the Jensen inequality in the case where $1 < p \leq 2$, the assumption 2, the Minkowski inequality and the Kerkycharian et al. (2005, Lemma 2) which is satisfied because v is small enough (see Definition 3.4). This ends the proof of Theorem 3.3. \square

Proof of Proposition 3.2. Let $p \geq 2$. Starting from the results of Theorems 3.2 and 3.3, we only need to show the following embedding :

$$\mathcal{B}_{p,\infty}^{\frac{\alpha}{2}} \cap \widehat{\mathcal{W}}_{\sigma}((1-\alpha)p, p) \subseteq \mathcal{B}_{p,\infty}^{\frac{\alpha}{2}} \cap \overline{\mathcal{W}}_{\sigma,\kappa,n_0}((1-\alpha)p, p)$$

for $\alpha \in]0, 1[$, $\kappa > \kappa_0$ with κ_0 large enough, and n_0 large enough. Thanks to the choice of the length of the blocks ($l_j \asymp \ln(n)^{\frac{p}{2}}$), for any k in $\mathcal{B}_{j,K}$ there exists a constant $C > 0$ such that :

$$\begin{aligned}
\left\{ b_{j,K}(p) \leq \kappa 2^m n^{-\frac{1}{2}} \sigma_j \right\} &= \left\{ \frac{1}{C \sqrt{\ln(n)}} \left(\sum_{l \in \mathcal{B}_{j,K}} |\beta_{j,l}|^p \right)^{\frac{1}{p}} \leq \kappa 2^m n^{-\frac{1}{2}} \sigma_j \right\} \\
&\subseteq \left\{ |\beta_{j,k}| \leq C \kappa 2^m \sqrt{\frac{\ln(n)}{n}} \sigma_j \right\}.
\end{aligned}$$

Thus, if $f \in \widehat{\mathcal{W}}_{\sigma}((1-\alpha)p, p)$, the inequality (2.1) and an inequality similar to (6.19) give us :

$$\begin{aligned}
L &= n^{\frac{\alpha p}{2}} \sum_{m \in \mathbb{N}} 2^{-mp} \left\| \sum_j \sum_K \sum_{(K)} \beta_{j,k} 1_{\{b_{j,K}(p) \leq \kappa 2^m n^{-\frac{1}{2}} \sigma_j\}} \psi_{j,k} \right\|_p^p \\
&\leq n^{\frac{\alpha p}{2}} \sum_{m \in \mathbb{N}} 2^{-mp} \left\| \sum_j \sum_{k \in \Delta_j} \beta_{j,k} 1_{\{|\beta_{j,k}| \leq C \kappa 2^m \sigma_j \sqrt{\frac{\ln(n)}{n}}\}} \psi_{j,k} \right\|_p^p \\
&\leq C n^{\frac{\alpha p}{2}} \sum_{m \in \mathbb{N}} 2^{-mp} \eta \left(\sqrt{\frac{\ln(n)}{n}} C \kappa 2^m \right)^{\alpha p} \leq C n^{\frac{\alpha p}{2}} \left(\frac{\ln(n)}{n} \right)^{\frac{\alpha p}{2}} \left(\ln \left((C \kappa \sqrt{\frac{\ln(n)}{n}} \wedge v)^{-1} \right) \right)^{-\frac{\alpha p}{2}} \leq C < \infty
\end{aligned}$$

for $\kappa > \kappa_0$ with κ_0 large enough. We conclude that $f \in \overline{\mathcal{W}}_{\sigma,\kappa,n_0}((1-\alpha)p, p)$. The proof of Proposition 3.2 is thus complete. \square

