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ABSTRACT. Let $\mathbf{G}$ be a connected reductive algebraic group defined over an algebraic closure of a finite field and let $F : \mathbf{G} \to \mathbf{G}$ be an endomorphism such that F^δ is a Frobenius endomorphism for some $\delta \geqslant 1$. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ admitting an F-stable Levi subgroup. We prove that the Deligne-Lusztig variety $\{g\mathbf{P} \mid g^{-1}F(g) \in \mathbf{P} \cdot F(\mathbf{P})\}$ is irreducible if and only if $\mathbf{P}$ is not contained in a proper F-stable parabolic subgroup of $\mathbf{G}$.

Let p be a prime number, let $\mathbb{F}$ be an algebraic closure of a finite field, let $\mathbf{G}$ be a connected reductive group over $\mathbb{F}$ and let $F : \mathbf{G} \to \mathbf{G}$ be an isogeny such that some power of F is a Frobenius endomorphism of $\mathbf{G}$. If $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$ admitting an F-stable Levi subgroup, we set

$$\mathbf{X}_\mathbf{P} = \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g^{-1}F(g) \in \mathbf{P} \cdot F(\mathbf{P})\}.$$

This is the Deligne-Lusztig variety associated to $\mathbf{P}$. The aim of this note is to prove the following result:

Theorem A. *Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ admitting an F-stable Levi subgroup. Then $\mathbf{X}_\mathbf{P}$ is irreducible if and only if $\mathbf{P}$ is not contained in a proper F-stable parabolic subgroup of $\mathbf{G}$.*

Note that this result has been obtained independently by Lusztig (unpublished) and Digne and Michel [DiMi2, Proposition 8.4] in the case where $\mathbf{P}$ is a Borel subgroup: both proofs are obtained by counting rational points of $\mathbf{X}_\mathbf{P}$ in terms of the Hecke algebra. We present here a geometric proof (inspired by an argument of Deligne [Lu, proof of Proposition 4.8]) which reduces the problem to the irreducibility of the Deligne-Lusztig variety associated to a Coxeter element: this case has been treated by Deligne and Lusztig [Lu, Proposition 4.8].

Before starting the proof of this theorem, we first describe an equivalent statement, based on an alternative description of Deligne-Lusztig varieties (see for instance [BoRo, §10.1]).

Let $\mathbf{B}$ be an F-stable Borel subgroup of $\mathbf{G}$, let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{B}$, let W be the Weyl group of $\mathbf{G}$ relative to $\mathbf{T}$ and let S be the set of simple reflections of W with respect to $\mathbf{B}$. We still denote by F the automorphism of W induced by F. If $I \subset S$, let W_I denote the standard parabolic subgroup of W generated by I, let W^I be the set of elements w of W which have minimal length in wW_I, let $\mathbf{P}_I = \mathbf{B}W_I\mathbf{B}$, let $\mathbf{V}_I$ be the unipotent radical of $\mathbf{P}_I$ and let $\mathbf{L}_I$ be the

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unique Levi complement of $\mathbf{P}_I$ containing $\mathbf{T}$. Given $w \in W$ such that $wF(I)w^{-1} = I$ and $w \in W^{F(I)} \cap (W^I)^{-1}$, we set

$$\mathbf{X}_I(w) = \{g\mathbf{P}_I \in \mathbf{G}/\mathbf{P}_I \mid g^{-1}F(g) \in \mathbf{P}_I w \mathbf{P}_{F(I)}\}.$$

By [BoRo, §11.2], Theorem A is equivalent to:

Theorem A'. *Let $I \subset S$ and let $w \in W^{F(I)} \cap (W^I)^{-1}$ with $wF(I)w^{-1} = I$. Then $\mathbf{X}_I(w)$ is irreducible if and only if $W_I w$ is not contained in a proper F-stable standard parabolic subgroup of W.*

The rest of this paper is devoted to the proof of Theorem A'. We fix a subset I of S and an element w of $W^{F(I)} \cap (W^I)^{-1}$ such that $wF(I)w^{-1} = I$. If necessary, the variety $\mathbf{X}_I(w)$ will be denoted by $\mathbf{X}_I^{\mathbf{G}}(w)$ or $\mathbf{X}_I^{\mathbf{G},F}(w)$. We fix a representative $\dot{w}$ of w in $N_{\mathbf{G}}(\mathbf{T})$. Note that $\mathbf{L}_I$ is $\dot{w}F$-stable.

