On block thresholding in nonparametric estimation: a maxiset approach under the Lp risk

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On Wavelet Block Thresholding in Nonparametric Estimation: A Maxiset Approach Under The $\mathbb{L}^p$ Risk.

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Abstract

Starting from a general statistical model, we investigate the performance of wavelet block thresholding procedures via the maxiset approach under the $\mathbb{L}^p$ risk ($p > 1$) for a rate of convergence of the form $n^{-\epsilon}$ (without logarithmic factor). We prove that such procedures can be better in the maxiset sense than the hard thresholding procedures. Moreover, we show that they can be optimal in the minimax sense over Besov balls.

Key Words: Maxiset, minimax, adaptive estimation, wavelet block thresholding, Besov spaces.

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1 Motivation

In this Section, let us suppose that we are given data $(Y_1, ..., Y_n)$ defined by

$$Y_i = f(x_i) + z_i, \qquad i = 1, ..., n, \tag{1.1}$$

where $x_i = i/n$ and z_i are i.i.d normal variables with mean zero and variance one. The standard wavelet method to recover f consists in transforming the data into empirical wavelet coefficients and in proceeding to an individual selection : we keep only those which are greater than a fixed threshold. Such local constructions include the soft and hard thresholding procedures which have been studied by many authors starting from Donoho and Johnstone (1994, 1995). More recently, Hall et al. (1998) and Cai (1998) have developed wavelet procedures based on thresholding several empirical wavelet coefficients simultaneously rather than individually as in the local approach.

For the regression problem as given in (1.1), Cai and Silverman (2001) and Cai (1999, 2002) proved that such procedures can enjoy better minimax properties than the hard (and soft) thresholding procedures if we consider the Besov balls and the $\mathbb{L}^2$ risk. More precisely, these adaptive procedures attain the optimal rate of convergence without logarithmic factor contrary to the hard thresholding procedures. Similar results have been established for other models than (1.1). For the regression model with nonequispaced data see Chicken (2003), for the density estimation see Hall et al. (1998), Pensky (1999) and Cai and Chicken (2005), for the nonparametric regression with long memory error see Li and Xiao (2004) and for the nonparametric density deconvolution see Pensky and Vidakovic (1998), to name a few.

By taking the problem under the $\mathbb{L}^p$ risk for $p > 1$, the purpose of the present paper is to investigate the performance of the block wavelet thresholding procedures via two statistical methods. Firstly, we adopt the maxiset approach for a rate of convergence of the form $n^{-\gamma p}$ (without logarithmic factor). This point of view consists in determining the function spaces $\mathcal{A}^\gamma$ which satisfy the following equivalence :

$$f \in \mathcal{A}^\gamma \quad \Longleftrightarrow \quad \sup_{n>0} \left(n^{\gamma p}\, \mathbb{E}_f^n\left(\int_0^1 |\hat{f}(t) - f(t)|^p dt\right)\right) < \infty,$$

where $\hat{f}$ is an estimate of f constructed from block thresholding rules and $\mathbb{E}_f^n$ is the expectation with respect to the law $\mathbb{P}_f^n$ of the observations. To exhibit such $\mathcal{A}^\gamma$, we use the powerful geometrical properties of certain wavelet bases for the $\mathbb{L}^p$ norms which have been pointed out by Kerkyacharian and Picard (2000) (unconditional nature and Temlyakov's property). The main conclusions of this first part are the following : The maxisets associated to the block thresholding rules for the rate of convergence $n^{-\gamma p}$ are big, and can be bigger than those associated to the hard thresholding rules for the same rate. This fact

has been mentioned by Autin (2005) in the framework of the white noise model and under the Besov risk.

Secondly, we use our maxiset results to investigate the minimax properties of the block thresholding procedures over Besov balls $B^s_{\pi,r}(R)$ (to be defined in Section 4). More precisely, we show that they are optimal in the minimax sense over $B^s_{\pi,r}(R)$ under the $\mathbb{L}^p$ risk in the case where $\pi \geq p > 1$, $r \geq 1$ and $s > 0$. If we consider the model (1.1) then, under the previous assumptions on s, π, r, there exists a constant $C > 0$ such that the block thresholding procedure $\hat{f}$ satisfies :

$$\sup_{f \in B^s_{\pi,r}(R)} \mathbb{E}^n_f \Big(\int_0^1 |\hat{f}(t) - f(t)|^p dt \Big) \leq C n^{-\frac{sp}{1+2s}}$$

for n large enough. It is important to notice that we provide general statistical results which can be apply for different models than (1.1) (including certain inverse problems, see Section 5 below). Moreover, our study contains $\mathbb{L}^p$ versions of some results developed by Cai (1998), Cohen et al. (2000) and Autin (2005).

The rest of the paper is organized as follows. Section 2 describes wavelet bases on the interval, some of their geometrical properties in the $\mathbb{L}^p$ norms, the block thresholding rules and the maxiset point of view. By considering a general statistical model, Section 3 isolates the maxisets associated to the block thresholding procedures and compares the maxiset properties between a particular family of block thresholding procedures and the hard thresholding procedures. Minimax results are developed in Section 4. In order to provide some applications to our general Theorems, Section 5 investigates two precise examples by considering the regression model (1.1) and a deconvolution in white noise studied by Johnstone et al. (2004). Section 6 contains proofs of Theorems and Propositions.

2 Methodology

Throughout this paper, we set :

$$\mathbb{L}^p([0,1]) = \big\{ f \text{ measurable on } [0,1]; \quad \|f\|_p^p = \int_0^1 |f(t)|^p dt < +\infty \big\}.$$

The notation $a \asymp b$ means : there exist two constants $C > 0$ and $c > 0$ such that $cb \leq a \leq Cb$. The notations $a \wedge b$ and $a \vee b$ mean respectivly : $\min(a,b)$ and $\max(a,b)$.

2.1 Wavelet bases and some of their geometrical properties under the $\mathbb{L}^p$ norms

We summarize in this subsection the basics on wavelet bases on the unit interval $[0,1]$ and we introduce some of their powerful geometrical properties in the $\mathbb{L}^p$ norms : the unconditional property and the Temlakov's property.

Let us consider the wavelet basis of $[0,1]$ described by Cohen et al. (1993) : We consider ϕ a "father" wavelet of a multiresolution analysis on $\mathbb{R}$ and ψ the associated "mother" wavelet. Assume that $Supp(\phi) = Supp(\psi) = [1-N, N]$ and $\int_{1-N}^{N} \phi(t)dt = 1$, $\int_{1-N}^{N} t^l \psi(t)dt = 0$ for $l = 0, ..., N-1$. Let

$$\phi_{j,k}(x) = 2^{\frac{j}{2}} \phi(2^j x - k) \quad \text{and} \quad \psi_{j,k}(x) = 2^{\frac{j}{2}} \psi(2^j x - k).$$

Then there exists an integer τ satisfying $2^\tau \geq 2N$ such that the collection ζ defined by :

$$\zeta = \{\phi_{\tau,k}(.),\ k = 0, ..., 2^\tau - 1;\ \psi_{j,k}(.);\ j \geq \tau,\ k = 0, ..., 2^j - 1\}$$

with an appropriate treatments at the boundaries, is an orthonormal basis of $\mathbb{L}^2([0,1])$. Another wavelet basis on the interval $[0,1]$ shall be introduced in Section 5.

Let $1 \leq p < \infty$. Any function f of $\mathbb{L}^p([0,1])$ can be decomposed on ζ as :

$$f(x) = \sum_{k \in \Delta_\tau} \alpha_{\tau,k} \phi_{\tau,k}(x) + \sum_{j \geq \tau} \sum_{k \in \Delta_j} \beta_{j,k} \psi_{j,k}(x), \qquad x \in [0,1],$$

where $\alpha_{j,k} = \int_0^1 f(t)\phi_{j,k}(t)dt$, $\beta_{j,k} = \int_0^1 f(t)\psi_{j,k}(t)dt$ and $\Delta_j = \{0, ..., 2^j - 1\}$.

Lemma 2.1 (Unconditional nature-Temlyakov's property). *Let $p>1$. Let us set $\psi_{\tau-1,k}=\phi_{\tau,k}$. Then the basis ζ is unconditional for $\mathbb{L}^p([0,1])$ i.e for any sequence $u=(u_{j,k})_{j,k}$ we have :*

$$\|\sum_{j\geq\tau-1}\sum_{k\in\Delta_j}u_{j,k}\psi_{j,k}\|_p^p\asymp\|(\sum_{j\geq\tau-1}\sum_{k\in\Delta_j}|u_{j,k}\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p. \tag{2.1}$$

Moreover, if for any subset Λ of $\mathbb{N}$ there exists a constant $C>0$ such that the positive sequence $\sigma=(\sigma_j)_j$ satisfies the following inequalities :

$$\sum_{j\in\Lambda}2^j\sigma_j^2\leq C\sup_{j\in\Lambda}(2^j\sigma_j^2), \tag{2.2}$$

then the weighted basis ξ_σ defined by $\xi_\sigma=\{\sigma_\tau\phi_{\tau,k}(.),\ k\in\Delta_\tau;\ \sigma_j\psi_{j,k}(.),\ j\geq\tau,\ k\in\Delta_j\}$ satisfies the Temlyakov property for $p>2$ i.e for any subset $A\subseteq\{\tau-1,...\}$ and any subset $C\subseteq\Delta_j$ we have :

$$\|(\sum_{j\in A}\sum_{k\in C}|\sigma_j\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p\asymp\sum_{j\in A}\sum_{k\in C}\sigma_j^p\|\psi_{j,k}\|_p^p. \tag{2.3}$$

Let us precise that for $p=2$, the inequality (2.3) holds without any condition on σ.

For exact references concerning the previous properties, we refer the reader to Kerkyacharian and Picard (2000, Subsection 4.1.1) and Johnstone et al. (2004, Theorem 2).

The following Lemma is standard :

Lemma 2.2. *Let $p\geq1$. For any sequence $u=(u_{j,k})_{j,k}$ and any $j\geq\tau$ there exists a constant $C>0$ such that :*

$$\|\sum_{k\in\Delta_j}u_{j,k}\phi_{j,k}\|_p^p\leq C2^{j(\frac{p}{2}-1)}\sum_{k\in\Delta_j}|u_{j,k}|^p.$$

This inequality holds if we exchange ϕ by ψ.

2.2 Block thresholding procedures, hard thresholding procedures and maxisets

Here and after, we assume that we observe a sequence of models E_n in which we are able to produce estimates $\hat\beta_{j,k}$ (resp. $\hat\alpha_{j,k}$) of the wavelets coefficients $\beta_{j,k}$ (resp. $\alpha_{j,k}$) of an unknown function f. The following procedures will be at the heart of our statistical study.

Definition 2.1 (Block thresholding procedures). *Let $1<p<\infty$ and $0<\nu\leq2$. Let j_1 be an integer satisfying :*

$$2^{j_1}\asymp\ln(n)^{\frac{p}{2}}\quad\text{and}\quad\sigma_{j_1}\asymp\ln(n)^{\upsilon},\quad\upsilon\geq0,$$

and let j_2 be an integer satisfying $2^{j_2}\asymp n^{\frac{\nu}{2}}$. For all j in $\{j_1,...,j_2-1\}$, let us divide Δ_j into consecutive nonoverlapping blocks $\mathcal{B}_{j,K}$ of length l_j (non decreasing in j) i.e :

$$\mathcal{B}_{j,K}=\{k\in\Delta_j:\ \ (K-1)l_j\leq k\leq Kl_j-1\},\quad K\in\mathcal{A}_j,$$

where the sets $\mathcal{A}_j$ are defined by :

$$\mathcal{A}_j=\left\{1,...,2^jl_j^{-1}\right\}$$

(for convenience we suppose that $2^jl_j^{-1}\in\mathbb{N}^*$ for all j in $\{j_1,...,j_2-1\}$). We define the block thresholding procedure $\hat f$ by :

$$\hat f(x)=\sum_{j\leq j_1}\hat\alpha_{j_1,k}\phi_{j_1,k}(x)+\sum_{j_1\leq j<j_2}\sum_{K\in\mathcal{A}_j}\sum_{k\in\mathcal{B}_{j,K}}\hat\beta_{j,k}1_{\left\{\hat b_{j,K}(p)\geq\kappa\sigma_jn^{-\frac{1}{2}}\right\}}\psi_{j,k}(x),\qquad x\in[0,1], \tag{2.4}$$

where κ is a positive real number, $\sigma=(\sigma_j)_j$ is a known increasing positive sequence and $\hat b_{j,K}(p)$ is the normalized l_p-norm of estimators $(\hat\beta_{j,k})_{k\in\mathcal{B}_{j,K}}$ i.e :

$$\hat b_{j,K}(p)=(l_j^{-1}\sum_{k\in\mathcal{B}_{j,K}}|\hat\beta_{j,k}|^p)^{\frac{1}{p}}.$$

Starting from this general definition, we distinguish two procedures :

– *the* ***global thresholding procedure*** $\hat{f}^g$ *which corresponds to the procedure* $\hat{f}$ *described by (2.4) with* $l_j = |\Delta_j| = 2^j$ *(i.e* $\mathcal{B}_{j,K}$ *is reduced to the set* Δ_j*),*

– *the* ***optimal block thresholding procedure*** $\hat{f}^o$ *which corresponds to the procedure* $\hat{f}$ *described by (2.4) with* $l_j \asymp \ln(n)^{\frac{p}{2}} (\asymp 2^{j_1} \asymp j_2^{\frac{p}{2}})$.

Definition 2.2 (Hard thresholding procedures). *Let* $\nu > 0$ *and* $\sigma = (\sigma_j)_j$ *be an increasing positive sequence. We define the hard thresholding procedure* $\hat{f}^h$ *by :*

$$\hat{f}^h(x) = \sum_{k\in\Delta_\tau} \hat{\alpha}_{\tau,k}\phi_{\tau,k}(x) + \sum_{j<j_2}\sum_{k\in\Delta_j} \hat{\beta}_{j,k} 1_{\left\{|\hat{\beta}_{j,k}|\geq \kappa\sigma_j\sqrt{\frac{\ln(n)}{n}}\right\}} \psi_{j,k}(x), \qquad x\in[0,1], \tag{2.5}$$

where j_2 *is an integer satisfying* $2^{j_2} \asymp n^{\frac{\nu}{2}}$.

Typically, the sequence σ is of the form $(2^{\delta j})_j$ where $\delta \geq 0$ and it often appears in the literature of inverse problems (see for instance Kerkyacharian et al. (2005), Cavalier et al. (2003) and Pensky and Vidakovic (1998)). For a detailed example, see Section 5.

In the case where $\sigma = 1$, the previous block thresholding procedures have been developed by Kerkyacharian et al. (1996), Hall et al. (1998), Cai (1998) and Picard and Tribouley (2000). The main differences between the global thresholding procedures and the optimal block thresholding procedures are the length and the form of the blocks size. One depends on the level j and the other depends on the number of observations n independently of j.