Proof of Proposition 4.1. Let $R > 0$, $s > 0$, $\pi \geq p > 1$ and $1 \leq r \leq \infty$. Since the sequence $\sigma = (2^{\delta j})$ satisfies the condition (2.2) for $p > 2$, if we apply Theorem 3.2 with $\alpha = \frac{2s}{2s+2\delta+1}$ then it suffices to show that :

$$B_{\pi,r}^s(R) \subseteq \mathcal{B}_{p,\infty}^{\frac{s(1+2\delta)}{2s+2\delta+1}} \cap \mathcal{W}_{\sigma} \left(\left(1 - \frac{2s}{2s+2\delta+1} \right) p, p \right) \cap \overline{\mathcal{W}}_{\sigma,\kappa,n_0} \left(\left(1 - \frac{2s}{2s+2\delta+1} \right) p, p \right)$$

for $\kappa > \kappa_0$ with κ_0 large enough, and n_0 large enough. Let us focus on the case where $p \geq 2$:

- *Proof of the inclusion $B_{\pi,r}^s(R) \subseteq \mathcal{B}_{p,\infty}^{\frac{s(1+2\delta)}{2s+2\delta+1}}$.* If $f \in B_{\pi,r}^s(R) (\subseteq B_{p,\infty}^s(R))$ then Lemma 2.2 gives us :

$$\begin{aligned}
2^u \frac{s(1+2\delta)}{2s+2\delta+1} p \left\| \sum_{j \geq u} \sum_{k \in \Delta_j} \beta_{j,k} \psi_{j,k} \right\|_p^p &\leq C \left(\sum_{j \geq u} 2^{j(\frac{1}{2} - \frac{1}{p})} \left(\sum_{k \in \Delta_j} |\beta_{j,k}|^p \right)^{\frac{1}{p}} \right)^p 2^{u \frac{s(1+2\delta)}{2s+2\delta+1} p} \\
&\leq C \left(\sum_{j \geq u} 2^{-js + \frac{us(1+2\delta)}{2s+2\delta+1}} \right)^p \leq C \left(\sum_{j \geq u} 2^{-s(j-u)} \right)^p \leq C < \infty.
\end{aligned}$$

Hence $f \in \mathcal{B}_{p,\infty}^{\frac{s(1+2\delta)}{2s+2\delta+1}}$ and the desired inclusion holds.

• *Proof of the inclusion $B_{\pi,r}^s(R) \subseteq \mathcal{W}_\sigma((1 - \frac{2s}{2s+2\delta+1})p, p)$.* Let us consider an integer j_3 such that $2^{j_3} \asymp u^{-\frac{2}{2s+2\delta+1}}$. Minkowski's inequality combined with the inequality (6.2) gives us :

$$\begin{aligned} \left\| \sum_j \sum_{k \in \Delta_j} \beta_{j,k} 1_{\{b_j(p) \leq 2^{\delta j} u\}} \psi_{j,k} \right\|_p^p &\leq \left(\sum_j \left\| \sum_{k \in \Delta_j} \beta_{j,k} 1_{\{b_j(p) \leq 2^{\delta j} u\}} \psi_{j,k} \right\|_p \right)^p \\ &\leq C \left(\sum_j 2^{j(\frac{1}{2} - \frac{1}{p})} \left(\sum_{k \in \Delta_j} |\beta_{j,k}|^p 1_{\{b_j(p) \leq 2^{\delta j} u\}} \right)^{\frac{1}{p}} \right)^p \\ &\leq C(F_1 + F_2) \end{aligned}$$

where :

$$F_1 = \left(\sum_{j < j_3} 2^{j(\frac{1}{2} - \frac{1}{p})} \left(\sum_{k \in \Delta_j} |\beta_{j,k}|^p 1_{\{b_j(p) \leq 2^{\delta j} u\}} \right)^{\frac{1}{p}} \right)^p \quad \text{and} \quad F_2 = \left(\sum_{j \geq j_3} 2^{j(\frac{1}{2} - \frac{1}{p})} \left(\sum_{k \in \Delta_j} |\beta_{j,k}|^p \right)^{\frac{1}{p}} \right)^p.$$

