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Recurrence matrices

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1 Introduction

The subject of this paper are $p-$*recurrence matrices* (or recurrence matrices for short) over a fixed ground field $\mathbf{K}$. A recurrence matrix is an element of the product

$$\prod_{l=0}^{\infty} \mathbf{K}^{p^l \times q^l}$$

(where $\mathbf{K}^{p^l \times q^l}$ denotes the vector-space of $p^l \times q^l$ matrices with coefficients in $\mathbf{K}$) satisfying finiteness conditions which are suitable for computations. Concretely, a recurrence matrix has a finite description involving finitely many elements in $\mathbf{K}$, the set $\mathrm{Rec}_{p\times q}(\mathbf{K})$ of all recurrence matrices in $\prod_{l=0}^{\infty} \mathbf{K}^{p^l \times q^l}$ is a vector space and the obvious product $AB \in \prod_{l=0}^{\infty} \mathbf{K}^{p^l \times q^l}$ of two recurrence matrices $A \in \mathrm{Rec}_{p\times r}(\mathbf{K}), B \in \mathrm{Rec}_{r\times q}(\mathbf{K})$ is again a recurrence matrix. The subset $\mathrm{Rec}_{p\times p}(\mathbf{K}) \subset \prod_{l=0}^{\infty} \mathbf{K}^{p^l \times p^l}$ of recurrence matrices of "square-size" is thus an algebra. We show how to do computations in this algebra and describe a few features of it.

The set of all invertible elements in the algebra $\mathrm{Rec}_{p\times p}(\mathbf{K})$ forms a group $\mathrm{GL}_{p-rec}(\mathbf{K})$ containing interesting subgroups. Indeed, a recurrence matrix $\mathrm{GL}_{p-rec}(\mathbf{K})$ is closely linked to a finite-dimensional matrix-representation of the free monoid on p^2 elements. Recurrence matrices for which the image of this representation is a finite monoid are in bijection with (suitably defined) "automatic functions" associated to finite-state automata. This implies easily that $\mathrm{GL}_{p-rec}(\mathbf{K})$ contains all "automata groups" or "$p-$self-similar groups" (formed by bijective finite-state transducers acting by automorphisms on the infinite plane rooted $p-$regular tree).

In particular, the group $\mathrm{GL}_{2-rec}(\mathbf{K})$ contains a famous group of Grigorchuk, see [7] (and all similarly defined groups) in a natural way.

Part of the present paper is contained in a condensed form in [2] whose main result was the initial motivation for developping the theory.

2 Monoids

Definition A *monoid* (or *a semi-group with identity*) is a set $\mathcal{M}$ endowed with an associative product $\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ admitting a two-sided identity.

Remark 2.1. *It is enough to require that a monoid admits a left identity ϵ and a right identity f for its associative product since then $ef = e = f$. This shows moreover unicity of the identity.*

In a *commutative* monoid, the product is commutative. A *morphism* of monoids $\mu : \mathcal{M} \longrightarrow \tilde{\mathcal{M}}$ is an application such that $\mu(e) = \tilde{e}$ and $\mu(ab) = \mu(a)\mu(b)$ where e is the identity of $\mathcal{M}$, $\tilde{e}$ the identity of $\tilde{\mathcal{M}}$ and where $a, b \in \mathcal{M}$ are arbitrary. A *submonoid* of a monoid $\mathcal{M}$ is a subset which is closed under the product and contains the identity of $\mathcal{M}$. A *(linear) representation* of a monoid $\mathcal{M}$ is a morphism $\rho : \mathcal{M} \longrightarrow \mathrm{End}(V)$ from $\mathcal{M}$ into the monoid (with product given by composition) of endomorphisms of a vector space V.

Given a subset $\mathcal{S} \subset M$ of a monoid, the submonoid $\mathcal{M}(\mathcal{S})$ generated by $\mathcal{S}$ is the smallest submonoid of $\mathcal{M}$ containing $\mathcal{S}$. A monoid $\mathcal{M}$ is finitely generated if $\mathcal{M} = \mathcal{M}(\mathcal{G})$ for some finite subset $\mathcal{G} \subset \mathcal{M}$, called a *generating set.*

The *free monoid* $\mathcal{M}_{\mathcal{A}}$ over a set (also called alphabet) $\mathcal{A}$ is the set of all finite words with letters in $\mathcal{A}$. We call $\mathcal{A}$ the free generating set of $\mathcal{A}$. Two free monoids over $\mathcal{A}$, respectively $\tilde{\mathcal{A}}$, are isomorphic if and only if $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are equipotent. In particular, there exists, up to isomorphism, a unique free monoid whose free generating set has a given cardinal number. One can thus speak of the free monoid on n letters for a natural integer $n \in \mathbb{N}$.

There is a natural notion of a monoid presented by generators and relations. A free monoid has no relations and an arbitrary finitely generated monoid $\mathcal{Q}$ can always be given in the form

$$\mathcal{Q} = \langle \mathcal{G} : \mathcal{R} \rangle$$

where $\mathcal{G}$ is a finite set of generators and $\mathcal{R} \subset \mathcal{G}^* \times \mathcal{G}^*$ a (perhaps infinite) set of relations of the form $L = R$ with $L, R \in \mathcal{G}^*$. The quotient monoid $\mathcal{Q}$ of the free monoid $\mathcal{G}^*$ by the relations $\mathcal{R}$ is the set of equivalence classes of words in $\mathcal{G}^*$ by the equivalence relation generated by $UL_iV \sim UR_iV$ for $U, V \in \mathcal{G}^*$ and $(L_i, R_i) \in \mathcal{R}$.

Remark 2.2. *Given a quotient monoid $\mathcal{Q} = \langle \mathcal{G} : \mathcal{R} \rangle$, the free monoid $\mathcal{F}_{\mathcal{G}}$ on $\mathcal{G}$ surjects onto $\mathcal{Q}$. The set $\pi^{-1}(\tilde{e}) \subset \mathcal{Q}$ of preimages of the identity $\tilde{e} \in \mathcal{Q}$ is of course a submonoid of $\mathcal{F}_{\mathcal{G}}$.*

However, unlike in the case of groups, the kernel $\pi^{-1}(\tilde{e}) \subset \mathcal{F}_{\mathcal{G}}$ does not characterize $\mathcal{Q}$. An example is given by the monoid $\mathcal{Q} = \{\tilde{e}, a\}$ with identity $\tilde{e}$ and product $aa = a$. The free monoid on 1 generator surjects onto $\mathcal{Q}$ but $\pi^{-1}(\tilde{e}) = \{\emptyset\} \in \mathcal{F}_1$ reduces to the trivial submonoid in the free monoid $\mathcal{F}_1$ on 1 generator.

Remark 2.3. *The composition law $\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ of a monoid $\mathcal{M}$ endows the free vector space generated by $\mathcal{M}$ with an associative algebra-structure,*

denoted $\mathbf{K}[\mathcal{M}]$ *and called the monoid-algebra of* $\mathcal{M}$*. The monoid algebra* $\mathbf{K}[\mathcal{M}]$ *of a free monoid is simply the polynomial algebra on free generators of* $\mathcal{M}$*, considered as non-commuting variables.*

A monoid $\mathcal{M}$ *has the finite-factorisation property if every element* $m \in \mathcal{M}$ *has only a finite number of distinct factorisations* $m = m_1 m_2$ *with* $m_1, m_2 \in \mathcal{M}$*. If a monoid* $\mathcal{M}$ *has the finite-factorisation property (eg. if* $\mathcal{M}$ *is a finitely generated quotient monoid such that* L_i *and* R_i *are of equal length for every relation* $(L_i, M_i) \in \mathcal{R}$*), then the algebraic dual* $\mathbf{K}[[\mathcal{M}]]$ *(which consists of all formal sums of elements in* $\mathcal{M}$*) is also an algebra (for the obvious product) and contains the monoid-algebra* $\mathbf{K}[\mathcal{M}]$*. In the case where* $\mathcal{M}$ *is a free monoid, the algebra* $\mathbf{K}[[\mathcal{M}]]$ *is the algebra of formal power-series with unknowns the free generators of* $\mathcal{M}$*, considered as non-commuting variables.*

3 The category $K^{\mathcal{M}}$

Given two natural integers $p, q \in \mathbb{N}$ and a natural integer $l \in \mathbb{N}$, we define $\mathcal{M}^l_{p\times q}$ as the set of all $p^l q^l$ pairs of words (U, W) of common length l with $U = u_1 \dots u_l \in \{0, \dots, p-1\}^l$ and $W = w_1 \dots w_l \in \{0, \dots, q-1\}^l$. The set $\mathcal{M}^0_{p\times q}$ contains by convention only $(\emptyset, \emptyset)$. We denote by $\mathcal{M}_{p\times q} = \bigcup_{l\in\mathbb{N}} \mathcal{M}^l_{p\times q}$ the union of all finite sets $\mathcal{M}^l_{p\times q}$. The common length $l = l(U) = l(W) \in \mathbb{N}$ is the *length* $l(U, W)$ of a *word* $(U, W) \in \mathcal{M}^l_{p\times q} \subset \mathcal{M}_{p\times q}$. We write $\mathcal{M}^{\leq l}_{p\times q}$ for the obvious set of all $1 + pq + \dots + p^l q^l = \frac{(pq)^{l+1}-1}{pq-1}$ words of length at most l in $\mathcal{M}_{p\times q}$.

The concatenation $(U, W)(U', W') = (UU', WW')$ turns $\mathcal{M}_{p\times q} = (\mathcal{M}^1_{p\times q})^*$ into a free monoid on the set $\mathcal{M}^1_{p\times q}$ of all pq words with length 1 in $\mathcal{M}_{p\times q}$.

Equivalently, $\mathcal{M}_{p\times q}$ can be described as the submonoid of the product-monoid $\mathcal{M}_p \times \mathcal{M}_q$ (where $\mathcal{M}_r$ stands for the free monoid on r generators) whose elements (U, W) are all pairs $U \in \mathcal{M}_p, W \in \mathcal{M}_q$ of the same length.

Remark 3.1. *Since the free cyclic monoid* $\{0\}^*$ *contains a unique word of length* l *for every* $l \in \mathbb{N}$*, we have an obvious isomorphism* $\mathcal{M}_{p\times 1} \sim \mathcal{M}_p = \{0, \dots, p-1\}^*$*. Moreover, since the free monoid* $\mathcal{M}_0$ *on the empty alphabet is the trivial monoid on one element, the monoid* $\mathcal{M}_{p\times q}$ *is reduced to the empty word* $(\emptyset, \emptyset)$ *if* $pq = 0$*.*

Remark 3.2. *Many constructions involving the monoid* $\mathcal{M}_{p\times q}$ *have generalizations to* $\mathcal{M}_{p_1\times p_2\times\dots\times p_k}$ *defined in the obvious way. In particular, using Remark 3.1, we write often simply* $\mathcal{M}_p$ *instead of* $\mathcal{M}_{p\times 1}$ *or* $\mathcal{M}_{1\times p}$*.*

In the sequel, we consider a fixed commutative field $\mathbf{K}$ (many results continue to hold for commutative rings). We denote by $\mathbf{K}^{\mathcal{M}_{p\times q}}$ the vector space of all functions from $\mathcal{M}_{p\times q}$ into $\mathbf{K}$. We denote by $A[\mathcal{S}]$ the restriction of $A \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ to a subset $\mathcal{S} \subset \mathcal{M}_{p\times q}$. In particular, we write $A[U, W]$ for the evaluation of A on a word $(U, W) \in \mathcal{M}_{p\times q}$. For $A \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ and $l \in \mathbb{N}$ we

consider the restriction $A[\mathcal{M}^l_{p\times q}]$ of A to $\mathcal{M}^l_{p\times q}$ as a matrix of size $p^l\times q^l$ with coefficients $A[U,W]$ indexed by all words $(U,W)=(u_1\dots u_l,w_1\dots w_l)\in\mathcal{M}^l_{p\times q}$.

For $A\in\mathbf{K}^{\mathcal{M}_{p\times r}}$ and $B\in\mathbf{K}^{\mathcal{M}_{r\times q}}$, we define the *matrix product* or *product* $AB\in\mathbf{K}^{\mathcal{M}_{p\times q}}$ of A and B by

$$(AB)[U,W]=\sum_{V\in\{0,\dots,r-1\}^l}A[U,V]B[V,W]$$

for $(U,W)\in\mathcal{M}^l_{p\times q}$ a word of length l. The matrix product is obviously bilinear and associative. We get thus a category $\mathbf{K}^{\mathcal{M}}$ (see for instance [9] for definitions) as follows: An *object* of $\mathbf{K}^{\mathcal{M}}$ is a vector space of the form $\mathbf{K}^{\mathcal{M}_p}=\mathbf{K}^{\mathcal{M}_{p\times 1}}$ for $p\in\mathbb{N}$. A *morphism* (or *arrow*) is given by $A\in\mathbf{K}^{\mathcal{M}_{q\times p}}$ and defines a linear application from $\mathbf{K}^{\mathcal{M}_p}$ to $\mathbf{K}^{\mathcal{M}_q}$ by matrix-multiplication.

The matrix-product turns the set $\mathbf{K}^{\mathcal{M}_{p\times p}}$ of endomorphisms of an object $\mathbf{K}^{\mathcal{M}_p}$ into an algebra.

Remark 3.3. *The category $\mathbf{K}^{\mathcal{M}}$ has also the following slightly different realization: Associate to an object $\mathbf{K}^{\mathcal{M}_p}$ corresponding to the natural integer $p\in\mathbb{N}$ the $\mathbb{N}-$graded vector space $\mathcal{FS}_p=\bigoplus_{l=0}^{\infty}\mathbf{K}^{p^l}$, identified with the subspace of $\mathbf{K}^{\mathcal{M}_p}$ of all functions with finite support. Morphisms are linear applications $\mathcal{FS}_p\longrightarrow\mathcal{FS}_q$ preserving the grading (and are given by a product $\prod_{l=0}^{\infty}\mathbf{K}^{q^l\times p^l}$ of linear maps $\mathbf{K}^{p^l}\longrightarrow\mathbf{K}^{q^l}$).*

Remark 3.4. *The vector-spaces $\mathcal{FS}_{p\times q}\subset\mathbf{K}^{\mathcal{M}_{p\times q}}$ can be identified with the vector-spaces $\mathbf{K}[X_{0,0},\dots,X_{p-1,q-1}]\subset\mathbf{K}[[X_{0,0},\dots,X_{p-1,q-1}]]$ of polynomials and formal power-series in pq non-commuting variables $X_{u,w},(u,w)\in\mathcal{M}^1_{p\times q}$. The vector-space $\mathbf{K}^{\mathcal{M}_{p\times q}}$ can also be considered as the algebraic dual of $\mathcal{FS}_{p\times q}$.*

Remark 3.5. *A vector $X\in\mathbf{K}^{\mathcal{M}_{p\times q}}$ can be given as a projective limit by considering the projection*

$$\mathbf{K}^{\mathcal{M}^{\leq l+1}_{p\times q}}\longrightarrow\mathbf{K}^{\mathcal{M}^{\leq l}_{p\times q}}$$

obtained by restricting the function $X[\mathcal{M}^{\leq l+1}_{p\times q}]$ to the subset $\mathcal{M}^{\leq l}_{p\times q}\subset\mathcal{M}^{\leq l+1}_{p\times q}$.

Remark 3.6. *The following analogue of the tensor product yields a natural functor of the category $\mathbf{K}^{\mathcal{M}}$: For $A\in\mathbf{K}^{\mathcal{M}_{p\times q}}$ and $B\in\mathbf{K}^{\mathcal{M}_{p'\times q'}}$, define $A\otimes B\in\mathbf{K}^{\mathcal{M}_{pp'\times qq'}}$ in the obvious way by considering tensor products $(A\otimes B)[\mathcal{M}^l_{pp'\times qq'}]=A[\mathcal{M}^l_{p\times q}]\otimes B[\mathcal{M}^l_{p'\times q'}]$ of the graded parts. This "tensor product" is bilinear and natural with respect to most constructions of this paper. The main difference with the usual tensor product is however the fact that $A\otimes B$ can be zero even if A and B are both non-zero: Consider A and B such that $A[U,W]=0$ if (U,W) is of even length and $B[U',W']=0$ if (U',W') is of odd length.*

Remark 3.7. *The category $\mathbf{K}^{\mathcal{M}}$, realized as in Remark 3.3 can be embedded as a full subcategory into a larger category with objects given by graded vector spaces $\mathcal{FS}_{(d_0,d_1,\dots)} = \bigoplus_{l=0}^{\infty} \mathbf{K}^{d_l}$ indexed by arbitrary sequences $d = (d_0, d_1, d_2, \dots) \in \mathbb{N}^{\mathbb{N}}$. Morphisms from $\mathcal{FS}_{(d_0,d_1,\dots)}$ to $\mathcal{FS}_{(e_0,e_1,\dots)}$ are linear applications preserving the grading and correspond to elements in the direct product of matrices $\prod_{l=0}^{\infty} \mathbf{K}^{e_l \times d_l}$. The category $\mathbf{K}^{\mathcal{M}}$ corresponds to the full subcategory with objects indexed by geometric progressions.*

4 The category Rec(K)

A word $(S,T) \in \mathcal{M}_{p\times q}$ defines an endomorphism $\rho(S,T) \in \mathrm{End}(\mathbf{K}^{\mathcal{M}_{p\times q}})$ of the vector space $\mathbf{K}^{\mathcal{M}_{p\times q}}$ by setting

$$(\rho(S,T)A)[U,W] = A[US, WT]$$

for $A \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ and $(U,W) \in \mathcal{M}_{p\times q}$. The easy computation

$$\begin{aligned}\rho(S,T)\big(\rho(S',T')A)[U,W] &= \rho(S',T')A[US,WT] \\ &= A[USS', WTT'] = \rho(SS',TT')A[U,W]\end{aligned}$$

shows that $\rho : \mathcal{M}_{p\times q} \longrightarrow \mathrm{End}(\mathbf{K}^{\mathcal{M}_{p\times q}})$ is a linear representation from the free monoid $\mathcal{M}_{p\times q}$ into the monoid $\mathrm{End}(\mathbf{K}^{\mathcal{M}_{p\times q}})$ of all linear endomorphisms of $\mathbf{K}^{\mathcal{M}_{p\times q}}$.

Definition 4.1. *We call the monoid $\rho(\mathcal{M}_{p\times q}) \subset End(\mathbf{K}^{\mathcal{M}_{p\times q}})$ the shift-monoid. An element $\rho(S,T) \in \rho(\mathcal{M}_{p\times q})$ is a shift-map.*

Remark 4.2. *The terminology is motivated by the special case $p = q = 1$: An element $A \in \mathbf{K}^{\mathcal{M}_{1\times 1}}$ is completely described by the sequence $\alpha_0, \alpha_1, \dots \in \mathbf{K}^{\mathbb{N}}$ defined by the evaluation $\alpha_l = A[0^l, 0^l]$ on the unique word $(0^l, 0^l) \in \mathcal{M}^l_{1\times 1}$ of length l. The generator $\rho(0,0) \in \rho(\mathcal{M}_{1\times 1})$ acts on A by the usual shift $(\alpha_0, \alpha_1, \alpha_2, \dots) \longmapsto (\alpha_1, \alpha_2, \alpha_3, \dots)$ which erases the first element α_0 of the sequence $\alpha_0, \alpha_1, \dots$ corresponding to A.*

Proposition 4.3. *The linear representation $\rho : \mathcal{M}_{p\times q} \longrightarrow End(\mathbf{K}^{\mathcal{M}_{p\times q}})$ is faithful. The shift-monoid $\rho(\mathcal{M}_{p\times q}) \subset End(\mathbf{K}^{\mathcal{M}_{p\times q}})$ is thus isomorphic to the free monoid $\mathcal{M}_{p\times q}$.*

Proof Otherwise there exists $(S,T) \neq (S',T') \in \mathcal{M}_{p\times q}$ such that $\rho(S,T) = \rho(S',T')$. Consider $A \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ such that $A[S,T] = 1$ and $A[U,W] = 0$ otherwise. The inequality

$$(\rho(S,T)A)[\emptyset,\emptyset] = 1 \neq 0 = (\rho(S',T')A)[\emptyset,\emptyset]$$

yields then a contradiction. □

Given subsets $\mathcal{S} \subset \mathcal{M}_{p\times q}$ and $\mathcal{A} \subset \mathbf{K}^{\mathcal{M}_{p\times q}}$, we write

$$\rho(\mathcal{S})\mathcal{A} = \{\rho(S,T)A \mid (S,T) \in \mathcal{S}, A \in \mathcal{A}\} \subset \mathbf{K}^{\mathcal{M}_{p\times q}}.$$

Since $\rho(\emptyset,\emptyset)$ acts as the identity, we have $\mathcal{A} \subset \rho(\mathcal{S})\mathcal{A}$ for any subset $\mathcal{S} \subset \mathcal{M}_{p\times q}$ containing the empty word $(\emptyset,\emptyset)$ of length 0.

Definition 4.4. *A subset $\mathcal{X} \subset \mathbf{K}^{\mathcal{M}_{p\times q}}$ is a recursively closed set if $\rho(\mathcal{M}_{p\times q})\mathcal{X} = \mathcal{X}$. The recursive set-closure of $\mathcal{X} \subset \mathbf{K}^{\mathcal{M}_{p\times q}}$ is given by $\rho(\mathcal{M}_{p\times q})\mathcal{X}$ and is the smallest recursively closed subset of $\mathbf{K}^{\mathcal{M}_{p\times q}}$ containing $\mathcal{X}$. The recursive closure $\overline{\mathcal{X}}^{rec}$ is the linear span of $\rho(\mathcal{M}_{p\times q})\mathcal{X}$ and is the smallest linear subspace of $\mathbf{K}^{\mathcal{M}_{p\times q}}$ which contains $\mathcal{X}$ and is recursively closed.*

By definition, a subspace $\mathcal{A}$ is recursively closed if and only if $\mathcal{A} = \overline{\mathcal{A}}^{rec}$. Intersections and sums (unions) of recursively closed subspaces (subsets) in $\mathbf{K}^{\mathcal{M}_{p\times q}}$ are recursively closed (subsets).

Any recursively closed subspace $\mathcal{A} \subset \mathbf{K}^{\mathcal{M}_{p\times q}}$ is invariant under the shift-monoid and we call the restriction $\rho_{\mathcal{A}}(\mathcal{M}_{p\times q}) \in \mathrm{End}(\mathcal{A})$ of $\rho(\mathcal{M}_{p\times q})$ to the invariant subspace $\mathcal{A}$ the *shift-monoid* of $\mathcal{A}$.

Definition 4.5. *Given an element $A \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ with recursive closure $\overline{A}^{rec}$, we call the dimension $dim(\overline{A}^{rec}) \in \mathbb{N} \cup \{\infty\}$ the (recursive) complexity of A. A recurrence matrix is an element of the vector space*

$$Rec_{p\times q}(\mathbf{K}) = \{A \in \mathbf{K}^{\mathcal{M}_{p\times q}} \mid dim(\overline{A}^{rec}) < \infty\}$$

consisting of all elements having finite complexity.

It is easy to check that $\mathrm{Rec}_{p\times q}(\mathbf{K})$ is a recursively closed subspace of $\mathbf{K}^{\mathcal{M}_{p\times q}}$ containing the subspace $\mathcal{FS}_{p\times q}(\mathbf{K}) \subset \mathbf{K}^{\mathcal{M}_{p\times q}}$ consisting of all elements with finite support. An element $A \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ is a recurrence matrix, if and only if the shift-monoid $\rho_{\overline{A}^{rec}}(\mathcal{M}_{p\times q})$ of $\overline{A}^{rec}$ is a finite-dimensional linear representation of the free monoid $\mathcal{M}_{p\times q}$.

Proposition 4.6. *We have*

$$dim(\overline{AB}^{rec}) \leq dim(\overline{A}^{rec})\ dim(\overline{B}^{rec})$$

for the matrix-product $AB \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ of $A \in \mathbf{K}^{\mathcal{M}_{p\times r}}$ and $B \in \mathbf{K}^{\mathcal{M}_{r\times q}}$.

Corollary 4.7. *The matrix-product $AB \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ of two recurrence matrices $A \in Rec_{p\times r}(\mathbf{K}), B \in Rec_{r\times q}(\mathbf{K})$, is a recurrence matrix.*

Corollary 4.7 suggests the following definition.

Definition 4.8. *The category $Rec(\mathbf{K})$ of recurrence matrices is the subcategory of $\mathbf{K}^{\mathcal{M}}$ containing only recurrence matrices as arrows. Its objects can be restricted to $Rec_{p\times 1}(\mathbf{K})$ or even to the recursively closed subspaces $\mathcal{FS}_p = \bigoplus_{l=0}^{\infty} \mathbf{K}^{p^l}$ of elements with finite support.*

Proof of Proposition 4.6 Given bases $A_1, A_2, \dots$ of $\overline{A}^{rec} \subset \mathrm{Rec}_{p\times r}(\mathbf{K})$ and $B_1, B_2, \dots$ of $\overline{B}^{rec} \subset \mathrm{Rec}_{r\times q}(\mathbf{K})$, the computation

$$\begin{aligned}
(\rho(s,t)(A_iB_j))[U,W] &= (A_iB_j)[Us,Wt] \\
&= \sum_{v=0}^{r-1} \sum_{V\in\{0,\dots,r-1\}^l} A_i[Us,Vv]B_j[Vv,Wt] \\
&= \sum_{v=0}^{r-1} \sum_{V\in\{0,\dots,r-1\}^l} (\rho(s,v)A_i)[U,V](\rho(v,t)B_j)[V,W] \\
&= \left(\sum_{v=0}^{r-1} (\rho(s,v)A_i)(\rho(v,t)B_j)\right)[U,W]
\end{aligned}$$

(with $(U,W) \in \mathcal{M}^l_{p\times q}$) shows that $\mathcal{C} = \sum_{1\le i,j} \mathbf{K}A_iB_j$ is recursively closed in $\mathbf{K}^{\mathcal{M}_{p\times p}}$. The obvious inclusion $\overline{AB}^{rec} \subset \mathcal{C}$ finishes the proof. □

4.1 Other ring-structures on $\mathbf{Rec}_{p\times q}(\mathbf{K})$

The vector-space $\mathbf{K}^{\mathcal{M}_{p\times q}}$ carries two natural ring-structures which are both inherited by $\mathrm{Rec}_{p\times q}(\mathbf{K})$.

A first ring-structure on $\mathbf{K}^{\mathcal{M}_{p\times q}}$ comes from the usual commutative product of functions $(A\circ B)[U,W] = (A[U,W])(B[U,W])$.

A second product (non-commutative if $pq > 1$) is given by associating to $A \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ the formal power series

$$\sum_{(U,W)=(u_1\dots u_n,w_1,w_n)\in\mathcal{M}_{p\times q}} A[u_1\dots u_n, w_1\dots w_n]\, X_{u_1,w_1}\cdots X_{u_n,w_n} \in \mathbf{K}[[\mathcal{M}_{p\times q}]]$$

in pq non-commuting variables $X_{u,w}, (u,w) \in \mathcal{M}^1_{p\times q}$. Multiplication of non-commutative formal power-series endows $\mathbf{K}^{\mathcal{M}_{p\times q}}$ with the *convolution-product*

$$(A * B)[U,W] = \sum_{(U,W)=(U_1,W_1)(U_2,W_2)} A[U_1,W_1]B[U_2,W_2]$$

where the sum is over all $l+1$ factorizations $(U,W) = (U_1,W_1)(U_2,W_2)$ of a word $(U,W) \in \mathcal{M}^l_{p\times q}$.

The following result shows that both ring-structures restrict to $\mathrm{Rec}_{p\times q}(\mathbf{K})$.

Proposition 4.9. *(i) $Rec_{p\times q}(\mathbf{K})$ is a commutative ring for the ordinary product of functions (given by) $(A\circ B)[U,W] = A[U,W]B[U,W]$. More precisely, we have $\overline{A\circ B}^{rec} \subset \sum \mathbf{K}A_i\circ B_j$ where $\overline{A}^{rec} = \sum \mathbf{K}A_i$ and $\overline{B}^{rec} = \sum \mathbf{K}B_j$.*

(ii) $Rec_{p\times q}(\mathbf{K})$ is a ring for the convolution-product $(A*B)[U,W] = \sum_{(U,W)=(U_1,W_1)(U_2,W_2)} A[U_1,W_1]B[U_2,W_2]$. More precisely, we have $\overline{A*B}^{rec} \subset \sum \mathbf{K}A_i + \sum \mathbf{K}A*B_j$ where $\overline{A}^{rec} = \sum \mathbf{K}A_i$ and $\overline{B}^{rec} = \sum \mathbf{K}B_j$.

Proof Assertion (i) follows from Corollary 4.7 and the existence of a diagonal embedding of $\mathbf{K}^{\mathcal{M}_{p\times q}}$ into $\mathbf{K}^{\mathcal{M}_{(pq)\times(pq)}}$ preserving the complexity.

