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ABSTRACT. Let $\mathbf{G}$ be a connected reductive algebraic group defined over an algebraic closure of a finite field and let $F : \mathbf{G} \to \mathbf{G}$ be an endomorphism such that F^δ is a Frobenius endomorphism for some $\delta \geqslant 1$. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ admitting an F-stable Levi subgroup. We prove that the Deligne-Lusztig variety $\{g\mathbf{P} \mid g^{-1}F(g) \in \mathbf{P} \cdot F(\mathbf{P})\}$ is irreducible if and only if $\mathbf{P}$ is not contained in a proper F-stable parabolic subgroup of $\mathbf{G}$.

Let p be a prime number, let $\mathbb{F}$ be an algebraic closure of a finite field, let $\mathbf{G}$ be a connected reductive group over $\mathbb{F}$ and let $F : \mathbf{G} \to \mathbf{G}$ be an isogeny such that some power of F is a Frobenius endomorphism of $\mathbf{G}$. If $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$ admitting an F-stable Levi subgroup, we set

$$\mathbf{X}_\mathbf{P} = \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g^{-1}F(g) \in \mathbf{P} \cdot F(\mathbf{P})\}.$$

This is the Deligne-Lusztig variety associated to $\mathbf{P}$. The aim of this note is to prove the following result:

Theorem A. *Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ admitting an F-stable Levi subgroup. Then $\mathbf{X}_\mathbf{P}$ is irreducible if and only if $\mathbf{P}$ is not contained in a proper F-stable parabolic subgroup of $\mathbf{G}$.*

Note that this result has been obtained independently by Lusztig (unpublished). We present a proof (inspired by an argument of Deligne [Lu, proof of Proposition 4.8]) which reduces the problem to the irreducibility of the Deligne-Lusztig variety associated to a Coxeter element: this case has been treated by Deligne and Lusztig [Lu, Proposition 4.8].

In §2, we give a different proof, obtained by counting rational points of Deligne-Lusztig varieties in terms of the Hecke algebra (Lusztig's proof is along these lines).

Before starting the proof of this theorem, we first describe an equivalent statement, based on an alternative description of Deligne-Lusztig varieties (see for instance [BoRo, §10.1]).

Let $\mathbf{B}$ be an F-stable Borel subgroup of $\mathbf{G}$, let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{B}$, let W be the Weyl group of $\mathbf{G}$ relative to $\mathbf{T}$ and let S be the set of simple reflections of W with respect to $\mathbf{B}$. We still denote by F the automorphism of W induced by F. If $I \subset S$, let W_I denote the standard parabolic subgroup of W generated by I, let W^I be the set of elements w of W which have minimal length in wW_I, let $\mathbf{P}_I = \mathbf{B}W_I\mathbf{B}$, let $\mathbf{V}_I$ be the unipotent radical of $\mathbf{P}_I$ and let $\mathbf{L}_I$ be the unique Levi complement of $\mathbf{P}_I$ containing $\mathbf{T}$. Given $w \in W$ such that $wF(I)w^{-1} = I$

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and $w \in W^{F(I)} \cap (W^I)^{-1}$, we set

$$\mathbf{X}_I(w) = \{g\mathbf{P}_I \in \mathbf{G}/\mathbf{P}_I \mid g^{-1}F(g) \in \mathbf{P}_I w \mathbf{P}_{F(I)}\}.$$

By [BoRo, §11.2], Theorem A is equivalent to:

Theorem A'. *Let $I \subset S$ and let $w \in W^{F(I)} \cap (W^I)^{-1}$ with $wF(I)w^{-1} = I$. Then $\mathbf{X}_I(w)$ is irreducible if and only if $W_I w$ is not contained in a proper F-stable standard parabolic subgroup of W.*

The rest of this paper is devoted to the proof of Theorem A'. We fix a subset I of S and an element w of $W^{F(I)} \cap (W^I)^{-1}$ such that $wF(I)w^{-1} = I$. If necessary, the variety $\mathbf{X}_I(w)$ will be denoted by $\mathbf{X}_I^{\mathbf{G}}(w)$ or $\mathbf{X}_I^{\mathbf{G},F}(w)$. We fix a representative $\dot{w}$ of w in $N_{\mathbf{G}}(\mathbf{T})$. Note that $\mathbf{L}_I$ is $\dot{w}F$-stable.