On the set $\{b_j(p) \leq 2^{\delta j} u\}$, we have $(\sum_{k \in \Delta_j} |\beta_{j,k}|^p)^{\frac{1}{p}} \leq u 2^{\delta j} 2^{\frac{j}{p}}$ so F_1 can be bounded by :

$$F_1 \leq C u^p \left(\sum_{j < j_3} 2^{j(\frac{1}{2} + \delta)} \right)^p \leq u^p 2^{j_3(\frac{1}{2} + \delta)p} \leq C u^{\frac{2sp}{2s+2\delta+1}}.$$

If we assume that $f \in B_{\pi,r}^s(R) (\subseteq B_{p,\infty}^s(R))$ since $\pi \geq p$ and $1 \leq r \leq \infty$) then the Minkowski inequality gives us :

$$F_2 \leq C \left(\sum_{j \geq j_3} 2^{j(\frac{1}{2} - \frac{1}{p})} \left(\sum_{k \in \Delta_j} |\beta_{j,k}|^p \right)^{\frac{1}{p}} \right)^p \leq C \left(\sum_{j \geq j_3} 2^{-js} \right)^p \leq C 2^{-j_3 sp} \leq C u^{\frac{2sp}{2s+2\delta+1}}.$$

We deduce that :

$$\sup_{n > 0} u^{-\frac{2sp}{2s+2\delta+1}} \left\| \sum_j \sum_{k \in \Delta_j} \beta_{j,k} 1_{\{b_j(p) \leq 2^{\delta j} u\}} \psi_{j,k} \right\|_p^p < \infty$$

and, a fortiori, $f \in \mathcal{W}_\sigma((1 - \frac{2s}{2s+2\delta+1})p, p)$.

• *Proof of the inclusion $B_{\pi,r}^s(R) \subseteq \overline{\mathcal{W}}_{\sigma,\kappa,n_0}((1 - \frac{2s}{2s+2\delta+1})p, p)$.* For any $m \in \mathbb{N}$, let us consider an integer j_4 such that

$$2^{j_4} \asymp 2^{-\frac{m}{2s}} n^{\frac{1}{2s+2\delta+1}}.$$

Minkowski's inequality combined with the inequality (6.2) gives us :

$$S = \sum_{m \in \mathbb{N}} 2^{-mp} \left\| \sum_j \sum_K \sum_{(K)} \beta_{j,k} 1_{\{b_{j,K}(p) \leq \kappa 2^{\delta j} 2^m n^{-\frac{1}{2}}\}} \psi_{j,k} \right\|_p^p \leq C(S_1 + S_2)$$

where :

$$S_1 = \sum_{m \in \mathbb{N}} 2^{-mp} \left(\sum_{j < j_4} 2^{j(\frac{1}{2} - \frac{1}{p})} \left(\sum_K \sum_{(K)} |\beta_{j,k}|^p 1_{\{b_{j,K}(p) \leq \kappa 2^{\delta j} 2^m n^{-\frac{1}{2}}\}} \right)^{\frac{1}{p}} \right)^p$$

and

$$S_2 = \sum_{m \in \mathbb{N}} 2^{-mp} \left(\sum_{j \geq j_4} 2^{j(\frac{1}{2} - \frac{1}{p})} \left(\sum_{k \in \Delta_j} |\beta_{j,k}|^p \right)^{\frac{1}{p}} \right)^p.$$