Definition 2.3 (Maxisets). *Let* $1 \leq p < \infty$ *and* $n_0 \in \mathbb{N}^*$. *Let* $\hat{f}$ *be an estimate of* f. *The maxiset of* $\hat{f}$ *at the rate* u_n *under the* $\mathbb{L}^p$ *risk is the set of functions* f *such that there exists* $R > 0$ *satisfying :*

$$\sup_{n\geq n_0} u_n^{-1}\mathbb{E}_f^n(\|\hat{f} - f\|_p^p) \leq R < \infty.$$

Such maxiset is denoted $\mathcal{M}_{n_0}(\hat{f}, p, u_n)$.

Such a point of view provides a functional set which is authentically connected to the procedure and the model. The interest of a maxiset is twofold. Firstly, it allows to compare the performance of several procedures at a given rate of convergence. Secondly, it provides upper bounds over 'standard' function spaces. Several maxiset results can be found in Cohen et al. (2000), Kerkyacharian and Picard (2000), Rivoirard (2004) and Autin (2005).

3 Maxiset results

3.1 Function spaces

Let us introduce the function spaces which shall appear in the expression of our maxisets. For sake of legibility, we shall adopt the following notations : $\sum_K = \sum_{K\in\mathcal{A}_j}$ and $\sum_{(K)} = \sum_{k\in\mathcal{B}_{j,K}}$.

Definition 3.1 (Strong Besov spaces). *Let* $1 \leq p < \infty$. *We say that a function* f *of* $\mathbb{L}^p([0,1])$ *belongs to* $\mathcal{B}^\alpha_{p,\infty}$ *if and only if there exists* $R > 0$ *such that :*

$$\sup_{u>0} \; 2^{u\alpha p}\|\sum_{j\geq u}\sum_{k\in\Delta_j}\beta_{j,k}\psi_{j,k}\|_p^p \leq R < \infty.$$

Definition 3.2 ($\mathcal{W}$-spaces). *Let* $0 < r < p < \infty$ *and* $\sigma = (\sigma_j)_j$ *be a positive sequence. We say that :*

– *a function* f *of* $\mathbb{L}^p([0,1])$ *belongs to* $\mathcal{W}_\sigma(r,p)$ *if and only if there exists* $R > 0$ *such that :*

$$\sup_{u>0} \; u^{r-p}\|\sum_{j}\sum_{k\in\Delta_j}\beta_{j,k}1_{\{b_j(p)\leq u\sigma_j\}}\psi_{j,k}\|_p^p \leq R < \infty,$$

– *a function* f *of* $\mathbb{L}^p([0,1])$ *belongs to* $\mathcal{W}^*_\sigma(r,p)$ *if and only if there exists* $R > 0$ *such that :*

$$\sup_{u>0} \; u^{r-p}\sum_{j}\sum_{k\in\Delta_j}|\beta_{j,k}|^p 1_{\{b_j(p)\leq u\sigma_j\}}2^{j(\frac{p}{2}-1)} \leq R < \infty,$$

where $b_j(p)$ is the the normalized l_p-norm of wavelet coefficients $(\beta_{j,k})_{k\in\Delta_j}$ i.e :

$$b_j(p) = (2^{-j}\sum_{k\in\Delta_j}|\beta_{j,k}|^p)^{\frac{1}{p}}. \tag{3.1}$$

The previous function spaces can be viewed as a weighted $\mathbb{L}^p$ versions of those introduced by Autin (2005, Subsection 6.3.1). The following $\overline{W}$-spaces will play an intermediate role to prove Proposition 3.2 below.

Definition 3.3 ($\overline{\mathcal{W}}$-spaces). *Let $0<r<p<\infty$, $\kappa\in\mathbb{R}^{+*}$, $n_0\in\mathbb{N}^*$ and $\sigma=(\sigma_j)_j$ be a positive sequence. Let us consider the sets $\mathcal{B}_{j,K}$ with $l_j\asymp\ln(n)^{\frac{p}{2}}$. We say that :*

– *a function f of $\mathbb{L}^p([0,1])$ belongs to $\overline{\mathcal{W}}_{\sigma,\kappa,n_0}(r,p)$ if and only if there exists $R>0$ such that :*

$$\sup_{n>n_0} n^{\frac{p-r}{2}}\sum_{m\in\mathbb{N}}2^{-mp}\|\sum_j\sum_K\sum_{(K)}\beta_{j,k}1_{\left\{b_{j,K}(p)\le\kappa 2^m n^{-\frac{1}{2}}\sigma_j\right\}}\psi_{j,k}\|_p^p\le R<\infty,$$

– *a function f of $\mathbb{L}^p([0,1])$ belongs to $\overline{\mathcal{W}}^*_{\sigma,\kappa,n_0}(r,p)$ if and only if there exists $R>0$ such that :*

$$\sup_{n>n_0} n^{\frac{p-r}{2}}\sum_{m\in\mathbb{N}}2^{-mp}\sum_j\sum_K\sum_{(K)}|\beta_{j,k}|^p1_{\left\{b_{j,K}(p)\le\kappa 2^m n^{-\frac{1}{2}}\sigma_j\right\}}2^{j(\frac{p}{2}-1)}\le R<\infty,$$

– *a function f of $\mathbb{L}^p([0,1])$ belongs to $\overline{\mathcal{W}}^{**}_{\sigma,\kappa,n_0}(r,p)$ if and only if there exists $R>0$ such that :*

$$\sup_{n>n_0} n^{\frac{p-r}{2}}\|\sum_j\sum_K\sum_{(K)}\beta_{j,k}1_{\left\{b_{j,K}(p)\le\frac{\kappa}{2} n^{-\frac{1}{2}}\sigma_j\right\}}\psi_{j,k}\|_p^p\le R<\infty,$$

where $b_{j,K}(p)$ is the the normalized l_p-norm of wavelet coefficients $(\beta_{j,k})_{k\in\mathcal{B}_{j,K}}$ i.e :

$$b_{j,K}(p)=\frac{1}{\sqrt{\ln(n)}}(\sum_{(K)}|\beta_{j,k}|^p)^{\frac{1}{p}}. \tag{3.2}$$

The following spaces naturally appear when we study the maximal spaces of hard thresholding rules for a rate of convergence of the form $n^{-\gamma p}$.

Definition 3.4 ($\widehat{\mathcal{W}}_\sigma(r,p)$-spaces). *Let $0<r<p<\infty$, $\upsilon\in]0,1[$ and $\sigma=(\sigma_j)_j$ be a positive sequence. We say that a function f of $\mathbb{L}^p([0,1])$ belongs to $\widehat{\mathcal{W}}_\sigma(r,p)$ if and only if there exists $R>0$ such that :*

$$\sup_{u>0}\eta(u)^{r-p}\|\sum_j\sum_{k\in\Delta_j}\beta_{j,k}1_{\{|\beta_{j,k}|\le\sigma_j u\}}\psi_{j,k}\|_p^p\le R<\infty,$$

where η is the continuous non decreasing function such that $\eta(0)=0$, $\eta(u)=u\ln((u\wedge\upsilon)^{-1})^{-\frac{1}{2}}$ and υ is a real number such that $0<\upsilon\le\exp(-\frac{p-r}{2r})$.

3.2 Maxisets associated to block thresholding rules

Theorems 3.1 and 3.2 below investigate the maxiset properties of the block thresholding procedures $\hat{f}_a^g$ and $\hat{f}^o$ measured under the $\mathbb{L}^p$ risk for the rate of convergence $n^{-\frac{\alpha p}{2}}$.

Theorem 3.1. *Let $1<p<\infty$, $\sigma=(\sigma_j)_j$ be a known positive increasing sequence and $\hat{f}$ be the block thresholding procedure described by (2.4). Assume that there exists $C>0$ such that :*

– *$\hat{\alpha}_{j,k}$ and $\hat{\beta}_{j,k}$ satisfy the following moments conditions :*

$$\mathbb{E}_f^n(|\hat{\alpha}_{j_1,k}-\alpha_{j_1,k}|^p)\le C\sigma_{j_1}^p n^{-\frac{p}{2}}, \tag{3.3}$$

$$\mathbb{E}_f^n(|\hat{\beta}_{j,k}-\beta_{j,k}|^{2p})\le C\sigma_j^{2p}n^{-p},\qquad j_1\le j<j_2, \tag{3.4}$$

and the following concentration condition :

$$\mathbb{P}_f^n((l_j^{-1}\sum_{(K)}|\hat{\beta}_{j,k}-\beta_{j,k}|^\gamma)^{\frac{1}{\gamma}}\ge\lambda\sigma_j n^{-\frac{1}{2}})\le Cn^{-h(\lambda)},\qquad j_1\le j<j_2, \tag{3.5}$$

where $\gamma=p\vee 2$, λ is a large enough real number and h is a positive function such that $\lim_{x\to\infty}h(x)=\infty$.

– *σ and j_2 satisfy the following weighted inequality :*

$$\sum_{j<j_2}\sum_{k\in\Delta_j}\sigma_j^p\|\psi_{j,k}\|_p^p\leq Cn^{\frac{p}{2}}. \tag{3.6}$$

Then for any α in $]0,1[$, n_0 and κ large enough, the maxiset associated to the global thresholding procedure satisfies :

$$\mathcal{M}_{n_0}\left(\hat{f}^g,p,n^{-\frac{\alpha p}{2}}\right)\subseteq\mathcal{B}_{p,\infty}^{\frac{\alpha}{\nu}}\cap\mathcal{W}_\sigma((1-\alpha)p,p)$$

and the maxiset associated to the optimal block thresholding procedure satisfies :

$$\mathcal{M}_{n_0}\left(\hat{f}^o,p,n^{-\frac{\alpha p}{2}}\right)\subseteq\mathcal{B}_{p,\infty}^{\frac{\alpha}{\nu}}\cap\overline{\mathcal{W}}^{**}_{\sigma,\kappa,n_0}((1-\alpha)p,p).$$

Theorem 3.2. *Let $1<p<\infty$, $\sigma=(\sigma_j)_j$ be a known positive increasing sequence which satisfies the inequality (2.2) (only for $p>2$) and $\hat{f}$ be the block thresholding procedure described by (2.4). Under the assumptions (3.3), (3.4), (3.5) and (3.6) of Theorem 3.1, for any α in $]0,1[$, n_0 and κ large enough, the maxiset associated to the global thresholding procedure satisfies :*

$$\begin{cases}\mathcal{B}_{p,\infty}^{\frac{\alpha}{\nu}}\cap\mathcal{W}_\sigma((1-\alpha)p,p)\subseteq\mathcal{M}_{n_0}\left(\hat{f}^g,p,n^{-\frac{\alpha p}{2}}\right) & if\quad p\geq 2,\\ \mathcal{B}_{p,\infty}^{\frac{\alpha}{\nu}}\cap\mathcal{W}_\sigma^*((1-\alpha)p,p)\subseteq\mathcal{M}_{n_0}\left(\hat{f}^g,p,n^{-\frac{\alpha p}{2}}\right) & if\quad 1<p\leq 2,\end{cases}$$

and the maxiset associated to the optimal block thresholding procedure satisfies :

$$\begin{cases}\mathcal{B}_{p,\infty}^{\frac{\alpha}{\nu}}\cap\overline{\mathcal{W}}_{\sigma,\kappa,n_0}((1-\alpha)p,p)\subseteq\mathcal{M}_{n_0}\left(\hat{f}^o,p,n^{-\frac{\alpha p}{2}}\right) & if\quad p\geq 2,\\ \mathcal{B}_{p,\infty}^{\frac{\alpha}{\nu}}\cap\overline{\mathcal{W}}^*_{\sigma,\kappa,n_0}((1-\alpha)p,p)\subseteq\mathcal{M}_{n_0}\left(\hat{f}^o,p,n^{-\frac{\alpha p}{2}}\right) & if\quad 1<p\leq 2.\end{cases}$$

It is important to mention that the assumptions of Theorem 3.1 are not very restrictive (see Section 5). Proposition 3.1 below shows that under certain conditions on the factor α, we can reduce the maxiset associated to the global thresholding procedure.

Proposition 3.1. *Let $2\leq p<\infty$ and $\hat{f}^g$ be the global thresholding procedure. Under the assumptions of Theorem 3.2, for any α in $]0,1[$, n_0 and κ large enough we have :*

$$\mathcal{M}_{n_0}\left(\hat{f}^g,p,n^{-\frac{\alpha p}{2}}\right)=\begin{cases}\mathcal{B}_{p,\infty}^{\frac{\alpha}{\nu}}\cap\mathcal{W}_\sigma((1-\alpha)p,p) & if\quad \alpha\in]0,1[,\\ \mathcal{W}_\sigma((1-\alpha)p,p) & if\quad \alpha\in]1-\frac{\nu}{2},1[.\end{cases}$$

Precise that Proposition 3.1 above has been proved for the case $p=2$ and $\sigma=1$ in Cohen et al. (2000, Theorem 5).

Remark 3.1. *The maxiset associated to the optimal block thresholding procedure is difficult to determine because of the presence of the number n in the length of each block. Moreover, let us notice that the maxiset comparison of the block thresholding procedures is not always possible due to the different forms of the blocks.*

3.3 Maxisets comparison between hard and optimal block thresholding procedures

Theorem 3.3 below exhibits the maxisets associated to the hard thresholding procedures for the rate of convergence $n^{-\frac{\alpha p}{2}}$.

Theorem 3.3 (Maxiset associated to $\hat{f}^h$). *Let $1<p<\infty$, $\sigma=(\sigma_j)_j$ be a known positive sequence which satisfies the condition (2.2) (only for $p>2$) and $\hat{f}^h$ be the hard thresholding procedure described by (2.5). Suppose that $\hat{\alpha}_{j,k}$ and $\hat{\beta}_{j,k}$ satisfy the conditions (3.4), (3.3) with $j_1=\tau$ and that there exists $C>0$ such that the following concentration inequality holds :*

$$\mathbb{P}_f^n(|\hat{\beta}_{j,k}-\beta_{j,k}|\geq\lambda\sigma_j\sqrt{\frac{\ln(n)}{n}})\leq Cn^{-h(\lambda)},\qquad j_1\leq j<j_2, \tag{3.7}$$

where λ is a large enough real number and h is a function of $\mathbb{R}_+^*$ such that $\lim_{x\to\infty}h(x)=\infty$. Suppose that the weighted condition (3.6) holds. Then for any $\alpha\in]0,1[$, n_0 and κ large enough we have :

$$\mathcal{M}_{n_0}\left(\hat{f}^h,p,n^{-\frac{\alpha p}{2}}\right)=\mathcal{B}_{p,\infty}^{\frac{\alpha}{\nu}}\cap\widehat{\mathcal{W}}_\sigma((1-\alpha)p,p).$$

A similar maxiset Theorem has been proved in Kerkyacharian and Picard (2000) for the rate of convergence $(\frac{\ln(n)}{n})^{\frac{\alpha p}{2}}$. Proposition 3.2 below does the maxiset comparison between the optimal block thresholding procedures and the hard thresholding procedures for $p \geq 2$.