1. Geometric proof

First step: the "only if" part. Assume that there exists a proper F-stable subset J of S such that $W_I w \subset W_J$. Then we have

$$\mathbf{X}_I(w) \simeq (\mathbf{G}^F/\mathbf{V}_J^F) \times_{\mathbf{L}_J^F} \mathbf{X}_I^{\mathbf{L}_J}(w)$$

(see for instance [DiMi, proof of 11.5]). Therefore, $\mathbf{X}_I(w)$ is not irreducible. This shows the "only if" part of Theorem A'.

Second step: reduction to tori. By the previous subsection, we can concentrate on the "if" part. So, we assume that $W_I w$ is not contained in a proper F-stable parabolic subgroup of W. Write $r = |I|$ and $I = \{s_1, \dots, s_r\}$. The hypothesis on w implies that $w^{-1} \in W^I$. Therefore, $\ell(s_1 \cdots s_r w) = r + \ell(w)$, so $s_1 \cdots s_r w$ does not belong to a proper F-stable standard parabolic subgroup of W. Moreover, we have

$$\mathbf{X}_{\varnothing}^{\mathbf{G},F}(s_1 \cdots s_r w) \simeq \mathbf{Y}_I^{\mathbf{G},F}(\dot{w}) \times_{\mathbf{L}_I^{\dot{w}F}} \mathbf{X}_{\varnothing}^{\mathbf{L}_I,\dot{w}F}(s_1 \cdots s_r)$$

(see for instance [DiMi, proof of 11.5]), where

$$\mathbf{Y}_I^{\mathbf{G},F}(\dot{w}) = \{g \in \mathbf{G}/\mathbf{V}_I \mid g^{-1}F(g) \in \mathbf{V}_I \dot{w} \mathbf{V}_{F(I)}\}.$$

Recall that $\mathbf{Y}_I^{\mathbf{G},F}(\dot{w})/\mathbf{L}_I^{\dot{w}F} \simeq \mathbf{X}_I^{\mathbf{G},F}(w)$. Therefore, the projection on the first factor induces a surjective morphism of varieties $\mathbf{X}_{\varnothing}(s_1 \cdots s_r w) \to \mathbf{X}_I(w)$. So, if Theorem A' is true when $I = \varnothing$, then it is true for every subset I of S.

We assume now that w is not contained in a proper F-stable standard parabolic subgroup of W and that $I = \varnothing$.

For simplification, we set $\mathbf{X}(w) = \mathbf{X}_{\varnothing}(w)$. In order to prove Theorem A' (or Theorem A), we only need to show that $\mathbf{X}(w)$ is irreducible (see the previous discussions).

Third step: smooth compactification. Let $(s_1, \dots, s_n)$ be a finite sequence of elements of S. Let

$$\hat{\mathbf{X}}(s_1, \dots, s_n) = \{(g_1\mathbf{B}, \dots, g_n\mathbf{B}) \in (\mathbf{G}/\mathbf{B})^n \mid g_n^{-1}F(g_1) \in \mathbf{P}_{\{s_n\}} \text{ and } \forall\, i \in \{1, 2, \dots, n-1\},\ g_{i+1}^{-1}g_i \in \mathbf{P}_{\{s_i\}}\}.$$

If $\ell(s_1\cdots s_n)=n$, then $\hat{\mathbf{X}}(s_1,\dots,s_n)$ is a smooth compactification of $\mathbf{X}(s_1\cdots s_n)$ (see [DeLu, Lemma 9.11]): in this case,