The computation

$$\begin{aligned}
&\big(\rho(s,t)(A*B)\big)[U,W] = (A*B)[Us,Wt] \\
&= \sum_{(U,W)=(U_1,W_1)(U_2,W_2)} A[U_1,W_1]B[U_2s,W_2t] + A[Us,Wt]B[\emptyset,\emptyset] \\
&= A*(\rho(s,t)B)[U,W] + \rho(s,t)A[U,W]B[\emptyset,\emptyset]
\end{aligned}$$

shows the identity

$$\rho(s,t)(A*B) = A*(\rho(s,t)B) + B[\emptyset,\emptyset]\rho(s,t)A$$

and implies assertion (ii). □

4.2 Convergent elements in $\mathbf{K}^{\mathcal{M}_{p\times q}}$ and $\mathbf{Rec}_{p\times q}(\mathbf{K})$

Using the bijection

$$s_1\dots s_n \longmapsto \sum_{j=1}^{n} s_j p^{j-1}$$

between $\{0,\dots,p-1\}^n$ and $\{0,\dots,p^n-1\}$, an infinite matrix $\tilde{A}$ with coefficients $\tilde{A}_{s,t}, 0\le s,t\in\mathbb{N}$ gives rise to an element $A\in\mathbf{K}^{\mathcal{M}_{p\times q}}$ by setting

$$A[s_1\dots s_n,t_1\dots t_n] = \tilde{A}_{s,t} \text{ where } s=\sum_{j=1}^{n} s_j p^{j-1}, t=\sum_{j=1}^{n} t_j q^{j-1}.$$

We call such an element A *convergent* with limit (the infinite matrix) $\tilde{A}$. If $p=1$ or $q=1$, the limit matrix $\tilde{A}$ degenerates to a row or column-vector and it degenerates to a single coefficient $\tilde{A}_{0,0}$ if $pq=0$. Obviously, an element $A\in\mathbf{K}^{\mathcal{M}_{p\times q}}$ is convergent if and only if $\rho(0,0)A=A$. The vector space spanned by all converging elements in $\mathbf{K}^{\mathcal{M}_{p\times p}}$ (respectively in $\mathrm{Rec}_{p\times p}(\mathbf{K})$) is not preserved under matrix-products (except if $p\le 1$) but contains a few subalgebras given for instance by converging matrices with limit an infinite lower (or upper) triangular matrix.

Remark 4.10. *The vector-space of convergent elements contains (strictly) a unique maximal subspace which is recursively closed. This subspace is the set of all convergent elements $A\in Rec_{p\times q}(\mathbf{K})$ such that $\overline{A}^{rec}$ consists only of convergent elements.*

5 The quotient category $\mathbf{Rec}(\mathbf{K})/\mathcal{F}S$ modulo elements of finite support and stable complexity

We call a quotient $\mathcal{A}/\mathcal{B}$ with vector-spaces $\mathcal{B}\subset\mathcal{A}\subset\mathrm{Rec}_{p\times q}(\mathbf{K})$ a *quotient-space of recurrence matrices* if $\mathcal{A}$ and $\mathcal{B}$ are both recursively closed.

An important example is given by $\mathcal{A} = \mathrm{Rec}_{p\times q}(\mathbf{K})$ and $\mathcal{B} = \mathcal{FS}_{p\times q}$ the vector space of all functions $B \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ with finite support. The vector space $\mathcal{FS}_{p\times q}$ is recursively closed since it consists of all elements B such that the shift map $\rho(\mathcal{M}_{p\times q})$ acts in a nilpotent way on $\overline{B}^{rec}$ (more precisely, for every element $B \in \mathcal{FS}_{p\times q}$ there exists a natural integer N such that $\rho(U,W)B = 0$ if $(U,W) \in \mathcal{M}_{p\times q}$ is of length $\geq N$). Elements of $\mathcal{FS}_{p\times q}$ are in some sense trivial. Since the subset $\mathcal{FS} = \cup_{p,q\in\mathbb{N}}\mathcal{FS}_{p\times q} \subset \cup_{p,q\in\mathbb{N}}\mathrm{Rec}_{p\times q}(\mathbf{K})$ is in an obvious sense a "two-sided ideal" for the matrix-product (whenever defined), we get a functor from $\mathrm{Rec}(\mathbf{K})$ onto the "quotient-category" $\mathrm{Rec}(\mathbf{K})/\mathcal{FS}$ by considering the projection $\pi_{\mathcal{FS}} : \mathrm{Rec}(\mathbf{K}) \longrightarrow \mathrm{Rec}(\mathbf{K})/\mathcal{FS}$.

Given a recursively closed finite-dimensional subspace $\mathcal{A} \subset \mathrm{Rec}_{p\times q}(\mathbf{K})$, we define its *stable complexity* $\dim_{\mathcal{FS}}(\mathcal{A})$ as the dimension $\dim(\pi_{\mathcal{FS}}(\mathcal{A})) \leq \dim(\mathcal{A})$ of its projection onto the quotient space $\mathrm{Rec}_{p\times q}(\mathbf{K})/\mathcal{FS}_{p\times q}$. The *stable complexity* of an element $A \in \mathrm{Rec}_{p\times q}(\mathbf{K})$ is the stable complexity $\dim(\pi_{\mathcal{FS}}(\overline{A}^{rec}))$ of its recursive closure. The *complexity* of an element $\tilde{A} \in \mathrm{Rec}_{p\times q}(\mathbf{K})/\mathcal{FS}_{p\times q}$is defined as the stable complexity $\dim_{\mathcal{FS}}(A)$ of any lift $A \in \pi^{-1}_{\mathcal{FS}}(\tilde{A}) \subset \mathrm{Rec}_{p\times q}(\mathbf{K})$.

Proposition 5.1. *(i) We have*

$$dim_{\mathcal{FS}}(\overline{AB}^{rec}) \leq dim_{\mathcal{FS}}(\overline{A}^{rec})\ dim_{\mathcal{FS}}(\overline{B}^{rec})$$

for the matrix-product $AB \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ of $A \in \mathbf{K}^{\mathcal{M}_{p\times r}}$ and $B \in \mathbf{K}^{\mathcal{M}_{r\times q}}$.

(ii) We have

$$dim_{\mathcal{FS}}(\overline{\pi^{-1}_{\mathcal{FS}}(\tilde{AB})}^{rec}) \leq dim_{\mathcal{FS}}(\overline{\pi^{-1}_{\mathcal{FS}}(\tilde{A})}^{rec})\ dim_{\mathcal{FS}}(\overline{\pi^{-1}_{\mathcal{FS}}(\tilde{B})}^{rec})$$

for the product $\tilde{AB} \in \mathbf{K}^{\mathcal{M}_{p\times q}}/\mathcal{FS}_{p\times q}$ of $\tilde{A} \in \mathbf{K}^{\mathcal{M}_{p\times r}}/\mathcal{FS}_{p\times r}$ and $\tilde{B} \in \mathbf{K}^{\mathcal{M}_{r\times q}}/\mathcal{FS}_{r\times q}$.

The proof is the same as for Proposition 4.6.

Remark 5.2. *The subspace $\mathcal{FS}_{p\times q} = Rec_{p\times q}(\mathbf{K}) \cap \mathcal{FS}$ of all elements with finite support in $Rec_{p\times q}(\mathbf{K})$ is also an ideal for the commutative product of functions. Although $\mathcal{FS}_{p\times q}$ is also closed for the convolution-product (and corresponds to polynomials in non-commutative variables) it is not an ideal in the convolution-ring since it contains the convolutional identity Id_* given by $Id_*[\emptyset,\emptyset] = 1$ and $Id_*[U,W] = 0$ for $(U,W) \in \mathcal{M}_{p\times q} \setminus (\emptyset,\emptyset)$.*

Remark 5.3. *One checks easily that the set $\mathcal{FS}$ is also an ideal (in the obvious sense) of the category $\mathbf{K}^{\mathcal{M}}$. One can thus consider the quotient algebras $\mathbf{K}^{\mathcal{M}_{p\times p}}/\mathcal{FS}$ and the functor (still denoted) $\pi_{\mathcal{FS}} : \mathbf{K}^{\mathcal{M}} \longrightarrow \mathbf{K}^{\mathcal{M}}/\mathcal{FS}$ onto the quotient category $\mathbf{K}^{\mathcal{M}}/\mathcal{FS}$.*

6 Matrix algebras

Given $A \in \mathbf{K}^{\mathcal{M}_{p\times p}}$, the elements $\rho(S,T)A$ indexed by $(S,T) \in \mathcal{M}^l_{p\times p}$ can be considered as coefficients of a square matrix of size $p^l \times p^l$ with values in the ring $\mathbf{K}^{\mathcal{M}_{p\times p}}$. More precisely, the application

$$A \longmapsto \varphi^l(A) = (\rho(S,T)A)_{(S,T)\in\mathcal{M}^l_{p\times p}}$$

can be described as the restriction

$$A = \prod_{j=0}^{\infty} A[\mathcal{M}^j_{p\times p}] \longmapsto \varphi^l(A) = \prod_{j=l}^{\infty} A[[\mathcal{M}^j_{p\times p}]$$

obtained by removing the initial terms $A[\mathcal{M}^0_{p\times p}], A[\mathcal{M}^1_{p\times p}], \ldots, A[\mathcal{M}^{l-1}_{p\times p}]$ from the sequence of matrices $A[\mathcal{M}^0_{p\times p}], A[\mathcal{M}^1_{p\times p}], \ldots$ representing A.

We leave the proof of the following obvious assertions to the reader.

Proposition 6.1. *(i) We have $\varphi^{l+k} = \varphi^l \circ \varphi^k$.*

(ii) φ^l defines a morphism of rings between the ring $\mathbf{K}^{\mathcal{M}_{p\times p}}$ and the ring of $p^l \times p^l$ matrices with values in $\mathbf{K}^{\mathcal{M}_{p\times p}}$.

(iii) φ^l restricts to a morphism of rings between the ring $Rec_{p\times p}(\mathbf{K})$ and the ring of $p^l \times p^l$ matrices with values in $Rec_{p\times p}(\mathbf{K})$.

(iv) We have $\mathcal{FS}_{p\times p} = \cup_{l=0}^{\infty} \ker(\varphi^l) \subset Rec_{p\times p}(\mathbf{K}) \subset \mathbf{K}^{\mathcal{M}_{p\times p}}$ for the vector space $\mathcal{FS}_{p\times p}$ of elements with finite support in $Rec_{p\times p}(\mathbf{K})$ (see section 5).

Moreover, φ^l induces an injective morphism $\overline{\varphi}^l$ from the quotient ring $\mathbf{K}^{\mathcal{M}_{p\times p}}/\mathcal{FS}_{p\times p}$ (respectively $Rec_{p\times p}(\mathbf{K})/\mathcal{FS}_{p\times p}$) onto the ring of $p^l \times p^l$ matrices with values in $\mathbf{K}^{\mathcal{M}_{p\times p}}/\mathcal{FS}_{p\times p}$ (respectively in $Rec_{p\times p}(\mathbf{K})/\mathcal{FS}_{p\times p}$) (see Remark 5.3 for the definition of the quotient ring $\mathbf{K}^{\mathcal{M}_{p\times p}}/\mathcal{FS}_{p\times p}$).

Remark 6.2. *$\varphi^l(A)$ with $A \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ is defined for arbitrary $(p,q) \in \mathbb{N}^2$. It can be considered as a functor from the category $\mathbf{K}^{\mathcal{M}}$ (or $Rec(\mathbf{K})$) into a category with objects indexed by $p \in \mathbb{N}$ and arrows given by matrices of size $p^l \times q^l$ having coefficients in $\mathbf{K}^{\mathcal{M}_{p\times q}}$ (respectively in $Rec(\mathbf{K})$).*

7 Presentations

We describe in this chapter two mehtods of defining elements of $\mathrm{Rec}_{p\times q}(\mathbf{K})$ by finite amounts of data. The first method, *(monoidal) presentations*, puts emphasis on the shift monoid. The second method, *recursive presentations*, is often more intuitive and sometimes more concise.

7.1 Monoidal presentations

Let $\mathcal{A} = \bigoplus_{j=1}^{a} \mathbf{K} A_j \subset \mathrm{Rec}_{p\times q}(\mathbf{K})$ be a finite-dimensional recursively closed vector space with basis $A_1, \ldots, A_a$. The isomorphism $\mathbf{K}^a \longrightarrow \mathcal{A}$ defined by $(\alpha_1, \ldots, \alpha_a) \longmapsto \sum_{j=1}^{a} \alpha_j A_j$ realizes the shift-monoid $\rho_{\mathcal{A}}(\mathcal{M}_{p\times q}) \subset \mathbf{K}^{a\times a}$ of $\mathcal{A}$ as a matrix monoid in $\mathrm{End}(\mathbf{K}^a)$. The generators $\rho_{\mathcal{A}}(s,t) \in \mathbf{K}^{a\times a}, 0 \leq s < p, 0 \leq t < q$ (with respect to the fixed basis $A_1, \ldots A_a$ of $\mathcal{A}$) are called *shift-matrices*. Their coefficients $\rho_{\mathcal{A}}(s,t)_{j,k}$ (for $1 \leq j,k \leq a$) are given by

$$\rho(s,t)A_k = \sum_{j=1}^{a} \rho_{\mathcal{A}}(s,t)_{j,k} A_j \ .$$

More generally, we can define shift-matrices $\rho_{\mathcal{A}}(s,t) \in \mathbf{K}^{d\times d}$ with respect to any finite (not necessarily linearly independent) generating set $A_1, \ldots, A_d$ of a recursively closed finite-dimensional vector-space $\mathcal{A} \subset \mathrm{Rec}_{p\times q}(\mathbf{K})$ by requiring the identity $\rho(s,t)A_k = \sum_{j=1}^{d} \rho_{\mathcal{A}}(s,t)_{j,k} A_j$. These equations define the matrices $\rho_{\mathcal{A}}(s,t)$ up to linear applications $\mathbf{K}^d \longrightarrow K$ where $K = \{(\alpha_1, \ldots, \alpha_d) \in \mathbf{K}^d \mid \sum_{j=1}^{d} \alpha_j A_j = 0\}$ is the subspace of relations among the generators $A_1, \ldots A_d \in \mathcal{A}$ (or, equivalently, the kernel of the map $(\alpha_1, \ldots, \alpha_d) \in \mathbf{K}^d \longmapsto \sum_{j=1}^{d} \alpha_j A_j$).

Proposition 7.1. *Let $\mathcal{A} = \sum_{j=1}^{d} \mathbf{K} A_j \subset Rec_{p\times q}(\mathbf{K})$ be a recursively closed vector-space with shift-matrices $\rho_{\mathcal{A}}(s,t) \in \mathbf{K}^{d\times d}$ with respect to the (not necessarily free) generating set $A_1, \ldots A_d$. We have*

$$\begin{pmatrix} A_1[s_1 \ldots s_n, t_1 \ldots, t_n] \\ \vdots \\ A_d[s_1 \ldots s_n, t_1 \ldots, t_n] \end{pmatrix} = \rho_{\mathcal{A}}(s_n, t_n)^t \cdots \rho_{\mathcal{A}}(s_1, t_1)^t \begin{pmatrix} A_1[\emptyset, \emptyset] \\ \vdots \\ A_d[\emptyset, \emptyset] \end{pmatrix}.$$

Proof For $(U, W) = (s_1 \ldots s_n, t_1 \ldots t_n) \in \mathcal{M}_{p\times q}^n$, the formula

$$A[U, W] = (\rho_{\mathcal{A}}(U, W)A)[\emptyset, \emptyset] = (\rho_{\mathcal{A}}(s_1, t_1) \cdots \rho_{\mathcal{A}}(s_n, t_n)A)[\emptyset, \emptyset]$$

implies the result by duality.

A second proof is given by the computation

$$\begin{pmatrix} A_1[s_1 \ldots s_n, t_1 \ldots, t_n] \\ \vdots \\ A_d[s_1 \ldots s_n, t_1 \ldots, t_n] \end{pmatrix} = \begin{pmatrix} \rho_{\mathcal{A}}(s_n, t_n) A_1[s_1 \ldots s_{n-1}, t_1 \ldots, t_{n-1}] \\ \vdots \\ \rho_{\mathcal{A}}(s_n, t_n) A_d[s_1 \ldots s_{n-1}, t_1 \ldots, t_{n-1}] \end{pmatrix}$$

$$= \rho_{\mathcal{A}}(s_n, t_n)^t \begin{pmatrix} A_1[s_1 \ldots s_{n-1}, t_1 \ldots, t_{n-1}] \\ \vdots \\ A_d[s_1 \ldots s_{n-1}, t_1 \ldots, t_{n-1}] \end{pmatrix}$$

and induction on n. □

Definition 7.2. *A monoidal presentation or presentation $\mathcal{P}$ of complexity d is given by the following data:*

a vector $(\alpha_1, \dots, \alpha_d) \in \mathbf{K}^d$ of d initial values,

pq shift-matrices $\rho_{\mathcal{P}}(s,t) \in \mathbf{K}^{d\times d}$ of size $d\times d$ with coefficients $\rho_{\mathcal{P}}(s,t)_{k,j}$, $1 \leq k, j \leq d$.

In the sequel, a presentation will always denote a monoidal presentation.

Proposition 7.3. *A presentation $\mathcal{P}$ of complexity d as above defines a unique set $A_1, \dots, A_d \in Rec_{p\times q}$ of d recurrence matrices such that $A_k[\emptyset,\emptyset] = \alpha_k$ and $\rho(s,t)A_k = \sum_{j=1}^d \rho_{\mathcal{P}}(s,t)_{j,k}A_j$ for $1 \leq k \leq d$ and for all $(s,t) \in \mathcal{M}^1_{p\times q}$.*

Proof For $(U,W) = (s_1 \dots s_n, t_1 \dots t_n) \in \mathcal{M}^n_{p\times q}$, we define the evaluations $A_1[U,W], \dots A_d[U,W] \in \mathbf{K}$ by

$$\begin{pmatrix} A_1[U,W] \\ \vdots \\ A_d[U,W] \end{pmatrix} = \rho_{\mathcal{P}}(s_n,t_n)^t \cdots \rho_{\mathcal{P}}(s_1,t_1)^t \begin{pmatrix} A_1[\emptyset,\emptyset] \\ \vdots \\ A_d[\emptyset,\emptyset] \end{pmatrix}.$$

The result follows from Proposition 7.1. □

In the sequel, we will often drop the subscript $\mathcal{P}$ for shift-matrices of a presentation. Thus, we identify (abusively) shift-matrices with the corresponding shift-maps, restricted to the subspace defined by the presentation.

A presentation $\mathcal{P}$ is *reduced* if the elements $A_1, \dots, A_d \in \mathrm{Rec}_{p\times q}(\mathbf{K})$ defined by $\mathcal{P}$ are linearly independent. We say that a presentation $\mathcal{P}$ *presents* (or is a presentation of) the recurrence matrix $A = A_1 \in \mathrm{Rec}_{p\times q}(\mathbf{K})$. The empty presentation of complexity 0 presents by convention the zero-element of $\mathrm{Rec}_{p\times q}(\mathbf{K})$. A presentation of $A \in \mathrm{Rec}_{p\times q}(\mathbf{K})$ with complexity $a = \dim(\overline{A}^{rec})$ is *minimal*. Every recurrence matrix $A \in \mathrm{Rec}_{p\times q}(\mathbf{K})$ has a minimal presentation $\mathcal{P}$: Complete $0 \neq A \in \mathrm{Rec}_{p\times q}(\mathbf{K})$ to a basis $A_1 = A, \dots, A_a$ of its recursive closure $\overline{A}^{rec}$. For $1 \leq k \leq a$, set $\alpha_k = A_k[\emptyset,\emptyset] \in \mathbf{K}$ and define the shift matrices $\rho(s,t)_{\mathcal{P}} \in \mathrm{End}(\mathbf{K}^a)$ by

$$\rho(s,t)A_k = \sum_{j=1}^a \rho_{\mathcal{P}}(s,t)_{j,k}A_j \ .$$

Linear independency of $A_1, \dots, A_a$ implies that the shift matrices $\rho(s,t)$ are well-defined.

In the sequel, a presentation denotes often a finite set of recurrence matrices $A_1, \dots, A_d \in \mathrm{Rec}_{p\times q}(\mathbf{K})$ spanning a recursively closed subspace $\sum_{k=1}^d \mathbf{K}A_k$ together with pq suitable shift matrices in $\mathrm{End}(\mathbf{K}^d)$ (which are sometimes omitted if they are obvious). We denote by $(A_1, \dots, A_d)[\emptyset,\emptyset] \in \mathbf{K}^d$ the corresponding initial values.

Proposition 7.4. *Given presentations $\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}$ with respect to generators $A_1, \ldots, A_d, B_1, \ldots, B_e \in Rec_{p\times q}(\mathbf{K})$, a presentation of $\mathcal{C} = \mathcal{A} + \mathcal{B}$ with respect to the generators $A_1, \ldots, A_d, B_, \ldots, B_e$ is given by shift matrices*

$$\rho_\mathcal{C}(s,t) = \begin{pmatrix} \rho_\mathcal{A}(s,t) & 0 \\ 0 & \rho_\mathcal{B}(s,t) \end{pmatrix} \in \mathbf{K}^{(d+e)\times d+e)}$$

consisting of diagonal blocs.

The proof is obvious.

Proposition 7.5. *Given presentations $\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{B}$ with generators $A_1, \ldots, A_d \in Rec_{p\times r}(\mathbf{K}), B_1, \ldots, B_e \in Rec_{r\times q}(\mathbf{K})$, a presentation $\mathcal{P}_\mathcal{C}$ of the recursively closed vector space $\mathcal{C} \subset Rec_{p\times q}(\mathbf{K})$ with respect to the generators $C_{ij} = A_i B_j$ of all products among generators is given by the initial values*

$$C_{ij}[\emptyset,\emptyset] = A_i[\emptyset,\emptyset]B_j[\emptyset,\emptyset]$$

and shift matrices $\rho_\mathcal{C}(s,t) \in \mathbf{K}^{de\times de}$ with coefficients

$$\rho_\mathcal{C}(s,t)_{kl,ij} = \sum_{u=1}^{r} \rho_\mathcal{A}(s,u)_{k,i}\rho_\mathcal{B}(u,t)_{l,j} \ .$$

Proof There is nothing to prove for the initial values.
For the shift matrices, we have

$$\begin{aligned} \rho_\mathcal{C}(s,t)C_{ij} &= \rho(s,t)(A_iB_j) = \sum_{u=1}^{r} \rho_\mathcal{A}(s,u)A_i\rho_\mathcal{B}(u,t)B_j \\ &= \sum_{u=1}^{r}\sum_{k=1}^{d}\sum_{l=1}^{e} \rho_\mathcal{A}(s,u)_{k,i}\rho_\mathcal{B}(u,t)_{l,j}A_kB_l \\ &= \sum_{k=1}^{d}\sum_{l=1}^{e}\left(\sum_{u=1}^{r} \rho_\mathcal{A}(s,u)_{k,i}\rho_\mathcal{B}(u,t)_{l,j}\right)C_{kl} \end{aligned}$$

which ends the proof. □

Remark 7.6. *The presentation $\mathcal{P}_\mathcal{C}$ given by proposition 7.5 is in general not reduced, even if $\mathcal{P}_\mathcal{A}$ and $\mathcal{P}_\mathcal{B}$ are reduced presentations. The reason for this are (possible) multiplicities (up to isomorphism) of submonoids in the shift monoid $\rho_\mathcal{C}(\mathcal{M}_{p\times q})$.*

Proposition 7.7. *Given a presentation $\mathcal{P}$ of d recurrence matrices $A_1, \ldots A_d \in Rec_{p\times q}(\mathbf{K})$ spanning a recursively closed subspace, a presentation $\mathcal{P}^t$ of the transposed recurrence matrices $A_1^t, \ldots A_d^t \in Rec_{q\times p}(\mathbf{K})$ is given by the same initial data $(A_1^t, \ldots A_d^t)[\emptyset,\emptyset] = (A_1, \ldots A_d)[\emptyset,\emptyset]$ and the same shift matrices $\tilde{\rho}(t,s) = \rho(s,t)$, up to "transposition" of their labels.*

Proof Use an easy induction on l for the restricted functions $A_1[\mathcal{M}^l_{p\times q}], \ldots, A_d[\mathcal{M}^l_{p\times q}]$.
□

Remark 7.8. *Shift-matrices of transposed recurrence matrices are identical to the original shift-matrices (and are not to be transposed), only their labels are rearranged!*

Remark 7.9. *Endowing $\mathcal{M}_{p\times q}$ with a complete order, the recursive closure $\overline{A}^{rec}$ of an element $A \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ has a unique basis of the form $\rho(U_1, W_1)A, \rho(U_2, W_2)A, \ldots$ where the word $(U_j, W_j) \in \mathcal{M}_{p\times q}$ (if it exists) is inductively defined as the smallest element such that $\rho(U_1, W_1)A, \ldots, \rho(U_j, W_j)A$ are linearly independent.*

Remark 7.10. *The stable complexity $dim_{\mathcal{FS}}(\pi_{\mathcal{FS}}(\overline{A}^{rec}))$ introduced in chapter 5 of an element $A \in Rec_{p\times q}(\mathbf{K})$ equals $dim(\overline{A}^{rec}) - dim(\overline{A}^{rec} \cap \mathcal{FS})$ where $\overline{A}^{rec} \cap \mathcal{FS}$ is the maximal recursively closed subspace of $\overline{A}^{rec}$ with nilpotent action of $\rho(\mathcal{M}_{p\times q})$.*

7.2 Recursive presentations

We start by defining recursive symbols which are the key ingredients for recursive presentations.

Given a field (or ring) $\mathbf{K}$, fixed in the sequel, the set

$$\mathcal{RS}_{p\times q}(A_1, \ldots, A_d) = \cup_{d=0}^{\infty} \mathcal{RS}^{\leq d}_{p\times q}(A_1, \ldots, A_d)$$

of recursive $p \times q-$symbols over $A_1, \ldots, A_d$ is recursively defined as follows: $\mathcal{RS}^{\leq -1}_{p\times q}(A_1, \ldots, A_d) = \emptyset$ and $\mathcal{RS}^{\leq d}_{p\times q}(A_1, \ldots, A_d)$ is the set of symbols (ρ, R) where $\rho \in \mathbf{K}$ is a constant and R is a matrix of size $p \times q$ with coefficients $R_{s,t}, 0 \leq s < p, 0 \leq t < q$ in the free vector space spanned by $A_1, \ldots, A_d$ and elements of $\mathcal{RS}^{\leq d-1}_{p\times q}(A_1, \ldots, A_d)$.

A symbol (ρ, R) has *depth* $d(\rho, R) = d$ if $(\rho, R) \in \mathcal{RS}^{d}_{p\times q}(A_1, \ldots, A_d) = \mathcal{RS}^{\leq d}_{p\times q}(A_1, \ldots, A_d) \setminus \mathcal{RS}^{\leq d-1}_{p\times q}(A_1, \ldots, A_d)$.

Examples An example of a recursive $2\times 2-$symbol of depth 2 over A, B (with groundfield $\mathbb{Q}$) is for instance given by

$$(2, \begin{pmatrix} 3B - 2A & -A + (-1, \begin{pmatrix} B & A+B \\ -A & 2A-B \end{pmatrix}) \\ 5A - B + (0, \begin{pmatrix} A & B \\ -B & (\rho, R) \end{pmatrix}) & 5A - 3B \end{pmatrix})$$

where $(\rho, R) = (7, \begin{pmatrix} 2A - B & 3B \\ -A + B & 5A + B \end{pmatrix}) \in \mathcal{RS}^0_{2\times 2}(A, B)$.

The expresssion $R = (1, (\rho, -A + R))$ defines however no recursive $1 \times 1-$symbol over A.

Definition A *recursive presentation* for $A_1, \dots, A_d$ is given by identities

$$A_1 = (\rho_1, R_1),\ A_2 = (\rho_2, R_2), \dots, A_d = (\rho_d, R_d)$$

where $(\rho_1, R(1)), \dots, (\rho_d, R(d)) \in \mathcal{RS}_{p\times q}(A_1, \dots, A_d)$ are recursive $p\times q-$symbols over $A_1, \dots, A_d$.

With a hopefully understandable abuse of notation we say that $B, A_1, \dots, A_d \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ is a solution to the equation $B = (\rho, R)$ with (ρ, R) a $p \times q-$symbol over $A_1, \dots, A_d$ (here is the abuse) if B satisfies $B[\emptyset, \emptyset] = \rho$ and we have recursively the identities $\rho(s,t)B = R_{s,t}$ for $0 \le s < p, 0 \le t < q$.

Proposition 7.11. *The identities of a recursive presentation for $A_1, \dots, A_d$ have a unique common solution $A_1, \dots, A_d \in \mathbf{K}^{\mathcal{M}_{p\times q}}$.*

Moreover, the vector space spanned by

$$\rho(\mathcal{M}_{p\times q}^{\le d(\rho_1, R_1)})A_1, \dots, \rho(\mathcal{M}_{p\times q}^{\le d(\rho_d, R_d)})A_d$$

is recursively closed and we have thus $A_1, \dots, A_d \in Rec_{p\times q}(\mathbf{K})$.

Remark 7.12. *The vector space spanned by $A_1, \dots, A_d$ is generally not recursively closed except if all symbols $(\rho_1, R_1), \dots, (\rho_d, R_d)$ are of depth 0.*

Proof of Proposition 7.11 We have obviously $A_k[\emptyset, \emptyset] = \rho_k$ for $k = 1, \dots, d$. An easy induction on l shows now that the matrices

$$A_1[\mathcal{M}_{p\times q}^{l+1}], \dots, A_d[\mathcal{M}_{p\times q}^{l+1}]$$

are uniquely defined by

$$A_1[\mathcal{M}_{p\times q}^{\le l}], \dots, A_d[\mathcal{M}_{p\times q}^{\le l}]\ .$$

The second part of the Proposition is obvious. □

Recursive presentations of depth 0 are particularly nice: Given such a recursive presentation $A_1 = (\rho_1, R(1)), \dots, A_d = (\rho_d, R(d))$, the subspace spanned by $A_1, \dots, A_d \subset \mathrm{Rec}_{p\times q}(\mathbf{K})$ is already recursively closed and the identity

$$R(k)_{s,t} = \rho(s,t)A_k = \sum_{j=1}^{d} \rho(s,t)_{j,k} A_j$$

shows that the matrices $R(1), \dots, R(d)$ encode the same information as shift matrices with respect to the generating set $A_1, \dots, A_d$. In particular, every element $A \in \mathrm{Rec}_{p\times q}(\mathbf{K})$ of complexity d admits a recursive presentation of depth 0 for $A_1 = A, \dots, A_d$ a basis of $\overline{A}^{rec}$.