Using the fact that $|\mathcal{A}_j| \asymp 2^j \ln(n)^{-\frac{p}{2}}$ and that, on the set $\{b_{j,K}(p) \leq \kappa 2^{\delta j} 2^m n^{-\frac{1}{2}}\}$, we have the inequality $(\sum_{(K)} |\beta_{j,k}|^p)^{\frac{1}{p}} \leq \kappa n^{-\frac{1}{2}} 2^m 2^{\delta j} \ln(n)^{\frac{1}{2}}$, we can dominate S_1 by :

$$\begin{aligned} S_1 &\leq C n^{-\frac{p}{2}} \sum_{m \in \mathbb{N}} \left(\sum_{j < j_4} 2^{j(\frac{1}{2} - \frac{1}{p})} (|\mathcal{A}_j| 2^{\delta j p} \ln(n)^{\frac{p}{2}})^{\frac{1}{p}} \right)^p \leq C n^{-\frac{p}{2}} \sum_{m \in \mathbb{N}} \left(\sum_{j < j_4} 2^{j(\frac{1}{2} + \delta)} \right)^p \leq C n^{-\frac{p}{2}} \sum_{m \in \mathbb{N}} 2^{j_4(\frac{1}{2} + \delta)p} \\ &\leq C n^{-\frac{sp}{2s+2\delta+1}} \sum_{m \in \mathbb{N}} 2^{-mp(\frac{1+2\delta}{4s})} \leq C n^{-\frac{sp}{2s+2\delta+1}}. \end{aligned}$$

If we assume that $f \in B_{\pi,r}^s(R) (\subseteq B_{p,\infty}^s(R))$ since $\pi \geq p$ and $1 \leq r \leq \infty$) then we have :

$$\begin{aligned} S_2 &\leq C \sum_{m \in \mathbb{N}} 2^{-mp} \left(\sum_{j \geq j_4} 2^{j(\frac{1}{2} - \frac{1}{p})} \left(\sum_{k \in \Delta_j} |\beta_{j,k}|^p \right)^{\frac{1}{p}} \right)^p \leq C \sum_{m \in \mathbb{N}} 2^{-mp} \left(\sum_{j \geq j_4} 2^{-js} \right)^p \\ &\leq C \sum_{m \in \mathbb{N}} 2^{-mp} 2^{-j_4 sp} \leq C n^{-\frac{sp}{2s+2\delta+1}} \sum_{m \in \mathbb{N}} 2^{-\frac{mp}{2}} \leq C n^{-\frac{sp}{2s+2\delta+1}}. \end{aligned}$$

We deduce that there exists a constant $C > 0$ such that $S \leq C < \infty$. This implies that $f \in \overline{W}_{\sigma, \kappa, n_0}((1 - \frac{2s}{2s+2\delta+1})p, p)$.

By using the same arguments as before, it is easy to prove the previous inclusion in the case where $1 < p \leq 2$. By taking $\kappa > \kappa_0$ with κ_0 large enough, and n_0 large enough, Theorem 3.2 allows us to conclude. \square

Proof of Theorem 4.2. We denote

$$\epsilon = \pi s + (\delta + \frac{1}{2})(\pi - p).$$

We recall that the integers j_1 and j_2 satisfy $2^{j_1} \asymp \ln(n)^{\frac{2}{\pi} \vee 1}$ and $2^{j_2} \asymp n^{\frac{1}{2+\delta}}$. If we consider the proof of Theorem 3.2, it suffices to show that if $f \in B_{\pi, r}^s(R)$ with $s > 0$, $p \geq 2$, $1 < \pi < p$ and $r \geq 1$ then there exists a constant $C > 0$ such that :

1.

$$\| \sum_{j \geq j_2} \sum_{k \in \Delta_j} \beta_{j,k} \psi_{j,k} \|_p^p \begin{cases} \leq C \left(\frac{\ln(n)}{n} \right)^{\frac{sp}{1+2s+2\delta}}, & \text{if } \epsilon > 0, \\ \leq C \left(\frac{\ln(n)}{n} \right)^{\frac{(s - \frac{1}{\pi} + \frac{1}{p})p}{2(s - \frac{1}{\pi} + \delta) + 1}}, & \text{if } \epsilon < 0, \\ \leq C \left(\frac{\ln(n)}{n} \right)^{\frac{(s - \frac{1}{\pi} + \frac{1}{p})p}{2(s - \frac{1}{\pi} + \delta) + 1}} \ln(n)^{(p - \frac{\pi}{r})_+}, & \text{if } \epsilon = 0. \end{cases}$$