Proposition 3.2 (Maxisets comparison). *Let $2 \leq p < \infty$, $\hat{f}^o$ be the optimal block thresholding procedure and $\hat{f}^h$ be the hard thresholding procedure. Under the assumptions of Theorems 3.2 and 3.3, for any α in $]0,1[$, n_0 and κ large enough we have :*

$$\mathcal{M}_{n_0}\left(\hat{f}^h, p, n^{-\frac{\alpha p}{2}}\right) \subseteq \mathcal{M}_{n_0}\left(\hat{f}^o, p, n^{-\frac{\alpha p}{2}}\right).$$

In other words, $\hat{f}^o$ is better in the maxiset sense than $\hat{f}^h$.

We conclude that the optimal block thresholding procedures $\hat{f}^o$ can enjoy better maxiset properties than the hard thresholding procedures when we take the problem under the $\mathbb{L}^p$ risk. A similar result has been pointed out by Autin (2005, Chapter 6, Proposition 6.3) for the white noise model under the Besov risk. In practice, numerous simulations show that the optimal block estimator has excellent numerical performance relative to more traditional wavelet estimators (see for instance http://stat.wharton.upenn.edu/~tcai/paper/html/Neighblock.html).

4 Minimax results over Besov balls

We investigate in this subsection the upper bounds of the block thresholding rules over Besov balls under the $\mathbb{L}^p$ for general statistical models (including certain inverse problems).

Definition 4.1 (Besov balls). *Let $N \in \mathbb{N}^*$, $R > 0$, $0 < s < N$, $1 \leq r \leq \infty$ and $1 \leq \pi \leq \infty$. For any function f measurable on $[0,1]$, we denote the associated N-th order modulus of smoothness as*

$$\rho^N(t, f, \pi) = \sup_{|h| \leq t} \left(\int_{J_{Nh}} \left| \sum_{k=0}^{N} \binom{N}{k} (-1)^k f(u + kh) \right|^{\pi} du \right)^{\frac{1}{\pi}}$$

where $J_{Nh} = \{x \in [0,1] : \ x + Nh \in [0,1]\}$. We say that a function f of $\mathbb{L}^\pi([0,1])$ belongs to the Besov balls $B^s_{\pi,r}(R)$ if and only if

$$\left(\int_0^1 \left(\frac{\rho^N(t, f, \pi)}{t^s} \right)^r \frac{1}{t} dt \right)^{\frac{1}{r}} \leq R < \infty$$

with the usual modification if $r = \infty$.

The equivalence below shows the link which exists between the Besov balls and the wavelet basis ζ. Let $0 < s < N$ and $1 \leq \pi \leq \infty$. We have

$$f \in B^s_{\pi,r}(R) \Longleftrightarrow \begin{cases} (\sum_{j \geq \tau - 1} (2^{j(s + \frac{1}{2} - \frac{1}{\pi})} (\sum_{k \in \Delta_j} |\beta_{j,k}|^\pi)^{\frac{1}{\pi}})^r)^{\frac{1}{r}} \leq R < \infty & if \quad r < \infty, \\ \sup_j 2^{j(s + \frac{1}{2} - \frac{1}{\pi})} (\sum_{k \in \Delta_j} |\beta_{j,k}|^\pi)^{\frac{1}{\pi}} \leq R < \infty & if \quad r = \infty. \end{cases}$$

The minimax results presented in Theorem 4.1 below are direct consequences of the maxiset results exhibit in Theorem 3.2.

Theorem 4.1 (Minimax results over Besov balls). *Let $1 < p < \infty$, $R > 0$, $\delta \geq 0$, $\sigma = (2^{\delta j})_j$ and $\hat{f}$ be either the global block thresholding procedure or the optimal block thresholding procedure taken with $\nu = \frac{2}{1+2\delta}$ (see Definition 2.1). Under the assumption of Theorem 3.2, for $s > 0$, $r \geq 1$ and $\pi \geq p$ there exists a constant $C > 0$ such that :*

$$\sup_{f \in B^s_{\pi,r}(R)} \mathbb{E}^n_f(\|\hat{f} - f\|_p^p) \leq C n^{-\frac{sp}{1+2s+2\delta}}, \qquad n \geq n_0,$$

for n_0 and κ large enough.

The rate of convergence exhibit in the previous theorem can be minimax for several statistical models under the $\mathbb{L}^p$ risk over Besov balls. It is important to notice the assumptions of Theorem 3.1

are often satisfied with $\delta = 0$ for the majority of the 'standard' models (white noise, regression with equispaced data, for instance). The choice of $\sigma > 0$ often appears for the inverse problems. If we compare the previous upper bound with that reached by the hard thresholding procedures under the $\mathbb{L}^p$ risk over Besov balls, we remark that it is better in the sense where it is without logarithmic factor (see for instance Kerkyacharian and Picard (2000, Theorem 6.1)).

Remark 4.1. *Theorem 4.1 generalizes several minimax results concerning the block thresholding procedures obtained under the $\mathbb{L}^2$ risk. See for instance Cai (1998) and Pensky and Vidakovic (1998).*

Section 5 below gives two applications of the previous results by considering the regression model with Gaussian errors and a deconvolution in white noise.

5 Applications

Here and later, we shall note C to design a constant (independent of n and f) which may be different from one term to the other.

5.1 Equispaced regression model with Gaussian errors

In this subsection, assume that we observe the random variables $(Y_1, ..., Y_n)$ governed by the equation (1.1). For minimax studies under various setting concerning this model, we refer to the book of Tsybakov (2004). Let $\hat{f}$ be the block thresholding procedures described by (2.4) with $\sigma = 1$, $\delta = 0$, $\nu = 2$ and the estimators :

$$\hat{\alpha}_{j,k} = \frac{1}{n}\sum_{i=1}^{n} Y_i \phi_{j,k}(\frac{i}{n}) \quad \text{and} \quad \hat{\beta}_{j,k} = \frac{1}{n}\sum_{i=1}^{n} Y_i \psi_{j,k}(\frac{i}{n}).$$

For any b in $]0,1]$, let us defined the Hölder balls $\mathcal{H}^b(L)$ by :

$$\mathcal{H}^b(L) = \left\{h \text{ measurable on } [0,1];\ |h(x) - h(y)| \leq L|x-y|^b, \quad x, y \in [0,1]\right\}.$$

Let us introduce an immediate consequence of Donoho and Johnstone (1999, Lemma 5.1) :

Lemma 5.1. *For any $j \geq \tau$ and $k, k' \in \Delta_j$, there exists a constant $C > 0$ such that*

$$n^{-1}\sum_{i=1}^{n} \phi_{j,k}(\frac{i}{n})\phi_{j,k'}(\frac{i}{n}) \leq C\left(1_{\{k=k'\}} + 2^j n^{-1} 1_{\{S_{j,k}\cap S_{j,k'} \neq \emptyset\}}\right)$$

where $S_{j,k}$ denotes the support of $\psi_{j,k}$. This inequality holds if we replace ϕ by ψ.

Let us investigate the moments conditions (3.3) and (3.4). Under the assumption that $f \in \mathcal{H}^{\frac{1}{2}}(L)$, the inequalities obtained in Picard and Tribouley (2000, Subsection 9.1.2) and Lemma 5.1 give us :

$$\begin{aligned}
\mathbb{E}_f^n(|\hat{\beta}_{j,k} - \beta_{j,k}|^{2p}) &\leq 2^{p-1}\left(|\frac{1}{n}\sum_{i=1}^{n}(f(\frac{i}{n})\psi_{j_1,k}(\frac{i}{n}) - \beta_{j_1,k})|^{2p} + \mathbb{E}_f^n(|\frac{1}{n}\sum_{i=1}^{n}\psi_{j_1,k}(\frac{i}{n})z_i|^{2p})\right) \\
&\leq C\left(n^{-p}2^{-j_1 p} + 2^{jp}n^{-2p} + n^{-p}(\frac{1}{n}\sum_{i=1}^{n}|\psi_{j,k}(\frac{i}{n})|^2)^p\right) \leq Cn^{-p}
\end{aligned}$$

for any $j_1 \leq j < j_2$. Thus the conditions (3.3) and (3.4) hold. Since $2^{j_2} \asymp n$, it is obvious that :

$$\sum_{j<j_2}\sum_{k\in\Delta_j} \|\psi_{j,k}\|_p^p = \sum_{j<j_2} 2^{\frac{jp}{2}} \leq C2^{\frac{j_2 p}{2}} \leq Cn^{\frac{p}{2}}.$$

The weighted condition (3.6) is hence satisfied. Now, let us focus on the concentration condition (3.5) in the case where $p \geq 2$.

Lemma 5.2 (Cirelson, Ibragimov, Sudakov's inequality (Cirelson et al. (1976))). *Let $(\eta_t,\ t\in T)$ be a Gaussian process. Let N and W be respectively defined by $\mathbb{E}(\sup_{t\in T}\eta_t)$ and $\sup_{t\in T}Var(\eta_t)$. Then for all $c>0$ we have :*

$$\mathbb{P}(\sup_{t\in T}\eta_t\geq c+N)\leq \exp(-\frac{c^2}{2W}). \tag{5.1}$$

Let us set $e_{j,k,n}=(\hat{\beta}_{j,k}-\beta_{j,k})=\frac{1}{n}\sum_{i=1}^n\psi_{j,k}(\frac{i}{n})z_i$, $\mathcal{C}_q=\left\{a=(a_{j,k});\ \ \sum_{(K)}|a_{j,k}|^q\leq 1\right\}$ where q is the real number satisfying $q^{-1}+p^{-1}=1$, and $\{\mathcal{Z}(a),\ \ a\in\mathcal{C}_q\}$ the centered Gaussian process defined by :

$$\mathcal{Z}(a)=\sum_{(K)}a_{j,k}e_{j,k,n}.$$

An argument of duality gives us :

$$\sup_{a\in\mathcal{C}_q}\mathcal{Z}(a)=(\sum_{(K)}|e_{j,k,n}|^p)^{\frac{1}{p}}=(\sum_{(K)}|\hat{\beta}_{j,k}-\beta_{j,k}|^p)^{\frac{1}{p}}.$$

This allows us to apply the inequality (5.1). Using Holder's inequality and the moments condition (3.4), one gets :

$$N=\mathbb{E}_f^n(\sup_{a\in\mathcal{C}_q}\mathcal{Z}(a))\leq(\sum_{(K)}\mathbb{E}_f^n(|\hat{\beta}_{j,k}-\beta_{j,k}|^p))^{\frac{1}{p}}\leq Cn^{-\frac{1}{2}}l_j^{\frac{1}{p}}. \tag{5.2}$$

Since $\mathbb{E}_f^n(e_{j,k,n})=0$ and $\mathbb{E}_f^n(z_iz_{i'})=1$ if $i=i'$ and 0 otherwise, it follows from Lemma 5.1 and l_p-Hölder's inequality that :

$$\begin{aligned}
W &= \sup_{a\in\mathcal{C}_q}Var_f^n(\mathcal{Z}(a))=\sup_{a\in\mathcal{C}_q}\mathbb{E}_f^n(\sum_{k\in\mathcal{B}_{j,K}}\sum_{k'\in\mathcal{B}_{j,K}}a_{j,k}a_{j,k'}e_{j,k,n}e_{j,k',n})\\
&= \sup_{a\in\mathcal{C}_q}(n^{-2}\sum_{k\in\mathcal{B}_{j,K}}\sum_{k'\in\mathcal{B}_{j,K}}a_{j,k}a_{j,k'}\sum_{i=1}^n\sum_{i'=1}^n\psi_{j,k}(\frac{i}{n})\psi_{j,k'}(\frac{i'}{n})\mathbb{E}_f^n(z_iz_{i'}))\\
&= n^{-1}\sup_{a\in\mathcal{C}_q}\sum_{k\in\mathcal{B}_{j,K}}\sum_{k'\in\mathcal{B}_{j,K}}a_{j,k}a_{j,k'}(n^{-1}\sum_{i=1}^n\psi_{j,k}(\frac{i}{n})\psi_{j,k'}(\frac{i}{n})))\\
&\leq Cn^{-1}\sup_{a\in\mathcal{C}_q}(\sum_{k\in\mathcal{B}_{j,K}}a_{j,k}^2+\sum_{k\in\mathcal{B}_{j,K}}\sum_{k'\in\mathcal{B}_{j,K}}a_{j,k}a_{j,k'}1_{\{S_{j,k}\cap S_{j,k'}\neq\emptyset\}}).
\end{aligned}\tag{5.3}$$

Let us denote $\tilde{S}_{j,k}(x)=\{k;\ x\in S_{j,k}\}$. If $S_{j,k}\cap S_{j,k'}\neq\emptyset$ then there exists x of $[0,1]$ such that $x\in S_{j,k}$ and $x\in S_{j,k'}$. Considering such x, Hölder's inequality, the fact that ψ is compactly supported and that $q\leq 2$ yield :

$$\begin{aligned}
\sum_{k\in\mathcal{B}_{j,K}}\sum_{k'\in\mathcal{B}_{j,K}}a_{j,k}a_{j,k'}1_{\{S_{j,k}\cap S_{j,k'}\neq\emptyset\}} &\leq (\sum_{k\in\mathcal{B}_{j,K}}a_{j,k}1_{\{\tilde{S}_{j,k}(x)\}})^2\leq\sum_{k\in\mathcal{B}_{j,K}}a_{j,k}^2\sum_{k\in\mathcal{B}_{j,K}}1_{\{\tilde{S}_{j,k}(x)\}}\\
&\leq C\sum_{k\in\mathcal{B}_{j,K}}a_{j,k}^2\leq(\sum_{k\in\mathcal{B}_{j,K}}|a_{j,k}|^q)^{\frac{1}{q}}.
\end{aligned}\tag{5.4}$$

Putting (5.3) and (5.4) together, we observe that $W\leq Cn^{-1}$. Taking $\lambda>0$ large enough and c of the form $(\lambda-C)n^{-\frac{1}{2}}l_j^{\frac{1}{p}}$, the inequality (5.1) says that there exists a positive function h which satisfies $\lim_{x\to+\infty}h(x)=\infty$ and :

$$\mathcal{V}_n=\mathbb{P}_f^n((l_j^{-1}\sum_{(K)}|\hat{\beta}_{j,k}-\beta_{j,k}|^p)^{\frac{1}{p}}\geq\lambda n^{-\frac{1}{2}})\leq\mathbb{P}_f^n(\sup_{a\in\mathcal{C}_q}\mathcal{Z}(a)\geq c+N)\leq\exp(-l_j^{\frac{2}{p}}h(\lambda)).$$

Thus, if we work with the global thresholding procedure (i.e $l_j\asymp 2^j$ and $2^{j_1}\asymp\ln(n)^{\frac{p}{2}}$) then for $j_1<j$ we have :

$$\mathcal{V}_n\leq\exp(-2^{\frac{2j_1}{p}}h(\lambda))\leq n^{-h(\lambda)}.$$

If we work with the optimal block thresholding procedure (i.e $l_j \asymp \ln(n)^{\frac{p}{2}}$) then we have clearly :

$$\mathcal{V}_n \leq n^{-h(\lambda)}.$$

We deduce that the condition (3.5) holds. By virtue of a Gaussian inequality and Lemma 5.1, one gets :

$$\begin{aligned}\mathbb{P}_f^n(|\hat{\beta}_{j,k}-\beta_{j,k}|\geq\lambda\sqrt{\frac{\ln(n)}{n}}) &\leq \mathbb{P}_f^n(|\frac{1}{n}\sum_{i=1}^n\psi_{j,k}(\frac{i}{n})z_i|\geq\lambda\sqrt{\frac{\ln(n)}{n}})\leq 2\exp\left(-\frac{\lambda^2\ln(n)}{\frac{2}{n}\sum_{i=1}^n|\psi_{j,k}(\frac{i}{n})|^2}\right)\\ &\leq Cn^{-h(\lambda)}\end{aligned}$$

for any $\lambda>0$ so the concentration condition (3.7) holds. Therefore, if we suppose that $f\in\mathcal{H}^{\frac{1}{2}}(L)$ then we can apply Theorem 3.1, Theorem 3.2, Proposition 3.1, Proposition 3.2 and Theorem 4.1.