Remark 7.13. *Since the importance of the role played by the shift monoid is not apparent in the definition of recursive presentations they are less natural than monoidal presentations from a theoretical point of view. They are however often more "intuitive" and more compact (see Remark 7.12 for the reason) than monoidal presentations. Moreover, several interesting examples are defined in a natural way in terms of recursion matrices while a definition using shift matrices looks more artificial.*

8 Saturation

The aim of this section is to present finiteness results for computations in the category Rec($\mathbf{K}$). More precisely, given presentations of two suitable recurrence matrices A, B, we show how to compute a presentation of the sum $A+B$, or of the matrix-product AB (whenever defined) of A and B in a finite number of steps.

For a vector space $\mathcal{A} \subset \mathbf{K}^{\mathcal{M}_{p\times q}}$ we denote by $\mathcal{A}[\mathcal{M}_{p\times q}^{\leq l}] \subset \mathbf{K}^{\mathcal{M}_{p\times q}^{\leq l}}$ the image of $\mathcal{A}$ under the projection $A \longmapsto A[\mathcal{M}_{p\times q}^{\leq l}]$ associating to $A \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ its restriction $A[\mathcal{M}_{p\times q}^{\leq l}] \in \mathbf{K}^{\mathcal{M}_{p\times q}^{\leq l}}$ to the subset of all $\frac{(pq)^{l+1}-1}{pq-1}$ words of length at most l in $\mathcal{M}_{p\times q}$.

Definition 8.1. *The saturation level of a non-zero vector space $\mathcal{A} \subset \mathbf{K}^{\mathcal{M}_{p\times q}}$ is the smallest natural integer $N \geq 0$, if it exists, for which the obvious projection*

$$\mathcal{A}[\mathcal{M}_{p\times q}^{\leq N+1}] \longrightarrow \mathcal{A}[\mathcal{M}_{p\times q}^{\leq N}]$$

is an isomorphism. A vector-space $\mathcal{A}$ has saturation level ∞ if such an integer N does not exist and the trivial vector space $\{0\}$ has saturation level -1. The saturation level of $A \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ is the saturation level of its recursive closure $\overline{A}^{rec}$.

Proposition 8.2. *Let $\mathcal{A} \subset \mathbf{K}^{\mathcal{M}_{p\times q}}$ be a recursively closed vector space of finite saturation level N. Then $\mathcal{A}$ and $\mathcal{A}[\mathcal{M}_{p\times q}^{\leq N}]$ are isomorphic.*

In particular, $\mathcal{A}$ is of finite dimension and contained in $Rec_{p\times q}(\mathbf{K})$.

Corollary 8.3. *A finite set $A_1, \ldots, A_d \subset Rec_{p\times q}(\mathbf{K})$ spanning a recursively closed subspace $\mathcal{A} = \sum_{j=1}^{d} \mathbf{K}A_j \subset Rec_{p\times q}(\mathbf{K})$ is linearly independent if and only if $A_1[\mathcal{M}_{p\times q}^{\leq N}], \ldots, A_d[\mathcal{M}_{p\times q}^{\leq N}] \in \mathbf{K}^{\mathcal{M}_r^{\leq N}}$ are linearly independent where $N \leq dim(\mathcal{A}) - 1 < d$ denotes the saturation level of $\mathcal{A}$.*

Proof of Proposition 8.2 We denote by $K_l = \{A \in \mathcal{A} \mid A[U,W] = 0 \text{ for all } (U,W) \in \mathcal{M}_{p\times q}^{\leq l}\} \subset \mathcal{A}$ the kernel of the natural projection $\mathcal{A} \longrightarrow \mathcal{A}[\mathcal{M}_{p\times q}^{\leq l}]$. We have

$$K_{N+1} = \{A \in K_N \mid \rho(\mathcal{M}_{p\times q}^{1})A \subset K_N\} = K_N$$

which shows the equality $\rho(\mathcal{M}_{p\times q})K_N = K_N$. For $A \in K_N \subset K_0$ we have thus $A[U,W] = \rho(U,W)A[\emptyset,\emptyset] = 0$ for all $(U,W) \in \mathcal{M}_{p\times q}$ which implies $A = 0$ and shows the result. □

Corollary 8.3 follows immediately.

Remark 8.4. *The proof of Proposition 8.2 shows the inequality $N+1 \leq dim(\overline{A}^{rec})$ for the saturation level N of $A \in Rec_{p\times q}(\mathbf{K})$. Equality is achieved eg. for the recurrence matrix defined by $A[U,W] = 1$ if $(U,W) \in \mathcal{M}_{p\times q}^{N}$ and $A[U,W] = 0$ otherwise.*

9 Algorithms

It is easy to extract a minimal presentation from a presentation $\mathcal{P}$ of a recurrence matrix $A \in \mathrm{Rec}_{p\times q}(\mathbf{K})$: Compute the saturation level N of the recursively closed space $\mathcal{A} = \sum_{j=1}^{d} \mathbf{K}A_j$ defined by the presentation $\mathcal{P}$. This allows to detect linear dependencies among $A_1, \dots A_d$.

Using Proposition 7.1, these computations can be done in polynomial time with respect to the complexity d of the presentation $\mathcal{P}$ for A.

The algorithm described in subsection 9.1 is useful for computing the saturation level and a few other items attached to a recurrence matrix.

Adding two recurrence matrices A, B in $\mathrm{Rec}_{p\times q}(\mathbf{K})$ is now easy: given presentations $A_1, \dots, A_a$ and $B_1, \dots, B_b$ of A and B, one can write down a presention $C_1 = A_1 + B_1, C_2 = A_2, \dots, C_a = A_a, C_{a+1} = B_1, C_{a+2} = B_2, \dots, C_{a+b} = B_b$ of $A + B$, eg. by using Proposition 7.4.

Similarly, using Proposition 7.5 a presentation $A_1, \dots, A_a$ and $B_1, \dots, B_b$ of $A \in \mathrm{Rec}_{p\times r}(\mathbf{K})$ and $B \in \mathrm{Rec}_{r\times q}(\mathbf{K})$ yields a presentation $C_{11} = A_1B_1, \dots, C_{ab} = A_aB_b$ of $C = C_{11} = A_1B_1$.

Remark 9.1. *Computing a presentation of AB for $A \in Rec_{p\times r}(\mathbf{K})$, $B \in Rec_{r\times q}(\mathbf{K})$ is also possible using the matrices $(A_iB_j)[\mathcal{M}^l_{p\times q}]$ for l up to the saturation level N of $\overline{A}^{rec}\overline{B}^{rec}$. This method is however quickly infeasible for for $r \geq 2$ and N not too small. Using Proposition 7.5 as suggested above, one gets around this difficulty and obtains essentially polynomial algorithms for computations in the algebra $Rec_{p\times p}(\mathbf{K})$.*

9.1 An algorithm for computing the saturation level

Proposition 9.2. *Given a recursively closed vector space $\mathcal{A} \subset Rec_{p\times q}(\mathbf{K})$ of dimension a, there exists a subset $\mathcal{S} = \{S_1, \dots, S_a\} \subset \mathcal{M}_{p\times q}$ of words such that the restriction*

$$X \longmapsto X[\mathcal{S}] = (X[S_1], \dots, X[S_a])$$

of X onto $\mathcal{S}$ induces an isomorphism between $\mathcal{A}$ and $\mathbf{K}^a$. Moreover, one can choose $\mathcal{S}$ in order to have

$$\mathcal{S} =\subset (\emptyset, \emptyset) \cup \mathcal{S}\mathcal{M}^1_{p\times q} \ .$$

Proof The first part of the proof is obvious. In order to have the inclusion $\mathcal{S} \subset (\emptyset, \emptyset) \cup \mathcal{S}\mathcal{M}^1_{p\times q}$ suppose that the evaluation of X on $\{S_1, \dots, S_{k_j}\} \subset \mathcal{S} \cap \mathcal{M}^{\leq j}_{p\times q}$ induces a bijection between $\mathcal{A}[\mathcal{M}^{\leq j}_{p\times q}]$ and $\mathbf{K}^{k_j}$ for $j \leq N$ with N the saturation level of $\mathcal{A}$. Choosing a presentation $A_1, \dots, A_d, \rho(s,t) \in \mathbf{K}^{d\times d}, 0 \leq s < p, 0 \leq t < q$ and writing

$$\rho^t(\overline{S}) = \rho^t(s_{j_n}, t_{j_n}) \cdots \rho^t(s_{j_1}, t_{j_1})$$

for $S = (s_{j_1} \cdots s_{j_n}, t_{j_1}, \cdots t_{j_n}) \in \mathcal{M}^n_{p\times q}$, Proposition 7.1 shows that the k_j vectors

$$\rho^t(S_j)\left((A_1, \ldots, A_d)[\emptyset, \emptyset]\right)^t, 1 \leq j \leq k_j$$

form a basis of the vector space spanned by

$$\rho^t(\mathcal{M}^{\leq j}_{p\times q})\left((A_1, \ldots, A_d)[\emptyset, \emptyset]\right)^t \subset \mathbf{K}^d \ .$$

This implies easily the result. □

Proposition 9.2 implies that the following algorithme computes the saturation level of a presentation.

Input: a finite set $A_1, \ldots, A_d \subset \mathrm{Rec}_{p\times q}(\mathbf{K})$ spanning a recursively closed subspace (eg. given by a finite presentation).

Set $a := 0,\ N := -1,\ \mathcal{S} := \emptyset$,

If $(A_1, \ldots, A_d)[\emptyset, \emptyset] = (0, \ldots, 0)$ then $a := 0,\ N := -1,\ \mathcal{S} := \emptyset$, stop else $a := 1,\ k := 1,\ u := 1,\ N := 0,\ S_1 := \{(\emptyset, \emptyset)\}$ endif

For $j \in \{k, k+1, \ldots, u\}$ do

For $(s, t) \in \mathcal{M}^1_{p\times q}$ do

If $\rho^t(s, t)\rho^t(\overline{S}_j)\left((A_1, \ldots, A_d)[\emptyset, \emptyset]\right)^t$ is not in the linear span of $\rho^t(\overline{S})\left((A_1, \ldots, A_d)[\emptyset, \emptyset]\right)^t$, $S \in \{S_1, \ldots, S_u\}$ then $a := a+1,\ S_a := S_j(s, t)$ endif (where $\overline{(s_1 s_2 \ldots s_m, t_1 t_2 \ldots t_m)} = (s_m s_{m-1} \ldots s_2 s_1, t_m t_{m-1} \ldots t_2 t_1)$)

If $a = d$ then stop endif

endfor

If $a = u$ then stop else $k := u + 1, u := a, N := N + 1$ endif

endfor

The final value of a in this algorithm is the dimension of the recursively closed vector space $\mathcal{A} = \sum_{j=1}^d \mathbf{K}A_j$, the final value of N is the saturation level $\mathcal{A}$ and the set $\mathcal{S} = \{S_1, \ldots, S_a\}$ satisfies the conditions of Proposition 9.2.

10 Topologies and a metric

For $\mathbf{K}$ a topological field, the vector space $\mathbf{K}^{\mathcal{M}_{p\times q}}$ carries two natural "obvious" topologies.

The first one is the product topology on $\prod_{l=0}^\infty \mathbf{K}^{\mathcal{M}^l_{p\times q}}$. It is defined as the coarsest topology for which all projections $A \longmapsto A[\mathcal{M}^l_{p\times q}]$ are continuous. Its open subsets are generated by $\mathcal{O}_{\leq m} \times \prod_{l>m} \mathbf{K}^{\mathcal{M}^l_{p\times q}}$ with $\mathcal{O}_{\leq m} \subset \mathbf{K}^{\mathcal{M}^{\leq m}_{p\times q}}$ open for the natural topology on the finite-dimensional vector space $\mathbf{K}^{\mathcal{M}^{\leq m}_{p\times q}}$.

The second topology is the strong topology or box topology. Its open subsets are generated by $\prod_{l=0}^\infty \mathcal{O}_l$, with $\mathcal{O}_l \subset \mathbf{K}^{\mathcal{M}^l_{p\times q}}$ open for all $l \in \mathbb{N}$.

The restrictions to $\mathrm{Rec}_{p\times q}(\mathbf{K})$ of both topologies are not very interesting: The product topology is very coarse, the strong topology is too fine: it gives rise to the discrete topology on the quotient $\mathrm{Rec}_{p\times q}(\mathbb{C})/\mathcal{F}S_{p\times q}$ considered in Chapter 5.

For a recurrence matrix $A \in \mathrm{Rec}_{p\times q}(\mathbb{C})$ defined over $\mathbb{C}$ we set

$$\| A \|_\infty^\infty = \mathrm{limsup}_{l\to\infty}\mathrm{sup}_{(U,W)\in\mathcal{M}^l_{p\times q}}|A[U,W]|^{1/l}.$$

Remark 10.1. *$\| A \|_\infty^\infty$ equals the supremum over $\mathcal{M}_{p\times q}$ of the numbers $\sigma(S,T)^{1/l}$ where $(S,T) \in \mathcal{M}^l_{p\times q}$ has length l and where $\sigma(S,T)$ is the spectral radius (largest modulus among eigenvalues) of the shift matrix $\rho_{\overline{A}^{rec}}(S,T)$ with respect to a minimal presentation of A. We have thus the inequality $\sigma_{s,t} \leq \| A \|_\infty^\infty$ (which is in general strict) where $\sigma_{s,t}$ are the spectral radii of shift-matrices $\rho(s,t) \in \mathbb{C}^{d\times d}$ associated to a minimal presentation of A.*

Proposition 10.2. *The application $A \longmapsto \| A \|_\infty^\infty$ defines a metric on $\mathbb{C}^*\backslash Rec_{p\times q}(\mathbb{C})/\mathcal{FS}_{p\times q}$ such that*

$$\| A + \lambda B \|_\infty^\infty \leq sup(\| A \|_\infty^\infty, \| B \|_\infty^\infty)$$

for $\lambda \in \mathbb{C}^*, A, B \in Rec_{p\times q}(\mathbb{C})$ (with equality holding generically) and

$$\| AB \|_\infty^\infty \leq r \| A \|_\infty^\infty \| B \|_\infty^\infty$$

for $A \in Rec_{p\times r}(\mathbb{C}), B \in Rec_{r\times q}(\mathbb{C})$.

Proof The proof of the first inequality is easy and left to the reader. The second inequality follows from

$$\| AB \|_\infty \leq \| \tilde{A}\tilde{B} \|_\infty = r \| A \|_\infty \| B \|_\infty$$

where $\tilde{A}, \tilde{B}$ are of complexity 1 and have coefficients $\tilde{A}[S,T] = \| A \|_\infty^l$ for $(S,T) \in \mathcal{M}^l_{p\times r}$ and $\tilde{B}[S,T] = \| B \|_\infty^l$ for $(S,T) \in \mathcal{M}^l_{r\times q}$ which depend only on the length of (S,T). The inequality $\| AB \|_\infty \leq \| \tilde{A}\tilde{B} \|_\infty$ follows now easily from the equalities $\| A \|_\infty = \| \tilde{A} \|_\infty$, $\| B \|_\infty = \| \tilde{B} \|_\infty$ and from the observation that all coefficients of $\tilde{A}$ (respectively $\tilde{B}$) are upper bounds for the corresponding coefficients of A (respectively B). □

Remark 10.3. *The set of all elements $A \in \mathbb{C}^{\mathcal{M}_{p\times q}}, p, q \in \mathbb{N}$, such that $\| A \|_\infty^\infty < \infty$ form a subcategory containing $Rec(\mathbb{C})$ of $\mathbb{C}^{\mathcal{M}}$.*

The vector space $Rec_{p\times q}(\mathbb{C})$ can be normed by

$$\| A \| = \sum_{l=0}^{\infty} \frac{\| A[\mathcal{M}^l_{p\times q}] \|_\infty}{l!}$$

where $\| A[\mathcal{M}^l_{p\times q}] \|_\infty$ denotes the largest absolute value of all coefficients $A[U,W], (U,W) \in \mathcal{M}^l_{p\times q}$. However, matrix-multiplication is unfortunately not continuous for this norm.

This norm has different obvious variations:

The factorials $l!$ of the denominators can be replaced by any other sequence $s_0, s_1, \ldots$ of strictly positive numbers such that $lim_{n\to\infty}\lambda^n/s_n = 0$ for all $\lambda > 0$.

The sup-norm $\| A[\mathcal{M}^l_{p\times q}] \|_\infty$ can be replaced by many other "reasonable" norms (like l_1 or l_2 norms) on $\mathbb{C}^{\mathcal{M}^l_{p\times q}}$.

It would be interesting to find a norm on the algebra $Rec_{p\times p}(\mathbb{C})$ for which matrix products are continuous.

11 Criteria for non-recurrence matrices

This short section lists a few easy properties (which have sometimes obvious generalizations, eg by replacing $\mathbb{Q}$ with a number field) of recurrence matrices and non-recurrence matrices in $\mathbf{K}^{\mathcal{M}_{p\times q}}$. Proofs are straightforward and left to the reader or only sketched.

Proposition 11.1. *An element $A \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ is not in $Rec_{p\times q}(\mathbf{K})$ if and only if there exist two sequences $(S_i, T_i), (U_j, W_j) \in (\mathcal{M}_{p\times q})^{\mathbb{N}}$ such that for all $n \in \mathbb{N}$, the $n \times n$ matrix with coefficients $c_{i,j} = \rho(S_i, T_i)A[U_j, W_j] = A[U_jS_i, W_jT_i], 1 \le i, j \le n$, has non-zero determinant.*

Proposition 11.2. *Consider $A \in Rec_{p\times q}(\mathbb{C})$. Then there exists a constant $C \ge 0$ such that $|A[U, W]| \le C^{n+1}$ for all $(U, W) \in \mathcal{M}^n_{p\times q}$.*

Proof Given a finite presentation $A_1, \ldots A_d$ with shift matrices $\rho(s, t) \in \mathbb{C}^{d\times d}$, choose $C \ge 0$ such that $C \ge |A_i[\emptyset, \emptyset]|$ for $i = 1, \ldots, d$ and $C \ge d \| \rho(s, t) \|_\infty$ for $0 \le s < p, 0 \le t < q$. □

Remark 11.3. *Associate to $A \in \mathbb{C}^{\mathcal{M}_p}$ the formal power series*

$$f_A = A[\emptyset] + \sum_{n=1}^{\infty} \sum_{0\le u_1,\ldots,u_n<p} A[u_1 \ldots u_n] Z_{u_1} \cdots Z_{u_n}$$

in p commuting variables $Z_0, \ldots, Z_{p-1}$. Proposition 11.2 shows that f_A defines a holomorphic function in a neighbourhood of $(0, \ldots, 0) \in \mathbb{C}^p$ if $A \in Rec_p(\mathbb{C})$.

Proposition 11.4. *The set of values $\{A[U, W] \mid (U, W) \in \mathcal{M}_{p\times q}\}$ of a recurrence matrix $A \in Rec_{p\times q}(\mathbf{K})$ is contained in a subring $\tilde{\mathbf{K}} \subset \mathbf{K}$ which is finitely generated (as a ring).*

Proof The subring $\tilde{\mathbf{K}}$ generated by all values and coefficients involved in a finite presentation $A = A_1, \ldots, A_d$ of A works. □

Proposition 11.5. *For $A \in Rec_{p\times q}(\mathbb{Q})$, there exists a natural integer N such that $N^{n+1}A[U, W] \in \mathbb{Z}$ for all $(U, W) \in \mathcal{M}^n_{p\times q}$.*

Proof Given a presentation $A_1, \ldots, A_d \in \mathrm{Rec}_{p\times q}(\mathbb{Q})$, a non-zero integer $N \in \mathbb{N}$ such that $N(A_1, \ldots, A_d)[\emptyset, \emptyset] \in \mathbb{Z}^d$ and $N\rho(s, t) \in \mathbb{Z}^{d\times d}$ for all $0 \le s < p, 0 \le t < q$ works. □

12 Diagonal and lower triangular subalgebras in $\mathbf{K}^{\mathcal{M}_{p\times p}}$ and $\mathrm{Rec}_{p\times p}(\mathbf{K})$

This section describes a maximal commutative subalgebra formed by diagonal elements, the center (which is contained as a subalgebra in the diagonal algebra) and the lower triangular subalgebra of the algebras $\mathbf{K}^{\mathcal{M}_{p\times p}}$ and $\mathrm{Rec}_{p\times p}(\mathbf{K})$.

12.1 The diagonal subalgebra and the center

An element $A \in \mathbf{K}^{\mathcal{M}_{p\times p}}$ is *diagonal* if $A[U,W] = 0$ for all $(U,W) \in \mathcal{M}_{p\times p}$ such that $U \neq W$. We denote by $\mathcal{D}_p(\mathbf{K}) \subset \mathbf{K}^{\mathcal{M}_{p\times p}}$ the vector space of all diagonal elements. It is easy to show that $\mathcal{D}_p(\mathbf{K})$ is a commutative algebra which is isomorphic to the function ring underlying $\mathbf{K}^{\mathcal{M}_p}$.

The center of the algebra $\mathbf{K}^{\mathcal{M}_{p\times p}}$ is the subalgebra $\mathcal{C}_p(\mathbf{K}) \subset \mathcal{D}_p(\mathbf{K})$ formed by all diagonal matrices A with diagonal coefficients $A[U,U]$ depending only on the length l of $(U,U) \in \mathcal{M}^l_{p\times p}$. For $p \geq 1$, the map

$$\mathcal{C}_p(\mathbf{K}) \ni C \longmapsto (C[\emptyset,\emptyset], C[0,0], C[0^2,0^2], \dots)$$

defines an isomorphism between $\mathcal{C}_p(\mathbf{K})$ and the algebra $\mathbf{K}^{\mathcal{M}_{1\times 1}}$ corresponding to all sequences $\mathbb{N} \longrightarrow \mathbf{K}$, endowed with the coefficient-wise (or Hadamard) product.

We denote by $\mathcal{D}_{p-rec}(\mathbf{K}) = \mathcal{D}_p(\mathbf{K}) \cap \mathrm{Rec}_{p\times p}(\mathbf{K})$ and by $\mathcal{C}_{p-rec}(\mathbf{K}) = \mathcal{C}_p(\mathbf{K}) \cap \mathrm{Rec}_{p\times p}(\mathbf{K})$ the subalgebras of $\mathcal{D}_p(\mathbf{K})$ and $\mathcal{C}_p(\mathbf{K})$ formed by all recurrence matrices. Associating to $C \in \mathcal{C}_{p-rec}(\mathbf{K})$ the generating series $\sum_{l=0}^{\infty} C[0^l,0^l]z^l$ (where $0^0 = \emptyset$) yields an isomorphism between the algebra $\mathcal{C}_{p-rec}(\mathbf{K})$ and the algebra $\mathrm{Rec}_{1\times 1}(\mathbf{K})$ corresponding to the vector-space of rational functions in one variable without singularity at the origin. The product is the coefficientwise product $\left(\sum_{l=0}^{\infty} \alpha_l z^l\right) \cdot \left(\sum_{l=0}^{\infty} \beta_l z^l\right) = \left(\sum_{l=0}^{\infty} \alpha_l \beta_l z^l\right)$ of the corresponding series-expansions.

Diagonal and central recurrence matrices in $\mathrm{Rec}_{p\times p}(\mathbf{K})$ can be characterized by the following result.

Proposition 12.1. *(i) A recurrence matrix $A \in Rec_{p\times p}(\mathbf{K})$ is diagonal if and only if it can be given by a presentation with shift-matrices $\rho(s,t) = 0$ whenever $s \neq t, 0 \leq s,t < p$.*

(ii) A diagonal recurrence matrix $A \in Rec_{p\times p}(\mathbf{K})$ is central if and only if it can be given by a presentation with shift-matrices $\rho(s,t) = 0$ whenever $s \neq t, 0 \leq s,t < p$ and $\rho(s,s) = \rho(0,0)$ for all $s, 0 \leq s < p$.

We leave the easy proof to the reader.

Remark 12.2. *The vector spaces $\mathcal{D}_p, \mathcal{D}_{p-rec}, \mathcal{C}_p, \mathcal{C}_{p-rec}$ are also (non-commutative for $\mathcal{D}_p$ and $\mathcal{D}_{p-rec}$, if $p > 1$) algebras for the convolution-product.*

12.2 Lower triangular subalgebras

Lower (or upper) triangular elements in $\mathbf{K}^{\mathcal{M}_{p\times p}}$ can be defined using the bijection

$$(u_1 \dots u_l, w_1 \dots w_l) \longmapsto (\sum_{j=1}^{l} u_j p^{j-1}, \sum_{j=1}^{l} w_j p^{j-1})$$

between $\mathcal{M}^l_{p\times p}$ and $\{0, \dots, p^l - 1\} \times \{0, \dots, p^l - 1\}$. More precisely, $A \in \mathbf{K}^{\mathcal{M}_{p\times p}}$ is *lower triangular* if for all $l \in \mathbb{N}$ the equality $A[U, W] = 0$ holds for $(U, W) = (u_1 \dots u_l, w_1 \dots w_l) \in \mathcal{M}_{p\times p}$ such that $\sum_{j=1}^{l} u_j p^{j-1} < \sum_{j=1}^{l} w_j p^{j-1}$.

Similarly, an element $A \in \mathbf{K}^{\mathcal{M}_{p\times p}}$ is *upper triangular* if the transposed element A^t (defined by $A^t[U, W] = A[W, U]$) is lower triangular.

We denote by $\mathcal{L}_p(\mathbf{K}) \subset \mathbf{K}^{\mathcal{M}_{p\times p}}$ the vector-space of all lower triangular elements in $\mathbf{K}^{\mathcal{M}_{p\times p}}$. It is easy to check that $\mathcal{L}_p(\mathbf{K})$ is closed under the matrix-product and the vector space $\mathcal{L}_p(\mathbf{K})$ is thus an algebra. The subspace of all convergent elements in $\mathcal{L}_p(\mathbf{K})$ is also closed under the matrix-product and forms a subalgebra.

A lower triangular matrix $A \in \mathcal{L}_p(\mathbf{K})$ is *unipotent* if $A[U, U] = 1$ for all "diagonal words" $(U, U) \in \mathcal{M}_{p\times p}$ and *strictly lower triangular* if $A[U, U] = 0$ for all diagonal words $(U, U) \in \mathcal{M}_{p\times p}$. The subset $\mathcal{N}_p(\mathbf{K})$ of all lower strictly-triangular matrices is a two-sided ideal in $\mathcal{L}_p(\mathbf{K})$. The associated quotient $\mathcal{L}_p(\mathbf{K})/\mathcal{N}_p(\mathbf{K})$ is isomorphic to the commutative algebra $\mathcal{D}_p(\mathbf{K})$ of diagonal elements. Unipotent lower triangular matrices form a multiplicative subgroup and correspond to the (multiplicative) identity of the quotient algebra $\mathcal{L}_p(\mathbf{K})/\mathcal{N}_p(\mathbf{K})$.

The following proposition is useful for recognizing triangular recurrence matrices:

Proposition 12.3. *A recurrence matrix $A \in End_{p-rec}(\mathbf{K})$ is triangular if and only if it admits a presentation of the form $A = A_1, \dots, A_k, A_{k+1}, \dots, A_a$ such that $\rho(s, s)A_1, \dots, \rho(s, s)A_k \in \sum_{j=1}^{k} A_j$ for $0 \le s < p$ and $\rho(s, t)A_1, \dots, \rho(s, t)A_k = 0$ for $0 \le s < t < p$.*

Such a recurrence matrix A is unipotent if and only if it admits a presentation as above which satisfies moreover $A_1[\emptyset, \emptyset] = 1, A_2[\emptyset, \emptyset] = \dots = A_k[\emptyset, \emptyset] = 0$, $\rho(s, s)A_1$ is in the affine space $A_1 + \sum_{j=2}^{k} \mathbf{K}A_j$ for $0 \le s < p$, and $\rho(s, s)A_h \in \sum_{j=2}^{k} \mathbf{K}A_j$ for $2 \le h \le k$ and $0 \le s < p$.

Proof These conditions are clearly sufficient since they imply by an easy induction on l that all matrices $A_1[\mathcal{M}^l_{p\times p}], \dots, A_k[\mathcal{M}^l_{p\times p}]$ are lower triangular. In the unipotent case they imply that $A_1[\mathcal{M}^l_{p\times p}]$ is unipotent and that $A_2[\mathcal{M}^l_{p\times p}], \dots, A_k[\mathcal{M}^l_{p\times p}]$ are strictly lower triangular for all $l \in \mathbb{N}$.

In order to see that they are also necessary, consider a basis $A_1 = A, A_k$ of the vector space $\mathcal{T} \subset \overline{A}^{rec}$ spanned by all lower triangular recurrence matrices of the form $\rho(S, S)A$, with $(S, S) \in \mathcal{M}_{p\times p}$. In the unipotent case,

all diagonal coefficients of an element $X \in \mathcal{T}$ are equal and we can consider a basis $A_2, \ldots, A_k$ spanning the subspace of all strictly inferior triangular matrices in $\mathcal{T}$. $\square$

Remark 12.4. *The proof of Proposition 12.3 constructs the subspace $\mathcal{T} \subset \overline{A}^{rec}$ spanned by all elements of the form $\rho(S,S)A \in \overline{A}^{rec}$. This subspace is contained in the maximal vector space spanned by all lower triangular matrices in $\overline{A}^{rec}$.*

13 Elements of Toeplitz type

In this section we identify the set $\mathcal{M}_p^l$ with $\{0, \ldots, p^l - 1\}$ using the bijection

$$u_1 \ldots u_l \longmapsto \sum_{j=1}^{l} u_j p^{j-1}.$$

Analogously, we identify $\mathcal{M}_{p\times p}$ in the obvious way with $\{0, \ldots, p^l - 1\} \times \{0, \ldots, p^l - 1\}$. This identification yields a natural isomorphism between $\mathbf{K}^{\mathcal{M}_p^l}$ and vectors $(\alpha_0^l, \ldots, \alpha_{p^l-1}^l) \in \mathbf{K}^{p^l}$, respectively between $\mathbf{K}^{\mathcal{M}_{p\times p}^l}$ and matrices with coefficients $(\alpha_{i,j}^l)_{0\le i,j<p^l}$.