2.

$$\sum_{m \in \mathbb{N}} 2^{-mp} \| \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \beta_{j,k} 1_{\{b_{j,K}(p) \leq \kappa 2^{\delta j} 2^m n^{-\frac{1}{2}}\}} \psi_{j,k} \|_p^p \begin{cases} \leq C \left(\frac{\ln(n)}{n} \right)^{\frac{sp}{1+2s+2\delta}}, & \text{if } \epsilon > 0, \\ \leq C \left(\frac{\ln(n)}{n} \right)^{\frac{(s - \frac{1}{\pi} + \frac{1}{p})p}{2(s - \frac{1}{\pi} + \delta) + 1}}, & \text{if } \epsilon < 0, \\ \leq C \left(\frac{\ln(n)}{n} \right)^{\frac{(s - \frac{1}{\pi} + \frac{1}{p})p}{2(s - \frac{1}{\pi} + \delta) + 1}} \ln(n)^{(p - \frac{\pi}{r})_+}, & \text{if } \epsilon = 0. \end{cases}$$

• Let us show the first point.

If $p > \pi$ then we have the following embedding $B_{\pi, r}^s(R) \subseteq B_{p, r}^{s - \frac{1}{\pi} + \frac{1}{p}}(R)$. Using this with Lemma 2.2, one gets :

$$\| \sum_{j \geq j_2} \sum_{k \in \Delta_j} \beta_{j,k} \psi_{j,k} \|_p^p \leq C \left(\sum_{j \geq j_2} 2^{j(\frac{1}{2} - \frac{1}{p})} \left(\sum_{k \in \Delta_j} |\beta_{j,k}|^p \right)^{\frac{1}{p}} \right)^p \leq C \left(\sum_{j \geq j_2} 2^{-j(s - \frac{1}{\pi} + \frac{1}{p})} \right)^p \leq C 2^{-j_2(s - \frac{1}{\pi} + \frac{1}{p})p}.$$

By definition of j_2 , we obtain the desired upper bounds according to the sign of ϵ .

• Let us show the second point. Let us denote j_5 an integer such that :

$$2^{j_5} \asymp 2^{-\frac{m}{2s}} \left(\frac{n}{\ln(n)} \right)^{\frac{1}{2s+2\delta+1}}.$$

– Firstly, let us focus on the case $\epsilon > 0$.

We have :

$$\begin{aligned} X &= \sum_{m \in \mathbb{N}} 2^{-mp} \| \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \beta_{j,k} 1_{\{b_{j,K}(p) \leq \kappa 2^{\delta j} 2^m n^{-\frac{1}{2}}\}} \psi_{j,k} \|_p^p \\ &\leq C \sum_{m \in \mathbb{N}} 2^{-mp} \left(\sum_{j_1 \leq j < j_2} 2^{j(\frac{1}{2} - \frac{1}{p})} \left(\sum_K \sum_{(K)} |\beta_{j,k}|^p 1_{\{b_{j,K}(p) \leq \kappa 2^{\delta j} 2^m n^{-\frac{1}{2}}\}} \right)^{\frac{1}{p}} \right)^p \\ &= C \sum_{m \in \mathbb{N}} 2^{-mp} \left(\sum_{j_1 \leq j < j_2} 2^{j(\frac{1}{2} - \frac{1}{p})} l_j^{\frac{1}{p}} \left(\sum_K b_{j,K}(p)^p 1_{\{b_{j,K}(p) \leq \kappa 2^{\delta j} 2^m n^{-\frac{1}{2}}\}} \right)^{\frac{1}{p}} \right)^p \\ &= C(X_1 + X_2) \end{aligned}$$