5.2 A deconvolution in white noise

Here, we consider an application of our maxiset results to a deconvolution problem which appears in Johnstone et al. (2004). Suppose that we observe the random process $(Y_t)_{t\in[0,1]}$ defined by :

$$dY_t = f\star g(t)dt + n^{-\frac{1}{2}}dB_t, \qquad t\in[0,1],$$

where B_t is a standard Brownian motion on $[0,1]$ and

$$f\star g(t)=\int_0^1 f(t-u)g(u)du.$$

We wish to recover the unknown function f via the the observations $(Y_t)_{t\in[0,1]}$. The function g is assumed to be known. Further, we assume that f and g are periodic on the unit interval and that the Fourier coefficients of g decay in a polynomial fashion i.e there exists a real number $\delta>2^{-1}$ satisfying :

$$\mathcal{F}(g)(l)\asymp|l|^{-\delta},\quad l\in\mathbb{Z}, \tag{5.5}$$

where $\mathcal{F}$ is the Fourier transform defined by $\mathcal{F}(h)(l)=\int_0^1 h(x)e^{-2i\pi lx}dx$ for $h\in\mathbb{L}^1([0,1])$.

Instead of working with the basis ζ as before, we consider a basis constructed from Meyer-type wavelet adapted to the unit interval by periodization. We denote this family by :

$$\zeta^M=\{\phi^M_{\tau,k}(.),\ k=0,...,2^\tau-1;\ \psi^M_{j,k}(.);\ j\geq\tau,\ k=0,...,2^j-1\}.$$

The main advantage of this choice is that the Fourier transforms of ψ^M and ϕ^M are compactly supported. Moreover, the Temlyakov property and the unconditional property described in Lemma 2.1 hold for ζ^M and ζ^M_σ (see Johnstone et al. (2004)).

Let $\hat{f}$ be the block thresholding procedure defined by (2.4) with $\sigma=(2^{\delta j})_j$, $\nu=\frac{2}{1+2\delta}$ and the following unbiased estimators of $\alpha_{j,k}$ and $\beta_{j,k}$:

$$\hat{\alpha}_{j,k}=\sum_{l\in C_j}\frac{\mathcal{F}^*(Y)(l)}{\mathcal{F}(g)(l)}\mathcal{F}(\phi^M_{j,k})(l)\quad\text{and}\quad\hat{\beta}_{j,k}=\sum_{l\in C_j}\frac{\mathcal{F}^*(Y)(l)}{\mathcal{F}(g)(l)}\mathcal{F}(\psi^M_{j,k})(l)$$

where $C_j=\{l;\ \mathcal{F}(\psi^M_{j,k})(l)\neq0\}=\{l;\ |l|\in[2\pi3^{-1}2^j,8\pi3^{-1}2^j]\}$ and, starting from a process $(R_t)_{t\in[0,1]}$ which satisfies $\mathbb{E}(|R_t|)<\infty$, the operator $\mathcal{F}^*(R)$ is defined by $\mathcal{F}^*(R)(l)=\int_0^1 e^{-2i\pi lx}dR_x$. The moments conditions (3.3), (3.4) and the concentration condition (3.7) have been shown in Johnstone et al. (2004, Section 5). Since $2^{j_2}\asymp n^{\frac{1}{1+2\delta}}$, it is obvious that :

$$\sum_{j<j_2}\sum_{k\in\Delta_j}\sigma_j^p\|\psi^M_{j,k}\|_p^p=\sum_{j<j_2}2^{jp(\delta+\frac{1}{2})}\leq C2^{j_2p(\delta+\frac{1}{2})}\leq Cn^{\frac{p}{2}}.$$

Hence the weighted condition (3.6) holds. Let us prove that the concentration condition (3.5) is satisfied in the case $p\geq2$ via the inequality (5.1). Let us set

$$\tilde{e}_{j,k,n}=\hat{\beta}_{j,k}-\beta_{j,k}=n^{-\frac{1}{2}}\sum_l\frac{\mathcal{F}^*(B)(l)}{\mathcal{F}(g)(l)}\mathcal{F}(\psi^M_{j,k})(l)$$

and let us observe that $\mathcal{F}^*(B)(l) \sim \mathcal{N}(0,1)$ and $\mathbb{E}_f^n(\mathcal{F}^*(B)(l)\mathcal{F}^*(B)(l')) = \int_0^1 e^{-2i\pi(l-l')t}dt = 1$ if $l = l'$ and 0 otherwise. Proceeding in the same way as in (5.2), let us consider the centered Gaussian process $\{\tilde{\mathcal{Z}}(a), \quad a \in \mathcal{C}_q\}$ defined by :

$$\tilde{\mathcal{Z}}(a) = \sum_{(K)} a_{j,k}\tilde{e}_{j,k,n}.$$

By virtue of the Holder inequality, it is easy to see that the moments condition (3.3) implies that $N = \mathbb{E}_f^n(\sup_{a\in\mathcal{C}_q} \tilde{\mathcal{Z}}(a)) \leq Cn^{-\frac{1}{2}} l_j^{\frac{1}{p}} 2^{\delta j}$.

Moreover, under the assumption (5.5), let us remark that :

$$\sup_{l\in C_j} |\mathcal{F}(g)(l)|^{-2} \leq C \sup_{l\in C_j} |l|^{2\delta} \leq C2^{2\delta j} (= C\sigma_j^2).$$

Considering the previous remarks and applying Plancherel's inequality, we can dominate W by :

$$\begin{aligned}
W &= \sup_{a\in\mathcal{C}_q} Var_f^n(\tilde{\mathcal{Z}}(a)) = \sup_{a\in\mathcal{C}_q} \mathbb{E}_f^n\Big(\sum_{k\in\mathcal{B}_{j,K}} \sum_{k'\in\mathcal{B}_{j,K}} a_{j,k}a_{j,k'}\tilde{e}_{j,k,n}\tilde{e}_{j,k',n}\Big) \\
&= n^{-1} \sup_{a\in\mathcal{C}_q}\Big(\sum_{k\in\mathcal{B}_{j,K}} \sum_{k'\in\mathcal{B}_{j,K}} a_{j,k}\overline{a_{j,k'}} \sum_l \sum_{l'} \mathcal{F}(g)(l)^{-1}\mathcal{F}(\psi_{j,k}^M)(l) \ \ldots \\
&\ldots \quad (\overline{\mathcal{F}(g)(l')})^{-1}\overline{\mathcal{F}(\psi_{j,k'}^M)(l')}\mathbb{E}_f^n(\mathcal{F}^*(B)(l)\mathcal{F}^*(B)(l'))\Big) \\
&= n^{-1} \sup_{a\in\mathcal{C}_q}\Big(\sum_{k\in\mathcal{B}_{j,K}} \sum_{k'\in\mathcal{B}_{j,K}} a_{j,k}\overline{a_{j,k'}} \sum_l |\mathcal{F}(g)(l)|^{-2}\mathcal{F}(\psi_{j,k}^M)(l)\overline{\mathcal{F}(\psi_{j,k'}^M)(l)}\Big) \\
&\leq n^{-1} \sup_{a\in\mathcal{C}_q}\Big(\sum_{k\in\mathcal{B}_{j,K}} \sum_{k'\in\mathcal{B}_{j,K}} a_{j,k}\overline{a_{j,k'}} \sup_{l\in C_j} |\mathcal{F}(g)(l)|^{-2} \sum_l \mathcal{F}(\psi_{j,k}^M)(l)\overline{\mathcal{F}(\psi_{j,k'}^M)(l)}\Big) \\
&= n^{-1} \sup_{l\in C_j} |\mathcal{F}(g)(l)|^{-2} \sup_{a\in\mathcal{C}_q}\Big(\sum_{k\in\mathcal{B}_{j,K}} \sum_{k'\in\mathcal{B}_{j,K}} a_{j,k}\overline{a_{j,k'}} \int \psi_{j,k}^M(x)\overline{\psi_{j,k'}^M(x)}dx\Big) \\
&= n^{-1} \sup_{l\in C_j} |\mathcal{F}(g)(l)|^{-2} \sup_{a\in\mathcal{C}_q}\Big(\sum_{k\in\mathcal{B}_{j,K}} |a_{j,k}|^2\Big) \leq C2^{2\delta j}n^{-1}.
\end{aligned}$$

Taking λ large enough and c of the form $(\lambda - C)n^{-\frac{1}{2}} l_j^{\frac{1}{p}} 2^{\delta j}$, the inequality (5.1) says that there exists a positive function h which satisfies $\lim_{x\to+\infty} h(x) = \infty$ and :

$$\mathcal{V}_n = \mathbb{P}_f^n\Big((l_j^{-1}\sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^p)^{\frac{1}{p}} \geq \sigma_j\lambda n^{-\frac{1}{2}}\Big) \leq \mathbb{P}_f^n\Big(\sup_{a\in\mathcal{C}_q} \tilde{\mathcal{Z}}(a) \geq c + N\Big) \leq \exp(-l_j^{\frac{2}{p}} h(\lambda)).$$

By using arguments similar to the previous subsection, we establish that the concentration condition (3.5) holds. Therefore, if we suppose that g satisfies the condition (5.5) then we can apply Theorem 3.1, Theorem (3.2), Proposition 3.1, Proposition 3.2 and Theorem 4.1.

Conclusion : In the two previous statistical models, we have proved that the corresponding optimal block thresholding procedure

1. is better in the maxiset sense than the corresponding hard thresholding procedure under the $\mathbb{L}^p$ risk and for the rate of convergence $n^{-\frac{\alpha p}{2}}$,
2. is optimal in the minimax sense over Besov balls $B_{\pi,r}^s(r)$ under the $\mathbb{L}^p$ risk if $\pi \geq p > 1$, $r \geq 1$ and $s > 0$ contrary to the corresponding hard thresholding procedure.

6 Appendix : proofs of Theorems and Propositions

Before beginning the proofs of our Theorems and Propositions, let us set several important remarks which will be intensively used in the sequel.

Preliminary remarks : For any sequence $u = (u_{j,k})_{j,k}$, $v = (v_{j,K})_{j,K}$ and any $j \geq \tau$, we can write :

$$\sum_K \sum_{(K)} u_{j,k}v_{j,K}\psi_{j,k}(x) = \sum_{k\in\Delta_j} u_{j,k}^*\psi_{j,k}(x),$$

where the coefficients $(u^*_{j,0}, u^*_{j,1}, ..., u^*_{j,2^j})$ are respectively :

$$(\underbrace{u_{j,0}v_{j,1}, u_{j,1}v_{j,1}, ..., u_{j,l_j-1}v_{j,1}}_{\text{length } l_j}, \underbrace{u_{j,l_j}v_{j,2}, ...}_{\text{length } l_j} ... ..., \underbrace{u_{j,2^j}v_{j,2^j l_j^{-1}}}_{\text{length } l_j}).$$

So, the unconditional nature of ζ (see Lemma 2.1) implies that :

$$\begin{aligned} \|\sum_{j\geq\tau-1}\sum_{K}\sum_{(K)} u_{j,k}v_{j,K}\psi_{j,k}\|_p^p &\asymp \|(\sum_{j\geq\tau-1}\sum_{k\in\Delta_j}|u^*_{j,k}\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p \\ &= \|(\sum_{j\geq\tau-1}\sum_{K}\sum_{(K)}|u_{j,k}v_{j,K}\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p. \end{aligned} \tag{6.1}$$

Moreover, Lemma 2.2 says that for any sequence $u=(u_{j,k})_{j,k}$, $v=(v_{j,K})_{j,K}$ and any $j\geq\tau$ we have :

$$\|\sum_{K}\sum_{(K)} u_{j,k}v_{j,K}\psi_{j,k}\|_p^p \leq C2^{j(\frac{p}{2}-1)}\sum_{k\in\Delta_j}|u^*_{j,k}|^p = C2^{j(\frac{p}{2}-1)}\sum_{K}\sum_{(K)}|u_{j,k}v_{j,K}|^p \tag{6.2}$$

For sake of simplicity in exposition, we shall refer to 'the inequality (2.1)' (see Lemma 2.1) or to 'the inequality (6.1)' when we use the unconditional nature of ζ. The proofs of Theorem 3.1 and Theorem 3.2 below are based on mathematical arguments similar to Kerkyacharian and Picard (2000, Proofs of Theorems 5.1 and 5.2).

Proof of Theorem 3.1. Let $p>1$. Let us show that for any constant $C>0$ satisfying :

$$\mathbb{E}_f^n(\|\hat{f}^g-f\|_p^p)\leq Cn^{-\frac{\alpha p}{2}} \quad \text{and} \quad \mathbb{E}_f^n(\|\hat{f}^o-f\|_p^p)\leq Cn^{-\frac{\alpha p}{2}}, \quad n\geq n_0,$$

we have respectively :

$$f\in\mathcal{B}^{\frac{\alpha}{\nu}}_{p,\infty}\cap\mathcal{W}_\sigma((1-\alpha)p,p) \quad \text{and} \quad f\in\mathcal{B}^{\frac{\alpha}{\nu}}_{p,\infty}\cap\overline{\mathcal{W}}^{**}_{\sigma,\kappa,n_0}((1-\alpha)p,p)$$

for n_0 and κ large enough.