13.1 Toeplitz matrices

A (finite or infinite) matrix T of square-order $n \in \mathbb{N} \cup \{\infty\}$ with coefficients $t_{u,w}, 0 \le u, w < n$ is a *Toeplitz matrix* if $t_{u,w}$ depends only on the difference $u - w$ of its indices. A Toeplitz matrix is thus described by a (finite or infinite) sequence $\ldots, \alpha_{-2}, \alpha_{-1}, \alpha_0, \alpha_1, \ldots$ defined by $\alpha_{u-w} = t_{u,w}$.

We call a matrix algebra $\mathcal{A} \subset M_{n\times n}(\mathbf{K})$ of $n \times n$ square matrices (with $n \in \mathbb{N} \cup \{\infty\}$ finite or infinite) an algebra of *Toeplitz type* if all elements of $\mathcal{A}$ are Toeplitz matrices. For finite n, the typical example is the $n-$dimensional commutative algebra $\mathrm{Toep}_\rho(n) = \sum_{j=0}^{n-1} \mathbf{K} T_\rho^j$ generated by the $n \times n$ matrix

$$T_\rho = \begin{pmatrix} 0 & & \ldots & 0 & \rho \\ 1 & 0 & & & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & & \ddots & & \\ 0 & & \ldots & 1 & 0 \end{pmatrix}$$

with minimal polynomial $T_\rho^n - \rho$ obtained by considering the product $T_1 D_{n,\rho}$ where T_1 is the obvious cyclic permutation matrix of order n and where $D_{n,\rho}$ is the diagonal matrix with diagonal entries $1, 1, \ldots, 1, \rho$.

An example of an element in $\mathrm{Toep}_1(2^n)$ is the matrix with coefficients $t_{u,w} = \gamma_{v_2(u-w)}$, $0 \le u, w < 2^n$ depending only on the highest power

$2^{v_2(u-w)} \in \{1,2,4,8,16,\dots,\} \cup \infty$ dividing $u-w$. If such a matrix

$$\begin{pmatrix} \gamma_\infty & \gamma_0 & \gamma_1 & \gamma_0 & \gamma_2 & \gamma_0 & \cdots \\ \gamma_0 & \gamma_\infty & \gamma_0 & \gamma_1 & \gamma_0 & \gamma_2 & \\ & & \ddots & & & & \end{pmatrix}$$

is invertible, then its inverse is of the same type.

In the sequel, we will mainly consider the algebra $\mathrm{Toep}(n) = \mathrm{Toep}_0(n)$ of all lower triangular Toeplitz matrices having finite or infinite square order $n \in \mathbb{N} \cup \{\infty\}$. We call $\mathrm{Toep}(n)$ the *(lower triangular) Toeplitz algebra* of order n.

The proof of the following well-known result is easy and left to the reader.

Proposition 13.1. *For finite n, the algebra Toep(n) is isomorphic to the ring $\mathbf{K}[x] \pmod{x^n}$ of polynomials modulo x^n and Toep(∞) is isomorphic to the ring $\mathbf{K}[[x]]$ of formal power series.*

In both cases, the isomorphism is given by considering the generating series $\sum_{j=0}^{n-1} t_{j,0}x^j$ associated to the first column of a lower triangular Toeplitz matrix $t_{i,j}, 0 \le i,j$.

An element $A \in \mathbf{K}^{\mathcal{M}_{p\times p}}$ is of *Toeplitz type* if all matrices $A[\mathcal{M}^l_{p\times p}]$ are Toeplitz matrices. *Algebras of Toeplitz type* in $\mathbf{K}^{\mathcal{M}_{p\times p}}$ or $\mathrm{Rec}_{p\times p}(\mathbf{K})$ are defined in the obvious way as containing only elements of Toeplitz type.

The algebra $\mathcal{T}_p(\mathbf{K})$ of lower triangular elements of Toeplitz type in $\mathbf{K}^{\mathcal{M}_{p\times p}}$ will be studied below.

Shift maps $\rho(s,t)$ preserve the vector space of (recurrence) matrices of Toeplitz type in $\mathbf{K}^{\mathcal{M}_{p\times p}}$. The recursive closure $\overline{T}^{rec}$ of an element of Toeplitz type in $\mathbf{K}^{\mathcal{M}_{p\times p}}$ contains thus only elements of Toeplitz type. We have the following result:

Proposition 13.2. *All recurrence matrices $T_1,\dots,T_d \in Rec_{p\times p}(\mathbf{K})$ of a reduced presentation are of Toeplitz type if and only if the shift matrices $\rho(s,t)$ satisfy the following two conditions:*

(1) The shift matrices $\rho(s,t),\ 0 \le s,t < p$ depend only on $s-t$.

(2) Assuming condition (1) and writing (somewhat abusively) $\rho(s-t)$ for a shift matrix $\rho(s,t)$ we have the identities

$$\rho(s+p)\rho(t) = \rho(s)\rho(t+1)$$

for $1-p \le s < 0$ and $1-p \le t < p-1$.

For $p=2$, the conditions of Proposition 13.2 boil down to the identities $\rho(0,0) = \rho(1,1)$ and

$$\begin{aligned} \rho(1,0)\rho(0,0) &= \rho(0,1)\rho(1,0), \\ \rho(1,0)\rho(0,1) &= \rho(0,1)\rho(0,0). \end{aligned}$$

Proof of Proposition 13.2 We show by induction on l that all matrices $T_i[\mathcal{M}^l_{p\times p}]$ are Toeplitz matrices.

For $l=0$, there is nothing to do. Condition (1) implies that $T_1[\mathcal{M}^1_{p\times p}],\dots,T_d[\mathcal{M}^1_{p\times p}]$ are Toeplitz matrices. Setting $T=T_i$ and denoting $(\rho(s,t)T)[\mathcal{M}^l_{p\times p}]$ by $\rho(s-t)T$ we get

$$T[\mathcal{M}^{l+1}_{p\times p}]=\begin{pmatrix} \rho(0)T & \rho(-1)T & \rho(-2)T & \cdots & \rho(1-p)T \\ \rho(1)T & \rho(0)T & \rho(-1)T & \cdots & \rho(2-p)T \\ \vdots & & \ddots & & \vdots \\ \rho(p-2)T & \rho(p-3)T & \rho(p-4)T & \cdots & \rho(1) \\ \rho(p-1)T & \rho(p-2)T & \rho(p-3)T & \cdots & \rho(0)T \end{pmatrix}$$

where all matrices $\rho(1-p)T,\dots,\rho(p-1)T$ are Toeplitz matrices by induction. We have to show that two horizontally adjacent blocks $(\rho(t+1)T\ \rho(t)T)$, $1-p\le t<p-1$ define a $p^l\times(2\cdot p^l)$ matrix of "Toeplitz-type" (the case of vertically adjacent blocks gives rise to the same conditions and is left to the reader). We have

$$\big(\rho(t+1)T|\rho(t)T\big)=\left(\begin{array}{cc|cc} \cdots & \rho(1-p)\rho(t+1)T & \rho(0)\rho(t)T & \cdots \\ \cdots & \rho(2-p)\rho(t+1)T & \rho(1)\rho(t)T & \cdots \\ & \vdots & \rho(2)\rho(t)T & \cdots \\ & \vdots & \vdots & \\ \cdots & \rho(-1)\rho(t+1)T & \vdots & \\ \cdots & \rho(0)\rho(t+1)T & \rho(p-1)\rho(t)T & \cdots \end{array}\right)$$

(the vertical line separates the matrix into two square blocks) where $\rho(\alpha)\rho(\beta)T=(\rho(\alpha)\rho(\beta)T)[\mathcal{M}^{l-1}_{p\times p}]$. If

$$\rho(s-p)\rho(t+1)T=\rho(s)\rho(t)T,\ 1\le s<p-1,$$

the matrix $T=T_i[\mathcal{M}^{l+1}_{p\times p}]$ is of Toeplitz type.

The opposite direction is easy and left to the reader. □

13.2 The algebra $\mathcal{T}_p(\mathbf{K})\subset\mathbf{K}^{\mathcal{M}_{p\times p}}$ formed by lower triangular elements of Toeplitz type

We denote by $\mathcal{T}_p(\mathbf{K})$ the algebra given by all lower triangular elements of Toeplitz type in $\mathbf{K}^{\mathcal{M}_{p\times p}}$. This algebra is isomorphic to $\prod_{l=0}^{\infty}\Big(\mathbf{K}[x]\pmod{x^{p^l}}\Big)$ by Proposition 13.1. The subalgebra of converging elements in $\mathcal{T}_p(\mathbf{K})$ can thus be identified with the algebra $\mathbf{K}[[x]]$ of formal power series. We denote by $\mathcal{T}_{p-rec}(\mathbf{K})=\mathcal{T}_p(\mathbf{K})\cap\mathrm{Rec}_{p\times p}(\mathbf{K})$ the subalgebra of recurrence matrices in $\mathcal{T}_p(\mathbf{K})$.

Using the sequence of vectors

$$A[\mathcal{M}^l_p]=(\alpha^l_0,\dots,\alpha^l_{p^l-1})\in\mathbf{K}^{p^l}$$

defined by $A \in \mathbf{K}^{\mathcal{M}_p}$, we associate to $A \in \mathbf{K}^{\mathcal{M}_p}$, the lower triangular element $L_A \in \mathbf{K}^{\mathcal{M}_{p\times p}}$ of Toeplitz type given by

$$L_A[\mathcal{M}^l_{p\times p}] = \begin{pmatrix} \alpha^l_0 & & & & \\ \alpha^l_1 & \alpha^l_0 & & & \\ \alpha^l_2 & \alpha^l_1 & \alpha^l_0 & & \\ \vdots & & & \ddots & \\ \alpha^l_{p^l-1} & \dots & & \alpha^l_1 & \alpha^l_0 \end{pmatrix}.$$

The map $A \longmapsto L_A$ is clearly an isomorphism of vector spaces between $\mathbf{K}^{\mathcal{M}_p}$ and the algebra $\mathcal{T}_p(\mathbf{K})$.

Proposition 13.3. *The application $A \longmapsto L_A$ satisfies*

$$dim(\overline{A}^{rec}) \leq dim(\overline{L_A}^{rec}) \leq 2\ dim(\overline{A}^{rec})$$

and induces thus an isomorphism of vector spaces between $Rec_p(\mathbf{K})$ and the algebra $\mathcal{T}_{p-rec}(\mathbf{K}) \subset Rec_{p\times p}(\mathbf{K})$.

Corollary 13.4. *The map $Rec_p(\mathbf{K}) \times Rec_p(\mathbf{K}) \longrightarrow Rec_{p\times p}(\mathbf{K})$ defined by $(A, B) \longmapsto L_A + L^t_B$ with L_A, L_B as above for $A, B \in Rec_p(\mathbf{K})$ is a surjection onto the vector space of all recurrence matrices of Toeplitz type with kernel the subspace of all pairs $(A, B) \in (Rec_p(\mathbf{K}))^2$ such that $A[U] = B[U] = 0$ for $U \in \mathcal{M}_p \setminus \{0^*\}$ and $A[0^l] = -B[0^l]$ for all $l \in \mathbb{N}$.*

In particular, an element of Toeplitz type in $\mathbf{K}^{\mathcal{M}_{p\times p}}$ is a recurrence matrix if and only if its first row and column vectors are in $Rec_p(\mathbf{K})$.

Remark 13.5. *The inequality $dim(\overline{A}^{rec}) \leq dim(\overline{L_A}^{rec})$ in Proposition 13.3 is in general strict as illustrated by the example $A \in \mathbb{Q}^{\mathcal{M}_{2\times 2}}$ with coefficients $A[U] = 1$ for all $U \in \mathcal{M}_2$. The recurrence vector $A \in Rec_2(\mathbb{Q})$ is thus of complexity 1 while L_A has complexity 2 since $\rho(0,0)L_A = \rho(1,1)L_A = L_A$ and $\rho(1,0)L_A$ (with coefficients $(\rho(1,0)L_A)[U,W] = 1$ for all $(U,W) \in \mathcal{M}_{2\times 2}$) are linearly independent.*

Proof of Proposition 13.3 The inequality $\dim(\overline{A}^{rec}) \leq \dim(\overline{L_A}^{rec})$ follows easily from the observation that the first column of $L_A[\mathcal{M}^l_{p\times p}]$ is given by $A[\mathcal{M}^l_p]$.

For $A \in \mathbf{K}^{\mathcal{M}_p}$, we denote by $U_A \in \mathbf{K}^{\mathcal{M}_{p\times p}}$ the strictly upper triangular matrix of Toeplitz type defined by

$$U_A[\mathcal{M}^l_{p\times p}] = \begin{pmatrix} 0 & \alpha^l_{p^l-1} & \dots & \alpha^l_3 & \alpha^l_2 & \alpha^l_1 \\ & 0 & \alpha^l_{p^l-1} & \dots & \alpha^l_3 & \alpha^l_2 \\ & & \ddots & & & \alpha^l_3 \\ & & & \ddots & & \\ & & & & 0 & \alpha^l_{p^l-1} \\ & & & & & 0 \end{pmatrix}$$

where $A[\mathcal{M}_p^l] = (\alpha_0^l, \alpha_1^l, \ldots, \alpha_{p^l-1}^l)$.

We consider now the vector spaces $\mathcal{L}_{\overline{A}^{rec}} = \{L_A \mid A \in \overline{A}^{rec}\}$ and $\mathcal{U}_{\overline{A}^{rec}} = \{U_A \mid A \in \overline{A}^{rec}\}$ spanned by all lower triangular, respectively strictly upper triangular elements of Toeplitz type in $\mathbf{K}^{\mathcal{M}_{p\times p}}$.

It is now straightforward to check that $\mathcal{L}_{\overline{A}^{rec}} \oplus \mathcal{U}_{\overline{A}^{rec}} \subset \mathbf{K}^{\mathcal{M}_{p\times p}}$ is recursively closed. This shows the inequalities

$$\dim(\overline{L_{\mathcal{A}}}^{rec}) \leq \dim(\mathcal{L}_{\overline{A}^{rec}} \oplus \mathcal{U}_{\overline{A}^{rec}}) \leq 2\ \dim(\overline{A}^{rec}).$$

The last part of Proposition 13.3 follows from the observation that the map $A \longmapsto L_A$ defines an isomorphism of vector-spaces between $K^{\mathcal{M}_p}$ and $\mathcal{T}_p(\mathbf{K})$. □

The proof of Corollary 13.4 is immediate using the observation that the sequence $X[0^l, 0^l]$ defines an element of $\mathrm{Rec}_1(\mathbf{K})$ for $X \in \mathrm{Rec}_{p\times p}(\mathbf{K})$.

13.3 The polynomial ring-structure on $\mathbf{K}^{\mathcal{M}_p}$ and $\mathbf{Rec}_p(\mathbf{K})$

The isomorphisms of vector spaces $\mathbf{K}^{\mathcal{M}_p} \sim \mathcal{T}_p \sim \prod_{l=0}^{\infty}\left(\mathbf{K}[x] \mod x^{p^l}\right)$ given by the map $A \longmapsto L_A$ considered above and Proposition 13.1 endow $\mathbf{K}^{\mathcal{M}_p}$ with the *polynomial product*. More precisely, we consider the map given by

$$A \longmapsto \psi(A) = \prod_{l=0}^{\infty} \psi_l(A) \in \prod_{l=0}^{\infty}\left(\mathbf{K}[x] \pmod{x^{p^l}}\right)$$

where $\psi_l(A) = \sum_{k=0}^{p^l-1} \alpha_k^l x^k$ if $A[\mathcal{M}_p^l] = (\alpha_0^l, \ldots, \alpha_{p^l-1}^l)$ for $A \in \mathbf{K}^{\mathcal{M}_p}$. We have then $\psi(C) = \psi(A)\psi(B) \in \prod_{l=0}^{\infty}\left(\mathbf{K}[x] \mod x^{p^l}\right)$ if and only if $\psi_l(C) \equiv \psi_l(A)\psi_l(B) \pmod{x^{p^l}}$ for all $l \in \mathbb{N}$ or equivalently if and only if $L_C = L_A L_B \in \mathbf{K}^{\mathcal{M}_{p\times p}}$.

In particular, if A and B are convergent and correspond to formal power series g_A, g_B, then the polynomial product $\psi(C) = \psi(A)\psi(B)$ corresponds to a convergent element $C \in \mathbf{K}^{\mathcal{M}_p}$ with associated formal power series $g_C = g_A g_B$ defined as the usual product of the formal power series g_A and g_B.

Using the isomorphism $A \longmapsto L_A$ of the previous section, we identify the polynomial product algebra $\mathbf{K}^{\mathcal{M}_p}$ with the subalgebra $\mathcal{T}_p(\mathbf{K}) \subset \mathbf{K}^{\mathcal{M}_{p\times p}}$ of lower triangular matrices of Toeplitz type. Similarly, we identify $\mathrm{Rec}_p(\mathbf{K})$ with the commutative subalgebra $\mathcal{T}_{p-rec}(\mathbf{K}) \subset \mathrm{Rec}_{p\times p}(\mathbf{K})$.

We denote by $\mathcal{IT}_p \subset \mathcal{T}_p(\mathbf{K})$ the subspace corresponding to all elements $A \in \mathbf{K}^{\mathcal{M}_p}$ such that $\rho(s)A$ is of finite support for $s = 0, 1, \ldots p-2$ and $\rho(p-1)A \in \mathcal{IT}_p$ (where we identify $\mathbf{K}^{\mathcal{M}_p}$ and $\mathcal{T}_p(\mathbf{K})$ using the map $A \longmapsto L_A$). Although this definition is recursive, it makes sense: An element $A \in \mathcal{IT}_p$ corresponds to a sequences $\psi_0(A), \psi_1(A), \ldots$ of polynomials where $\psi_k(A) \in$

$\mathbf{K}[x]$ is of degree $< p^l$ and almost all polynomials $\psi_l(A)$ satisfy $\psi_l(A) \equiv 0 \pmod{x^{p^l - p^k}}$ for every fixed natural integer $k \in \mathbb{N}$.

It is easy to check that $\mathcal{IT}_p$ is an ideal of the algebra $\mathcal{T}_p(\mathbf{K})$. We denote by $\tilde{\mathcal{T}}_p(\mathbf{K}) = \mathcal{T}_p(\mathbf{K})/\mathcal{IT}_p$ and $\tilde{\mathcal{T}}_{p-rec}(\mathbf{K}) = \mathcal{T}_{p-rec}(\mathbf{K})/\mathcal{IT}_{p-rec}$ (where $\mathcal{IT}_{p-rec} = \mathcal{IT}_p \cap \mathcal{T}_{p-rec}(\mathbf{K})$) the obvious quotient algebras. It is also easy to check that the differential operator $\frac{d}{dx}$ (acting in the obvious way on the polynomial sequence $\psi(A) = (\psi_0(A), \psi_1(A), \dots)$ associated to $A \in \mathbf{K}^{\mathcal{M}_p}$) is well-defined on the quotient algebra $\tilde{\mathcal{T}}_p(\mathbf{K})$.

Proposition 13.6. *The algebra $\tilde{\mathcal{T}}_{p-rec}(\mathbf{K})$ is a differential subalgebra of the differential algebra $\tilde{\mathcal{T}}_p(\mathbf{K})$.*

In particular, converging recurrence vectors correspond to a differential subring of the differential ring $(\mathbf{K}[[x]], \frac{d}{dx})$ of formal power-series.

Proof Consider the factorization $\frac{d}{dx} = \frac{1}{x}\left(x\frac{d}{dx}\right)$ of the differential operator $\frac{d}{dx}$ into the differential operator $(x\frac{d}{dx})$, followed by multiplication by x^{-1}. Given an element $A \in \mathbf{K}^{\mathcal{M}_p}$ (identified in the obvious way with the corresponding sequence $\psi(A) = \prod_{l=0}^{\infty} \psi_l(A)$ of polynomials where $\psi_l(A) \in \mathbf{K}[x]$ is of degree $< p^l$), the differential operator $\psi(A) \longmapsto (x\frac{d}{dx})\psi(A)$ corresponds to the map $A \longmapsto B = NA$ multiplying a row-vector $A \in \mathbf{K}^{\mathcal{M}_{p\times 1}}$ on the left with the converging recurrence matrix N with limit the diagonal matrix having diagonal entries $0, 1, 2, 3, \dots$. Multiplication of $\psi(B)$ by $^{-1}$ corresponds now to the map $B \longmapsto C = PB$ where

$$P[\mathcal{M}^l_{p\times p}] = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ \vdots & & \ddots & \ddots & \\ 1 & & & & 0 \end{pmatrix}$$

is the obvious cyclic permutation of order p^l. Since N and P are both of complexity 2 in $\mathrm{Rec}_{p\times p}(\mathbf{K})$ this shows the inequality

$$\dim(\overline{\frac{d}{dx}A}^{rec}) \leq 4\ \dim(\overline{A}^{rec})$$

for $A \in \mathrm{Rec}_p(\mathbf{K})$ and implies the first part.

The second part follows from the obvious observation that the quotient map $\mathcal{T}_p(\mathbf{K}) \longmapsto \tilde{\mathcal{T}}_p(\mathbf{K})$ restricts to an injection on the subalgebra of converging elements in $\mathcal{T}_p(\mathbf{K})$. □

Remark 13.7. *Over the field $\mathbf{K} = \mathbb{C}$ of complex numbers and for $p \geq 2$, the formal power series g_A associated to a converging recurrence vector $A \in Rec_p(\mathbb{C})$ defines by Proposition 11.2 a holomorphic function in the open unit disc.*

14 Elements of Hankel type

A (finite or infinite) matrix H with coefficients $h_{s,t}$, $0 \leq s,t$ is a *Hankel matrix* if $h_{s,t} = \alpha_{s+t}$ for some sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$. An element $H \in \mathbf{K}^{\mathcal{M}_{p\times p}}$ is *of Hankel type* if for all $l \in \mathbb{N}$, the matrix $H[\mathcal{M}^l_{p\times p}]$ is a Hankel matrix (for the usual total order on rows and columns induced by $(u_1 \ldots u_l) \longmapsto \sum_{j=1}^{l} u_j p^{j-1}$).

Consider the involutive element $S \in \mathrm{Rec}_{p\times p}(\mathbf{K})$ where the coefficients of $S[\mathcal{M}^l_{p\times p}]$ are 1 on the antidiagonal and zero elsewhere. More precisely, $S[u_1 \ldots u_n, w_1 \ldots w_n] = 1$ if $w_1 = p-1-u_1, \ldots, w_n = p-1-u_n$ and $S[U,W] = 0$ otherwise. (In particular, S is a recurrence matrix of Hankel type with complexity 1.) The following result reduces the study of Hankel matrices to the study of Toeplitz matrices.

Proposition 14.1. *(Left- or right-)multiplication by the involution $S \in Rec_{p\times p}(\mathbf{K})$ defines a bijection preserving complexities between elements of Hankel type and elements of Toeplitz type in $\mathbf{K}^{\mathcal{M}_{p\times p}}$. In particular, the maps $A \longmapsto SA$ and $A \longmapsto AS$ yield bijections between recurrence matrices of Hankel type and recurrence matrices of Toeplitz type in $Rec_{p\times p}(\mathbf{K})$.*

The easy proof is left to the reader.

Elements of Hankel type satisfy $H[U,W] = H[W,U]$ for all $(U,W) \in \mathcal{M}_{p\times p}$ and are thus examples of symmetric elements in $\mathbf{K}^{\mathcal{M}_{p\times p}}$.

Let $H \in \mathbf{K}^{\mathcal{M}_{p\times p}}$ be an element of Hankel type. Since $\rho(s,t)H$, $0 \leq s,t < r$ is of Hankel type, the recursive closure $\overline{H}^{rec}$ contains only elements of Hankel type. The following result is the exact analogue of Proposition 13.2:

Proposition 14.2. *All recurrence matrices $H_1, \ldots, H_d \in Rec_{p\times p}(\mathbf{K})$ of a reduced presentation are of Hankel type if and only if the shift matrices $\rho(s,t)$ satisfy the following two conditions:*

(1) The shift matrices $\rho(s,t)$, $0 \leq s,t < p$ depend only on $s+t$.

(2) Assuming condition (1) and writing $\rho(s+t)$ for a shift matrix $\rho(s,t)$, we have the identities

$$\rho(s+p)\rho(t) = \rho(s)\rho(t+1)$$

for $0 \leq s \leq p-2$ and $0 \leq t \leq 2p-3$.

For $p = 2$, Proposition 14.2 boils down to the identities $\rho(1,0) = \rho(0,1)$ and

$$\rho(0,0)\rho(1,1) = \rho(1,1)\rho(0,1),$$
$$\rho(1,1)\rho(0,0) = \rho(0,0)\rho(0,1)$$

for shift-matrices of a presentation containing only recurrence matrices of Hankel type.

Proof of Proposition 14.2 We show by induction on l that all matrices $H_i[\mathcal{M}^l_{p\times p}]$ are Hankel matrices.

For $l=0$, there is nothing to do. Condition (1) implies that $H_1[\mathcal{M}^1_{p\times p}],\dots,H_d[\mathcal{M}^1_{p\times p}]$ are Hankel matrices. Setting $H=H_i$ and denoting $(\rho(s,t)H)[\mathcal{M}^l_{p\times p}]$ by $\rho(s+t)H$ we get

$$H[\mathcal{M}^{l+1}_{p\times p}] = \begin{pmatrix} \rho(0)H & \rho(1)H & \rho(2)H & \cdots & \rho(p-1)H \\ \rho(1)H & \rho(2)H & \rho(3)H & \cdots & \rho(p)H \\ \vdots & & & & \vdots \\ \rho(p-1)H & \rho(p)H & \rho(p+1)H & \cdots & \rho(2p-2)H \end{pmatrix}$$

where all matrices $\rho(0)H,\dots,\rho(2p-2)H$ are Hankel matrices by induction. We have to show that two horizontally adjacent blocks $(\rho(t)H\ \rho(t+1)H)$, $0\le t<2p-2$ define a $p^l\times(2\cdot p^l)$ matrix of "Hankel-type" (the argument for vertically adjacent blocks gives rise to the same conditions and is left to the reader). We have

$$\big(\rho(t)H|\rho(t+1)H\big) = \left(\begin{array}{cc|cc} \cdots & \rho(p-1)\rho(t)H & \rho(0)\rho(t+1)H & \cdots \\ \cdots & \rho(p)\rho(t)H & \rho(1)\rho(t+1)H & \cdots \\ \cdots & \rho(p+1)\rho(t)H & \vdots & \\ & \vdots & \vdots & \\ & \vdots & \rho(p-2)\rho(t+1)H & \cdots \\ \cdots & \rho(2p-2)\rho(t)H & \rho(p-1)\rho(t+1)H & \cdots \end{array}\right)$$

(the vertical line separates the matrix into two square blocks) using the shorthand notation $\rho(\alpha)\rho(\beta)H=(\rho(\alpha)\rho(\beta)H)[\mathcal{M}^{l-1}_{p\times p}]$. Such a matrix is of Hankel-type if

$$\rho(s+p)\rho(t)H=\rho(s)\rho(t+1)H \text{ for } 0\le s\le p-2\ .$$

The opposite direction is left to the reader. □

Given two vectors $A,B\in\mathbf{K}^{\mathcal{M}_{p\times p}}$ with coefficients

$$A[\mathcal{M}^l_p]=\left(\alpha^l_0,\dots,\alpha^l_{p^l-1}\right)\in\mathbf{K}^{p^l}$$
$$B[\mathcal{M}^l_p]=\left(\beta^l_0,\dots,\beta^l_{p^l-1}\right)\in\mathbf{K}^{p^l}$$

we consider the Hankel matrices $H_A,\tilde{H}_B\in\mathbf{K}^{\mathcal{M}_{p\times p}}$ with coefficients

$$H_A[\mathcal{M}^l_{p\times p}] = \begin{pmatrix} \alpha^l_0 & \alpha^l_1 & \dots & \alpha^l_{p^l-1} \\ \alpha^l_1 & \alpha^l_2 & & 0 \\ \vdots & & & \vdots \\ \alpha^l_{p^l-1} & 0\dots & 0 & \end{pmatrix},$$

$$\tilde{H}_B[\mathcal{M}^l_{p\times p}] = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & & & & 0 & \beta^l_0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \beta^l_0 & & \beta^l_{p^l-4} & \beta^l_{p^l-3} \\ 0 & \beta^l_0 & \beta^l_1 & \dots & \beta^l_{p^l-3} & \beta^l_{p^l-2} \end{pmatrix} .$$

The following results are close analogues of Proposition 13.3 and Corollary 13.4.

Proposition 14.3. *The applications $A \longmapsto H_A, \tilde{H}_A$ defined above satisfy the inequalities*

$$dim(\overline{A}^{rec}) \leq dim(\overline{H_A}^{rec})$$

and

$$dim(\overline{H_A}^{rec}), dim(\overline{\tilde{H}_A}^{rec}) \leq 2\ dim(\overline{A}^{rec}) .$$

Corollary 14.4. *The application $Rec_p(\mathbf{K}) \times Rec_p(\mathbf{K}) \longrightarrow Rec_{p\times p}(\mathbf{K})$ defined by $(A, B) \longmapsto H_A + \tilde{H}_B$ with $H_A, \tilde{H}_B \in Rec_{p\times p}(\mathbf{K})$ as above for $A, B \in Rec_p(\mathbf{K})$ is a surjection onto the vector space of all Hankel matrices in $Rec_{p\times p}(\mathbf{K})$ with kernel spanned by $(A, B) \in (Rec_p(\mathbf{K}))^2$ such that $A = 0$ and $B[U] = 0$ for $U \in \mathcal{M}_p \setminus \{(p-1)^*\}$.*

In particular, a Hankel matrix in $\mathbf{K}^{\mathcal{M}_{p\times p}}$ is a recurrence matrix if and only if its first column vector and its last row vector are elements of $Rec_p(\mathbf{K})$.