where

$$X_1 = \sum_{m \in \mathbb{N}} 2^{-mp} \left(\sum_{j_1 \leq j < j_5} 2^{j(\frac{1}{2} - \frac{1}{p})} l_j^{\frac{1}{p}} \left(\sum_K b_{j,K}(p)^p 1_{\{b_{j,K}(p) \leq \kappa 2^{\delta j} 2^m n^{-\frac{1}{2}}\}} \right)^{\frac{1}{p}} \right)^p$$

and

$$X_2 = \sum_{m \in \mathbb{N}} 2^{-mp} \left(\sum_{j_5 \leq j < j_2} 2^{j(\frac{1}{2} - \frac{1}{p})} l_j^{\frac{1}{p}} \left(\sum_K b_{j,K}(p)^p 1_{\{b_{j,K}(p) \leq \kappa 2^{\delta j} 2^m n^{-\frac{1}{2}}\}} \right)^{\frac{1}{p}} \right)^p.$$

Let us analyze the terms X_1 and X_2 in turn.

The bound for X_1 . If we follow the bound for the term S of the proof of Theorem 4.1, we easily obtain :

$$\begin{aligned} X_1 &\leq C \sum_{m \in \mathbb{N}} n^{-\frac{p}{2}} \left(\sum_{j < j_5} 2^{j(\frac{1}{2} - \frac{1}{p})} (|\mathcal{A}_j| 2^{\delta j p} \ln(n)^{\frac{p}{2}})^{\frac{1}{p}} \right)^p \leq C n^{-\frac{p}{2}} \sum_{m \in \mathbb{N}} \left(\sum_{j < j_5} 2^{j(\frac{1}{2} + \delta)} \right)^p \leq C n^{-\frac{p}{2}} \sum_{m \in \mathbb{N}} 2^{j_5(\frac{1}{2} + \delta)p} \\ &\leq C \left(\frac{\ln(n)}{n} \right)^{\frac{sp}{1+2s+2\delta}} \sum_{m \in \mathbb{N}} 2^{-mp(\frac{1+2\delta}{4s})} \leq C \left(\frac{\ln(n)}{n} \right)^{\frac{sp}{1+2s+2\delta}}. \end{aligned}$$

The bound for X_2 . Since $p > \pi$, an elementary inequality and Lemma 6.3 give us :

$$\begin{aligned} \sum_K b_{j,k}(p)^p \mathbf{1}_{\{b_{j,K}(p) \leq \kappa 2^{\delta j} 2^m n^{-\frac{1}{2}}\}} &\leq C n^{\frac{\pi-p}{2}} 2^{\delta j(p-\pi)} 2^{m(p-\pi)} \sum_K b_{j,K}(p)^\pi \\ &\leq C n^{\frac{\pi-p}{2}} 2^{\delta j(p-\pi)} 2^{m(p-\pi)} l_j^{-\frac{\pi}{p}} \sum_{k \in \Delta_j} |\beta_{j,k}|^\pi. \end{aligned}$$

Using this, the fact that $l_j = \ln(n)^{\frac{p}{2}}$ for $p \geq 2$, $\epsilon > 0$, $p > \pi$ and the embedding $B_{\pi,r}^s(R) \subseteq B_{p,r}^{s-\frac{1}{\pi}+\frac{1}{p}}(R)$, one gets :