Let $\hat{f}$ be the block thresholding procedure defined by (2.4) (including $\hat{f}^g$ and $\hat{f}^o$). For any constant $C>0$ such that $\mathbb{E}_f^n(\|\hat{f}-f\|_p^p)\leq Cn^{-\frac{\alpha p}{2}}$, the inequality (6.1) yields :

$$\begin{aligned} n^{-\frac{\alpha p}{2}}C &\geq \mathbb{E}_f^n(\|\hat{f}-f\|_p^p) \\ &\geq C\mathbb{E}_f^n(\|(\sum_{j_1\leq j<j_2}\sum_{K}\sum_{(K)}|\hat{\beta}_{j,k}1_{\left\{\hat{b}_{j,K}(p)\geq\sigma_j\kappa n^{-\frac{1}{2}}\right\}}-\beta_{j,k}|^2|\psi_{j,k}|^2+\sum_{j\geq j_2}\sum_{k\in\Delta_j}|\beta_{j,k}\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p) \\ &\geq C\max(T_1,T_2) \end{aligned}$$

where

$$T_1=\mathbb{E}_f^n(\|(\sum_{j_1\leq j<j_2}\sum_{K}\sum_{(K)}|\beta_{j,k}|^2 1_{\left\{\hat{b}_{j,K}(p)<\sigma_j\kappa n^{-\frac{1}{2}}\right\}}|\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p) \quad \text{and} \quad T_2=\|(\sum_{j\geq j_2}\sum_{k\in\Delta_j}|\beta_{j,k}\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p.$$

Thus we have :

$$\max(T_1,T_2)\leq Cn^{-\frac{\alpha p}{2}}. \tag{6.3}$$

Let us show that $f\in\mathcal{B}^{\frac{\alpha}{\nu}}_{p,\infty}$. Using the inequalities (2.1) and (6.3), one gets :

$$\|\sum_{j\geq j_2}\sum_{k\in\Delta_j}\beta_{j,k}\psi_{j,k}\|_p^p\leq C\|(\sum_{j\geq j_2}\sum_{k\in\Delta_j}|\beta_{j,k}\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p=CT_2\leq Cn^{-\frac{\alpha p}{2}}. \tag{6.4}$$

So, if $u\leq c$ where c denotes a small constant then the inequality (2.1) gives us :

$$\|\sum_{j\geq u}\sum_{k\in\Delta_j}\beta_{j,k}\psi_{j,k}\|_p^p\leq C\|f\|_p^p\leq C2^{-u\frac{\alpha p}{\nu}}.$$

For $u > c$, let us remark that there exists an integer n satisfying $j_2 < u < j_2 + 1$. Using the inequalities (2.1) and (6.4), one gets :

$$\|\sum_{j\geq u}\sum_{k\in\Delta_j}\beta_{j,k}\psi_{j,k}\|_p^p \leq C\|\sum_{j\geq j_2}\sum_{k\in\Delta_j}\beta_{j,k}\psi_{j,k}\|_p^p \leq Cn^{-\frac{\alpha p}{2}} \leq C2^{-j_2\frac{\alpha p}{\nu}} \leq C2^{-u\frac{\alpha p}{\nu}}.$$

We conclude that $f \in \mathcal{B}^{\frac{\alpha}{\nu}}_{p,\infty}$.

Let us show that $f \in \mathcal{W}_\sigma((1-\alpha)p,p)$ if $\hat{f} = \hat{f}^g$ and $f \in \overline{W}^{**}_{\sigma,\kappa,n_0}((1-\alpha)p,p)$ if $\hat{f} = \hat{f}^o$. Applying Minkowski's inequality with the elementary inequality :

$$|\sum_{i=1}^m a_i|^r \leq m^{r-1}(\sum_{i=1}^m |a_i|^r), \quad 0 \leq a_i < \infty, \quad 1 \leq m < \infty, \quad r \geq 1, \tag{6.5}$$

one gets :

$$\|\sum_j\sum_K\sum_{(K)}\beta_{j,k}1_{\left\{b_{j,K}(p)\leq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}}\psi_{j,k}\|_p^p \leq 3^{p-1}(W_1+W_2+W_3) \tag{6.6}$$

where

$$W_1 = \|\sum_{j<j_1}\sum_K\sum_{(K)}\beta_{j,k}1_{\left\{b_{j,K}(p)\leq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}}\psi_{j,k}\|_p^p, \quad W_2 = \|\sum_{j_1\leq j<j_2}\sum_K\sum_{(K)}\beta_{j,k}1_{\left\{b_{j,K}(p)\leq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}}\psi_{j,k}\|_p^p$$

and

$$W_3 = \|\sum_{j\geq j_2}\sum_K\sum_{(K)}\beta_{j,k}1_{\left\{b_{j,K}(p)\leq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}}\psi_{j,k}\|_p^p.$$

The upper bound for the term W_1. By virtue of Minkowski's inequality, the inequality (6.2), the fact that $|\mathcal{A}_j| = 2^j l_j^{-1}$ and that $\alpha \in]0,1[$, one gets :

$$\begin{aligned}
W_1 &\leq (\sum_{j<j_1}\|\sum_K\sum_{(K)}\beta_{j,k}\psi_{j,k}1_{\left\{b_{j,K}(p)\leq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}}\|_p)^p \\
&\leq C(\sum_{j<j_1}2^{j(\frac{1}{2}-\frac{1}{p})}(\sum_K\sum_{(K)}|\beta_{j,k}|^p 1_{\left\{b_{j,K}(p)\leq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}})^{\frac{1}{p}})^p \\
&= C(\sum_{j<j_1}2^{j(\frac{1}{2}-\frac{1}{p})}l_j^{\frac{1}{p}}(\sum_K b_{j,K}(p)^p 1_{\left\{b_{j,K}(p)\leq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}})^{\frac{1}{p}})^p \\
&\leq Cn^{-\frac{p}{2}}(\sum_{j<j_1}2^{\frac{j}{2}}\sigma_j)^p \leq Cn^{-\frac{p}{2}}2^{\frac{j_1 p}{2}}\sigma_{j_1}^p \leq Cn^{-\frac{p}{2}}\ln(n)^{\upsilon p+\frac{p^2}{4}} \leq Cn^{-\frac{\alpha p}{2}}
\end{aligned} \tag{6.7}$$

for n large enough.

The upper bound for the term W_2. Minkowski's inequality and the inequality (6.5) yield :

$$W_2 \leq 2^{p-1}(U_1+U_2) \tag{6.8}$$

where

$$U_1 = \mathbb{E}_f^n(\|\sum_{j_1\leq j<j_2}\sum_K\sum_{(K)}\beta_{j,k}1_{\left\{b_{j,K}(p)\leq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}}1_{\left\{\hat{b}_{j,K}(p)<\sigma_j\kappa n^{-\frac{1}{2}}\right\}}\psi_{j,k}\|_p^p)$$

and

$$U_2 = \mathbb{E}_f^n(\|\sum_{j_1\leq j<j_2}\sum_K\sum_{(K)}\beta_{j,k}1_{\left\{b_{j,K}(p)\leq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}}1_{\left\{\hat{b}_{j,K}(p)\geq\sigma_j\kappa n^{-\frac{1}{2}}\right\}}\psi_{j,k}\|_p^p).$$

Using the inequalities (6.1) and (6.3), one obtains :

$$U_1 \leq C\mathbb{E}_f^n(\|(\sum_{j_1\leq j<j_2}\sum_K\sum_{(K)}|\beta_{j,k}|^2 1_{\left\{\hat{b}_{j,K}(p)<\sigma_j\kappa n^{-\frac{1}{2}}\right\}}|\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p) = CT_1 \leq Cn^{-\frac{\alpha p}{2}}. \tag{6.9}$$

In order to bound the term U_2, let us remark that l_p-Minkowski's inequality gives us :

$$\begin{aligned}
1_{\left\{b_{j,K}(p)\leq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}}1_{\left\{\hat{b}_{j,K}(p)\geq\sigma_j\kappa n^{-\frac{1}{2}}\right\}} &\leq 1_{\left\{|\hat{b}_{j,K}(p)-b_{j,K}(p)|\geq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}} \\
&\leq 1_{\left\{(l_j^{-1}\sum_{(K)}|\hat{\beta}_{j,k}-\beta_{j,k}|^p)^{\frac{1}{p}}\geq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}}.
\end{aligned} \tag{6.10}$$

Now, let us investigate separately the case where $p \geq 2$ and the case where $1 < p \leq 2$.

For $p \geq 2$: Using the inequality (6.1), the generalized Minkowski inequality, the inequality (6.10), the concentration condition (3.5) and the inequality (2.1), it comes :

$$\begin{aligned}
U_2 &\leq C\mathbb{E}_f^n(\|(\sum_{j_1\leq j<j_2}\sum_K\sum_{(K)}|\beta_{j,k}|^2 1_{\left\{b_{j,K}(p)\leq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}}1_{\left\{\hat{b}_{j,K}(p)\geq\sigma_j\kappa n^{-\frac{1}{2}}\right\}}|\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p) \\
&\leq C\|(\sum_{j_1\leq j<j_2}\sum_K\sum_{(K)}|\beta_{j,k}|^2\mathbb{E}_f^n(1_{\left\{b_{j,K}(p)\leq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}}1_{\left\{\hat{b}_{j,K}(p)\geq\sigma_j\kappa n^{-\frac{1}{2}}\right\}})^{\frac{2}{p}}|\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p \\
&\leq C\|(\sum_{j_1\leq j<j_2}\sum_K\sum_{(K)}|\beta_{j,k}|^2\mathbb{P}_f^n((l_j^{-1}\sum_{(K)}|\hat{\beta}_{j,k}-\beta_{j,k}|^p)^{\frac{1}{p}}\geq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2})^{\frac{2}{p}}|\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p \\
&\leq Cn^{-h(\frac{\kappa}{2})}\|(\sum_{j<j_2}\sum_{k\in\Delta_j}|\beta_{j,k}|^2|\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p \leq C\|f\|_p^p n^{-h(\frac{\kappa}{2})} \leq Cn^{-\frac{\alpha p}{2}}
\end{aligned} \tag{6.11}$$

for κ large enough.

For $1 < p \leq 2$: Using the inequality (6.1), the Jensen inequality, the inequality (6.10), the fact that :

$$(l_j^{-1}\sum_{(K)}|\hat{\beta}_{j,k}-\beta_{j,k}|^p)^{\frac{1}{p}} \leq (l_j^{-1}\sum_{(K)}|\hat{\beta}_{j,k}-\beta_{j,k}|^2)^{\frac{1}{2}} \tag{6.12}$$

and the inequality (2.1), one gets :

$$\begin{aligned}
U_2 &\leq C\mathbb{E}_f^n(\|(\sum_{j_1\leq j<j_2}\sum_K\sum_{(K)}|\beta_{j,k}|^2 1_{\left\{b_{j,K}(p)\leq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}}1_{\left\{\hat{b}_{j,K}(p)\geq\sigma_j\kappa n^{-\frac{1}{2}}\right\}}|\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p) \\
&\leq C\|(\sum_{j_1\leq j<j_2}\sum_K\sum_{(K)}|\beta_{j,k}|^2\mathbb{E}_f^n(1_{\left\{b_{j,K}(p)\leq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}}1_{\left\{\hat{b}_{j,K}(p)\geq\sigma_j\kappa n^{-\frac{1}{2}}\right\}})|\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p \\
&\leq C\|(\sum_{j_1\leq j<j_2}\sum_K\sum_{(K)}|\beta_{j,k}|^2\mathbb{P}_f^n((l_j^{-1}\sum_{(K)}|\hat{\beta}_{j,k}-\beta_{j,k}|^2)^{\frac{1}{2}}\geq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2})|\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p \\
&\leq Cn^{-h(\frac{\kappa}{2})\frac{p}{2}}\|(\sum_{j<j_2}\sum_{k\in\Delta_j}|\beta_{j,k}|^2|\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p \leq C\|f\|_p^p n^{-h(\frac{\kappa}{2})\frac{p}{2}} \leq Cn^{-\frac{\alpha p}{2}}
\end{aligned} \tag{6.13}$$

Combining (6.8), (6.9), (6.11) and (6.13) we deduce that for $p > 1$:

$$W_2 \leq Cn^{-\frac{\alpha p}{2}}. \tag{6.14}$$

The upper bound for the term W_3. By virtue of the inequality (6.1), one gets :

$$W_3 \leq C\|\sum_{j\geq j_2}\sum_{k\in\Delta_j}\beta_{j,k}\psi_{j,k}\|_p^p \leq CT_2 \leq Cn^{-\frac{\alpha p}{2}}. \tag{6.15}$$

Putting (6.6), (6.7), (6.14) and (6.15) together, we deduce that :

$$\|\sum_j\sum_K\sum_{(K)}\beta_{j,k}1_{\left\{b_{j,K}(p)\leq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}}\psi_{j,k}\|_p^p \leq Cn^{-\frac{\alpha p}{2}} \tag{6.16}$$

for $p > 1$. It follows that $f \in \overline{\mathcal{W}}^{**}_{\sigma,\kappa,n_0}((1-\alpha)p,p)$ if $\hat{f} = \hat{f}^o$ (i.e $l_j = \ln(n)^{\frac{p}{2}}$).

In order to show that $f \in \mathcal{W}_\sigma((1-\alpha)p,p)$ if $\hat{f} = \hat{f}^g$ (i.e $l_j = 2^j$), let us disntiguish two cases :

In the case $u > c$ where c denotes a small constant, the inequality (6.1) gives us :

$$\|\sum_j\sum_K\sum_{(K)}\beta_{j,k}1_{\{b_{j,K}(p)\leq\sigma_j u\}}\psi_{j,k}\|_p^p \leq C\|f\|_p^p \leq Cu^{\alpha p}.$$

For $u < c$, let us remark that there exists an integer n such that $\frac{\kappa(n+1)^{-\frac{1}{2}}}{2} < u \leq \frac{\kappa n^{-\frac{1}{2}}}{2}$. Using the inequalities (6.1) and (6.16), one gets :

$$\begin{aligned}
\|\sum_j\sum_K\sum_{(K)}\beta_{j,k}1_{\{b_{j,K}(p)\leq\sigma_j u\}}\psi_{j,k}\|_p^p &\leq \|\sum_j\sum_K\sum_{(K)}\beta_{j,k}1_{\left\{b_{j,K}(p)\leq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}}\psi_{j,k}\|_p^p \leq Cn^{-\frac{\alpha p}{2}} \\
&\leq Cu^{\alpha p}(\frac{n}{n+1})^{-\frac{\alpha p}{2}} \leq Cu^{\alpha p}.
\end{aligned}$$

This ends the proof of the theorem. □

Proof of Theorem 3.2. Let us prove that :

- For $p \geq 2$, any $f \in \mathcal{B}^{\frac{\alpha}{\nu}}_{p,\infty} \cap \mathcal{W}_\sigma((1-\alpha)p,p)$ and any $f \in \mathcal{B}^{\frac{\alpha}{\nu}}_{p,\infty} \cap \overline{\mathcal{W}}_{\sigma,\kappa,n_0}((1-\alpha)p,p)$ there exists a constant $C>0$ such that we have respectively :

$$\mathbb{E}^n_f(\|\hat{f}^g - f\|^p_p) \leq Cn^{-\frac{\alpha p}{2}} \quad \text{and} \quad \mathbb{E}^n_f(\|\hat{f}^o - f\|^p_p) \leq Cn^{-\frac{\alpha p}{2}}, \quad n \geq n_0,$$

  for n_0 and κ large enough.
- For $2 \geq p > 1$, any $f \in \mathcal{B}^{\frac{\alpha}{\nu}}_{p,\infty} \cap \mathcal{W}^*_\sigma((1-\alpha)p,p)$ and any $f \in \mathcal{B}^{\frac{\alpha}{\nu}}_{p,\infty} \cap \overline{\mathcal{W}}^*_{\sigma,\kappa,n_0}((1-\alpha)p,p)$ there exists a constant $C>0$ such that we have respectively :

$$\mathbb{E}^n_f(\|\hat{f}^g - f\|^p_p) \leq Cn^{-\frac{\alpha p}{2}} \quad \text{and} \quad \mathbb{E}^n_f(\|\hat{f}^o - f\|^p_p) \leq Cn^{-\frac{\alpha p}{2}}, \quad n \geq n_0,$$

  for n_0 and κ large enough.