First step: the "only if" part. Assume that there exists a proper F-stable subset J of S such that $W_I w \subset W_J$. Then we have

$$\mathbf{X}_I(w) \simeq (\mathbf{G}^F/\mathbf{V}_J^F) \times_{\mathbf{L}_J^F} \mathbf{X}_I^{\mathbf{L}_J}(w)$$

(see for instance [DiMi1, proof of 11.5]). Therefore, $\mathbf{X}_I(w)$ is not irreducible. This shows the "only if" part of Theorem A'.

Second step: reduction to tori. By the previous subsection, we can concentrate on the "if" part. So, we assume that $W_I w$ is not contained in a proper F-stable parabolic subgroup of W. Write $r = |I|$ and $I = \{s_1, \dots, s_r\}$. The hypothesis on w implies that $w^{-1} \in W^I$. Therefore, $\ell(s_1 \cdots s_r w) = r + \ell(w)$, so $s_1 \cdots s_r w$ does not belong to a proper F-stable standard parabolic subgroup of W. Moreover, we have

$$\mathbf{X}_{\varnothing}^{\mathbf{G},F}(s_1 \cdots s_r w) \simeq \mathbf{Y}_I^{\mathbf{G},F}(\dot{w}) \times_{\mathbf{L}_I^{\dot{w}F}} \mathbf{X}_{\varnothing}^{\mathbf{L}_I,\dot{w}F}(s_1 \cdots s_r)$$

(see for instance [DiMi1, proof of 11.5]), where

$$\mathbf{Y}_I^{\mathbf{G},F}(\dot{w}) = \{g \in \mathbf{G}/\mathbf{V}_I \mid g^{-1}F(g) \in \mathbf{V}_I \dot{w} \mathbf{V}_{F(I)}\}.$$

Recall that $\mathbf{Y}_I^{\mathbf{G},F}(\dot{w})/\mathbf{L}_I^{\dot{w}F} \simeq \mathbf{X}_I^{\mathbf{G},F}(w)$. Therefore, the projection on the first factor induces a surjective morphism of varieties $\mathbf{X}_{\varnothing}(s_1 \cdots s_r w) \to \mathbf{X}_I(w)$. So, if Theorem A' is true when $I = \varnothing$, then it is true for every subset I of S.

We assume now that w is not contained in a proper F-stable standard parabolic subgroup of W and that $I = \varnothing$.

For simplification, we set $\mathbf{X}(w) = \mathbf{X}_{\varnothing}(w)$. In order to prove Theorem A' (or Theorem A), we only need to show that $\mathbf{X}(w)$ is irreducible (see the previous discussions).

Third step: smooth compactification. Let $(s_1, \dots, s_n)$ be a finite sequence of elements of S. Let

$$\hat{\mathbf{X}}(s_1, \dots, s_n) = \{(g_1\mathbf{B}, \dots, g_n\mathbf{B}) \in (\mathbf{G}/\mathbf{B})^n \mid g_n^{-1}F(g_1) \in \mathbf{P}_{\{s_n\}} \text{ and } \forall\, i \in \{1, 2, \dots, n-1\},\ g_{i+1}^{-1}g_i \in \mathbf{P}_{\{s_i\}}\}.$$

If $\ell(s_1 \cdots s_n) = n$, then $\hat{\mathbf{X}}(s_1, \ldots, s_n)$ is a smooth compactification of $\mathbf{X}(s_1 \cdots s_n)$ (see [DeLu, Lemma 9.11]): in this case,