$$\begin{aligned} X_2 &\leq C \ln(n)^{\frac{p-\pi}{2}} n^{\frac{\pi-p}{2}} \sum_{m \in \mathbb{N}} 2^{-m\pi} \left(\sum_{j_5 \leq j < j_2} 2^{j(\frac{1}{2} - \frac{1}{p})} 2^{\delta j(\frac{p-\pi}{p})} 2^{-j(s+\frac{1}{2} - \frac{1}{\pi})\frac{\pi}{p}} \right)^p \\ &\leq C \ln(n)^{\frac{p-\pi}{2}} n^{\frac{\pi-p}{2}} \sum_{m \in \mathbb{N}} 2^{-m\pi} \left(\sum_{j_5 \leq j < j_2} 2^{-j\frac{\epsilon}{p}} \right)^p \leq C \ln(n)^{\frac{p-\pi}{2}} n^{\frac{\pi-p}{2}} \sum_{m \in \mathbb{N}} 2^{-m\pi} 2^{-j_5 \epsilon} \\ &\leq C \ln(n)^{\frac{p-\pi}{2}} n^{\frac{\pi-p}{2}} \left(\frac{\ln(n)}{n} \right)^{\frac{\epsilon}{2s+2\delta+1}} \sum_{m \in \mathbb{N}} 2^{-m\frac{\pi}{2} + m(\delta + \frac{1}{2})(\frac{\pi-p}{4s})} \leq C \left(\frac{\ln(n)}{n} \right)^{\frac{sp}{1+2s+2\delta}}, \end{aligned}$$

– Let us now prove the upper bound obtained for $\epsilon < 0$. Let us introduce the integer j_6 defined by :

$$2^{j_6} \asymp 2^{-\frac{m}{2s}} \left(\frac{n}{\ln(n)} \right)^{\frac{1}{2(s-\frac{1}{\pi}+\delta)+1}}.$$

We have :

$$Z = \sum_{m \in \mathbb{N}} 2^{-mp} \left\| \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \beta_{j,k} \mathbf{1}_{\{b_{j,K}(p) \leq \kappa 2^{\delta j} 2^m n^{-\frac{1}{2}}\}} \psi_{j,k} \right\|_p^p \leq C(Z_1 + Z_2)$$

where :

$$Z_1 = \sum_{m \in \mathbb{N}} 2^{-mp} \left\| \sum_{j_1 \leq j < j_6} \sum_K \sum_{(K)} \beta_{j,k} \mathbf{1}_{\{b_{j,K}(p) \leq \kappa 2^{\delta j} 2^m n^{-\frac{1}{2}}\}} \psi_{j,k} \right\|_p^p$$

and

$$Z_2 = \sum_{m \in \mathbb{N}} 2^{-mp} \left\| \sum_{j \geq j_6} \sum_K \sum_{(K)} \beta_{j,k} \psi_{j,k} \right\|_p^p.$$

Proceeding as before, we find :

$$\begin{aligned} Z_1 &\leq C \ln(n)^{\frac{p-\pi}{2}} n^{\frac{\pi-p}{2}} \sum_{m \in \mathbb{N}} 2^{-m\pi} \left(\sum_{j_1 \leq j < j_6} 2^{-j\frac{\epsilon}{p}} \right)^p \leq C \ln(n)^{\frac{p-\pi}{2}} n^{\frac{\pi-p}{2}} \sum_{m \in \mathbb{N}} 2^{-m\pi} 2^{-j_6 \epsilon} \\ &\leq C \left(\frac{\ln(n)}{n} \right)^{\frac{(s-\frac{1}{\pi}+\frac{1}{p})p}{2(s+\delta-\frac{1}{\pi})+1}} \sum_{m \in \mathbb{N}} 2^{-m\frac{\pi}{2} + m(\delta + \frac{1}{2})(\frac{\pi-p}{4s})} \leq C \left(\frac{\ln(n)}{n} \right)^{\frac{(s-\frac{1}{\pi}+\frac{1}{p})p}{2(s+\delta-\frac{1}{\pi})+1}}. \end{aligned}$$

and

$$Z_2 \leq C \sum_{m \in \mathbb{N}} 2^{-mp} 2^{-j_6(s-\frac{1}{\pi}+\frac{1}{p})p} \leq C \left(\frac{\ln(n)}{n} \right)^{\frac{(s-\frac{1}{\pi}+\frac{1}{p})p}{2(s+\delta-\frac{1}{\pi})+1}} \sum_{m \in \mathbb{N}} 2^{-\frac{mp}{2}} 2^{m(\frac{\pi-p}{2s\pi p})} \leq C \left(\frac{\ln(n)}{n} \right)^{\frac{(s-\frac{1}{\pi}+\frac{1}{p})p}{2(s+\delta-\frac{1}{\pi})+1}}.$$