Remark 14.5. *The "missing" inequality*

$$dim(\overline{A}^{rec}) \leq dim(\overline{\tilde{H}_A}^{rec})$$

in Proposition 14.3 does not necessarily hold. Its possible failure is due to the fact that the last coefficient of $A[\mathcal{M}^l_p]$ is not involved in $\tilde{H}_A[\mathcal{M}^l_{p\times p}]$.

Proof of Proposition 14.3 Check that the vector space spanned by all elements $H_X, \tilde{H}_X$ for $X \in \overline{A}^{rec}$ is recursively closed. □

The proof of Corollary 14.4 is obvious.

14.1 Hankel matrices and continued fractions of Jacobi type

An infinite Hankel matrix H associated to the generating function $\gamma = \sum_{n=0}^{\infty} \gamma_n x^n \in \mathbf{K}[[x]]$ (where we suppose $\gamma_0 = 1$ in order to simplify subsequent statements) is *non-degenerate* if all finite Hankel matrices $H(n)$ with coefficients $H(n)_{i,j} = \gamma_{i+j}, 0 \leq i, j < n$ are invertible. It is then well-known (see for instance [6]) that we have a continued fraction of Jacobi type

$$\gamma = \frac{1}{1 - \alpha_0 x - \beta_0 \frac{x^2}{1-\alpha_1 x - \beta_1 \frac{x^2}{1-\dots}}}$$

for $\gamma = 1 + \gamma_1 x + \dots$ where the sequences $\alpha_0, \alpha_1, \dots, \beta_0, \beta_1, \dots$ appear also in the recursive definition

$$p_{n+1} = (x - \alpha_n)p_n - \beta_n p_{n-1}$$

of the monic orthogonal polynomials $p_0 = 1, \dots$ for H. We have then the following result.

Proposition 14.6. *Let $H \in Rec_{p\times p}(\mathbf{K})$ be a converging non-degenerate Hadamard matrix such that $H = LDL^t$ with $L \in Rec_{p\times p}(\mathbf{K})$ lower triangular unipotent and invertible in $Rec_{p\times p}(\mathbf{K})$. Then the sequences $\alpha_0, \alpha_1, \dots$ and $\beta_0, \beta_1, \dots$ involved in the continued fraction of Jacobi type associated to H define converging elements of $Rec_p(\mathbf{K})$.*

Proof Since $L^{-1}H\left(L^{-1}\right)^t = D$ is a diagonal matrix with invertible diagonal coefficients, the $n-$th row $(m_{n,0}, \dots, m_{n,n}, 0, \dots$ of $M = L^{-1}$ (identified with its limit) consists of the coefficients of the $n-$th orthogonal polynomial $p_n = \sum_{k=0}^{n} m_{n,k}x^k$ for H. Since $\langle xp_n, p_k\rangle = \langle p_n, xp_k\rangle$, there exists constants α_n, β_n such that we have the identities

$$xp_n = p_{n+1} + \alpha_n p_n + \beta_n p_n$$

which can be written in matrix-form as

$$\tilde{M} = \begin{pmatrix} \alpha_0 & 1 & & \\ \beta_1 & \alpha_1 & 1 & \\ & \beta_2 & \alpha_2 & 1 \\ & & & \ddots \end{pmatrix} M$$

where $\tilde{M}$ is defined by adding a first zero column to the infinite lower triangular matrix $M = L^{-1}$. The tridiagonal matrix

$$S = \begin{pmatrix} \alpha_0 & 1 & & \\ \beta_1 & \alpha_1 & 1 & \\ & \beta_2 & \alpha_2 & 1 \\ & & & \ddots \end{pmatrix}$$

is called the *Stiltjes matrix* of H and a small computation shows that it is also given by $S = L^{-1}\tilde{L}$ where $\tilde{L}$ is obtained by removing the first row of the infinite lower triangular matrix L. For L invertible in $\mathrm{Rec}_{p\times p}(\mathbf{K})$ both matrices $\tilde{L}$ and L^{-1} are in $\mathrm{Rec}_{p\times p}(\mathbf{K})$. This shows $S \in \mathrm{Rec}_{p\times p}(\mathbf{K})$ and implies the result. □

15 Groups of recurrence matrices

The (multiplicative) identity Id of the algebra $\mathbf{K}^{\mathcal{M}_{p\times p}}$ satisfies $\rho(s,s)Id = Id, 0 \le s < p$ and $\rho(s,t)Id = 0, 0 \le s \ne t < p$, and is thus an element of complexity 1 in $\mathrm{Rec}_{p\times p}(\mathbf{K})$.

Definition 15.1. *We denote by $GL_{p-rec}(\mathbf{K})$ the general linear group of all recurrence matrices in $Rec_{p\times p}(\mathbf{K})$ which are invertible (for the matrix-product) in $Rec_{p\times p}(\mathbf{K})$.*

A group of recurrence matrices is a subgroup of $GL_{p-rec}(\mathbf{K})$ for some $p \in \mathbb{N}$.

Proposition 15.2. *The set $(GL_{p-rec}(\mathbf{K}))^p \times (Rec_{p\times p}(\mathbf{K}))^{\binom{p}{2}}$ injects into $GL_{p-rec}(\mathbf{K})$.*

Proof For $A_0, \ldots, A_{p-1} \in \mathrm{GL}_{p-rec}(\mathbf{K})$ and $B_{s,t} \in \mathrm{Rec}_{p\times p}(\mathbf{K}), 0 \leq t < s < p$, consider the matrix $G \in \mathrm{Rec}_{p\times p}(\mathbf{K})$ defined by $\rho(s,s)G = A_s$, $\rho(s,t)G = B_{s,t}$ for $0 \leq t < s < p$ and $\rho(s,t)G = 0$ for $s < t$ and make an arbitrary choice in $\mathbf{K}^*$ for $G[\emptyset,\emptyset]$. This defines an element $G \in \mathrm{Rec}_{p\times p}(\mathbf{K})$ which is invertible. □

For $p \geq 2$, the group $\mathrm{GL}_{p-rec}(\mathbf{K})$ is complicated: Proposition 15.2 shows already that it is very huge. One can also show that it contains every finite group and every finite-dimensional matrix group over $\mathbf{K}$. Moreover, the following result shows that there is (in general) no relation between the complexity of an element $G \in \mathrm{GL}_{p-rec}(\mathbf{K})$ and the complexity of its inverse element $G^{-1} \in \mathrm{GL}_{p-rec}(\mathbf{K})$.

Proposition 15.3. *For $p > 1$ and $N \in \mathbb{N}$ arbitrary, the group $GL_{p-rec}(\mathbb{C})$ contains an element G of complexity 2 with inverse of complexity $\geq N$.*

Proof For a natural integer $p > 1$, the determinant of the symmetric $N \times N-$Hankel matrix

$$\begin{pmatrix} x^p & x^{p^2} & x^{p^3} & \ldots & x^{p^N} \\ x^{p^2} & & & & \vdots \\ \vdots & & & & \vdots \\ x^{p^N} & & & & x^{p^{2N-1}} \end{pmatrix}$$

is a monic polynomial P of degree $p+p^2+p^{2N-1} = \frac{p^{2N}-p}{p-1} > 0$. Choose $n \in \mathbb{N}$ such that $\omega = e^{2i\pi/n} \in \mathbb{C}$ is not a root of P and consider the converging element $G \in \mathrm{Rec}_{p\times p}(\mathbb{C})$ defined by the lower triangular Toeplitz matrix

$$\begin{pmatrix} 1 & & & \\ -\omega & 1 & & \\ 0 & -\omega & 1 & \\ & & & \ddots \end{pmatrix}$$

associated to $1 - \omega x \in \mathbb{C}[x] \subset \mathbb{C}[[x]]$ as in section 13.3. It is easy to check that G is of complexity 2. Since ω is of finite order, G admits an inverse

element $G^{-1} \in \mathrm{Rec}_{p\times p}(\mathbb{C})$, given by the converging lower triangular Toeplitz matrix with limit

$$\begin{pmatrix} 1 & & & & \\ \omega & 1 & & & \\ \omega^2 & \omega & 1 & & \\ \omega^3 & \omega^2 & \omega & 1 & \\ & & & & \ddots \end{pmatrix}$$

associated to $\sum_{j=0}^{\infty}(\omega x)^j \in \mathbb{C}[[x]]$. For $l \geq 1$, we consider the sequence

$$\begin{array}{rcl} S(l) & = & ((\rho(1^l,0^l)G^{-1})[\emptyset,\emptyset],(\rho(1^l,0^l)G^{-1})[1,0],(\rho(1^l,0^l)G^{-1})[1^2,0^2],\dots) \\ & = & (\omega^{p^l},\omega^{p^{l+1}},\omega^{p^{l+2}},\dots) \end{array}.$$

The submatrix

$$\begin{pmatrix} \omega^p & \omega^{p^2} & \omega^{p^3} & \dots \\ \omega^{p^2} & \omega^{p^4} & \omega^{p^3} & \dots \\ \vdots & & & \end{pmatrix}$$

formed by the N initial coefficients of the sequences $S(1),\dots,S(N)$ has non-zero determinant by construction. This shows that G^{-1} has complexity $\geq N$. □

Remark 15.4. *Proposition 15.3 holds also for instance for the algebraic closure of finite fields. It is however trivially wrong (for p fixed) over finite fields: Since there are only finitely many elements of complexity $\leq b$ in $Rec_{p\times p}(\mathbb{F})$ for $\mathbb{F}$ a finite field, the number of invertible elements in $Rec_{p\times p}(\mathbb{F})$ having complexity $\leq b$ is also finite and there is thus an upper bound for the complexities of their inverses.*

Remark 15.5. *An element $X \in Rec_{p\times p}(\mathbf{K})$ having a multiplicative inverse $X^{-1} \in \mathbf{K}^{\mathcal{M}_{p\times p}}$ is not necessarily in $GL_{p-rec}(\mathbf{K})$ since X^{-1} has in general infinite complexity. A simple example is the "diagonal" matrix $X \in \mathbb{Q}^{\mathcal{M}_{p\times p}}$ which is in the center of the algebra $\mathbb{Q}^{\mathcal{M}_{p\times p}}$ and has diagonal coefficients $X[U,U] = n+1 \in \{1,2,\dots\}$ depending only on the length n of $(U,U) \in \mathcal{M}^n_{p\times q}$ (and off-diagonal coefficients $X[U,W]=0$ for $U \neq W$ with $(U,W) \in \mathcal{M}_{p\times p}$). We have $\rho(s,t)X = 0$ for $0 \leq s \neq t < p$ and $\rho(s,s)X = X + Id$ where $Id \in Rec_{p\times p}(\mathbb{Q})$ denotes the multiplicative identity. The element X has thus complexity 2 and is in $Rec_{p\times p}(\mathbf{K})$. Since the denominators of X^{-1} involve all primes of $\mathbb{Z}$, it cannot be a recurrence matrix by Proposition 11.4 or 11.5. A second proof of this fact is given by the observation that the infinite matrix H with coefficients $c_{i,j} = (\rho(0^i,0^i)X^{-1})[0^j,0^j] = \frac{1}{i+j+1}, 0 \leq i,j$ (using the convention $0^0 = \emptyset$) is the Hilbert matrix*

$$\begin{pmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \dots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \dots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The finite submatrix formed by the first n rows and columns of H is non-singular for all n. This implies that X^{-1} is of infinite complexity.

Another simple example for $p \geq 2$ is the converging lower triangular unipotent recurrence matrix $L \in Rec_{2\times 2}(\mathbb{Q})$ of Toeplitz type with associated generating series $1-2x$. Its coefficients are 1 on the diagonal, -2 on the subdiagonal and 0 everywhere else. Its inverse $L^{-1} \in \mathbb{Q}^{\mathcal{M}_{2\times 2}}$ is lower triangular of Toeplitz type with first row given by successive powers of 2 and corresponds to the generating series $\frac{1}{1-2x}$. Proposition 11.2 shows that L^{-1} cannot be a recurrence matrix (see also Remark 13.7). However, the element L^{-1} is a recurrence matrix over a field of positive characteristic. (In the case of odd characteristic $\wp$, its complexity depends on the multiplicative order of $2 \in (\mathbb{Z}/\wp\mathbb{Z})^*$.)

Remark 15.6. *For $\mathbf{K}$ a topological field, the group of invertible elements in $\mathbf{K}^{\mathcal{M}_{p\times p}}$ is a topological group for both topologies considered in §10. The subgroup $GL_{p-rec}(\mathbf{K})$ is thus also a topological group (with discrete topology for the second topology described in §10).*

It would be interesting to have answers to the following questions: Does $GL_{p-rec}(\mathbf{K})$ admit other, more interesting topologies? Is the map $X \longmapsto X^{-1}$ of invertible elements in the quotient algebra $Rec_{p\times p}(\mathbf{K})/\mathcal{FS}_{p\times p}$ continuous for the metric $\| X \|_\infty^\infty$ described in §10?

15.1 A few homomorphisms and characters

Given an element $A \in \mathrm{GL}_{p-rec}(\mathbf{K})$, the projection

$$A \longmapsto A[\mathcal{M}^l_{p\times p}] \in \mathrm{GL}(\mathbf{K}^{p^l})$$

yields a homomorphism of groups. All these homomorphisms are surjective and have sections. In particular, $\mathrm{GL}_{p-rec}(\mathbf{K})$ contains (an isomorphic image of) every finite-dimensional matrix-group over $\mathbf{K}$ for $p \geq 2$.

The *determinant* is the homomorphism $\mathrm{GL}_{p-rec}(\mathbf{K}) \longrightarrow \prod_{l=0}^\infty \mathbf{K}^*$ defined by

$$A \longmapsto \det(A) = (\det(A[\mathcal{M}^0_{p\times p}]), \det(A[\mathcal{M}^1_{p\times p}]), \det(A[\mathcal{M}^2_{p\times p}]), \dots) \in \prod_{l=0}^\infty \mathbf{K}^* \ .$$

Its image is in general not surjective: For a finite (countable) field $\mathbf{K}$ there exist only finitely (countably) many elements of complexity $\leq d$ in $\mathrm{End}_{p-rec}(\mathbf{K})$ and the group $\mathrm{GL}_{p-rec}(\mathbf{K})$ is thus countable. If $\mathbf{K}^*$ contains at least two elements, the subgroup $\det(\mathrm{GL}_{p-rec}(\mathbf{K}))$ is thus a proper subgroup of the uncountable group $\prod_{l=0}^\infty \mathbf{K}^*$.

Remark 15.7. *It would probably be interesting to have a description of the countable subgroup*

$$\det(GL_{p-rec}(\mathbf{K})) \subset \prod_{l=0}^\infty \mathbf{K}^* \ .$$

An obvious restriction is for instance given by the trivial remark that all numerators and denominators of det(A) involve only a finite number of distinct prime-factors for $A \in GL_{p-rec}(\mathbb{Q})$ (see Proposition 11.5). The largest subgroup of $\prod_{l=0}^{\infty} \mathbb{Q}^*$ with this property is however still uncountable.

Setting $\tau_n(A) = \sum_{U \in \mathcal{M}_p^n} A[U,U]$ we get the *trace-map* $A \longmapsto \mathrm{tr}(A) = (\tau_0(A), \tau_1(A), \dots, \dots)$ defining a linear application

$$\mathrm{tr} : \mathrm{Rec}_{p\times p}(\mathbf{K}) \longrightarrow \mathrm{Rec}_{1\times 1}(\mathbf{K}).$$

The trace map satisfies the identity $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and is thus constant on conjugacy classes under the action of $\mathrm{GL}_{p-rec}(\mathbf{K})$. The trace $\mathrm{tr}(A) \in \mathrm{Rec}_{1\times 1}(\mathbf{K})$ of a recurrence matrix $A \in \mathrm{Rec}_{p\times p}(\mathbf{K})$ is easy to compute from a presentation $(A_1, \dots, A_a)[\emptyset, \emptyset], \rho_{\overline{A}^{rec}}(s,t),\ 0 \le s,t < p$: It admits the presentation $(\mathrm{tr}(A_1), \dots, \mathrm{tr}(A_a))[\emptyset, \emptyset] = (A_1, \dots, A_a)[\emptyset, \emptyset]$ with shift-matrix $\sum_{s=0}^{p-1} \rho_{\overline{A}^{rec}}(s,s)$. The associated generating series

$$\sum_{n=0}^{\infty} \tau_n(A) t^n \in \mathbf{K}[[t]]$$

is always a rational function.

Remark 15.8. *The properties of the traces $tr(A^0), tr(A^1), tr(A^2), tr(A^3), \dots$ are essentially shared by the coefficients of x^{p^l-k} of the characteristic polynomial of $A[\mathcal{M}_{p\times p}^l]$.*

This suggests the following question: What can be said of the spectra of the matrices $A[\mathcal{M}_{p\times p}^l], l = 0, 1, 2, \dots$? Is there sometimes (perhaps after a suitable normalization) a "spectral limit", eg. a nice spectral measure, etc? (The answer to the last question is trivially yes for recurrence matrices of complexity 1 since they are tensor-powers, up to a scalar factor.)

15.2 A few other properties

Recall that a group Γ is *residually finite* if for every element $\gamma \in \Gamma$ different from the identity there exists a homomorphism $\pi : \Gamma \longrightarrow F$ into a finite group F such that $\pi(\gamma) \neq 1 \in F$.

Proposition 15.9. *(i) For $\mathbf{K}$ a finite field, the group $GL_{p-rec}(\mathbf{K})$ is residually finite.*

(ii) For $\mathbf{K}$ an arbitrary field, a finitely generated group $\Gamma \subset GL_{p-rec}(\mathbf{K})$ is residually finite.

Proof. Given a finite field $\mathbf{K}$ and an element $1 \neq A \in \mathrm{GL}_{p-rec}(\mathbf{K})$, there exists l such that $A[\mathcal{M}_{p\times p}^l] \neq 1 \in \mathrm{Aut}(\mathbf{K}^{p^l})$ thus proving assertion (i).

For proving assertion (ii), we remark that presentations for a finite set of generators $\gamma_1^{\pm 1}, \dots, \gamma_m^{\pm 1}$ generating $\Gamma \subset \mathrm{GL}_{p-rec}(\mathbf{K})$ involve only finitely

many elements $k_1, \ldots, k_N \in \mathbf{K}$ (appearing in the initial data and as coefficients of the shift-matrices). The group Γ is thus defined over the finitely generated field extension $\mathbf{F}$ containing $k_1, \ldots, k_N$ over the primary field $\mathbf{k}$ (which is either $\mathbb{Q}$ or the finite primary field $\mathbb{F}_\wp$ with $\wp$ elements for $\wp$ a prime number) of $\mathbf{K}$. Given an element $\gamma \neq 1\mathrm{d}$ in Γ, choose $l \in \mathbb{N}$ such that $\gamma[\mathcal{M}^l_{p\times p}] \neq 1\mathrm{d}$. Choose a transcendental basis $t_1, \ldots, t_f$ of $\mathbf{F}$ such that $\mathbf{F} = \tilde{\mathbf{F}}(t_1, \ldots, t_f)$ where $\tilde{\mathbf{F}}$ is the maximal subfield of $\mathbf{F}$ which is algebraic over $\mathbf{k}$. Choose now a maximal ideal $\mathcal{J} \subset \mathbf{A} = \tilde{\mathbf{F}}[t_1, \ldots, t_f]$ such that Γ is defined over $\mathbf{A}/\mathcal{J}$ and $\gamma[\mathcal{M}^l_{p\times p}] \not\equiv 1\mathrm{d} \pmod{\mathcal{J}}$. The quotient field $\mathbf{A}/\mathcal{J}$ is either finite or a number field. Reducing modulo a suitable prime $\wp \in \mathbf{A}/\mathcal{J}$ in the case of a number field we get a quotient group (with $\overline{\gamma} \neq 1$ in the quotient) of $p^l \times p^l-$matrices defined over a finite field. $\square$

Remark 15.10. *It follows from Proposition 11.5 or from the proof of Proposition 15.9, that any finitely generated group $\Gamma \subset GL_{p-rec}(\mathbb{Q})$ (or more generally $\Gamma \subset GL_{p-rec}(\mathbf{K})$ where $\mathbf{K}$ is a number field) can be reduced modulo $\wp$ to a quotient-group $\Gamma_\wp \subset GL_{p-rec}(\mathbb{Z}/\wp\mathbb{Z})$ for almost all primes $\wp$ (or prime-ideals $\wp$ of the number field $\mathbf{K}$).*

Remark 15.11. *Considering the group $\Big(Rec_{p\times p}(\mathbf{K})\Big/\mathcal{FS}_{p\times p}\Big)^*$ of invertible elements in the quotient algebra $Rec_{p\times p}(\mathbf{K})\Big/\mathcal{FS}_{p\times p}$ (where $\mathcal{FS}_{p\times p} \subset Rec_{p\times p}(\mathbf{K})$ denotes the two-sided ideal of all finitely supported elements $X \in Rec_{p\times p}(\mathbf{K})$ with $X[U,W] = 0$ except for finitely many words $(U,W) \in \mathcal{M}_{p\times p}$) we get a group homomorphism*

$$GL_{p-rec}(\mathbf{K}) \longrightarrow \Big(Rec_{p\times p}(\mathbf{K})\Big/\mathcal{FS}_{p\times p}\Big)^*$$

with kernel $\oplus_{n=0}^{\infty} GL(\mathbf{K}^{p^n})$.

Remark 15.12. *There are three projective versions of the group $GL_{p-rec}(\mathbf{K})$. One can either consider the quotient-group $GL_{p-rec}(\mathbf{K})/(\mathbf{K}^* Id)$ or the quotient-group $GL_{p-rec}(\mathbf{K})/(GL_{p-rec}(\mathbf{K}) \cap \mathcal{C}_{p-rec}(\mathbf{K}))$ where $\mathcal{C}_{p-rec}(\mathbf{K}) \subset Rec_{p\times p}(\mathbf{K})$ denotes the center of $Rec_{p\times p}(\mathbf{K})$. Finally, one can also consider equivalence classes by invertible central elements in $\mathbf{K}^{\mathcal{M}_{p\times p}}$ of all recurrence matrices $X \in Rec_{p\times p}(\mathbf{K})$ for which there exists a recurrence matrix $Y \in Rec_{p\times p}(\mathbf{K})$ such that XY is central and invertible in $\mathbf{K}^{\mathcal{M}_{p\times p}}$.*

The obvious homomorphism $GL_{p-rec}(\mathbf{K})$ into this last group is in general neither injective nor surjective as can be seen as follows: For $p \geq 2$, we denote by $Id \in Rec_{p\times p}(\mathbf{K})$ the identity recurrence matrix and by $J \in Rec_{p\times p}(\mathbf{K})$ the recurrence matrix of complexity 1 defined by $J[U,W] = 1$ for all $(U,W) \in \mathcal{M}_{p\times p}$. Choose $\alpha, \beta \in \mathbb{C}$ such that $\alpha(\alpha + p^l\beta) \neq 0$ for all $l \in \mathbb{N}$ and define $X \in Rec_{p\times p}(\mathbb{C})$ by

$$X[\mathcal{M}^l_{p\times p}] = \alpha\ Id[\mathcal{M}^l_{p\times p}] + \beta\ J[\mathcal{M}^l_{p\times p}].$$

The computation

$$\Big(\alpha\ Id[\mathcal{M}^l_{p\times p}] + \beta\ J[\mathcal{M}^l_{p\times p}]\Big)\Big((\alpha + p^l\beta)Id[\mathcal{M}^l_{p\times p}] - \beta\ J[\mathcal{M}^l_{p\times p}]\Big)$$
$$= \alpha(\alpha + p^l\beta)\ Id[\mathcal{M}^l_{p\times p}]$$

shows that X is invertible in the quotient of $Rec_{p\times p}(\mathbb{C})$ by central elements of $\mathbf{K}^{\mathcal{M}_{p\times p}}$.

Choosing $\alpha = \beta = 1$, the eigenvalues of $(Id + J)\,[\mathcal{M}^l_{p\times p}]$ are $1 + p^l$ and 1 with multiplicity $p^l - 1$. We have thus $\det((Id + J)\,[\mathcal{M}^l_{p\times p}]) = 1 + p^l$. Choosing an odd prime $\wp$ such that $p \pmod{\wp}$ is a non-square in $\mathbb{Z}/\wp\mathbb{Z}$ and setting $l = (\wp - 1)/2$, we have $\det((Id + J)\,[\mathcal{M}^l_{p\times p}]) \equiv 0 \pmod{\wp}$. Since the number of such primes $\wp$ is infinite, Proposition 11.5 or Remark 15.10 imply that $(Id + J) \notin GL_{p-rec}(\mathbb{Q})$. A similar argument implies finally that the class of X in the projective quotient corresponds to no element in $GL_{p-rec}(\mathbb{Q})$.

16 Examples of groups of recurrence matrices

16.1 $\mathbf{GL}_p(\mathbf{K}) \subset \mathbf{GL}_{p-rec}(\mathbf{K})$

Any matrix g of size $p \times p$ with coefficients $g_{u,w}, 0 \leq u, w < p$ in $\mathbf{K}$ gives rise to a recurrence matrix $\mu(g) = G \in \mathrm{Rec}_{p\times p}(\mathbf{K})$ of complexity 1 by considering the $n-$th tensor-power $g \otimes g \otimes \cdots \otimes g$ and setting

$$G[u_1 \dots u_n, w_1 \dots w_n] = g_{u_1,w_1} g_{u_2,w_2} \cdots g_{u_n,w_n}.$$

Since $\mu(g)\mu(g') = \mu(gg')$ (and $\mu(\mathrm{Id}(\mathrm{GL}_p)) = \mathrm{Id}(\mathrm{GL}_{p-rec})$), the map $g \longmapsto \mu(g)$ induces an injective homomorphisme $\mathrm{GL}_p(\mathbf{K}) \longmapsto \mathrm{GL}_{p-rec}(\mathbf{K})$.

16.2 An infinite cyclic group related to the shift

The recurrence matrix $A = A_1 \in \mathrm{Rec}_{2\times 2}(\mathbf{K})$ presented by $(A_1, A_2)[\emptyset, \emptyset] = (1, 1)$ and shift-matrices

$$\rho(0,0) = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \qquad \rho(0,1) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$
$$\rho(1,1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \rho(1,1) = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}.$$

yields permutation matrices $A_1[\mathcal{M}^l_{2\times 2}]$ associated to the cyclic permutation $(0\ 1\ 2\ \dots 2^l - 1)$ defined by addition of 1 modulo 2^l. The first few matrices $A_1[\mathcal{M}^l_{2\times 2}], l = 0, 1, 2$ (using the bijection $s_1 \dots s_n \longmapsto \sum_{j=1}^n s_j 2^{j-1}$ for rows and columns) are

$$\begin{pmatrix} 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

All matrices $A_2[\mathcal{M}^l_{2\times 2}]$ have a coefficient 1 in the upper-right corner and zero coefficients everywhere else.

The inverse of A_1 is given by A_1^t. The two converging elements $\rho(0,0)A_1^t, \rho(0,0)A_1 \in \mathrm{End}_{2-rec}(\mathbb{Q})$ correspond to the shift $(x_0, x_1, \dots) \longmapsto (x_1, x_2, \dots)$ and its section $(x_0, x_1, \dots) \longmapsto (0, x_0, x_1, \dots)$ on converging elements in $\mathbf{K}^{\mathcal{M}_2}$. This example has an obvious generalization to $\mathrm{GL}_{p-rec}(\mathbf{K})$ for all $p \geq 2$.

16.3 Diagonal groups

The set $\mathcal{D}_{p-rec}(\mathbf{K}) \cap \mathrm{GL}_{p-rec}(\mathbf{K})$ of all diagonal recurrence matrices in $\mathrm{GL}_{p-rec}(\mathbf{K})$ forms a commutative subgroup containing the maximal central subgroup $\mathcal{C}^* = \mathcal{C}_{p-rec}(\mathbf{K}) \cap \mathrm{GL}_{p-rec}(\mathbf{K})$ of $\mathrm{GL}_{p-rec}(\mathbf{K})$. For $\mathbf{K}$ of characteristic $\neq p$ the trace map establishes a bijection between $\mathcal{C}^*$ and $\mathrm{GL}_{1-rec}(\mathbf{K})$. An element of $\mathcal{C}^*$ (in characteristic $\neq p$) corresponds to an element $A \in \mathrm{Rec}_{1\times 1}(\mathbf{K})$ which is invertible in the algebra (or, equivalently, in the function-ring) $\mathrm{Rec}_{1\times 1}(\mathbf{K})$. Such an element is thus encoded by a rational function $\sum_{n=0}^{\infty} \alpha_n x^n \in \mathbf{K}(x)$ having only non-zero coefficients such that $\sum_{n=0}^{\infty} \frac{x^n}{\alpha_n}$ is also rational. Examples of such functions are eg. $\frac{1}{1-\lambda z}$ with $\lambda \neq 0$, functions having only non-zero, ultimately periodic coefficients $\alpha_0, \alpha_1, \dots$, and, more generally, generating functions of recurrence matrices in $\mathrm{Rec}_{1\times 1}(\mathbf{K})$ with values in a finite subset of $\mathbf{K}^*$.

16.4 Lower (or upper) triangular groups

Lower (or upper) triangular elements in $\mathrm{GL}_{p-rec}(\mathbf{K})$ form a group $\mathcal{L}^* = \mathcal{L}_{p\times p}(\mathbf{K}) \cap \mathrm{GL}_{p-rec}(\mathbf{K})$ containing the subgroup consisting of all converging lower triangular elements in $\mathrm{GL}_{p-rec}(\mathbf{K})$. A still smaller subgroup is given by considering all lower triangular elements of Toeplitz type in $\mathrm{GL}_{p-rec}(\mathbf{K})$. A slight modification of the proof of Proposition 15.2 shows that the set $(\mathcal{L}^*)^p \times (\mathrm{Rec}_{p\times p}(\mathbf{K}))^{\binom{p}{2}}$ injects into $\mathcal{L}^*$. It would be interesting to know if every conjugacy class in $\mathrm{GL}_{p-rec}(\mathbf{K})$ intersects $\mathcal{L}^*\,(\mathcal{L}^*)^t$ where $(\mathcal{L}^*)^t$ denotes the group of upper triangular recurrence matrices obtained by transposing $\mathcal{L}^*$.