(1) *$\mathbf{X}(s_1\cdots s_n)$ is irreducible if and only if $\hat{\mathbf{X}}(s_1,\dots,s_n)$ is irreducible.*

Note that $(\mathbf{B},\dots,\mathbf{B})\in\hat{\mathbf{X}}(s_1,\dots,s_n)$: we denote by $\hat{\mathbf{X}}^\circ(s_1,\dots,s_n)$ the connected (i.e. irreducible) component of $\hat{\mathbf{X}}(s_1,\dots,s_n)$ containing $(\mathbf{B},\dots,\mathbf{B})$. Let $H(s_1,\dots,s_n)\subset\mathbf{G}^F$ be the stabilizer of $\hat{\mathbf{X}}^\circ(s_1,\dots,s_n)$. Let us now prove the following fact:

(2) *if $1\leqslant i_1<\cdots<i_r\leqslant n$, then $H(s_{i_1},\dots,s_{i_r})\subset H(s_1,\dots,s_n)$.*

Proof of (2) - The map $f:\hat{\mathbf{X}}(s_{i_1},\dots,s_{i_r})\longrightarrow\hat{\mathbf{X}}(s_1,\dots,s_n)$ defined by

$$f(g_1\mathbf{B},\dots,g_r\mathbf{B})=$$
$$(g_1\mathbf{B},\dots,\underbrace{g_1\mathbf{B}}_{\substack{i_1\text{-th}\\ \text{position}}},g_2\mathbf{B},\dots,\underbrace{g_{r-1}\mathbf{B}}_{\substack{i_{r-1}\text{-th}\\ \text{position}}},g_r\mathbf{B},\dots,\underbrace{g_r\mathbf{B}}_{\substack{i_r\text{-th}\\ \text{position}}},F(g_1)\mathbf{B},\dots,F(g_1)\mathbf{B})$$

is a well-defined $\mathbf{G}^F$-equivariant morphism of varieties. Moreover,

$$f(\underbrace{\mathbf{B},\dots,\mathbf{B}}_{r\text{ times}})=(\underbrace{\mathbf{B},\dots,\mathbf{B}}_{n\text{ times}}).$$

In particular, $f(\hat{\mathbf{X}}^\circ(s_{i_1},\dots,s_{i_r}))$ is contained in $\hat{\mathbf{X}}^\circ(s_1,\dots,s_n)$. This proves the expected inclusion between stabilizers. ■

Last step: twisted Coxeter element. The quotient variety

$$\mathbf{G}^F\backslash\{g\in\mathbf{G}|g^{-1}F(g)\in\mathbf{B}w\mathbf{B}\}$$

is irreducible (it is isomorphic to $\mathbf{B}w\mathbf{B}$ through the Lang map $\mathbf{G}^Fg\mapsto g^{-1}F(g)$), hence $\mathbf{G}^F\backslash\mathbf{X}(w)$ is irreducible as well. So,

(3) *$\mathbf{G}^F$ permutes transitively the components of $\mathbf{X}(w)$.*

Let $w=s_1\cdots s_n$ be a reduced decomposition of W as a product of elements of S. By (1) and (3), we need to show that $H(s_1,\dots,s_n)=\mathbf{G}^F$. Since w does not belong to any F-stable proper parabolic subgroup of W, there exists a sequence $1\leqslant i_1<\cdots<i_r\leqslant n$ such that $(s_{i_k})_{1\leqslant k\leqslant r}$ is a family of representatives of F-orbits in S. By (2), we have $H(s_{i_1},\dots,s_{i_r})\subset H(s_1,\dots,s_n)$. But, by [Lu, Proposition 4.8], $\mathbf{X}(s_{i_1}\cdots,s_{i_r})$ is irreducible so, again by (1) and (3), $H(s_{i_1},\dots,s_{i_r})=\mathbf{G}^F$. Therefore, $H(s_1,\dots,s_n)=\mathbf{G}^F$, as expected.