We deduce that for $p \geq 2$:

$$Z \leq C(Z_1 + Z_2) \leq C \left(\frac{\ln(n)}{n} \right)^{\frac{(s-\frac{1}{\pi}+\frac{1}{p})p}{2(s+\delta-\frac{1}{\pi})+1}}.$$

– Let us now prove the upper bound obtained for $\epsilon = 0$. Adopting the notations of the previous paragraph. We only need to bound the term Z_1 defined above. Let us notice that :

$$Z_1 \leq C n^{\frac{\pi-p}{2}} \ln(n)^{\frac{p-\pi}{2}} \sum_{m \in \mathbb{N}} 2^{-mp} \left(\sum_{j_1 \leq j < j_6} l_j \right)^p$$

where

$$l_j = (2^{j(s+\frac{1}{2}-\frac{1}{\pi})\pi} \sum_{k \in \Delta_j} |\beta_{j,k}|^\pi)^{\frac{1}{p}}.$$

Now, let us distinguish the case where $\pi \geq rp$ and the case where $\pi < rp$. If $\pi \geq rp$ then $B_{\pi,r}^s(L)$ is included in $B_{\pi,\frac{\pi}{p}}^s(L)$ so $\sum_j l_j \leq C$ and :

$$Z \leq C(Z_1 + Z_2) \leq C(n^{\frac{\pi-p}{2}} \ln(n)^{\frac{p-\pi}{2}} + (\frac{\ln(n)}{n})^{\frac{(s-\frac{1}{\pi}+\frac{1}{p})p}{2(s-\frac{1}{\pi}+\delta)+1}}) \leq C(\frac{\ln(n)}{n})^{\frac{(s-\frac{1}{\pi}+\frac{1}{p})p}{2(s-\frac{1}{\pi}+\delta)+1}}.$$

Let us consider the case where $\pi < rp$. Using Hölder's inequality and the fact that $f \in B_{\pi,r}^s(L) \subseteq B_{\pi,\infty}^s(L)$, it comes :

$$\begin{aligned} \sum_{m \in \mathbb{N}} 2^{-mp} \left(\sum_{j_1 \leq j < j_6} l_j \right)^p &\leq \sum_{m \in \mathbb{N}} 2^{-mp} \left(\sum_{j_1 \leq j < j_6} l_j^{\frac{pr}{\pi}} \right)^{\frac{\pi}{r}} \left(\sum_{j_0 \leq j < j_6} l_j^{\frac{1-\frac{\pi}{rp}}{r}} \right)^{p-\frac{\pi}{r}} \leq C \sum_{m \in \mathbb{N}} 2^{-mp} \left(\sum_{j_1 \leq j < j_6} 1 \right)^{p-\frac{\pi}{r}} \\ &\leq C \sum_{m \in \mathbb{N}} 2^{-mp} j_6^{(p-\frac{\pi}{r})} \leq C \ln(n)^{(p-\frac{\pi}{r})}. \end{aligned}$$

and a fortiori :

$$Z \leq C(Z_1 + Z_2) \leq C(j_6^{(p-\frac{\pi}{r})} n^{\frac{\pi-p}{2}} \ln(n)^{\frac{p-\pi}{2}} + (\frac{\ln(n)}{n})^{\frac{(s-\frac{1}{\pi}+\frac{1}{p})p}{2(s-\frac{1}{\pi}+\delta)+1}}) \leq C(\frac{\ln(n)}{n})^{\frac{(s-\frac{1}{\pi}+\frac{1}{p})p}{2(s-\frac{1}{\pi}+\delta)+1}} \ln(n)^{(p-\frac{\pi}{r})}.$$

We obtain the desired upper bound.

The proof of Theorem 4.2 is achieved. \square

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