Let $\hat{f}$ be the block thresholding procedure defined by (2.4). We distinguish the case where $\hat{f} = \hat{f}^g$ and the case where $\hat{f} = \hat{f}^g$ when it is necessary. Combining Minkowski's inequality with (6.5), the $\mathbb{L}^p$ risk of $\hat{f}$ can be bounded by a sum of four components :

$$\mathbb{E}^n_f(\|\hat{f} - f\|^p_p) \leq 4^{p-1}(E_1 + E_2 + E_3 + E_4) \tag{6.17}$$

where

$$E_1 = \mathbb{E}^n_f(\| \sum_{k \in \Delta_{j_1}} (\hat{\alpha}_{j_1,k} - \alpha_{j_1,k})\psi_{j_1,k}\|^p_p), \quad E_2 = \mathbb{E}^n_f(\| \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \beta_{j,k} 1_{\left\{\hat{b}_{j,K}(p) < \sigma_j \kappa n^{-\frac{1}{2}}\right\}} \psi_{j,k}\|^p_p),$$

$$E_3 = \mathbb{E}^n_f(\| \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} (\hat{\beta}_{j,k} - \beta_{j,k}) 1_{\left\{\hat{b}_{j,K}(p) \geq \sigma_j \kappa n^{-\frac{1}{2}}\right\}} \psi_{j,k}\|^p_p) \quad \text{and} \quad E_4 = \| \sum_{j \geq j_2} \sum_{k \in \Delta_j} \beta_{j,k}\psi_{j,k}\|^p_p.$$

Let us analyze each term E_i, i=1,2,3,4.

The upper bound for the term E_1. It follows from the moments condition (3.3) and Lemma 2.2 that :

$$E_1 \leq C 2^{j_1(\frac{p}{2}-1)} \sum_{k \in \Delta_{j_1}} \mathbb{E}^n_f(|\hat{\alpha}_{j_1,k} - \alpha_{j_1,k}|^p) \leq n^{-\frac{p}{2}} 2^{\frac{j_1 p}{2}} \sigma^p_{j_1}.$$

Since j_1 is chosen such that $2^{j_1} \asymp \ln(n)^{\frac{p}{2}}$, the sequence σ satisfies $\sigma_{j_1} \asymp \ln(n)^\upsilon$ with $\upsilon > 0$ and that $\alpha \in [0,1]$, we see that :

$$E_1 \leq Cn^{-\frac{\alpha p}{2}} \tag{6.18}$$

for n large enough.

The upper bound for the term E_4. Since $f \in B^{\frac{\alpha}{\nu}}_{p,\infty}$:

$$E_4 \leq Cn^{-\frac{\alpha p}{2}}. \tag{6.19}$$

The upper bound for the term E_2. By virtue of Minkowski's inequality and the inequality (6.5), we have :

$$E_2 \leq 2^{p-1}(E_{2,1} + E_{2,2}) \tag{6.20}$$

where

$$E_{2,1} = \mathbb{E}^n_f(\| \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \beta_{j,k} 1_{\left\{\hat{b}_{j,K}(p) < \sigma_j \kappa n^{-\frac{1}{2}}\right\}} 1_{\left\{b_{j,K}(p) \leq 2\sigma_j \kappa n^{-\frac{1}{2}}\right\}} \psi_{j,k}\|^p_p)$$

and

$$E_{2,2} = \mathbb{E}^n_f(\| \sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} \beta_{j,k} 1_{\left\{\hat{b}_{j,K}(p) < \sigma_j \kappa n^{-\frac{1}{2}}\right\}} 1_{\left\{b_{j,K}(p) > 2\sigma_j \kappa n^{-\frac{1}{2}}\right\}} \psi_{j,k}\|^p_p).$$

In order to majorize $E_{2,1}$, let us investigate separately the case where $p \geq 2$ and the case where $1 < p \leq 2$.

For $p \geq 2$: Using the inequality (6.1) and the fact that $f \in \mathcal{W}_\sigma((1-\alpha)p,p)$ or $f \in \overline{\mathcal{W}}_{\sigma,\kappa,n_0}((1-\alpha)p,p)$ (according to the form of l_j), the term $E_{2,1}$ can be bounded by :

$$E_{2,1} \leq C\|\sum_j \sum_K \sum_{(K)} \beta_{j,k} 1_{\left\{b_{j,K}(p) \leq 2\sigma_j \kappa n^{-\frac{1}{2}}\right\}} \psi_{j,k}\|_p^p \leq Cn^{-\frac{\alpha p}{2}}. \tag{6.21}$$

For $1 < p \leq 2$: By virtue of the inequality (6.1), the comparison between the l_p norms and the fact that $f \in \mathcal{W}_\sigma^*((1-\alpha)p,p)$ or $f \in \overline{\mathcal{W}}_{\sigma,\kappa,n_0}^*((1-\alpha)p,p)$, we observe that :

$$\begin{aligned} E_{2,1} &\leq C\|\sum_j \sum_K \sum_{(K)} \beta_{j,k} 1_{\left\{b_{j,K}(p) \leq 2\sigma_j \kappa n^{-\frac{1}{2}}\right\}} \psi_{j,k}\|_p^p \leq C\|(\sum_j \sum_K \sum_{(K)} \beta_{j,k} 1_{\left\{b_{j,K}(p) \leq 2\sigma_j \kappa n^{-\frac{1}{2}}\right\}} |\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p \\ &\leq \sum_j \sum_K \sum_{(K)} |\beta_{j,k}|^p 1_{\left\{b_{j,K}(p) \leq 2\sigma_j \kappa n^{-\frac{1}{2}}\right\}} 2^{j(\frac{p}{2}-1)} \leq Cn^{-\frac{\alpha p}{2}}. \end{aligned} \tag{6.22}$$

In order to dominate the term $E_{2,2}$, let us remark that the l_p-Minkowski inequality gives us :

$$\begin{aligned} 1_{\left\{b_{j,K}(p) > 2\sigma_j \kappa n^{-\frac{1}{2}}\right\}} 1_{\left\{\hat{b}_{j,K}(p) < \sigma_j \kappa n^{-\frac{1}{2}}\right\}} &\leq 1_{\left\{|\hat{b}_{j,K}(p) - b_{j,K}(p)| \geq \sigma_j \kappa n^{-\frac{1}{2}}\right\}} \\ &\leq 1_{\left\{(l_j^{-1} \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^p)^{\frac{1}{p}} \geq \sigma_j \kappa n^{-\frac{1}{2}}\right\}}. \end{aligned} \tag{6.23}$$

Let us distinguish the case where $p \geq 2$ and the case where $1 < p \leq 2$.

For $p \geq 2$: Using the inequality (6.1), the generalized Minkowski inequality, the inequality (6.23), the concentration condition (3.5) and the inequality (2.1), we have :

$$\begin{aligned} E_{2,2} &\leq C\mathbb{E}_f^n(\|(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\beta_{j,k}|^2 1_{\left\{b_{j,K}(p) > 2\sigma_j \kappa n^{-\frac{1}{2}}\right\}} 1_{\left\{\hat{b}_{j,K}(p) < \sigma_j \kappa n^{-\frac{1}{2}}\right\}} |\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p) \\ &\leq C\|(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\beta_{j,k}|^2 \mathbb{E}_f^n(1_{\left\{b_{j,K}(p) > 2\sigma_j \kappa n^{-\frac{1}{2}}\right\}} 1_{\left\{\hat{b}_{j,K}(p) < \sigma_j \kappa n^{-\frac{1}{2}}\right\}})^{\frac{2}{p}} |\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p \\ &\leq C\|(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\beta_{j,k}|^2 \mathbb{P}_f^n((l_j^{-1} \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^p)^{\frac{1}{p}} \geq \sigma_j \kappa n^{-\frac{1}{2}})^{\frac{2}{p}} |\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p \\ &\leq Cn^{-h(\kappa)}\|(\sum_{j < j_2} \sum_{k \in \Delta_j} |\beta_{j,k}|^2 |\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p \leq C\|f\|_p^p n^{-\frac{\alpha p}{2}} \leq Cn^{-\frac{\alpha p}{2}} \end{aligned} \tag{6.24}$$

for κ large enough.

For $1 < p \leq 2$: Using the inequality (6.1), the Jensen inequality, the inequalities (6.23) and (6.12), the concentration condition (3.5) and the inequality (2.1), we have :

$$\begin{aligned} E_{2,2} &\leq C\mathbb{E}_f^n(\|(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\beta_{j,k}|^2 1_{\left\{b_{j,K}(p) > 2\sigma_j \kappa n^{-\frac{1}{2}}\right\}} 1_{\left\{\hat{b}_{j,K}(p) < \sigma_j \kappa n^{-\frac{1}{2}}\right\}} |\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p) \\ &\leq C\|(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\beta_{j,k}|^2 \mathbb{E}_f^n(1_{\left\{b_{j,K}(p) > 2\sigma_j \kappa n^{-\frac{1}{2}}\right\}} 1_{\left\{\hat{b}_{j,K}(p) < \sigma_j \kappa n^{-\frac{1}{2}}\right\}}) |\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p \\ &\leq C\|(\sum_{j_1 \leq j < j_2} \sum_K \sum_{(K)} |\beta_{j,k}|^2 \mathbb{P}_f^n((l_j^{-1} \sum_{(K)} |\hat{\beta}_{j,k} - \beta_{j,k}|^2)^{\frac{1}{2}} \geq \sigma_j \kappa n^{-\frac{1}{2}}) |\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p \\ &\leq Cn^{-h(\kappa)\frac{p}{2}}\|(\sum_{j < j_2} \sum_{k \in \Delta_j} |\beta_{j,k}|^2 |\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p \leq C\|f\|_p^p n^{-\frac{\alpha p}{2}} \leq Cn^{-\frac{\alpha p}{2}} \end{aligned} \tag{6.25}$$

for κ large enough.

It follows from (6.20), (6.21), (6.22), (6.24) and (6.25) that :

$$E_2 \leq Cn^{-\frac{\alpha p}{2}} \tag{6.26}$$

for $p > 1$, κ and n large enough.

The upper bound for the term E_3. Combining Minkowski's inequality with (6.5), one gets :

$$E_3 \leq 2^{p-1}(E_{3,1} + E_{3,2}) \tag{6.27}$$

where

$$E_{3,1} = \mathbb{E}_f^n(\| \sum_{j_1\le j<j_2} \sum_K \sum_{(K)} (\hat\beta_{j,k} - \beta_{j,k}) 1_{\left\{\hat b_{j,K}(p) \ge \sigma_j \kappa n^{-\frac12}\right\}} 1_{\left\{b_{j,K}(p) < \sigma_j \frac{\kappa n^{-\frac12}}{2}\right\}} \psi_{j,k}\|_p^p)$$

and

$$E_{3,2} = \mathbb{E}_f^n(\| \sum_{j_1\le j<j_2} \sum_K \sum_{(K)} (\hat\beta_{j,k} - \beta_{j,k}) 1_{\left\{\hat b_{j,K}(p) \ge \sigma_j \kappa n^{-\frac12}\right\}} 1_{\left\{b_{j,K}(p) \ge \sigma_j \frac{\kappa n^{-\frac12}}{2}\right\}} \psi_{j,k}\|_p^p).$$

To bound $E_{3,1}$, let us distinguish the case where $p \ge 2$ and the case where $1 < p \le 2$.

For $p \ge 2$: It follows from the inequality (6.1), the generalized Minkowski inequality, the inequality (6.10), the Cauchy-Schwartz inequality, the moments condition (3.4), the concentration property (3.5), the fact that ζ_σ satisfies the Temlyakov property (see Lemma 2.1, inequality (2.1)) and the weighted inequality (3.6) that :

$$\begin{aligned}
E_{3,1} &\le C\mathbb{E}_f^n(\|(\sum_{j_1\le j<j_2} \sum_K \sum_{(K)} |\hat\beta_{j,k} - \beta_{j,k}|^2 1_{\{\hat b_{j,K}(p) \ge \sigma_j \kappa n^{-\frac12}\}} 1_{\left\{b_{j,K}(p) < \sigma_j \frac{\kappa n^{-\frac12}}{2}\right\}} |\psi_{j,k}|^2)^{\frac12}\|_p^p) \\
&\le C\|(\sum_{j_1\le j<j_2} \sum_K \sum_{(K)} \mathbb{E}_f^n(|\hat\beta_{j,k} - \beta_{j,k}|^p 1_{\{\hat b_{j,K}(p) \ge \sigma_j \kappa n^{-\frac12}\}} 1_{\left\{b_{j,K}(p) < \sigma_j \frac{\kappa n^{-\frac12}}{2}\right\}})^{\frac2p} |\psi_{j,k}|^2)^{\frac12}\|_p^p \\
&\le C\|(\sum_{j_1\le j<j_2} \sum_K \sum_{(K)} \mathbb{E}_f^n(|\hat\beta_{j,k} - \beta_{j,k}|^{2p})^{\frac1p} \mathbb{P}_f^n((l_j^{-1} \sum_{(K)} |\hat\beta_{j,k} - \beta_{j,k}|^p)^{\frac1p} \ge \sigma_j \frac{\kappa n^{-\frac12}}{2})^{\frac1p} |\psi_{j,k}|^2)^{\frac12}\|_p^p \\
&\le Cn^{-\frac{p}{2}} n^{-h(\frac{\kappa}{2})} \|(\sum_{j<j_2} \sum_{k\in\Delta_j} \sigma_j^2 |\psi_{j,k}|^2)^{\frac12}\|_p^p \le Cn^{-\frac{p}{2}} n^{-h(\frac{\kappa}{2})} \sum_{j<j_2} \sum_{k\in\Delta_j} \sigma_j^p \|\psi_{j,k}\|_p^p \le Cn^{-\frac{\alpha p}{2}} \qquad (6.28)
\end{aligned}$$

for κ large enough.