2. Sketch of another proof

As explained in the introduction, we propose a second proof (due independently to Lusztig) of Theorem A (or Theorem A'). It relies on a formula for counting rational points of Deligne-Lusztig varieties. We present here a sketch of this proof.

Write $w=s_1\cdots s_n$ with $s_i\in S$ and $n=\ell(w)$. First, by using the first two steps of the geometric proof and the statement (1), it is enough to show that $\hat{\mathbf{X}}(s_1,\dots,s_n)$ is irreducible whenever $\{s_1,\dots,s_n\}$ meets every F-orbit in S. So, assume this last condition is satisfied.

Let $A = \mathbb{Q}[v]$ and let $K = \mathbb{Q}(v)$, where v is an indeterminate. Let $\mathcal{H}$ denote the Iwahori-Hecke A-algebra of (W, S) with parameters v^2 (i.e. the standard generators $(T_s)_{s\in S}$ of $\mathcal{H}$ satifisfy the relation $(T_s - v^2)(T_s + 1) = 0$). Then F induces an automorphism of $\mathcal{H}$, which we denote by σ. Let $\tilde{\mathcal{H}} = \mathcal{H} \rtimes \langle\sigma\rangle$. Let $\mathrm{ind} : K\mathcal{H} \to K$ (resp. $\varepsilon : K\mathcal{H} \to K$) be the unique morphism of algebras such that $\mathrm{ind}(T_s) = v^2$ (resp. $\varepsilon(T_s) = -1$) for every $s \in S$.

The variety $\hat{\mathbf{X}}(s_1, \ldots, s_n)$ is irreducible if and only if $H_c^{2n}(\hat{\mathbf{X}}(s_1, \ldots, s_n), \overline{\mathbb{Q}}_\ell)$ is one-dimensional. By [DiMiRou, Corollary 3.3.8], this amounts to say that, if γ is an irreducible character of $K\tilde{\mathcal{H}}$ such that $\gamma((T_{s_1} + 1)\ldots(T_{s_n} + 1)\sigma)$ is a polynomial in v of degree $2n$, then $\mathrm{Res}^{K\tilde{\mathcal{H}}}_{K\mathcal{H}}\gamma = \mathrm{ind}$. By [GePf, Proposition 9.4.1 (a) and Corollary 9.4.2], this is equivalent to say that, if γ is an irreducible character of $K\tilde{\mathcal{H}}$ such that $\gamma(T_w\sigma)$ has a non-zero constant term, then $\mathrm{Res}^{K\tilde{\mathcal{H}}}_{K\mathcal{H}}\gamma = \varepsilon$ (note that [GePf, Proposition 9.4.1 (a) and Corollary 9.4.2] are proved for the algebra $K\mathcal{H}$ but their proofs extend easily to the case of $K\tilde{\mathcal{H}}$).

So, let γ be an irreducible character of $K\tilde{\mathcal{H}}$ such that $\gamma(T_w\sigma)(0) \neq 0$. Let $\mathcal{H}_0 = \mathcal{H}/v\mathcal{H}$ and $\tilde{\mathcal{H}}_0 = \tilde{\mathcal{H}}/v\tilde{\mathcal{H}}$. Using the parametrization of the simple modules for $\tilde{\mathcal{H}}_0$ (this follows easily from [No] and Clifford theory), we deduce that the specialization $v \mapsto 0$ of $\mathrm{Res}^{K\tilde{\mathcal{H}}}_{K\mathcal{H}}\gamma$ contains the specialization $v \mapsto 0$ of ε. But it is well-known that ε is alone in its 0-block (it corresponds to the central idempotent $(1/P_W(v))\sum_{w\in W}(-1)^{\ell(w)}v^{2(\ell(w_0)-\ell(w))}T_w$ of $\tilde{\mathcal{H}}$, where $P_W(v) = \sum_{w\in W} v^{2\ell(w)}$ and w_0 is the longest element of W). This completes the second proof.

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