(1) *$\mathbf{X}(s_1 \cdots s_n)$ is irreducible if and only if $\hat{\mathbf{X}}(s_1, \ldots, s_n)$ is irreducible.*

Note that $(\mathbf{B}, \ldots, \mathbf{B}) \in \hat{\mathbf{X}}(s_1, \ldots, s_n)$: we denote by $\hat{\mathbf{X}}^\circ(s_1, \ldots, s_n)$ the connected (i.e. irreducible) component of $\hat{\mathbf{X}}(s_1, \ldots, s_n)$ containing $(\mathbf{B}, \ldots, \mathbf{B})$. Let $H(s_1, \ldots, s_n) \subset \mathbf{G}^F$ be the stabilizer of $\hat{\mathbf{X}}^\circ(s_1, \ldots, s_n)$. Let us now prove the following fact:

(2) *if $1 \leqslant i_1 < \cdots < i_r \leqslant n$, then $H(s_{i_1}, \ldots, s_{i_r}) \subset H(s_1, \ldots, s_n)$.*

Proof of (2) - The map $f : \hat{\mathbf{X}}(s_{i_1}, \ldots, s_{i_r}) \longrightarrow \hat{\mathbf{X}}(s_1, \ldots, s_n)$ defined by

$$f(g_1\mathbf{B}, \ldots, g_r\mathbf{B}) = (g_1\mathbf{B}, \ldots, \underbrace{g_1\mathbf{B}}_{\substack{i_1\text{-th}\\ \text{position}}}, g_2\mathbf{B}, \ldots, \underbrace{g_{r-1}\mathbf{B}}_{\substack{i_{r-1}\text{-th}\\ \text{position}}}, g_r\mathbf{B}, \ldots, \underbrace{g_r\mathbf{B}}_{\substack{i_r\text{-th}\\ \text{position}}}, F(g_1)\mathbf{B}, \ldots, F(g_1)\mathbf{B})$$

is a well-defined $\mathbf{G}^F$-equivariant morphism of varieties. Moreover,

$$f(\underbrace{\mathbf{B}, \ldots, \mathbf{B}}_{r \text{ times}}) = (\underbrace{\mathbf{B}, \ldots, \mathbf{B}}_{n \text{ times}}).$$

In particular, $f(\hat{\mathbf{X}}^\circ(s_{i_1}, \ldots, s_{i_r}))$ is contained in $\hat{\mathbf{X}}^\circ(s_1, \ldots, s_n)$. This proves the expected inclusion between stabilizers. ■

Last step: twisted Coxeter element. The quotient variety

$$\mathbf{G}^F \backslash \{g \in \mathbf{G} | g^{-1}F(g) \in \mathbf{B}w\mathbf{B}\}$$

is irreducible (it is isomorphic to $\mathbf{B}w\mathbf{B}$ through the Lang map $\mathbf{G}^F g \mapsto g^{-1}F(g)$), hence $\mathbf{G}^F \backslash \mathbf{X}(w)$ is irreducible as well. So,

(3) *$\mathbf{G}^F$ permutes transitively the components of $\mathbf{X}(w)$.*

Let $w = s_1 \cdots s_n$ be a reduced decomposition of W as a product of elements of S. By (1) and (3), we need to show that $H(s_1, \ldots, s_n) = \mathbf{G}^F$. Since w does not belong to any F-stable proper parabolic subgroup of W, there exists a sequence $1 \leqslant i_1 < \cdots < i_r \leqslant n$ such that $(s_{i_k})_{1 \leqslant k \leqslant r}$ is a family of representatives of F-orbits in S. By (2), we have $H(s_{i_1}, \ldots, s_{i_r}) \subset H(s_1, \ldots, s_n)$. But, by [Lu, Proposition 4.8], $\mathbf{X}(s_{i_1} \cdots, s_{i_r})$ is irreducible so, again by (1) and (3), $H(s_{i_1}, \ldots, s_{i_r}) = \mathbf{G}^F$. Therefore, $H(s_1, \ldots, s_n) = \mathbf{G}^F$, as expected.

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