For $1 < p \le 2$: Using the inequality (6.1), the comparison between the l_p norms, the Cauchy-Schwartz inequality, the moments condition (3.4), the concentration property (3.5) and the weighted inequality (3.6), we find :

$$\begin{aligned}
E_{3,1} &\le C\mathbb{E}_f^n(\|(\sum_{j_1\le j<j_2} \sum_K \sum_{(K)} |\hat\beta_{j,k} - \beta_{j,k}|^2 1_{\{\hat b_{j,K}(p) \ge \sigma_j \kappa n^{-\frac12}\}} 1_{\left\{b_{j,K}(p) < \sigma_j \frac{\kappa n^{-\frac12}}{2}\right\}} |\psi_{j,k}|^2)^{\frac12}\|_p^p) \\
&\le C \sum_{j_1\le j<j_2} \sum_K \sum_{(K)} \mathbb{E}_f^n(|\hat\beta_{j,k} - \beta_{j,k}|^p 1_{\{\hat b_{j,K}(p) \ge \sigma_j \kappa n^{-\frac12}\}} 1_{\left\{b_{j,K}(p) < \sigma_j \frac{\kappa n^{-\frac12}}{2}\right\}}) \|\psi_{j,k}\|_p^p \\
&\le C \sum_{j_1\le j<j_2} \sum_K \sum_{(K)} \mathbb{E}_f^n(|\hat\beta_{j,k} - \beta_{j,k}|^{2p})^{\frac12} \mathbb{P}_f^n((l_j^{-1} \sum_{(K)} |\hat\beta_{j,k} - \beta_{j,k}|^2)^{\frac12} \ge \sigma_j \frac{\kappa n^{-\frac12}}{2})^{\frac12} \|\psi_{j,k}\|_p^p \\
&\le Cn^{-\frac{p}{2}} n^{-\frac12 h(\frac{\kappa}{2})} \sum_{j<j_2} \sum_{k\in\Delta_j} \sigma_j^p \|\psi_{j,k}\|_p^p \le Cn^{-\frac{\alpha p}{2}} \qquad (6.29)
\end{aligned}$$

for κ large enough.

In order to dominate the term $E_{3,2}$, let us study separately the case where $p \ge 2$ and the case where $1 < p \le 2$.

For $p \ge 2$: By virtue of the inequality (6.1), the generalized Minkowski inequality, the moments condition (3.4), the fact that ζ_σ satisfies the Temlyakov property, and the Markov inequality, one gets :

$$\begin{aligned}
E_{3,2} &\le C\mathbb{E}_f^n(\|(\sum_{j_1\le j<j_2} \sum_K \sum_{(K)} |\hat\beta_{j,k} - \beta_{j,k}|^2 1_{\left\{b_{j,K}(p) \ge \sigma_j \frac{\kappa n^{-\frac12}}{2}\right\}} |\psi_{j,k}|^2)^{\frac12}\|_p^p) \\
&\le C\|(\sum_{j_1\le j<j_2} \sum_K \sum_{(K)} \mathbb{E}_f^n(|\hat\beta_{j,k} - \beta_{j,k}|^p)^{\frac2p} 1_{\left\{b_{j,K}(p) \ge \sigma_j \frac{\kappa n^{-\frac12}}{2}\right\}} |\psi_{j,k}|^2)^{\frac12}\|_p^p \\
&\le Cn^{-\frac{p}{2}} \|(\sum_{j_1\le j<j_2} \sum_K \sum_{(K)} 1_{\left\{b_{j,K}(p) \ge \sigma_j \frac{\kappa n^{-\frac12}}{2}\right\}} \sigma_j^2 |\psi_{j,k}|^2)^{\frac12}\|_p^p \\
&\le Cn^{-\frac{p}{2}} \sum_j \sum_K \sum_{(K)} 1_{\left\{b_{j,K}(p) \ge \sigma_j \frac{\kappa n^{-\frac12}}{2}\right\}} \sigma_j^p \|\psi_{j,k}\|_p^p \\
&= Cn^{-\frac{p}{2}} \sum_{m\in\mathbb{N}} \sum_j \sum_K 1_{\left\{\frac{\kappa n^{-\frac12}}{2} \sigma_j 2^m \le b_{j,K}(p) < \sigma_j \frac{\kappa n^{-\frac12}}{2} 2^{m+1}\right\}} \sigma_j^p l_j 2^{j(\frac{p}{2}-1)}
\end{aligned}$$

$$\begin{aligned} &\leq C\sum_{m\in\mathbb{N}}2^{-mp}\sum_j\sum_K(b_{j,K}(p))^p 1_{\left\{b_{j,K}(p)<\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}2^{m+1}\right\}}l_j 2^{j(\frac{p}{2}-1)}\\ &= C\sum_{m\in\mathbb{N}}2^{-mp}\sum_j\sum_K\sum_{(K)}|\beta_{j,k}|^p 1_{\left\{b_{j,K}(p)\leq\frac{\kappa n^{-\frac{1}{2}}}{2}\sigma_j 2^{m+1}\right\}}2^{j(\frac{p}{2}-1)}=J\end{aligned}$$

Applying the Markov inequality, the comparison between the l_p-norms and the inequality (2.1), J can be majorized by :

$$\begin{aligned} J &= C\int_0^1\sum_{m\in\mathbb{N}}2^{-mp}\sum_j\sum_K\sum_{(K)}|\beta_{j,k}|^p 1_{\left\{b_{j,K}(p)\leq\frac{\kappa n^{-\frac{1}{2}}}{2}\sigma_j 2^{m+1}\right\}}|\psi_{j,k}|^p\\ &\leq C\sum_{m\in\mathbb{N}}2^{-mp}\int_0^1(\sum_j\sum_K\sum_{(K)}|\beta_{j,k}|^2 1_{\left\{b_{j,K}(p)\leq\frac{\kappa n^{-\frac{1}{2}}}{2}\sigma_j 2^{m+1}\right\}}|\psi_{j,k}|^2)^{\frac{p}{2}}\\ &\leq C\sum_{m\in\mathbb{N}}2^{-mp}\|\sum_j\sum_K\sum_{(K)}\beta_{j,k}1_{\left\{b_{j,K}(p)\leq\frac{\kappa n^{-\frac{1}{2}}}{2}\sigma_j 2^{m+1}\right\}}\psi_{j,k}\|_p^p \end{aligned}\tag{6.30}$$

Thus, if $l_j\asymp 2^j$ and $f\in\mathcal{W}_\sigma((1-\alpha)p,p)$ with $\alpha\in]0,1[$ then we have :

$$E_{3,2}\leq Cn^{-\frac{\alpha p}{2}}\sum_{m\in\mathbb{N}}2^{(\alpha-1)mp}\leq Cn^{-\frac{\alpha p}{2}}$$

and if $l_j\asymp\ln(n)^{\frac{p}{2}}$ and $f\in\overline{\mathcal{W}}_{\sigma,\kappa,n_0}((1-\alpha)p,p)$ then we have immediately :

$$E_{3,2}\leq Cn^{-\frac{\alpha p}{2}}.$$

For $1<p\leq 2$: Using the inequality (6.1), the comparison between the l_p-norms, the moments condition (3.4) and the Markov inequality in the same way that for the case $p\geq 2$, we find :

$$\begin{aligned} E_{3,2} &\leq C\mathbb{E}_f^n(\|(\sum_{j_1\leq j<j_2}\sum_K\sum_{(K)}|\hat{\beta}_{j,k}-\beta_{j,k}|^2 1_{\left\{b_{j,K}(p)\geq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}}|\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p)\\ &\leq C\sum_{j_1\leq j<j_2}\sum_K\sum_{(K)}\mathbb{E}_f^n(|\hat{\beta}_{j,k}-\beta_{j,k}|^p)1_{\left\{b_{j,K}(p)\geq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}}\|\psi_{j,k}\|_p^p\\ &\leq Cn^{-\frac{p}{2}}\sum_j\sum_K\sum_{(K)}1_{\left\{b_{j,K}(p)\geq\sigma_j\frac{\kappa n^{-\frac{1}{2}}}{2}\right\}}\sigma_j^p\|\psi_{j,k}\|_p^p\\ &\leq C\sum_{m\in\mathbb{N}}2^{-mp}\sum_j\sum_K\sum_{(K)}|\beta_{j,k}|^p 1_{\left\{b_{j,K}(p)\leq\frac{\kappa n^{-\frac{1}{2}}}{2}\sigma_j 2^{m+1}\right\}}2^{j(\frac{p}{2}-1)}=K \end{aligned}$$

Starting from the expression of K, it is easy to see that if $l_j\asymp 2^j$ and $f\in\mathcal{W}_\sigma^*((1-\alpha)p,p)$ then :

$$E_{3,2}\leq Cn^{-\frac{\alpha p}{2}}\sum_{m\in\mathbb{N}}2^{(\alpha-1)mp}\leq Cn^{-\frac{\alpha p}{2}}$$

and if $l_j\asymp\ln(n)^{\frac{p}{2}}$ and $f\in\overline{\mathcal{W}}^*_{\sigma,\kappa,n_0}((1-\alpha)p,p)$ then :

$$E_{3,2}\leq Cn^{-\frac{\alpha p}{2}}.$$

Combining (6.27), (6.28), (6.29) and (6.30), we deduce that for $p>1$, κ and n are large enough we have :

$$E_3\leq Cn^{-\frac{\alpha p}{2}}.\tag{6.31}$$

Putting (6.17), (6.18), (6.19), (6.26) and (6.31) together, this achieved the proof of Theorem 3.2. □

Proof of Proposition 3.1. If $p\geq 2$, the following identity :

$$\mathcal{M}_{n_0}\left(\hat{f}^g,p,n^{-\frac{\alpha p}{2}}\right)=\mathcal{B}^{\frac{\alpha}{\nu}}_{p,\infty}\cap\mathcal{W}_\sigma((1-\alpha)p,p)$$

is provided by Theorem 3.1 and Theorem 3.2. Thus, we only need to show that that for any $\nu > 0$, $\alpha \in]1-\frac{\nu}{2}, 1[$, κ and n_0 large enough, we have the inclusion $\mathcal{W}_\sigma((1-\alpha)p,p) \subseteq \mathcal{B}^{\frac{\alpha}{\nu}}_{p,\infty}$.

If $f \in \mathcal{W}_\sigma((1-\alpha)p,p)$ and $\alpha > 1-\frac{\nu}{2}$ then there exists a constant $C>0$ such that the integer j_2 described in Definition 2.1 satisfies :

$$\begin{aligned} 2^{\frac{j_2 p}{2}} \sum_{j\geq j_2} 1_{\left\{b_j(p)\geq Cn^{-\frac{1}{2}}\sigma_j\right\}} &\leq \sum_{j\geq j_2} 1_{\left\{b_j(p)\geq Cn^{-\frac{1}{2}}\sigma_j\right\}} 2^{\frac{jp}{2}} \leq \sum_j \sum_{k\in\Delta_j} 1_{\left\{b_j(p)\geq Cn^{-\frac{1}{2}}\sigma_j\right\}} \sigma_j^p \|\psi_{j,k}\|_p^p \\ &\leq n^{(1-\alpha)\frac{p}{2}} < n^{\frac{\nu p}{4}} \end{aligned}$$

and $2^{j_2} \geq n^{\frac{\nu}{2}}$. We used an inequality similar to (6.30) and the fact that $\inf_j \sigma_j \geq 1$ (without loss of generality). We deduce that if $j \geq j_2$ then we have necessarily $b_j(p) < Cn^{-\frac{1}{2}}\sigma_j$. Therefore, for any f of $\mathcal{W}_\sigma((1-\alpha)p,p)$, the inequality (2.1) implies that :

$$\|\sum_{j\geq j_2}\sum_{k\in\Delta_j}\beta_{j,k}\psi_{j,k}\|_p^p \leq \|\sum_{j\geq j_2}\sum_{k\in\Delta_j}\beta_{j,k}1_{\left\{b_j(p)<Cn^{-\frac{1}{2}}\sigma_j\right\}}\psi_{j,k}\|_p^p \leq Cn^{-\frac{\alpha p}{2}}.$$

We conclude that $f \in \mathcal{B}^{\frac{\alpha}{\nu}}_{p,\infty}$. Hence the inclusion $\mathcal{W}_\sigma((1-\alpha)p,p) \subseteq \mathcal{B}^{\frac{\alpha}{\nu}}_{p,\infty}$ holds. □

Proof of Theorem 3.3. The proof of this theorem is identical to the proof of Theorem 3.1. It suffices to remplace j_1 by τ, the threshold $n^{-\frac{1}{2}}$ by $\sqrt{\frac{\ln(n)}{n}}$, $\sum_K \sum_{(K)}$ by $\sum_{k\in\Delta_j}$, $\hat{b}_{j,K}(p)$ by $|\hat{\beta}_{j,k}|$, $b_{j,K}(p)$ by $|\beta_{j,k}|$ and to remark that :

$$\eta(\sqrt{\frac{\ln(n)}{n}}) \asymp n^{-\frac{1}{2}}.$$

Let us just mention that if $f \in \widehat{\mathcal{W}}_\sigma((1-\alpha)p,p)$ then the term analog to $E_{3,2}$ can be dominated by :

$$\begin{aligned} E_{3,2} &= \mathbb{E}_f^n(\|\sum_{j_1\leq j<j_2}\sum_{k\in\Delta_j}(\hat{\beta}_{j,k}-\beta_{j,k})1_{\left\{|\hat{\beta}_{j,k}|\geq\sigma_j\kappa\sqrt{\frac{\ln(n)}{n}}\right\}}1_{\left\{|\beta_{j,k}|\geq\sigma_j\frac{\kappa}{2}\sqrt{\frac{\ln(n)}{n}}\right\}}\psi_{j,k}\|_p^p) \\ &\leq \|(\sum_{j_1\leq j<j_2}\sum_{k\in\Delta_j}\mathbb{E}_f^n(|\hat{\beta}_{j,k}-\beta_{j,k}|^{2\vee p})^{\frac{2}{p}\wedge 1}1_{\left\{|\beta_{j,k}|\geq\sigma_j\frac{\kappa}{2}\sqrt{\frac{\ln(n)}{n}}\right\}}|\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p) \\ &\leq Cn^{-\frac{p}{2}}\|(\sum_{j_1\leq j<j_2}\sum_K\sum_{(K)}\sigma_j^2 1_{\left\{|\beta_{j,k}|\geq\sigma_j\frac{\kappa}{2}\sqrt{\frac{\ln(n)}{n}}\right\}}|\psi_{j,k}|^2)^{\frac{1}{2}}\|_p^p \\ &\leq Cn^{-\frac{p}{2}}\|\sum_{j_1\leq j<j_2}\sum_K\sum_{(K)}\sigma_j 1_{\left\{|\beta_{j,k}|\geq\sigma_j\frac{\kappa}{2}\sqrt{\frac{\ln(n)}{n}}\right\}}\psi_{j,k}\|_p^p \\ &\leq Cn^{-\frac{p}{2}}(\sum_{m\in\mathbb{N}}\|\sum_{j_1\leq j<j_2}\sum_K\sum_{(K)}\sigma_j 1_{\left\{\sigma_j 2^{m+1}\frac{\kappa}{2}\sqrt{\frac{\ln(n)}{n}}>|\beta_{j,k}|\geq\sigma_j 2^m\frac{\kappa}{2}\sqrt{\frac{\ln(n)}{n}}\right\}}\psi_{j,k}\|_p)^p \\ &\leq C\ln(n)^{-\frac{p}{2}}(\sum_{m\in\mathbb{N}}2^{-m}\|\sum_j\sum_{k\in\Delta_j}\beta_{j,k}1_{\left\{|\beta_{j,k}|\leq\sigma_j\frac{\kappa}{2}\sqrt{\frac{\ln(n)}{n}}2^{m+1}\right\}}\psi_{j,k}\|_p)^p \\ &\leq C\ln(n)^{-\frac{p}{2}}(\sum_{m\in\mathbb{N}}2^{-m}\eta(\frac{\kappa}{2}\sqrt{\frac{\ln(n)}{n}}2^{m+1})^\alpha)^p \\ &\leq Cn^{-\frac{p}{2}}(\sum_{m\in\mathbb{N}}2^{(\alpha-1)m}(\frac{\ln(n)}{n})^{\frac{(\alpha-1)}{2}}(\ln((2^{m+1}\frac{\kappa}{2}\sqrt{\frac{\ln(n)}{n}}\wedge\upsilon)^{-1}))^{-\frac{\alpha}{2}})^p \\ &\leq Cn^{-\frac{p}{2}}(\frac{\ln(n)}{n})^{\frac{(\alpha-1)p}{2}}(\ln((\frac{\kappa}{2}\sqrt{\frac{\ln(n)}{n}}\wedge\upsilon)^{-1}))^{-\frac{\alpha p}{2}} \leq Cn^{-\frac{\alpha p}{2}} \end{aligned} \tag{6.32}$$

for n large enough. We used the inequality (6.1), the generalized Minkowski inequality in the case where $p \geq 2$ and the Jensen inequality in the case where $1 < p \leq 2$, the Minkowski inequality and the Kerkyacharian et al. (2005, Lemma 2) which is satisfied because υ is small enough (see Definition 3.4). This ends the proof of Theorem 3.3. □