Consider the commutative subgroup $\mathcal{T}^*$ of all converging lower triangular elements of Toeplitz type in $\mathrm{GL}_{p-rec}(\mathbf{K})$. Converging elements of $\mathcal{T}_{p-rec}(\mathbf{K})$ form a differential subring of $\mathbf{K}[[x]]$, (see §12.2), and the logarithmic derivative $G \longmapsto G'/G$ defines a group homomorphism from $\mathcal{T}^*$ into an additive subgroup of $\mathcal{T}_{p\times p}(\mathbf{K})$. In the case $\mathbf{K} \subset \mathbb{C}$, elements of $\mathcal{T}^*$ correspond to some holomorphic functions on the open unit disc of $\mathbb{C}$ which have no zeroes or poles in the open unit disc, cf. Remark 13.7.

Examples of such elements in $\mathcal{T}^*$ are given by rational functions involving only cyclotomic polynomials. The rational function $\frac{1}{1-t}$ for instance corresponds to the invertible recurrence matrix of Toeplitz-type with limit the unipotent Toeplitz matrix consisting only of $1'$s below the diagonal. A

more exotic example is given by the element of $\mathcal{T}^*$ associated to the power series $\prod_{n=0}^{\infty}(1+it^{2^n}) \in \mathbb{C}[[t]]$ (where $i^2 = 1$ is a square root of 1) with inverse series $(1-t^2)\prod_{n=0}^{\infty}(1-it^{2^n})$.

If $\mathbf{K}$ is contained in the algebraic closure of a finite field, the group $\mathcal{T}^*$ contains the subring of all rational functions without zero or pole at the origin.

16.5 Orthogonal groups

A recurrence matrix $A \in \mathrm{Rec}_{p\times p}(\mathbf{K})$ is *symmetric* if $A[U,W] = A[W,U]$ for all $(U,W) \in \mathcal{M}_{p\times p}$. A complex recurrence matrix $A \in \mathrm{Rec}_{p\times p}(\mathbf{K})$ is *hermitian* if $A[U,W] = \overline{A[W,U]}$, for all $(U,W) \in \mathcal{M}_{p\times p}$, with $\overline{x}$ denoting the complex conjugate of $x \in \mathbb{C}$. (More generally, one can define "hermitian" matrices over a field admitting an involutive automorphism.) A real symmetric recurrence matrix is *positive definite* if all matrices $A[\mathcal{M}^l_{p\times p}]$ are positive definite.

A symmetric or hermitian matrix $A \in \mathrm{Rec}_{p\times p}(\mathbf{K})$ is *non-singular* if $\det(A[\mathcal{M}^l_{p\times p}]) \in \mathbf{K}^*$ for all $l \in \mathbb{N}$ and *non-degenerate* if $A \in \mathrm{GL}_{p-rec}(\mathbf{K})$. An example of a non-singular positive definite symmetric recurrence matrix is the diagonal recurrence matrix $A \in \mathrm{Rec}_{p\times p}(\mathbb{Q})$ with diagonal coefficients $A[U,U] = l+1$ for $(U,U) \in \mathcal{M}^l_{p\times p}$ of length l. An example of a positive definite non-degenerate symmetric recurrence matrix is the identity matrix $\mathrm{Id} \in \mathrm{Rec}_{p\times p}(\mathbb{Q})$.

Such a non-singular matrix $A \in \mathrm{Rec}_{p\times p}(\mathbf{K})$ defines the *orthogonal group*

$$\mathrm{O}(A) = \{B \in \mathrm{GL}_{p-rec}(\mathbf{K}) \mid B^t AB = A\} \subset \mathrm{GL}_{p-rec}(\mathbf{K})$$

of A in the symmetric case and the *unitary group*

$$\mathrm{U}(A) = \{B \in \mathrm{GL}_{p-rec}(\mathbf{K}) \mid \overline{B}^t AB = A\}$$

of A in the case of a hermitian recurrence matrix A. For $\mathbf{K}$ a real field, one speaks also of *Lorenzian groups* if the symmetric matrix A is not positive definite.

The following obvious proposition relating presentations of $A, A^t \in \mathrm{Rec}_{p\times p}(\mathbf{K})$ and $\overline{A}$ (over $\mathbf{K} = \mathbb{C}$) is mainly a restatment of Proposition 7.7, recalled for the convenience of the reader. Its easy proof is omitted.

Proposition 16.1. *The following assertions are equivalent:*

(i) $(A_1,\dots,A_a)[\emptyset,\emptyset] = (\alpha_1,\dots,\alpha_a) \in \mathbf{K}^a, \rho(s,t) \in \mathbf{K}^{a\times a}, 0 \le s,t < r$ *is a presentation of* $A = A_1$.

(ii) $(A_1,\dots,A_a)[\emptyset,\emptyset] = (\alpha_1,\dots,\alpha_a), \tilde{\rho}(s,t) \in \mathbf{K}^{a\times a}, 0 \le s,t < p$ *with* $\tilde{\rho}(s,t) = \rho(t,s)$ *is a presentation of* $A^t = A_1$.

(iii) (Over $\mathbf{K} = \mathbb{C}$*)* $(A_1,\dots,A_a)[\emptyset,\emptyset] = (\overline{\alpha_1},\dots,\overline{\alpha_a}), \overline{\rho}(s,t) \in \mathbf{K}^{a\times a}, 0 \le s,t < r$ *is a presentation of* $\overline{A} = A_1$ *with* $\overline{X}$ *denoting complex conjugation applied to all coefficients of* X.

16.6 Symplectic groups

A recurrence matrix $A \in \mathrm{Rec}_{p\times p}(\mathbf{K})$ is *antisymmetric* if $A[U,W] = -A[W,U]$ for all $(U,W) \in \mathcal{M}_{p\times p}$. An antisymmetric recurrence matrix $A \in \mathrm{Rec}_{p\times p}(\mathbf{K})$ is *symplectic* if $\det(A[\mathcal{M}^l_{p\times p}]) \in \mathbf{K}^*$ for all $l \geq 1$. If $\mathbf{K}$ is of characteristic 2, we require moreover $A[U,U] = 0$ for all $(U,U) \in \mathcal{M}_{p\times p}$. Symplectic recurrence matrices exist only for p even.

A symplectic recurrence matrix A is *non-degenerate* if $\tilde{A} \in \mathrm{GL}_{p-rec}(\mathbf{K})$ where $\tilde{A}[\emptyset,\emptyset] = 1$ and $\tilde{A}[U,W] = A[U,W]$ if $(U,W) \neq [\emptyset,\emptyset]$.

For $A \in \mathrm{Rec}_{p\times p}(\mathbf{K})$ a symplectic recurrence matrix, the associated *symplectic group* $Sp(A) \subset GL_{p-rec}(\mathbf{K})$ of recurrence matrices is defined as

$$\mathrm{Sp}(A) = \{B \in \mathrm{GL}_{p-rec}(\mathbf{K}) \mid B^t A B = A\}$$

where the value $B[\emptyset,\emptyset]$ can be neglected.

16.7 Groups generated by elements of bounded complexity

Denote by $\Gamma_{a,b} \subset \mathrm{GL}_{p-rec}(\mathbf{K})$ the group generated by all elements $A \in \mathrm{GL}_{p-rec}(\mathbf{K})$ of complexity $\leq a$ with inverse $B = A^{-1}$ of complexity $\leq b$. We have $\Gamma_{a,b} = \Gamma_{b,a}$, $\Gamma_{a,b} \subset \Gamma_{a',b'}$ if $a \leq a', b \leq b'$ and Proposition 15.3 implies that many of these inclusions are strict.

Moreover, the set of generators of $\Gamma_{a,b}$ (elements in $\mathrm{GL}_{p-rec}(\mathbf{K})$ of complexity $\leq a$ with inverse of complexity $\leq b$) is a union of algebraic sets since they can be described by (a finite union of) polynomial equations.

Remark 16.2. *The group $\Gamma_{1,1} = \Gamma_{1,2} = \Gamma_{1,3} = \dots$ is isomorphic to $\mathbf{K}^* \times GL(\mathbf{K}^p)$.*

For $\mathbf{K}$ a finite field, the group $\Gamma_{a,b}$ is finitely generated and the sequence

$$\Gamma_{a,1} \subset \Gamma_{a,2} \subset \Gamma_{a,3} \subset \dots$$

stabilizes. It would be interesting to determine the smallest integer $A = A(a,\mathbf{K})$ such that $\Gamma_{a,A} = \Gamma_{a,b}$ for all $b \geq A$. The first non-trivial case is the determination of $A(2,\mathbb{F}_2)$.

Remark 16.3. *One can similarly consider the subalgebra $\mathcal{R}_a \subset Rec_{p\times p}(\mathbf{K})$ generated as an algebra by all recurrence matrices of complexity $\leq a$ in $Rec_{p\times p}(\mathbf{K})$. The subalgebra $\mathcal{R}_a$ of $Rec_{p\times p}(\mathbf{K})$ is always recursively closed.*

The following example shows that many inclusions $\mathcal{R}_a \subset \mathcal{R}_{a+1}$ are strict for $Rec_{p\times p}(\mathbb{Q})$: Consider a central diagonal recurrence matrix with diagonal coefficients $A[U,U] = \alpha_l, (U,U) \in \mathcal{M}^l_{p\times p}$ for $\alpha_0, \alpha_1, \alpha_2, \dots \subset \mathbb{Q}$ a periodic sequence of minimal period-length a prime number $\wp$. Such an element has complexity $(\wp - 1)$ and cannot be contained in the sub-algebra generated by recurrence matrices of lower complexities in $Rec_{p\times p}(\mathbb{Q})$.

17 Computing G^{-1} for $G \in \mathbf{GL}_{p-rec}(\mathbf{K})$

The aim of this section is to discuss a few difficulties when computing inverses of recurrence matrices in $\mathrm{GL}_{p-rec}(\mathbf{K})$.

Definition 17.1. *The depth of $A \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ is the smallest element $D \in \mathbb{N} \cup \{\infty\}$ such that $\rho(\mathcal{M}^{\leq D}_{p\times q})A$ spans $\overline{A}^{rec}$ (where $\mathcal{M}^{\infty}_{p\times q} = \mathcal{M}$).*

It is easy to check that A is a recurrence matrix if and only if A has finite depth $D < \infty$.

The following result is closely related to Proposition 9.2.

Proposition 17.2. *We have*

$$dim\left(\sum_{(U,W)\in\mathcal{M}^{\leq k}_{p\times q}} \mathbf{K}\rho(U,W)A\right) < dim\left(\sum_{(U,W)\in\mathcal{M}^{\leq k+1}_{p\times q}} \mathbf{K}\rho(U,W)A\right)$$

if k is smaller than the depth D of A.

Corollary 17.3. *The depth of any non-zero recurrence matrix A is smaller than its complexity $dim(\overline{A}^{rec})$.*

Proof of Proposition 17.2 (See also the proof of Proposition 9.2.) The equality

$$\sum_{(U,W)\in\mathcal{M}^{\leq k}_{p\times q}} \mathbf{K}\rho(U,W)A = \sum_{(U,W)\in\mathcal{M}^{\leq k+1}_{p\times q}} \mathbf{K}\rho(U,W)A$$

implies that these vector-spaces are recursively closed and coincide thus with $\overline{A}^{rec}$. □

Given a presentation $\mathcal{P}$ of $G \in \mathrm{GL}_{p-rec}(\mathbf{K})$, there are two obvious methods for computing a presentation of its inverse G^{-1}. The first method analyzes the matrices

$$\left(G[\mathcal{M}^0_{p\times p}]\right)^{-1}, \left(G[\mathcal{M}^1_{p\times p}]\right)^{-1}, \ldots, \left(G[\mathcal{M}^a_{p\times p}]\right)^{-1}$$

with a huge enough in order to guess a presentation $\tilde{\mathcal{P}}$ of G^{-1}. This can be done if $a \geq 1 + D + N$ where D is the depth and N the saturation level of G^{-1}. It is then straightforward to check if the presentation $\tilde{P}$ is correct by computing the matrix-product of G and the recurrence matrix presented by $\tilde{\mathcal{P}}$. The limitation of this method is the need of inverting the square-matrix $G[\mathcal{M}^a_{p\times p}]$ of large order $p^a \times p^a$. Below, we will describe an algorithm based on this method.

The second method is to guess an upper bound b for the complexity of G^{-1}, to write down a generic presentation $\tilde{P}$ of complexity b where the initial data and shift matrices involve a set of $d + p^2d^2$ unknowns. Equating

the matrix-product of G with the recurrence matrix presented by $\tilde{\mathcal{P}}$ to the identity yields polynomial equations. We will omit a detailed discussion of this method since it seems to be even worse than the first one.

Remark 17.4. *Another important issue which we will not adress here is the existence of a finite algorithm which is able to tell if an element $A \in Rec_{p\times p}(\mathbf{K})$ (given, say, by a minimal presentation) is or is not invertible in $GL_{p\times p}(\mathbf{K})$? The algorithm presented below will always succeed (with finite time and memory requirements) in computing an inverse of A for $A \in GL_{p-rec}(\mathbf{K})$. It will however fail to stop (or more likely, use up all memory on your computing device) if $A \notin GL_{p-rec}(\mathbf{K})$ is invertible in $\mathbf{K}^{\mathcal{M}_{p\times p}}$.*

Proposition 15.3 is perhaps an obstruction to the existence of such an algorithm.

17.1 An algorithm for computing $G^{-1} \in \mathbf{GL}_{p-rec}(\mathbf{K})$

Given a presentation $\mathcal{P}$ of $G \in \mathrm{GL}_{p-rec}(\mathbf{K})$, the following algorithm computes a presentation $\tilde{\mathcal{P}}$ of G^{-1} in a finite number of steps.

Step 1 Set $D = 0$.

Step 2 Compute the saturation level N of the (not necessarily recursively closed) vector space spanned by $\rho(\mathcal{M}_{p\times p}^{\leq D})G^{-1}$. (This needs inversion of the $p^{D+N+1} \times p^{D+N+1}$ matrix $G[\mathcal{M}_{p\times p}^{\leq D+N+1}]$ where $N < \frac{p^{D+1}-1}{p-1}$ is the saturation level of the vector space spanned by $\rho(\mathcal{M}_{p\times p}^{\leq D})G^{-1}$.)

Step 3 Supposing the depth D correct, use the saturation level N of step 2 for computing a presentation $\tilde{P}_D$ using the finite-dimensional vector space spanned by $\left(\rho(\mathcal{M}_{p\times p}^{\leq D})G^{-1}\right)[\mathcal{M}_{p\times p}^{\leq N+1}]$.

Step 4 If the recurrence matrix $\tilde{G}$ defined by the presentation $\tilde{\mathcal{P}}_D$ satisfies $G\tilde{G} = \mathrm{Id}$, stop and print the presentation $\tilde{\mathcal{P}}$. Otherwise, increment D by 1 and return to step 2.

Remark 17.5. *The expensive part (from a computational view) of the algorithm are steps 2 and 3 and involve computations with large matrices (for $p \geq 2$). A slight improvement is to merge step 2 and 3 and to do the computations for guessing the presentation $\tilde{\mathcal{P}}_D$ of step 3 at once during step 2.*

One could avoid a lot of iterations by running step 2 simultaneously for D and $D+1$. The cost of this "improvement" is however an extra factor of p in the size of the involved matrices and should thus be avoided since step 4 is faster than step 2.

18 Lie algebras

The Lie bracket $[A, B] = AB - BA$ turns the algebra $\mathrm{Rec}_{p\times p}(\mathbf{K})$ (or $\mathbf{K}^{\mathcal{M}_{p\times p}}$) into a Lie algebra. For $\mathbf{K}$ a suitable complete topological field (say $\mathbf{K} = \mathbb{R}$ or

$\mathbf{K} = \mathbb{C}$), the exponential function $X \longmapsto \exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!}$ is well-defined and continuous for both topologies defined in §10. The exponential function does however not preserve the subspace $\mathrm{Rec}_{p\times p}(\mathbf{K})$ of recurrence matrices and the associated Lie group

$$\{\exp(A) \in \mathbf{K}^{\mathcal{M}_{p\times p}} \mid A \in \mathrm{Rec}_{p\times p}(\mathbf{K})\}$$

is thus only a group in $\mathbf{K}^{\mathcal{M}_{p\times p}}$.

The Lie algebra $\mathrm{Rec}_{p\times p}(\mathbb{C})$ contains analogues of the classical Lie algebras of type A, B, C and D.

The analogue of type A is given by the vector space of recurrence matrices $X \in \mathrm{Rec}_{p\times p}(\mathbb{C})$ such that $\mathrm{tr}(X) = 0 \in \mathrm{Rec}_{1\times 1}$.

The B and D series are defined as the set of all recursive antisymmetric matrices of $\mathrm{Rec}_{p\times p}(\mathbb{C}), p \geq 3$ odd, for the B series and of $\mathrm{Rec}_{p\times p}(\mathbb{C}), p \geq 4$ even, for the D series.

The C series is defined only for even p and consists of all recursive matrices $X \in \mathrm{Rec}_{p\times p}(\mathbb{C})$ such that $\Omega X = (\Omega X)^t$ where $\Omega \in \mathrm{Rec}_{p\times p}(\mathbb{C})$ has complexity 2 and $\Omega[\mathcal{M}^l_{p\times p}]$ is of the form $\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$ for $l \geq 1$ with Id and 0 denoting the identity matrix and the zero matrix of size $p^l/2 \times p^l/2$ (the value of $\Omega[\emptyset, \emptyset]$ is irrelevant).

A different and perhaps more natural way to define the C series is to consider triplets $A, B, C \in \mathrm{Rec}_{p\times p}(\mathbf{K})$ (for all $p \in \mathbb{N}$) of recurrence matrices with $B = B^t$ and $C = C^t$ symmetric. The C series is then the Lie subalgebra in $\prod_{l=0}^{\infty} \mathbf{K}^{2p^l \times 2p^l}$ of elements given by

$$\begin{pmatrix} A[\mathcal{M}^l_{p\times p}] & B[\mathcal{M}^l_{p\times p}] \\ C[\mathcal{M}^l_{p\times p}] & -A^t[\mathcal{M}^l_{p\times p}] \end{pmatrix}.$$

I ignore if there are natural "recursive" analogues of (some of) the exceptional simple Lie algebras.

Remark 18.1. *Analogues of type B, C, D Lie algebras can also be defined using arbitrary non-singular symmetric or symplectic recurrence matrices in $Rec_{p\times p}(\mathbf{K})$.*

Remark 18.2. *A Lie-algebra $\mathcal{L} \subset Rec_{p\times p}(\mathbf{K})$ is in general not a recursively closed subspace of $Rec_{p\times p}(\mathbf{K})$.*

Remark 18.3. *It would be interesting to understand the algebraic structure of Lie algebras in $Rec_{p\times p}(\mathbf{K})$. Given such a Lie algebra $\mathcal{L} \subset Rec_{p\times p}(\mathbf{K})$, the intersection $\mathcal{L} \cap \mathcal{FS}_{p\times p}$ with the vector-space $\mathcal{FS}_{p\times p}$ of all finitely supported elements defines an ideal in $\mathcal{L}$. The interesting object is thus probably the quotient Lie algebra $\mathcal{L}/(\mathcal{L} \cap \mathcal{FS}_{p\times p})$. What is the structure of this quotient algebra for the analogues in $Rec_{p\times p}(\mathbb{C})$ of the $A - D$ series?*

18.1 Lie algebras in the convolution ring $\mathbf{K}^{\mathcal{M}_p}$

Using the convolution-ring structure on $\mathbf{K}^{\mathcal{M}_{p\times q}}$ we get other, different Lie algebras. Since the convolution-structure depends only on the product pq, we restrict ourself to $\mathbf{K}^{\mathcal{M}_p}$ for simplicity.

The three obvious subalgebras in $\mathbf{K}^{\mathcal{M}_p}$ (corresponding to formal power series in p non-commuting variables) given by $\mathbf{K}^{\mathcal{M}_p}$, $\mathrm{Rec}_p(\mathbf{K})$ and the vector space $\mathcal{FS}_p \subset \mathrm{Rec}_{p-rec}(\mathbf{K})$ of finitely supported elements give rise to three Lie-algebras with Lie bracket $[A,B] = A*B - B*A$ (where $*$ stands for the convolution product in $\mathbf{K}^{\mathcal{M}_p}$).

The Lie algebra resulting from $\mathcal{FS}_p$ (where $\mathcal{FS}_p$ as a convolution algebra is isomorphic to the polynomial algebra in p non-commuting variables) contains the free Lie algebra on p generators.

All these Lie algebras are filtrated: We have $[A,B][\emptyset,\emptyset] = 0$ for all $A, B \in \mathbf{K}^{\mathcal{M}_p}$ and $[A,B][\mathcal{M}_p^{<\alpha+\beta}] = 0$ if $A[\mathcal{M}_p^{<\alpha}] = 0$ and $B[\mathcal{M}_p^{<\beta}] = 0$.

19 Integrality and lattices of recurrence vectors

A recurrence matrix $A \in \mathrm{Rec}_{p\times q}(\mathbb{Q})$ is *integral* if all its coefficients $A[U,W]$ are integers. This notion can easily be generalized by considering coefficients in the integral ring $\mathcal{O}_{\mathbf{K}}$ of algebraic integers over a number field $\mathbf{K}$.

We denote by $\mathrm{Rec}_{p\times q}(\mathbb{Z})$ the $\mathbb{Z}-$module of all integral recurrence matrices in $\mathrm{Rec}_{p\times q}(\mathbb{Q})$. Since products of integral matrices are integral, one can consider the subcategory $\mathbb{Z}^{\mathcal{M}} \subset \mathbb{Q}^{\mathcal{M}}$ having only integral recurrence matrices as arrows and the subalgebra $\mathrm{Rec}_{p\times p}(\mathbb{Z}) \subset \mathrm{Rec}_{p\times q}(\mathbb{Q})$ consisting of all integral recurrence matrices.

Call a presentation $(A_1,\ldots,A_d)[\emptyset,\emptyset]$ with shift matrices $\rho(s,t) \in \mathbb{Q}^{d\times d}$ of a finite-dimensional recursively closed subspace in $\mathrm{Rec}_{p\times q}(\mathbb{Q})$ *integral* if $(A_1,\ldots,A_d)[\emptyset,\emptyset] \in \mathbb{Z}^d$ and $\rho(s,t) \in \mathbb{Z}^{d\times d}$ for $0 \le s < p, 0 \le t < q$.

Proposition 19.1. *Every integral recurrence matrix $A \in Rec_{p\times q}(\mathbb{Z})$ admits an an integral minimal presentation of $\overline{A}^{rec}$ such that $A_1,\ldots,A_a$ is a $\mathbb{Z}-$basis of $Rec_{p\times q}(\mathbb{Z}) \cap \overline{A}^{rec}$. In particular, A can be written as $A = \sum_{j=1}^{a} \lambda_j A_j$ with $\lambda_1,\ldots,\lambda_a \in \mathbb{Z}$.*

Proof Proposition 8.2 shows that the set $(\mathrm{Rec}_{p\times q}(\mathbb{Z}) \cap \overline{A}^{rec}) \subset \overline{A}^{rec} \subset \mathrm{Rec}_{p\times q}(\mathbb{Q})$ is a free $\mathbb{Z}-$module. Since it contains the recursive set-closure $\rho(\mathcal{M}_{p\times q})A$ spanning $\overline{A}^{rec}$, it is of maximal rank $a = \dim(\overline{A}^{rec})$. A $\mathbb{Z}-$basis $A_1,\ldots,A_a$ of $(\mathrm{Rec}_{p\times q}(\mathbb{Z}) \cap \overline{A}^{rec})$ has the required properties. □

We call an integral recurrence matrix $A \in \mathrm{Rec}_{p\times p}(\mathbb{Z})$ *unimodular* if $\det(A[\mathcal{M}_{p\times p}^l]) \in \{\pm 1\}$ for all $l \in \mathbb{Z}$. The set of of all integral unimodular recurrence matrices in $\mathrm{GL}_{p-rec}(\mathbb{Q})$ is the *unimodular subgroup of integral recurrence matrices* in $\mathrm{Rec}_{p\times p}(\mathbb{Q})$.

Remark 19.2. *The $\mathbb{Q}-$vector space $Rec_{p\times q}(\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}$ is in general a strict subspace of $Rec_{p\times q}(\mathbb{Q})$: An element $A\in Rec_{p\times q}(\mathbb{Q})\setminus Rec_{p\times q}(\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}$ is for instance given by $A[\mathcal{M}^l_{p\times q}]=\frac{1}{2^l}$, $l\in\mathbb{N}$. For $q=p$, the vector space $Rec_{p\times p}(\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}$ is thus a proper subalgebra of the algebra $Rec_{p\times p}(\mathbb{Q})$. However, given an arbitrary recurrence matrix $A\in Rec_{p\times q}(\mathbb{Q})$, there exists by Proposition 11.5 an integer $\alpha\geq 1$ and an integer $\lambda\geq 1$ with $\alpha H_\lambda A\in Rec_{p\times q}(\mathbb{Z})$ where $H_\lambda\in\mathcal{C}_{p-rec}(\mathbb{Q})$ is the integral diagonal matrix in the center $\mathcal{C}_{p-rec}(\mathbb{Q})$ of $Rec_{p\times p}(\mathbb{Q})$ with diagonal coefficients $H_\lambda[U,U]=\lambda^l$ for $(U,U)\in\mathcal{M}^l_{p\times p}$.*

Remark 19.3. *An integral unimodular recurrence matrix in $Rec_{p\times p}(\mathbb{Z})$ is generally not invertible in $Rec_{p\times p}(\mathbb{Z})$. An example is given by the converging lower triangular recurrence matrix $A\in\mathcal{T}_2(\mathbb{Z})\subset Rec_{2\times 2}(\mathbb{Z})$ of Toeplitz type (cf. Section 13.2) with generating series $1-2z\in\mathbb{Z}[[z]]$ already mentioned in the second part of Remark 15.5. Its inverse $A^{-1}\in\mathcal{T}_2(\mathbb{Z})\subset\mathbb{Q}^{\mathcal{M}_{2\times 2}}$ corresponds to the generating series $\frac{1}{1-2z}=1+2z+4z^2+8z^3+\cdots\in\mathbb{Z}[[z]]$ and cannot be a recurrence matrix by Proposition 11.2.*

19.1 Finite-dimensional lattices

Definition 19.4. *A finite-dimensional lattice of $Rec_{p\times q}(\mathbb{C})$ is a free $\mathbb{Z}-$module of finite rank in $Rec_{p\times q}(\mathbb{C})$.*

A finite-dimensional lattice of $\mathrm{Rec}_{p\times q}(\mathbb{C})$ is discrete for both topologies introduced in §10.

Since the multiplicative structure is irrelevant for lattices, it is enough to consider lattices in $\mathrm{Rec}_p(\mathbb{C})$. A particularly beautiful set of lattices is given by lattices which are recursively closed sets (ie. they satisfy the inclusion $\rho(\mathcal{M}_p)\Lambda\subset\Lambda$). An example of such a lattice is the subset $\mathrm{Rec}_p(\mathbb{Z})\cap\overline{A}^{rec}\subset\overline{A}^{rec}$ for $A\in\mathrm{Rec}_p(\mathbb{Q})$.

Given two recurrence vectors $A,B\in\mathrm{Rec}_p(\mathbb{R})$, we define their *scalar-product* as the element

$$\langle A,B\rangle=A^tB\in\mathrm{Rec}_1(\mathbb{R}).$$

More precisely, such a scalar-product can be represented by the generating series

$$\langle A,B\rangle_z=\sum_{l=0}^{\infty}z^l\sum_{U\in\mathcal{M}^l_p}A[U]B[U]\in\mathbb{R}[[z]]$$

and defines a rational function. For $\Lambda\subset\mathrm{Rec}(\mathbb{R})$ a lattice, we call the rational function $z\longmapsto\det(\langle A_i,A_j\rangle_z)$ (with $A_1,A_2,\ldots$ a $\mathbb{Z}-$basis of Λ) the *determinant* of Λ. Given a finite dimensional subspace $\mathcal{A}\subset\mathrm{Rec}_p(\mathbb{R})$ (spanned eg. by a lattice) there exists an open interval $(0,\alpha(\mathcal{A}))\subset\mathbb{R}$ such that bilinear map $\mathcal{A}^2\ni(A,B)\longmapsto\langle A,B\rangle_{z_0}\in\mathbb{R}$ defines an Euclidean scalar-product on $\mathcal{A}$ for all $z_0\in(0,\alpha(\mathcal{A}))$.

In particular, such an evaluation yields an isometry between a lattice $\Lambda \subset \mathrm{Rec}_p(\mathbb{R})$ and a lattice of the ordinary Euclidean vector space.

Remark 19.5. *It is of course also possible to define scalar-products using a symmetric positive definite recurrence matrix of $Rec_{p\times p}(\mathbb{R})$ (the case considered above corresponds to the identity).*

One defines similarly Hermitian products and Hermitian lattices over $\mathbb{C}$.

20 Monoids and their linear representations

Any quotient monoid $\mathcal{Q} = \langle \mathcal{M}^1_{p\times q} : \mathcal{R} \subset \mathcal{M}_{p\times q} \times \mathcal{M}_{p\times q}\rangle$ of the monoid $\mathcal{M}_{p\times q}$ (see chapter 2 for definitions) defines a subspace $V_{\mathcal{Q}} \subset \mathbf{K}^{\mathcal{M}_{p\times q}}$ by setting

$$V_{\mathcal{Q}} = \{A \in \mathbf{K}^{\mathcal{M}_{p\times q}} \mid \rho(UL_iV)A = \rho(UR_iV)A, U, V \in \mathcal{M}_{p\times q}, (L_i, R_i) \in \mathcal{R}\}.$$

The space $V_{\mathcal{Q}}$ is by construction recursively closed, and the shift monoid $\rho_{\mathcal{Q}}(\mathcal{M}_{p\times q})$ acts on $V_{\mathcal{Q}}$ with a "kernel" generated by the relations $\mathcal{R}$.

In the case $p = q$, the subspace $V_{\mathcal{Q}}$ is in general not multiplicatively closed. In the next section we will however describe a particular case where this happens.

Let us recall here a few elementary facts already discussed in chapter 4:

A linear representation of a monoid is a morphisme $\pi : \mathcal{Q} \longrightarrow \mathrm{End}(V)$ of some abstract monoid $\mathcal{Q}$ into a submonoid of $\mathrm{End}(V)$ where $\mathrm{End}(V)$ denotes the monoid of all linear endomorphisms of a vector space V. As in the case of linear representations of groups, one can define indecomposable representations (without proper non-trivial invariant subspace), direct sums of representations, irreducible representations (not a non-trivial direct sum of representations) etc and a linear representation of a monoid $\mathcal{Q}$ gives rise to a linear representation of its monoid-algebra $\mathbf{K}[\mathcal{Q}]$.

A linear representation of the free monoid on r generators is simply a set $\pi(g_1), \ldots, \pi(g_r) \subset \mathrm{End}(V)$ of r endomorphisms corresponding to the free generators $g_1, \ldots, g_r$. If a quotient monoid $\mathcal{Q}$ of a free monoid has relations $\mathcal{R}$, then $\pi(L_i) = \pi(R_i)$ for every relation $(L_i, R_i) \in \mathcal{R}$. Conjugate linear representations of a monoid are in generally considered as equivalent. Obviously, every recursively closed subspace of $\mathbf{K}^{\mathcal{M}_{p\times q}}$ gives rise to a representation of the free monoid $\mathcal{M}_{p\times q}$ on pq generators. Reciprocally, every finite-dimensional linear representation $\rho : \mathcal{M}_{p\times q} \longrightarrow \mathrm{End}(\mathbf{K}^d)$ defines a recursively closed finite-dimensional subspace of $\mathrm{Rec}_{p\times q}(\mathbf{K})$ by considering the direct sum (of dimension $\leq d^2$) of all recursively closed subspaces with shift-matrices $\rho(\mathcal{M}^1_{p\times q})$ and arbitrary initial values. The precise description of such subspaces will be given in section 22.

Remark 20.1. *A minimal presentation $A_1, A_2, \ldots, A_a$ of an element $A = A_1 \in Rec_{p\times q}$ having complexity $dim(\overline{A}^{rec}) = a$ yields a linear representation*

$\rho(\mathcal{M}_{p\times q})$, up to conjugation by elements of $GL_a(\mathbf{K})$ fixing the first basis vector A_1.

21 Abelian monoids

For $p, q \in \mathbb{N}$ we consider the *free abelian* quotient monoid $\mathcal{A}b_{p\times q} \sim \mathbb{N}^{pq}$ generated by all pq elements of $\mathcal{M}^1_{p\times q}$ with relations $\mathcal{R} = \{(XY, YX) \,|X, Y \in \mathcal{M}^1_{p\times q}\}$. The corresponding subspace $V_{\mathcal{A}b_{p\times q}}$ consists of all functions $A \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ such that

$$A[u_1 \dots u_n, w_1 \dots w_n] = A[u_{\pi(1)} \dots u_{\pi(n)}, w_{\pi(1)} \dots w_{\pi(n)}]$$

for all $(u_1 \dots u_n, w_1 \dots w_n) \in \mathcal{M}_{p\times q}$ and all permutations π of $\{1, \dots, n\}$. The vector space $V_{\mathcal{A}b_{p\times q}}$ can thus be identified with the vector space $\mathbf{K}[[Z_{0,0}, \dots, Z_{p-1,q-1}]]$ of formal power series in pq commuting variables $Z_{u,w}, 0 \le u < p, 0 \le w < q$. We denote the vector space $V_{\mathcal{A}b_{p\times q}}$ by $\mathbf{K}^{\mathcal{A}b_{p\times q}}$.

For $A \in \mathbf{K}^{\mathcal{A}b_{p\times q}}$ we have

$$\rho(s,t)\rho(s',t')A = \rho(s',t')\rho(s,t)A$$

for all $0 \le s, s' < p, 0 \le t, t' < q$. The easy computation

$$\begin{aligned}
\rho(s,t)\rho(s',t')(AB) &= \rho(s,t)\left(\sum_{v'}(\rho(s',v')A)(\rho(v',t')B)\right)\\
&= \sum_{v,v'}(\rho(s,v)\rho(s',v')A)(\rho(v,t)\rho(v',t')B)\\
&= \sum_{v,v'}(\rho(s',v')\rho(s,v)A)(\rho(v',t')\rho(v,t)B)\\
&= \cdots = \rho(s',t')\rho(s,t)(AB)
\end{aligned}$$

shows that $AB \in \mathbf{K}^{\mathcal{A}b_{p\times q}}$ if $A \in \mathbf{K}^{\mathcal{A}b_{p\times r}}$, $B \in \mathbf{K}^{\mathcal{A}b_{r\times q}}$. We get thus subcategories $\mathbf{K}^{\mathcal{A}b}$ of $\mathbf{K}^{\mathcal{M}}$ and $\mathrm{Rec}(\mathbf{K}) \cap \mathbf{K}^{\mathcal{A}b}$ of $\mathrm{Rec}(\mathbf{K})$.

These subcategories contain all elements of complexity 1. The algebra formed by all recurrence matrices in $\mathbf{K}^{\mathcal{A}b_{p\times p}}$ is thus not commutative if $p > 1$.

It would be interesting to have other examples of "natural" quotient monoids of $\mathcal{M}_{p\times q}, p, q \in \mathbf{N}$, giving rise to subcategories in $\mathbf{K}^{\mathcal{M}}$ and $\mathrm{Rec}(\mathbf{K})$.

Remark 21.1. *The association*

$$A \longmapsto f_A = A[\emptyset] + \sum_{n=1}^{\infty} \sum_{0\le u_1,\dots,u_n<p} A[u_1 \dots u_n] Z_{u_1} \cdots Z_{u_n}$$

of a recurrence vector $A \in \mathbb{C}^{\mathcal{A}b_p}$ to the formal power series f_A in p commuting variables is here completely natural (cf. Remark 11.3). Does this have interesting analytic consequences for f_A (which is holomophic in a neighbourhood of $(0, \dots, 0)$ by Proposition 11.2)?

22 Birecursivity

The *palindromic involution*

$$\iota(s_1 s_2 \dots s_{n-1} s_n, t_1 \dots t_n) = (s_n s_{n-1} \dots s_2 s_1, t_n \dots t_1)$$

defines an involutive antiautomorphism of the monoid $\mathcal{M}_{p\times q}$ and we get the *palindromic automorphism* $\iota : \mathbf{K}^{\mathcal{M}_{p\times q}} \longrightarrow \mathbf{K}^{\mathcal{M}_{p\times q}}$ by considering the involutive automorphism defined by $(\iota X)[U,W] = X[\iota(U,W)]$ for $X \in \mathbf{K}^{\mathcal{M}_{p\times q}}$. Since we have the identity $(\iota X)(\iota(Y) = \iota(XY), X \in \mathbf{K}^{\mathcal{M}_{p\times r}}, Y \in \mathbf{K}^{\mathcal{M}_{r\times q}}$, the palindromic automorphism defines an involutive automorphic functor of the category $\mathbf{K}^{\mathcal{M}}$.

In characteristic $\neq 2$, the palindromic automorphism ι endows the algebra $\mathbf{K}^{\mathcal{M}_{p\times q}}$ with a $\mathbb{Z}/2\mathbb{Z}-$ grading in the usual way by considering the decomposition $X = X_+ + X_-$ into its *even* and *odd* (palindromic) parts defined by

$$X_+ = \frac{X + \iota X}{2} \text{ and } X_- = \frac{X - \iota X}{2}$$

for $X \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ and we have the sign-rules

$$(XY)_+ = X_+Y_+ + X_-Y_-, (XY)_- = X_+Y_- + X_-Y_+$$

whenever the matrix product XY is defined. In particular, we get a subcategory $\left(\mathbf{K}^{\mathcal{M}}\right)_+$ consisting of all even parts in $\mathbf{K}^{\mathcal{M}}$.

Conjugating the shift-monoid $\rho(\mathcal{M}_{p\times q})$ by ι, we get a second morphism $\lambda : \mathcal{M}_{p\times q} \longrightarrow \mathrm{End}(\mathbf{K}^{\mathcal{M}_{p\times q}})$, called the *left-shift-monoid*. It is defined by

$$(\lambda(s_1 \dots s_n, t_1 \dots t_n)X)[U,W] = X[s_n \dots s_1 U, t_n \dots t_1 W].$$

The obvious commutation rule $\rho(S,T)\lambda(S',T') = \lambda(S',T')\rho(S,T)$ yields an action of the (direct) product-monoid $\mathcal{M}_{p\times q} \times \mathcal{M}_{p\times q}$ on $\mathbf{K}^{\mathcal{M}_{p\times q}}$. An element $X \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ is a *birecurrence matrix* if the linear span $\overline{X}^{birec}$ of the orbit $\lambda(\mathcal{M}_{p\times q})\rho(\mathcal{M}_{p\times q})X$ has finite dimension. The *birecursive complexity* of X is defined as $\dim(\overline{X}^{birec}) \in \mathbb{N} \cup \{\infty\}$.

Proposition 22.1. *We have*

$$dim(\overline{A}^{birec}) = dim(\mathbf{K}[\rho_{\mathcal{A}}(\mathcal{M}_{p\times q})])$$

with $\mathbf{K}[\rho_{\mathcal{A}}(\mathcal{M}_{p\times q})] \subset End(\mathcal{A})$ *denoting the subalgebra of* $End(\mathcal{A})$ *generated by the shift-monoid acting on the recursive closure* $\mathcal{A} = \overline{A}^{rec}$ *of* A.

In particular, we have the inequalities

$$dim(\overline{A}^{rec}) \leq dim(\overline{A}^{birec}) \leq \left(dim(\overline{A}^{rec})\right)^2.$$

Remark 22.2. *The proof shows in fact that the action of the shift-monoid $\rho(\mathcal{M}_{p\times q})$ on $\overline{A}^{birec}$ is (isomophic to) the obvious linear action by right multiplication of $\rho_{\mathcal{A}}(\mathcal{M}_{p\times q})$ on $\mathbf{K}[\rho_{\mathcal{A}}(\mathcal{M}_{p\times q})]$.*

In particular, if $pq \geq 2$ we have generically the equality

$$dim(\overline{A}^{birec}) = \left(dim(\overline{A}^{rec})\right)^2$$

for $A \in Rec_{p\times q}(\mathbb{C})$.

Corollary 22.3. *(i) The vector spaces of recurrence matrices and birecurrence matrices in $\mathbf{K}^{\mathcal{M}_{p\times q}}$ coincide.*

(ii) The palindromic automorphism ι of $\mathbf{K}^{\mathcal{M}_{p\times q}}$ restricts to an involutive automorphism of $Rec_{p\times q}(\mathbf{K})$. The decomposition of a recurrence matrix A into its even and odd parts $A_+ = (A + \iota A)/2,\ A_- = (A - \iota A)/2$ holds thus in $Rec(\mathbf{K})$ for $\mathbf{K}$ of characteristic $\neq 2$.

(iii) The palindromic automorphism yields an involutive automorphic functor of the category $Rec(\mathbf{K})$.

Remark 22.4. *Defining the palindromic element $P_p^\iota \in \mathbf{K}^{\mathcal{M}_{p\times p}}$ by $P_p^\iota[s_1 \dots s_n, s_n \dots s_1] = 1$ and $P_p^\iota[s_1 \dots s_n, t_1 \dots t_n] = 0$ if $s_1 \dots s_n \neq t_n \dots t_1$, the palindromic automorphism of $\mathbf{K}^{\mathcal{M}_{p\times q}}$ or $Rec_{p\times q}$ is given by $X \longmapsto P_p^\iota X P_q^\iota$. In particular, the palindromic automorphism is an inner automorphism of the algebra $\mathbf{K}^{\mathcal{M}_{p\times p}}$.*

The palindromic element P_p^ι is however not a recurrence matrix if $p \geq 2$. Indeed, we have $(\rho(s_1 \dots s_n, t_1 \dots t_n)P_p^\iota)[u_1 \dots u_n, w_1 \dots w_n] = 1$ if $u_1 \dots u_n = t_n \dots t_1, w_1 \dots w_n = s_n \dots s_1$ and $(\rho(s_1 \dots s_n, t_1 \dots t_n)P_p^\iota)[u_1 \dots u_n, w_1 \dots w_n] = 0$ otherwise. This implies $dim(\overline{P_p^\iota}^{rec}) \geq p^{2n}$ for all $n \in \mathbb{N}$ and shows that the palindromic automorphism of the algebra $Rec_{p\times p}(\mathbf{K})$ is not inner for $p \geq 2$. (For $p = 1$ the palindromic automorphisms P_1^ι is trivial and thus inner in the commutative algebra $Rec_{1\times 1}(\mathbf{K})$.)

Let us finish this remark by adding that the algebras $Rec_{p\times p}(\mathbf{K})$ admit many more similar "exterior" automorphisms. An example is for instance given by the automorphism φ_k (for $k \geq 1$) which acts as ι on $A[\mathcal{M}_{p\times p}^{nk}]$ and as the identity on $A[\mathcal{M}_{p\times p}^{n}]$ if n is not divisible by k. It would perhaps be interesting to understand the algebraic structure of the group of outer automorphism which is defined as the quotient of all automorphisms of the algebra $Rec_{p\times p}(\mathbf{K})$ by the normal subgroup $GL_{p-rec}(\mathbf{K})/(GL_{p-rec}(\mathbf{K}) \cap \mathcal{C}_{p-rec})$ (where $\mathcal{C}_{p-rec}$ denotes the center of $Rec_{p\times p}(\mathbf{K})$) of inner automorphisms given by conjugations.

Proof of Proposition 22.1 Choose a presentation

$$(A_1 = A, A_2, \dots)[\emptyset, \emptyset], \rho(\mathcal{M}_{p\times q}^1) \subset \mathrm{End}(\mathcal{A})$$

of $\mathcal{A} = \overline{A}^{rec}$. The right action ρ of $\mathcal{M}_{p\times q}$ on A is obvious and Proposition 7.1 shows that $\tilde{A} = \lambda(s,t)A$ is presented by

$$(\tilde{A}_1 = \tilde{A}, \tilde{A}_2, \dots)[\emptyset, \emptyset] = ((A_1 = A, A_2, \dots)[\emptyset, \emptyset])\, \rho(s,t)$$

with shift-matrices $\rho(\mathcal{M}^1_{p\times q})$ as above.

The right action $\rho(\mathcal{M}_{p\times q})$ of the shift-monoid on the birecursive closure $\overline{A}^{birec}$ is thus (up to conjugacy) given by the right multiplication of $\rho_{\mathcal{A}}(\mathcal{M}_{p\times q})$ on the algebra $\mathbf{K}[\rho_{\mathcal{A}}(\mathcal{M}_{p\times q})] \subset \mathrm{End}(\mathcal{A})$ (where $\mathcal{A} = \overline{A}^{rec}$). This implies the equality

$$\dim(\overline{A}^{birec}) = \dim(\mathbf{K}[\rho_{\mathcal{A}}(\mathcal{M}_{p\times q})]) \ .$$

The inequalities are now obvious. □

Remark 22.5. *The content of this chapter can roughly be paraphrased as follows: Recursively closed subspaces of* $\mathbf{K}^{\mathcal{M}_{p\times q}}$ *are direct sums of linear vector spaces spanned by orbits of* $\rho(\mathcal{M}_{p\times q})$*. Birecursively closed vector spaces are suitable linear representations of the monoid algebra* $\mathbf{K}[\mathcal{M}_{p\times q}]$*. Notice that not every linear representation corresponds to a birecursively closed subspace of* $\mathbf{K}^{\mathcal{M}_{p\times q}}$*; for instance direct sums containing (up to isomorphism) a common irreducible linear subrepresentation of* $\mathcal{M}_{p\times q}$ *are forbidden.*

23 Virtual representations and birecursively closed subspaces

A finite-dimensional birecursively closed subspace $\mathcal{A} \subset \mathrm{Rec}_{p\times p}(\mathbf{K})$ gives rise to a finite-dimensional representation $\rho_{\mathcal{A}} : \mathcal{M}_{p,p} \longrightarrow \mathrm{End}(\mathcal{A})$. Choosing a basis of $\mathcal{A}$ yields a finite-dimensional linear representation of $\rho_{\mathcal{A}}(\mathcal{M}_{p\times p})$. Considering the equivalence relation on representations given by conjugation, birecursively closed subspaces of $\mathrm{Rec}_{p\times p}(\mathbf{K})$ are in bijection with suitable finite-dimensional matrix-representations of $\mathcal{M}_{p\times p}$. We denote by Rep the set of equivalence classes of all finite-dimensional representations of $\mathcal{M}_{p\times p}$ and introduce the free vector space $\mathbf{K}[\mathrm{Rep}]$ with basis Rep and elements formal linear combinations of finite-dimensional representations of $\mathcal{M}_{p\times p}$. Setting $\tau = \rho\sigma$ for $\rho, \sigma \in \mathrm{Rep}$ where

$$\tau(s,t)_{kl,ij} = \sum_{u=1}^{r} \rho(s,u)_{k,i}\sigma(u,t)_{l,j} \ ,$$

cf. Proposition 7.5, defines an associative bilinear product on $\mathbf{K}[\mathrm{Rep}]$ and turns it into an associative algebra.

This algebra has two interesting quotients. The first one is given by considering the quotient by the ideal generated by $\sigma = \rho \oplus \tau$ if the representation σ is the direct sum of subrepresentations ρ, τ. Its elements, sometimes called *virtual representations* are thus linear combinations of indecomposable (but not necessarily irreducible) finite-dimensional representations of the monoid $\mathcal{M}_{p\times p}$.

Another quotient of $\mathbf{K}[\mathrm{Rep}]$ is obtained by considering the free vector space generated by all finite-dimensional birecursively closed subspaces in $\mathrm{Rec}_{p\times p}(\mathbf{K})$. Given two such subspaces $\mathcal{A},\mathcal{B}$ the product $\mathcal{C}=\mathcal{A}\mathcal{B}$ is the smallest birecursively closed subspace containing (and in fact spanned by) all products XY with $X\in\mathcal{A}$ and $Y\in\mathcal{B}$.

Remark 23.1. *The usual tensor product of matrices endows the vector-space $\mathbf{K}[Rep]$ with a different, commutative (and associative) product.*

Remark 23.2. *All constructions can also be carried out on the quotient space $Rec_{p\times p}(\mathbf{K})/\mathcal{FS}_{p\times p}$.*

One can of course also consider not necessarily finite-dimensional representations and subspaces. In this case one might also work in the full algebra $\mathbf{K}^{\mathcal{M}_{p\times p}}$.

Let us also mention the obvious algebra structure on the free vector space spanned by all recursively (but not necessarily birecursively) closed finite-dimensional subspaces of $Rec_{p\times p}(\mathbf{K})$ (or of $Rec_{p\times p}(\mathbf{K})/\mathcal{FS}_{p\times p}$).

Computations in most algebras discussed in this paragraph can be done algorithmically and involve only finitely many operations on finite amounts of data.

24 Finite monoids and finite state automata

This section describes a connection between automatic functions or automatic sequences (defined by finite state automata) and recurrence matrices with finite shift-monoid.

24.1 Finite monoids

Proposition 24.1. *Given a recurrence matrix $A\in Rec_{p\times q}(\mathbf{K})$, the following assertions are equivalent:*

(i) The subset $\{A[U,W]\mid (U,W)\in\mathcal{M}_{p\times q}\}\subset\mathbf{K}$ of values of A is finite.

(ii) For all $B\in\overline{A}^{rec}$, the subset $\{B[U,W]\mid (U,W)\in\mathcal{M}_{p\times q}\}\subset\mathbf{K}$ of values of B is finite.

(iii) The recursive set-closure $\rho(\mathcal{M}_{p\times q})A$ of A is finite.

(iv) The shift-monoid $\rho_{\overline{A}^{rec}}(\mathcal{M}_{p\times q})\in End(\overline{A}^{rec})$ of A is finite.

Proof of Proposition 24.1 For a recurrence matrix $A\in\mathrm{Rec}_{p\times q}(\mathbf{K})$ satisfying assertion (i), consider the finite set $\mathcal{F}=\{A[U,W]\mid (U,W)\in\mathcal{M}_{p\times q}\}\subset\mathbf{K}$ containing all evaluations of A. The set $\mathcal{F}$ contains thus also all evaluations $\{X[U,W]\mid (U,W)\in\mathcal{M}_{p\times q}\}$ for $X\in\rho(\mathcal{M}_{p\times q})A$ in the set-closure of A. Choose $(U_1,W_1),\ldots,(U_a,W_a)\in\mathcal{M}_{p\times q}$ such that $\rho(U_1,W_1)A,\ldots,\rho(U_a,W_a)A$ form a basis of $\overline{A}^{rec}$. An arbitrary element $B\in\overline{A}^{rec}$ can thus be written as a linear combination $B=\sum_{j=1}^{a}\beta_j\rho(U_j,W_j)A$ and the set $\{B[U,W]\mid (U,W)\in\mathcal{M}_{p\times q}\}$ of all its evaluations is a subset of

the finite set $\sum_{j=1}^{a} \beta_j \mathcal{F}$ containing at most $1 + a\sharp\{\mathcal{F} \setminus \{0\}\} < \infty$ elements. This shows the equivalence of assertions (i) and (ii).

All elements in the set closure $\rho(\mathcal{M}_{p\times q})A \subset \overline{A}^{rec}$ of $A \in \mathrm{Rec}_{p\times q}(\mathbf{K})$ are determined by their projection onto $\overline{A}^{rec}[\mathcal{M}_{p\times q}^{\leq N}]$ where N denotes the saturation level of $\overline{A}^{rec}$. Assuming (i), the elements of $\rho(\mathcal{M}_{p\times q})A$ are thus in bijection with a subset of the finite set $\mathcal{F}^{\mathcal{M}_{p\times q}^{\leq N}}$ where $\mathcal{F} = \{A[U,W] \mid (U,W) \in \mathcal{M}_{p\times q}\} \subset \mathbf{K}$ is the finite set of all evalutions of A. This proves that (i) implies (iii).

The shift-monoid $\rho_{\overline{A}^{rec}}(\mathcal{M}_{p\times q}) \subset \mathrm{End}(\overline{A}^{rec})$ of A acts faithfully on the set-closure $\rho(\mathcal{M}_{p\times q})A$ of A. Finiteness of $\rho(\mathcal{M}_{p\times q})A$ implies thus (iv).

We have $\{A[U,W] \mid (U,W) \in \mathcal{M}_{p\times q}\} = \{(\rho(U,W)A)[\emptyset,\emptyset] \mid (U,W) \in \mathcal{M}_{p\times q}\}$ which shows that (iv) implies (i). □

24.2 Automata

An *initial automaton* with *input alphabet* X and *output alphabet* Y is a directed graph $\mathcal{A}$ with vertices or *states* $\mathcal{V}$ such that

- $\mathcal{V}$ contains a marked *initial state* v_*.
- The set of directed edges originating in a given state $v \in \mathcal{V}$ is bijectively labelled by the input alphabet X.
- We have an *output function* $w : \mathcal{V} \longrightarrow Y$ on the set $\mathcal{V}$ of states.

An initial automaton $\mathcal{A}$ is *finite state* if its input alphabet X and its set of states $\mathcal{V}$ are both finite.

An initial automaton defines an application $\alpha : \mathcal{M}_X \longrightarrow Y$ from the free monoid $\mathcal{M}_X$ generated by the input alphabet X into Y. Indeed, an element $x = x_1 \dots x_n \in \mathcal{M}_X$ corresponds to a unique continuous path $\gamma \subset \mathcal{A}$ of length n starting at v_* and running through n directed edges labelled $x_1, x_2, \dots, x_n$. We set $\alpha(x_1 \dots x_n) = w(v_x)$ where v_x is the endpoint of the path γ. Every function $\alpha \in Y^{\mathcal{M}_X}$ can be constructed in this way by a suitable initial automaton which is generally infinite. A function $\alpha \in Y^{\mathcal{M}_X}$ is *automatic* if it is realizable by a finite state initial automaton. Automatic functions (with $Y \subset \mathbf{K}$ a subset of a field) which are converging elements of $\mathbf{K}^{\mathcal{M}_p}$ are called $p-$*automatic* or *automatic sequences.*

Proposition 24.2. *Automatic functions with $X = \mathcal{M}_p$ and $Y = \mathbf{K}$ are in bijection with elements of $Rec_p(\mathbf{K})$ having finite shift-monoids.*

Proof We show first that an automatic function $\alpha \in \mathbf{K}^{\mathcal{M}_p}$ yields a recurrence vector with finite shift-monoid. Choose a finite state initial automaton $\mathcal{A} = (v_* \in \mathcal{V}, w \in \mathbf{K}^{\mathcal{V}})$ realizing α and having $a = \sharp(\mathcal{V})$ states. Given a word $S = s_1 \dots s_n \in \mathcal{M}_p^n$, we get an application $\gamma_S : \mathcal{V} \longrightarrow \mathcal{V}$

defined by $\gamma_S(v) = u$ if the path of length n starting at state $v \in \mathcal{V}$ and consisting of directed edges with labels $s_1, \ldots, s_n$, ends at state u. Associate to a word S the initial finite state automaton $\mathcal{A}_S = (v_* \in \mathcal{V}, \tilde{w} = w \circ \gamma_S)$ with the same set $\mathcal{V}$ of states and the same initial state v_* but with output function $\tilde{w}(v) = w(\gamma_S(v)) \in \mathbf{K}$. We have then by construction $\alpha(US) = \alpha_S(U)$ where α_S is the automatic function of the finite state initial automaton $\mathcal{A}_S$.

Given a fixed finite state initial automaton $\mathcal{A}$, the automatic function α_S depends only on the function γ_S which belongs to the finite set $\mathcal{V}^{\mathcal{V}}$ of all a^a functions from $\mathcal{V}$ to $\mathcal{V}$. The recursive set-closure

$$\rho(\mathcal{M}_p)\alpha = \{\alpha_S \mid S \in \mathcal{M}_p\}$$

of $\alpha \in \mathbf{K}^{\mathcal{M}_p}$ is thus finite. This proves that $\alpha \in \mathrm{Rec}_p(\mathbf{K})$ and Proposition 24.1 implies that the shift-monoid of α is finite.

Consider now a recurrence matrix $A \in \mathbf{K}^{\mathcal{M}_p}$ with finite shift-monoid. We have to show that $U \longmapsto A[U]$ (for $U \in \mathcal{M}_p$) is an automatic function. If A is identically zero, there is nothing to prove. Otherwise, complete A to a basis $A_1 = A, A_2, \ldots, A_a$ of its recursive closure $\overline{A}^{rec}$ of dimension a and consider the associated presentation with shift-matrices $\rho_{\overline{A}^{rec}}(s) \in \mathbf{K}^{a\times a}$ and initial values $W = (A_1[\emptyset], \ldots, A_a[\emptyset]) \in \mathbf{K}^a$. We get thus a matrix-realization $\mathcal{M}_A = \rho_{\overline{A}^{rec}}(\mathcal{M}_p) \subset \mathrm{End}(\mathbf{K}^a)$ of the finite shift-monoid of A. Consider now the Cayley graph Γ of $\mathcal{M}_A$ with respect to the generators $\rho_{\overline{A}^{rec}}(s) \in \rho_{\overline{A}^{rec}}(\mathcal{M}_p^1)$. The graph Γ is the finite graph with vertices indexed by all elements of $\mathcal{M}_A$. An oriented (or directed) edge labelled s joins X to Y if $Y = X\rho_{\overline{A}^{rec}}(s)$. Consider the graph Γ as an initial finite state automaton with initial state the identity $\rho_A(\emptyset)$ and weight function $w(X) = \left(X^tW^t\right)_1$ where $\left(X^tW^t\right)_1$ is the first coordinate of the row-vector $X^tW^t \in \mathbf{K}^a$. By construction, the automatic function $\alpha \in \mathbf{K}^{\mathcal{M}_p}$ associated to this initial automaton is given by $\alpha(U) = A_1[U]$ since we have

$$\begin{pmatrix} A_1[u_1 \ldots u_n] \\ A_2[u_1 \ldots u_n] \\ \vdots \\ A_a[u_1 \ldots u_n] \end{pmatrix} = \rho_{\overline{A}^{rec}}(u_n)^t \cdots \rho_{\overline{A}^{rec}}(u_1)^t \begin{pmatrix} A_1[\emptyset] \\ A_2[\emptyset] \\ \vdots \\ A_a[\emptyset] \end{pmatrix}$$

(cf. Proposition 7.1). □

Remark 24.3. *When dealing with automatic functions, the output alphabet is often a finite field* $\mathbf{K}$. *Shift-monoids of recurrence matrices are then always finite and* $\mathbf{K}-$*valued automatic functions on* $\mathcal{M}_p$ *are in bijection with* $Rec_p(\mathbf{K})$. *Automatic functions form thus a ring with respect to the Hadamard product (Proposition 4.9, assertion (i)) or with respect to the polynomial product (see §13.3). Converging automatic functions (also called automatic sequences,) form a subring for the Hadamard product (see also [1],*

Corollary 5.4.5 and Theorem 12.2.6) and a differential subring (after identification with the corresponding generating series, see Proposition 13.6) for the polynomial product (see also [1], Theorem 16.4.1).