Proof of Proposition 3.2. Let $p \geq 2$. Starting from Theorems 3.2 and 3.3, we only need to show the following embedding :

$$\left(\mathcal{M}_{n_0}(\hat{f}^h,p,n^{-\frac{\alpha p}{2}}) =\right) \mathcal{B}^{\frac{\alpha}{\nu}}_{p,\infty} \cap \widehat{\mathcal{W}}_\sigma((1-\alpha)p,p) \subseteq \mathcal{B}^{\frac{\alpha}{\nu}}_{p,\infty} \cap \overline{\mathcal{W}}_{\sigma,\kappa,n_0}((1-\alpha)p,p) \left(\subseteq \mathcal{M}_{n_0}(\hat{f}^o,p,n^{-\frac{\alpha p}{2}})\right)$$

for $\alpha \in]0,1[$, κ and n_0 large enough. Thanks to the choice of the length of the blocks ($l_j \asymp \ln(n)^{\frac{p}{2}}$), for any k of $\mathcal{B}_{j,K}$ there exists a constant $C>0$ such that :

$$\left\{b_{j,K}(p) \leq \kappa 2^m n^{-\frac{1}{2}}\sigma_j\right\} = \left\{\frac{1}{C\sqrt{\ln(n)}}\Big(\sum_{l\in\mathcal{B}_{j,K}}|\beta_{j,l}|^p\Big)^{\frac{1}{p}} \leq \kappa 2^m n^{-\frac{1}{2}}\sigma_j\right\} \subseteq \left\{|\beta_{j,k}| \leq C\kappa 2^m\sqrt{\frac{\ln(n)}{n}}\sigma_j\right\}.$$

Thus, if $f \in \widehat{\mathcal{W}}_\sigma((1-\alpha)p,p)$, the inequality (2.1) and an inequality similar to (6.32) give us :

$$\begin{aligned}
L &= n^{\frac{\alpha p}{2}}\sum_{m\in\mathbb{N}}2^{-mp}\|\sum_j\sum_K\sum_{(K)}\beta_{j,k}1_{\left\{b_{j,K}(p)\leq\kappa 2^m n^{-\frac{1}{2}}\sigma_j\right\}}\psi_{j,k}\|_p^p \\
&\leq n^{\frac{\alpha p}{2}}\sum_{m\in\mathbb{N}}2^{-mp}\|\sum_j\sum_{k\in\Delta_j}\beta_{j,k}1_{\left\{|\beta_{j,k}|\leq C\kappa 2^m\sigma_j\sqrt{\frac{\ln(n)}{n}}\right\}}\psi_{j,k}\|_p^p \\
&\leq C\, n^{\frac{\alpha p}{2}}\sum_{m\in\mathbb{N}}2^{-mp}\eta(\sqrt{\frac{\ln(n)}{n}}C\kappa 2^m)^{\alpha p} \leq Cn^{\frac{\alpha p}{2}}(\frac{\ln(n)}{n})^{\frac{\alpha p}{2}}(\ln((C\kappa\sqrt{\frac{\ln(n)}{n}}\wedge\upsilon)^{-1}))^{-\frac{\alpha p}{2}} \leq C<\infty
\end{aligned}$$

for n large enough. We conclude that $f \in \overline{\mathcal{W}}_{\sigma,\kappa,n_0}((1-\alpha)p,p)$. The proof of Proposition 3.2 is thus complete. □

Proof of Proposition 4.1. Let $R>0$, $s>0$, $\pi \geq p > 1$ and $1 \leq r \leq \infty$. Since the sequence $\sigma = (2^{\delta j})$ satisfies the condition (2.2) for $p>2$, if we apply Theorem 3.2 with $\alpha = \frac{2s}{2s+2\delta+1}$ then it suffices to show that :

$$\begin{cases}
B^s_{\pi,r}(R) \subseteq \mathcal{B}^{\frac{s(1+2\delta)}{2s+2\delta+1}}_{p,\infty} \cap \mathcal{W}_\sigma((1-\frac{2s}{2s+2\delta+1})p,p) \cap \overline{\mathcal{W}}_{\sigma,\kappa,n_0}((1-\frac{2s}{2s+2\delta+1})p,p) & \text{if} \quad p \geq 2, \\
B^s_{\pi,r}(R) \subseteq \mathcal{B}^{\frac{s(1+2\delta)}{2s+2\delta+1}}_{p,\infty} \cap \mathcal{W}^*_\sigma((1-\frac{2s}{2s+2\delta+1})p,p) \cap \overline{\mathcal{W}}^*_{\sigma,\kappa,n_0}((1-\frac{2s}{2s+2\delta+1})p,p) & \text{if} \quad 1<p\leq 2,
\end{cases}$$

for n_0 and κ large enough. Let us focus on the case where $p \geq 2$:

Proof of the inclusion $B^s_{\pi,r}(R) \subseteq \mathcal{B}^{\frac{s(1+2\delta)}{2s+2\delta+1}}_{p,\infty}$. If $f \in B^s_{\pi,r}(R)(\subseteq B^s_{p,\infty}(R))$ then Lemma 2.2 gives us :

$$\begin{aligned}
2^{u\frac{s(1+2\delta)}{2s+2\delta+1}p}\|\sum_{j\geq u}\sum_{k\in\Delta_j}\beta_{j,k}\psi_{j,k}\|_p^p &\leq C(\sum_{j\geq u}2^{j(\frac{1}{2}-\frac{1}{p})}(\sum_{k\in\Delta_j}|\beta_{j,k}|^p)^{\frac{1}{p}})^p 2^{u\frac{s(1+2\delta)}{2s+2\delta+1}p} \\
&\leq C(\sum_{j\geq u}2^{-js+\frac{us(1+2\delta)}{2s+2\delta+1}})^p \leq C(\sum_{j\geq u}2^{-s(j-u)})^p \leq C<\infty.
\end{aligned}$$

Hence $f \in \mathcal{B}^{\frac{s(1+2\delta)}{2s+2\delta+1}}_{p,\infty}$ and the desired inclusion holds.

Proof of the inclusion $B^s_{\pi,r}(R) \subseteq \mathcal{W}_\sigma((1-\frac{2s}{2s+2\delta+1})p,p)$. Let us consider an integer j_3 such that $2^{j_3} \asymp u^{-\frac{2}{2s+2\delta+1}}$. Minkowski's inequality combined with the inequality (6.2) gives us :

$$\begin{aligned}
\|\sum_j\sum_{k\in\Delta_j}\beta_{j,k}1_{\{b_j(p)\leq\kappa 2^{\delta j}u\}}\psi_{j,k}\|_p^p &\leq (\sum_j\|\sum_{k\in\Delta_j}\beta_{j,k}1_{\{b_j(p)\leq\kappa 2^{\delta j}u\}}\psi_{j,k}\|_p)^p \\
&\leq C(\sum_j 2^{j(\frac{1}{2}-\frac{1}{p})}(\sum_{k\in\Delta_j}|\beta_{j,k}|^p 1_{\{b_j(p)\leq\kappa 2^{\delta j}u\}})^{\frac{1}{p}})^p \\
&\leq C(F_1+F_2)
\end{aligned}$$

where :

$$F_1 = (\sum_{j<j_3}2^{j(\frac{1}{2}-\frac{1}{p})}(\sum_{k\in\Delta_j}|\beta_{j,k}|^p 1_{\{b_j(p)\leq\kappa 2^{\delta j}u\}})^{\frac{1}{p}})^p \quad \text{and} \quad F_2 = (\sum_{j\geq j_3}2^{j(\frac{1}{2}-\frac{1}{p})}(\sum_{k\in\Delta_j}|\beta_{j,k}|^p)^{\frac{1}{p}})^p.$$

On the set $\left\{b_j(p) \leq \kappa 2^{\delta j}u\right\}$, we have $(\sum_{k\in\Delta_j}|\beta_{j,k}|^p)^{\frac{1}{p}} \leq \kappa u 2^{\delta j}2^{\frac{j}{p}}$ so F_1 can be bounded by :

$$F_1 \leq Cu^p(\sum_{j<j_3}2^{j(\frac{1}{2}+\delta)})^p \leq u^p 2^{j_3(\frac{1}{2}+\delta)p} \leq Cu^{\frac{2sp}{2s+2\delta+1}}.$$

If we assume that $f \in B^s_{\pi,r}(R)$ ($\subseteq B^s_{p,\infty}(R)$ since $\pi \geq p$ and $1 \leq r \leq \infty$) then the Minkowski inequality gives us :

$$F_2 \leq C(\sum_{j\geq j_3} 2^{j(\frac{1}{2}-\frac{1}{p})}(\sum_{k\in\Delta_j}|\beta_{j,k}|^p)^{\frac{1}{p}})^p \leq C(\sum_{j\geq j_3}2^{-js})^p \leq C2^{-j_3 sp} \leq Cu^{\frac{2sp}{2s+2\delta+1}}.$$

We deduce that :

$$\sup_{n>0}\ u^{-\frac{2sp}{2s+2\delta+1}}\|\sum_j\sum_{k\in\Delta_j}\beta_{j,k}1_{\{b_j(p)\leq\kappa 2^{\delta j}u\}}\psi_{j,k}\|_p^p < \infty$$

and, a fortiori, $f \in \mathcal{W}_\sigma((1-\frac{2s}{2s+2\delta+1})p,p)$.

Proof of the inclusion $B^s_{\pi,r}(R) \subseteq \overline{\mathcal{W}}_{\sigma,\kappa,n_0}((1-\frac{2s}{2s+2\delta+1})p,p)$. For any $m \in \mathbb{N}$, let us consider an integer j_4 such that $2^{j_4} \asymp 2^{-\frac{m}{2s}}n^{\frac{1}{2s+2\delta+1}}$. Minkowski's inequality combined with the inequality (6.2) gives us :

$$\sum_{m\in\mathbb{N}}2^{-mp}\|\sum_j\sum_K\sum_{(K)}\beta_{j,k}1_{\left\{b_{j,K}(p)\leq\kappa 2^{\delta j}2^m n^{-\frac{1}{2}}\right\}}\psi_{j,k}\|_p^p \leq C\sum_{m\in\mathbb{N}}2^{-mp}(Q_1^m+Q_2^m)$$

where :

$$Q_1^m = (\sum_{j<j_4}2^{j(\frac{1}{2}-\frac{1}{p})}(\sum_K\sum_{(K)}|\beta_{j,k}|^p 1_{\left\{b_{j,K}(p)\leq\kappa 2^{\delta j}2^m n^{-\frac{1}{2}}\right\}})^{\frac{1}{p}})^p \quad \text{and} \quad Q_2^m = (\sum_{j\geq j_4}2^{j(\frac{1}{2}-\frac{1}{p})}(\sum_{k\in\Delta_j}|\beta_{j,k}|^p)^{\frac{1}{p}})^p.$$

Using the fact that $|\mathcal{A}_j| \asymp 2^j\ln(n)^{-\frac{p}{2}}$ and that, on the set $\left\{b_{j,K}(p) \leq \kappa 2^{\delta j}2^m\ln(n)^{\frac{1}{2}}n^{-\frac{1}{2}}\right\}$, we have the inequality $(\sum_{k\in\Delta_j}|\beta_{j,k}|^p)^{\frac{1}{p}} \leq \kappa n^{-\frac{1}{2}}2^m 2^{\delta j}\ln(n)^{\frac{1}{2}}$, we can majorize R_1 by :

$$Q_1^m \leq C2^{mp}n^{-\frac{p}{2}}(\sum_{j<j_4}2^{j(\frac{1}{2}+\delta)})^p \leq 2^{mp}n^{-\frac{p}{2}}2^{j_4(\frac{1}{2}+\delta)p} \leq C2^{mp}2^{-mp(\frac{1+2\delta}{4s})}n^{-\frac{sp}{2s+2\delta+1}}.$$

If we assume that $f \in B^s_{\pi,r}(R)$ ($\subseteq B^s_{p,\infty}(R)$ since $\pi \geq p$ and $1 \leq r \leq \infty$) then we have :

$$Q_2^m \leq C(\sum_{j\geq j_4}2^{j(\frac{1}{2}-\frac{1}{p})}(\sum_{k\in\Delta_j}|\beta_{j,k}|^p)^{\frac{1}{p}})^p \leq C(\sum_{j\geq j_4}2^{-js})^p \leq C2^{-j_4 sp} \leq C2^{\frac{mp}{2}}n^{-\frac{sp}{2s+2\delta+1}}.$$

We deduce that there exists a ω in $]0,1[$ such that :

$$\begin{aligned}\sup_{n>0}\ n^{\frac{sp}{2s+2\delta+1}}\sum_{m\in\mathbb{N}}2^{-mp}\|\sum_j\sum_K\sum_{(K)}\beta_{j,k}1_{\left\{b_{j,K}\leq\kappa 2^m 2^{\delta j}n^{-\frac{1}{2}}\right\}}\psi_{j,k}\|_p^p &\leq C\sum_{m\in\mathbb{N}}2^{-mp}(Q_1^m+Q_2^m)\\ &\leq C\sum_{m\in\mathbb{N}}2^{-m\omega p} < \infty.\end{aligned}$$

This implies that $f \in \overline{\mathcal{W}}_{\sigma,\kappa,n_0}((1-\frac{2s}{2s+2\delta+1})p,p)$.

By using the same arguments as before, it is easy to show that $B^s_{\pi,r}(R) \subseteq \mathcal{B}^{\frac{s(1+2\delta)}{2s+2\delta+1}}_{p,\infty} \cap \mathcal{W}^*_\sigma((1-\frac{2s}{2s+2\delta+1})p,p) \cap \overline{\mathcal{W}}^*_{\sigma,\kappa,n_0}((1-\frac{2s}{2s+2\delta+1})p,p)$ for $1 < p \leq 2$. By taking n_0 and κ large enough, Theorem 3.2 allows us to conclude. □

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