25 The categories of transducers and finite-state transducers

The *transducer* of an element $A \in Y^{\mathcal{M}_X}$ is the length-preserving application $\tau_A : \mathcal{M}_X \longrightarrow \mathcal{M}_Y$ defined by $\tau_A(\emptyset) = \emptyset$ and

$$\tau_A(u_1 \dots u_n) = A[u_1 u_2 \dots u_n] A[u_2 \dots u_n] A[u_3 \dots u_n] \dots A[u_{n-1} u_n] A[u_n].$$

A transducer τ_A is in general not a morphism of monoids (except if $A[u_1 \dots u_n] = A[u_1]$ for all $u_1 \dots u_n \in \mathcal{M}_X$ of length ≥ 1) and the identity $\tau_A = \tau_B$ holds if and only if $A[U, W] = B[U, W]$ for all $(U, W) \in \mathcal{M}_X \setminus \emptyset$.

Proposition 25.1. *A length-preserving function $\tau : \mathcal{M}_X \longrightarrow \mathcal{M}_Y$ is a transducer if and only if $\tau(uU)$ is of the form $*\tau(U)$ for all $u \in X$ and $U \in \mathcal{M}_X$.*

Corollary 25.2. *The composition $\tau_B \circ \tau_A : \mathcal{M}_X \longrightarrow \mathcal{M}_Z$ of two transducers $\tau_A : \mathcal{M}_X \longrightarrow \mathcal{M}_Y, \tau_B : \mathcal{M}_Y \longrightarrow \mathcal{M}_Z$ is a transducer.*

Proof of Proposition 25.1 Given a length-preserving function $\tau : \mathcal{M}_X \longrightarrow \mathcal{M}_Y$, set $A[\emptyset] = \alpha$ with $\alpha \in Y$ arbitrary and $A[uU] = y$ if $\tau(uU) = y\tau(U)$. The transducer τ_A associated to A coincides then with the function τ if and only if τ satisfies the conditions of Proposition 25.1. □

The easy proof of Corollary 25.2 is left to the reader.

Corollary 25.2 allows to define the *category of transducers* by considering free monoids $\mathcal{M}_p$ on $\{0, 1, \dots, p-1\}$ as objects with arrows from $\mathcal{M}_p$ to $\mathcal{M}_q$ given by transducers $\tau_A : \mathcal{M}_p \longrightarrow \mathcal{M}_q$ associated to $A \in \{0, \dots, q-1\}^{\mathcal{M}_p}$. The set of all transducers from $\mathcal{M}_p$ to $\mathcal{M}_q$ is in bijection with the set of functions $\{0, \dots, q-1\}^{\mathcal{M}_p \setminus \emptyset}$.

Given a transducer $\tau : \mathcal{M}_p \longrightarrow \mathcal{M}_q$, we define its *transducer-matrix* $M_\tau \in \mathbf{K}^{\mathcal{M}_{q \times p}}$ by $M_\tau[U, W] = 1$ if $\tau(W) = U$ and $M_\tau[U, W] = 0$ otherwise. The set of transducer-matrices is contained in $\mathbf{K}^{\mathcal{M}_{q \times p}}$ for any field $\mathbf{K}$. Since $M_{\tau \circ \tau'} = M_\tau M_{\tau'}$, the application $\tau \longmapsto M_\tau$ which associates to a transducer its transducer-matrix is a faithful functor from the category of transducers into the category $\mathbf{K}^{\mathcal{M}}$.

A transducer $\tau_A \in \mathcal{M}_q^{\mathcal{M}_p}$ is *finite-state* if $A \in \{0, \dots, q-1\}^{\mathcal{M}_p}$ is automatic.

Proposition 25.3. *A transducer-matrix $M_\tau \in \mathbf{K}^{\mathcal{M}_{q \times p}}$ is a recurrence matrix if and only if τ is a finite-state transducer.*

Corollary 25.4. *Finite-state transducers form a subcategory in the category of transducers.*

Corollary 25.5. *The category of finite-state transducers can be realized as a subcategory of the category Rec*($\mathbf{K}$) *of recurrence matrices over any field* $\mathbf{K}$.

Proof or Proposition 25.3 Consider a transducer $\tau = \tau_A : \mathcal{M}_p \longrightarrow \mathcal{M}_q$ associated to $A \in \{0, \dots, q-1\}^{\mathcal{M}_p} \subset \mathbb{Q}^{\mathcal{M}_p}$. For $T \in \mathcal{M}_p$, we denote by τ_T the finite state transducer and by M_T the transducer matrix associated to the function $\rho(T)A$ (defined in the usual way by $(\rho(T)A)[W] = A[WT]$). We have then

$$\tau(WT) = \tau_T(W)\tau(T)$$

where $\tau = \tau_\emptyset$. Moreover, the formula

$$(\rho(s,t)M_T)[U,W] = M_T[Us, Wt] = M[Us\tau(T), WtT]$$

shows the identities $\rho(s,t)M_T = M_{tT}$ if $\tau(tT) = s\tau(T)$ (or, equivalently, if $A[tT] = s$) and $\rho(s,t)M_T = 0$ otherwise, for all $(s,t) \in \mathcal{M}^1_{q\times p}$.

If A is automatic, then $A \in \mathrm{Rec}_p(\mathbb{Q})$ by Proposition 24.2. Since A takes all its values in the finite set $\{0, \dots, q-1\}$, Proposition 24.1 implies finiteness of its recursive set-closure $\rho(\mathcal{M}_p)A = \{\rho(T)A \mid T \in \mathcal{M}_p\}$. This proves finiteness of the set $\mathcal{S} = \{M_T \mid T \in \mathcal{M}_p\} \subset \mathbf{K}^{\mathcal{M}_{q\times p}}$. The recursive set-closure $\rho(\mathcal{M}_{q\times p})M \subset \{0 \cup \mathcal{S}\}$ (with $0 \in \mathbf{K}^{\mathcal{M}_{q\times p}}$ denoting the zero recurrence matrix) is thus also finite and $M = M_\emptyset = M_\tau$ is a recurrence matrix.

In the other direction, we consider a transducer matrix $M = M_\tau \in \mathrm{Rec}_{q\times p}(\mathbf{K})$ which is a recurrence matrix of a transducer $\tau = \tau_A : \mathcal{M}_p \longrightarrow \mathcal{M}_q$ associated to $A \in \{0, \dots, q-1\}^{\mathcal{M}_p}$. The recurrence matrix $\rho(S,T)M$ is then either 0 or the transducer matrix of the transducer $\tau_{\rho(T)A}$ associated to $\rho(T)A$. Finiteness of $\rho(\mathcal{M}_{q\times p})M$ (which follows from Proposition 24.1) implies finiteness of the recursive set-closure $\rho(\mathcal{M}_p)A$ and shows that A is a recurrence matrix with finite shift-monoid. The function A is thus automatic by Proposition 24.2 and $\tau = \tau_A$ is an automatic transducer. □

Proposition 25.6. *A recurrence matrix* $M \in \mathbf{K}^{\mathcal{M}_{q\times p}}$ *is a transducer-matrix of a finite-state transducer if and only if it can be given by a presentation with initial values* $(M = M_1, M_2, \dots, M_d)[\emptyset, \emptyset] = (1, 1, \dots, 1)$ *and shift matrices* $\rho(s,t) \in \{0,1\}^{d\times d}$ *with coefficients in* $\{0,1\}$ *such that all column-sums of the* p *matrices* $\sum_{s=0}^{q-1} \rho(s,t)$ *(with fixed* $t \in \{0, \dots p-1\}$*) are* 1.

Moreover, all finite-state transducers $\tau_1, \dots, \tau_d$ *defined by* d *transducer-matrices* $M_1, \dots M_d$ *as above are surjective if and only if all column-sums of the* q *matrices* $\sum_{t=0}^{p-1} \rho(s,t)$ *are strictly positive.*

Remark 25.7. *There are* $(qd)^{pd}$ *presentations of complexity* d *definining* d *finite-state transducer-matrices in* $\mathrm{Rec}_{q\times p}$ *as in Proposition 25.6. Indeed, these presentations are in bijection with matrices* $\{0,1\}^{qd\times pd}$ *having*

all column-sums equal to 1 *as can be seen by contemplating the matrix*

$$\begin{pmatrix} \rho(0,0) & \rho(0,1) & \dots & \rho(0,p-1) \\ \rho(1,0) & \rho(1,1) & \dots & \rho(1,p-1) \\ \vdots & & & \vdots \\ \rho(q-1,0) & \rho(q-1,1) & \dots & \rho(q-1,p-1) \end{pmatrix}$$

obtained by gluing all qp *shift-matrices* $\rho(s,t)$ *into a* $qd \times pd$ *matrix. Since all* pd *columns of such a matrix can be choosen independently with* qd *possibilities, there are* $(qd)^{pd}$ *such matrices.*

Proof of Proposition 25.6 Given a transducer-matrix $M \in \mathrm{Rec}_{q\times p}(\mathbf{K})$ associated to $A \in \mathrm{Rec}_p(\mathbb{Q})$, we have $\rho(s,t)M = 0$ if $A[t] \neq s$ and $\rho(s,t)M$ is the transducer matrix M_t associated to $\rho(t)A \in \mathrm{Rec}_p(\mathbb{Q})$ if $A[t] = s$. The first part of Proposition 25.6 follows by considering the presentation defined by all distinct transducer matrices in $\{M_T \mid T \in \mathcal{M}_p\} = \rho(\mathcal{M}_{q\times p})M \setminus \{0\}$ where M_T is the transducer matrix associated to $\rho(T)A \in \mathrm{Rec}_p(\mathbb{Q})$.

By induction on the length l of words in $\mathcal{M}_p^l$, the transducers associated to the transducer matrices M_T are all surjective if for each $s \in \{0,\dots,q-1\}$ there exists an integer $t = t_T, 0 \leq t < p$ such that $A[tT] = s$. The row corresponding to a transducer matrix M_T of the shift-matrix $\rho(s,t_T) \in \{0,A\}^{d\times d}$ contains thus at least one non-zero coefficient and this proves the last part of Proposition 25.6 . □

Remark 25.8. *Recursive presentation as introduced in chapter 7.2 are particularly useful for transducer-matrices: They admit recursive presentations of the form* $A_i = (\rho_i, R(i)), i = 1,\dots,d$ *of depth* 0 *such that all rows of all matrices* $R(i)$ *contain exactly one non-zero element which belongs to the set* $A_1,\dots A_d$.

As an example we consider the two finite-state transducer-matrices $M_1, M_2 \in \{0,1\}^{\mathcal{M}_{2\times 3}}$ *recursively defined by*

$$M_1 = (1, \begin{pmatrix} M_1 & M_2 & 0 \\ 0 & 0 & M_1 \end{pmatrix}), \quad M_2 = (1, \begin{pmatrix} 0 & M_2 & M_1 \\ M_2 & 0 & 0 \end{pmatrix}).$$

They span a recursively closed subspace in $Rec_{2\times 3}(\mathbb{Q})$ *with (monoidal) presentation given by* $(M_1,M_2)[\emptyset,\emptyset] = (1,1)$ *and*

$$\rho(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \rho(0,1) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \rho(0,2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$\rho(1,0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \rho(1,1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \rho(1,2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Remark 25.9. *Many authors define the transducer of a function* $A \in Y^{\mathcal{M}_X}$ *as the length-preserving application* $\tilde{\tau}_A : \mathcal{M}_X \longrightarrow \mathcal{M}_Y$ *defined by* $\tau_A(\emptyset) = \emptyset$ *and*

$$\tilde{\tau}_A(x_1\dots x_n) = A[x_1]A[x_1x_2]A[x_1x_2x_3]\dots A[x_1\dots x_n].$$

This definition yields an equivalent theory since we have $\tilde{\tau}_A = \iota \circ \tau_A \circ \iota$ with ι denoting the palindromic antiautomorphism $w_1 \dots w_n \longmapsto w_n \dots w_n$ of $\mathcal{M}_X$ or $\mathcal{M}_Y$ considered in chapter 22. The corresponding transducer matrices M_{τ_A} and $M_{\tilde{\tau}_A}$ (for $A \in \left(\mathcal{M}_q^1\right)^{\mathcal{M}_p}$ are thus related by $M_{\tilde{\tau}} = P_q^\iota M_\tau P_p^\iota$ where $P_n^\iota \in \mathbf{K}^{\mathcal{M}_{n\times n}}$ is the palindromic involution defined in chapter 22. By Proposition 22.1, the "transducer matrix" of $\tilde{\tau}_A$ is thus a recurrence matrix if and only if the transducer matrix of τ_A is a recurrence matrix.

25.1 Transducers and regular rooted planar trees

Definition 25.10. *The $p-$regular rooted planar tree is the infinite tree $\mathbf{T}_p$ with vertices in bijection with $\mathcal{M}_p$ and directed (or oriented) edges labelled by $s \in \{0, \dots, p-1\}$ joining a vertex $S \in \mathcal{M}_p$ to the vertex sS. The vertex $\emptyset$ corresponds to the root.*

Geometrically, a vertex $s_1 \dots s_l \in \mathcal{M}_p$ corresponds to the endpoint of the continuous path of length l starting at the root $\emptyset$ and running through l consecutive oriented edges labelled $s_l, s_{l-1}, \dots, s_2, s_1$.

A transducer $\tau : \mathcal{M}_p \longrightarrow \mathcal{M}_q$ induces an application $\tau : \mathbf{T}_p \longrightarrow \mathbf{T}_q$ by its action on vertices. In particular, a bijective transducer $\tau : \mathcal{M}_p \longrightarrow \mathcal{M}_p$ corresponds to an automorphism of the $p-$regular rooted tree $\mathbf{T}_p$ and the uncountable group Γ of all automorphisms of $\mathbf{T}_p$ is thus contained in $\{0,1\}^{\mathcal{M}_{p\times p}} \subset \mathbf{K}^{\mathcal{M}_{p\times p}}$ where $\{0,1\}^{\mathcal{M}_{p\times p}}$ denotes the set of all elements with values in $\{0,1\}$. This group contains the countable subgroup $\Gamma \cap \mathrm{Rec}_{p\times p}(\mathbf{K})$ of all automorphisms corresponding to bijective transducers which are automatic.

26 Automatic groups

Proposition 26.1. *The inverse of a bijective transducer $\tau : \mathcal{M}_p \longrightarrow \mathcal{M}_p$ is a transducer.*

Moreover, if a bijective transducer $\tau : \mathcal{M}_p \longrightarrow \mathcal{M}_p$ is finite-state, then its inverse transducer τ^{-1} is also finite-state.

Proof Suppose first that τ is a finite state transducer and consider a presentation $M_1, \dots, M_d$ as in Proposition 25.6 with shift-matrices $\rho(s,t) \in \{0,1\}^{d\times d}$ of the associated transducer-matrix $M = M_\tau = M_1$. By Proposition 25.6, the sum of all coefficients of the matrix $\sum_{0\le s,t<p} \rho(s,t) \in \mathbb{Z}^{d\times d}$ equals pd. The second part of Proposition 25.6 shows thus that for fixed $s, 0 \le s < p$, each matrix $\sum_{t=0}^{p} \rho(s,t)$ has all row-sums equal to 1. Proposition 25.6 can thus be applied to the transposed matrices $M_1^t, \dots M_t^t$ (given by the presentation $(M_1^t, \dots, M_d^t)[\emptyset, \emptyset] = (1, \dots, 1)$ and shift-matrices $\tilde{\rho}(s,t)$ defined by $\tilde{\rho}(s,t) = \rho(t,s)$) and shows that $M_1^t = M_1^{-1}, \dots, M_d^t = M_d^{-1}$ are transducer-matrices. Consider now the restriction $\tau(\mathcal{M}_p^{\le l})$ of an arbitrary

transducer τ to the set of words of length at most l in $\mathcal{M}_p$. This restriction coincides with the restriction to $\mathcal{M}_p^{\leq l}$ of a suitable finite-state transducer. Since the length l can be an arbitrary integer this implies the first part of the result.

The second part has already been proven. $\square$

The set of all bijective finite-state transducers from $\mathcal{M}_p$ to $\mathcal{M}_p$ forms a group which is isomorphic to the subgroup $\mathcal{TD}_{p-rec} \subset \mathrm{GL}_{p-rec}(\mathbf{K})$ formed by all invertible transducer-matrices in $\mathrm{Rec}_{p\times p}(\mathbf{K})$. We call a subgroup of $\mathcal{TD}_{p-rec}$ a $p-$*automaton group* or *automaton group* for short, also called branched groups by some authors. For $M \in \mathcal{TD}_{p-rec}$, the matrices $M[\mathcal{M}_{p\times p}^l]$ are permutation matrices. We have thus $M^{-1} = M^t$ and the element M belongs to the orthogonal subgroup (with respect to the scalar product given by the identity) of $\mathrm{GL}_{p-rec}(\mathbf{K})$ (see section 16.5).

Remark 26.2. *There are $(d^p\ p!)^d$ presentations of complexity d defining d bijective finite-state transducer-matrices $M_1, \ldots, M_d \in Rec_{p-rec}(\mathbb{Z})$. Indeed, using the obvious recursive presentation outlined in Remark 25.8, such a matrix $M_i = (1, R(i))$ is encoded by a "coloured" permutation matrix $R(i)$ of order $p \times p$ with all coefficients 1 "coloured" (independently) by a colour in the set $\{M_1, \ldots, M_d\}$. For each such matrix $R(i)$, there are thus $p!\ d^p$ possibilities.*

26.1 An example: The first Grigorchuk group Γ

This fascinating group appeared first in [7]. A few interesting properties (see for instance [5], page 211 or [8]) of Γ are:

- Γ has no faithful finite-dimensional representation.
- Γ is not finitely presented.
- Γ is of intermediate growth.
- Γ contains every finite $2-$group.

The group Γ is the subgroup generated by four bijections a, b, c, d of the set $\mathcal{S} = \bigcup_{l=0}^{\infty}\{\pm 1\}^l$. Since a, b, c, d preserve the subsets $\mathcal{S}_l = \{\pm 1\}^l$ we denote by a_l, b_l, c_l, d_l the restricted bijections induced by a, b, c, d on the finite subset $\mathcal{S}_l$. For $l = 0$, we have $\mathcal{S}_0 = \emptyset$ with trivial action of the permutations a_0, b_0, c_0, d_0. For $l > 0$ we write $(\epsilon, x) = (y_1, y_2, \ldots, y_l)$ with $\epsilon = y_1 \in \{\pm 1\}$ and $x = (y_2, y_3, \ldots, y_l) \in \{\pm 1\}^{l-1}$. The action of a_l, b_l, c_l, d_l is then recursively defined by

$$\begin{array}{ll}
a_l(1,x) = (-1,x) & a_l(-1,x) = (1,x) \\
b_l(1,x) = (1, a_{l-1}(x)) & b_l(-1,x) = (-1, c_{l-1}(x)) \\
c_l(1,x) = (1, a_{l-1}(x)) & c_l(-1,x) = (-1, d_{l-1}(x)) \\
d_l(1,x) = (1,x) & d_l(-1,x) = (-1, b_{l-1}(x))
\end{array}$$

(cf VIII.B, pages 217-218 of [5]).

The set $\mathcal{S}$ corresponds to the vertices of the $2-$regular rooted planar tree $\mathbf{T}_2$ and the four bijections a, b, c, d act as automorphisms on $\mathbf{T}_2$. They correspond thus to transducers $\tau_a, \tau_b, \tau_c, \tau_d$. The associated transducer-matrices M_a, M_b, M_c, M_d are recursively presented by

$$M_a = (1, \begin{pmatrix} 0 & 1\mathrm{d} \\ 1\mathrm{d} & 0 \end{pmatrix}), M_b = (1, \begin{pmatrix} M_a & 0 \\ 0 & M_c \end{pmatrix}),$$

$$M_c = (1, \begin{pmatrix} M_a & 0 \\ 0 & M_d \end{pmatrix}), M_d = (1, \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & M_b \end{pmatrix})$$

with $\mathrm{Id} \in \mathrm{GL}_{2-rec}(\mathbb{Z})$ denoting the identity matrix recursivley presented by $\mathrm{Id} = (1, \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix})$.

A (monoidal) presentation of M_a, M_b, M_c, M_d is given by $(\mathrm{Id}, M_a, M_b, M_c, M_d)[\emptyset, \emptyset] = (1, 1, 1, 1, 1)$ and shift-matrices

$$\rho(0,0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \rho(0,1) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\rho(1,0) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \rho(1,1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Remark 26.3. *The fact that the four generators a, b, c, d of the Grigorchuk group are of order 2 is equivalent to the identity $\rho(0,1) = \rho(1,0)$ of the shift-matrices described above.*

27 A few asymptotic problems in $\mathbf{Rec}_{p\times p}(\mathbf{K})$

This section introduces generating series counting dimensions associated to recurrence matrices. It contains mainly definitions and (open) problems.

Given a recurrence matrix $A \in \mathrm{Rec}_{p\times p}(\mathbf{K})$ one can consider the generating series

$$1 + \dim(\overline{A}^{rec})t + \cdots = \sum_{n=0}^{\infty} \dim(\overline{A^n}^{rec})t^n$$

associated to the complexities of its powers. One has the inequality $\dim(\overline{A^n}^{rec}) \leq \left(\dim(\overline{A}^{rec})\right)^n$ which implies convergency of the series for $\{t \in \mathbb{C} \mid |t| < 1/\dim(\overline{A}^{rec})\}$.

What can be said about the analytic properties of this generating series? In a few easy cases (nilpotent or of complexity 1 for instance), it is meromorphic in $\mathbb{C}$.

If $A \in \mathrm{GL}_{p-rec}(\mathbf{K})$ is invertible in $\mathrm{Rec}_{p\times p}(\mathbf{K})$, one can of course also consider the formal sum

$$\sum_{n=-\infty}^{\infty} \dim(\overline{A^n}^{rec})t^n$$

encoding the complexities of all integral powers of A.

Remark 27.1. *There are many variations for the generating function(s) defined above. One can replace the complexity by the stable complexity or the birecursive complexity of powers. One can also consider the generating function encoding the dimension α_n of the recursive closure $\overline{Id, A^1, \ldots, A^n}^{rec}$ containing all powers $A^0 = Id, A^1, \ldots, A^n$ etc.*

Similarly, given a monoid or group generated by a finite set $\mathcal{G} \subset \mathrm{Rec}_{p\times p}(\mathbf{K})$ of recurrence matrices, one can consider the generating series $\sum_{n=0}^{\infty} \dim(\mathcal{A}_n)t^n$ encoding the complexities of the recursive closures $\mathcal{A}_n$ associated to all products of (at most) n elements in $\mathcal{G}$.

A different kind of generating series is given by considering for $A \in \mathrm{Rec}_{p\times p}(\mathbb{C})$ the series

$$1+ \parallel A \parallel_\infty^\infty \ t + \cdots = \sum_{n=0}^{\infty} \parallel A^n \parallel_\infty^\infty \ t^n.$$

The inequality $\parallel A^n \parallel_\infty^\infty \leq p^{2n-1}(\parallel A \parallel_\infty^\infty)^n$ (which can be proven by considering a recursive matrix whose coefficients $A[U,W]$ depend only on the length l of $(U,W) \in \mathcal{M}_{p\times p}^l$) shows again convergency for $t \in \mathbb{C}$ small enough.

28 A generalization

Consider a ring $\mathbf{R}$ of functions $\mathbb{N} \longrightarrow \mathbf{K}$ with values in a commutative field $\mathbf{K}$. An element $A \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ is an $\mathbf{R}-$*recurrence matrix* (or simply a *recurrence matrix* if the underlying function ring $\mathbf{R}$ is obvious) if there exists a finite number of elements $A_1 = A, A_2, \ldots, A_a \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ and *pq shift-matrices* $\rho_A(s,t) \in \mathbf{R}^{a\times a}$ with coefficients $\rho_A(s,t)_{j,k} \in \mathbf{R}$ (for $1 \leq j,k \leq a$) in the function ring $\mathbf{R}$ such that

$$\big(\rho(s,t)A_k\big)[\mathcal{M}_{p\times q}^{l+1}] = \sum_{j=1}^{a} \rho_A(s,t)_{j,k}(l) \ A_j[\mathcal{M}_{p\times q}^l], \ 1 \leq k \leq a,$$

for all $0 \leq s < p, 0 \leq t < q$.

We denote by $\mathbf{R} - \mathrm{Rec}_{p\times q}(\mathbf{K})$ the vector-space of all $\mathbf{R}-$recurrence matrices in $\mathbf{K}^{\mathcal{M}_{p\times q}}$.

Proposition 4.6 and its proof can easily be modified in order to deal with $\mathbf{R}-$recurrence matrices and we get thus a category $\mathbf{R} - \mathrm{Rec}(\mathbf{K})$ with $\mathbf{R}-$recurrence matrices as morphisms. Moreover, the vector-space $\mathbf{R} - \mathrm{Rec}_{p\times q}(\mathbf{K})$ is invariant under the action of the shift-monoid if the function ring $\mathbf{R}$ is preserved by the translation $x \longmapsto x+1$ of the argument (ie. if $\alpha \in \mathbf{R}$ implies that the function $x \longmapsto \alpha(x+1)$ is also in $\mathbf{R}$).

A few analogies and differences between $\mathbf{R}-$recurrence matrices and (ordinary) recurrence matrices are:

- Elements of $\mathbf{R} - \mathrm{Rec}_{p\times q}(\mathbf{K})$ have finite descriptions involving a finite number of elements in the function ring $\mathbf{R}$.

- The notion of a presentation is in general more involved: Every presentation of an ordinary recurrence matrix A contains a basis of $\overline{A}^{rec}$ and can thus be used to construct a minimal presentation (defining a basis of $\overline{A}^{rec}$). This is no longer true in general for $\mathbf{R}-$recurrence matrices since $\mathbf{R}$ might contain non-zero elements which have no multiplicative inverse. One has thus to work with (not necessarily free) $\mathbf{R}-$modules when dealing with presentations.

- Proposition 7.1, slightly modified, remains valid. We have

$$\begin{pmatrix} A_1[U,W] \\ \vdots \\ A_d[U,W] \end{pmatrix} = \rho_{\mathcal{A}}(u_n,w_n)^t(n-1)\cdots\rho_{\mathcal{A}}(u_1,u_1)^t(0)\begin{pmatrix} A_1[\emptyset,\emptyset] \\ \vdots \\ A_d[\emptyset,\emptyset] \end{pmatrix}$$

  where $(U,W) = (u_1\dots u_n, w_1\dots w_n) \in \mathcal{M}^n_{p\times q}$. The matrices $\rho_{\mathcal{A}}(s,t) \in \mathbf{R}^{d\times d}$ are the obvious shift-matrices of the subspace $\mathcal{A} = \sum \mathbf{K}A_j$ with respect to the generators $A_1,\dots,A_d$.

- An analogue of the saturation level is no longer available in general. This makes automated computations impossible in the general case.

Remark 28.1. *Non-existence of a saturation level does not necessarily imply the impossibility of proving a few identities among $\mathbf{R}-$recurrence matrices, since such an identity can perhaps be proven by induction on the word-length l, after restriction to $\mathcal{M}^{\leq l}_{p\times q}$.*

28.1 Examples

The case where the function ring $\mathbf{R}$ consists of all constant functions $\mathbb{N} \longrightarrow \mathbf{K}$ corresponds of course to the case of (ordinary) recurrence matrices studied in this paper.

The case where the function ring $\mathbf{R}$ consists of ultimately periodic functions $\mathbb{N} \longrightarrow \mathbf{K}$ yields again only ordinary recurrence matrices.

The first interesting new case is given by considering the ring $\mathbf{R} = \mathbf{K}[x]$ of all polynomial functions. The category $\mathbb{C}[x] - \mathrm{Rec}(\mathbb{C})$ of $\mathbb{C}[x]-$recurrence matrices is indeed larger than the category $\mathrm{Rec}(\mathbb{C})$ of ordinary recurrence matrices since $\mathbb{C}[x] - \mathrm{Rec}_1(\mathbb{C})$ contains for instance the element $A[0^n] = n!$ which is not in $\mathrm{Rec}_1(\mathbb{C})$ by Proposition 11.2.

It would be interesting to have finiteness results in this case: Given $A, B, C \in \mathbf{K}[x]-\mathrm{Rec}_{p\times q}$ defined by presentations of complexity a, b, c (where the complexity is the minimal number of elements appearing in a finite presentation) with shift-matrices involving only polynomials of degree $\leq \alpha, \beta, \gamma$, can one give a bound $N_+ = N_+(a, b, c, \alpha, \beta, \gamma, p, q)$ such that the equality $(A + B)[\mathcal{M}_{p\times q}^{\leq N_+}] = C[\mathcal{M}_{p\times q}^{\leq N_+}]$ ensures the equality $A + B = C$ in $\mathbf{K}[x] - \mathrm{Rec}_{p\times q}$? Similarly, given $A \in \mathbf{K}[x]-\mathrm{Rec}_{p\times r}, B \in \mathbf{K}[x]-\mathrm{Rec}_{r\times q}, C \in \mathbf{K}[x]-\mathrm{Rec}_{p\times q}$ defined by presentations of complexity a, b, c with shift-matrices involving only polynomials of degree $\leq \alpha, \beta, \gamma$, can one give a bound $N_\times = N_\times(a, b, c, \alpha, \beta, \gamma, p, q, r)$ such that the equality $(AB)[\mathcal{M}_{p\times q}^{\leq N_+}] = C[\mathcal{M}_{p\times q}^{\leq N_+}]$ ensures the equality $AB = C$ in $\mathbf{K}[x] - \mathrm{Rec}_{p\times q}$? Are there natural and "interesting" examples of $\mathbf{K}[x]-$recurrence matrices which are not (ordinary) recurrence matrices?

Another interesting example is given by considering the ring $\mathbf{R}$ of all functions $\mathbb{N} \longrightarrow \mathbf{K}$ which are linear combinations involving terms of the form

$$n \longmapsto n^k \lambda^n$$

for $\lambda \in \mathbf{K}$. For $p \geq 2$ and $\mathbf{K}$ algebraically closed, the group of lower triangular convergent Toeplitz matrices in $\mathbf{R} - \mathrm{GL}_{p-rec}(\mathbf{K})$ contains then the multiplicative group $\mathbf{K}(x)^*$ of all invertible rational power-series (this is not the case for ordinary recurrence matrices over $\mathbb{C}$, see the last part of Remark 15.5 and Example 19.3).

28.2 An intermediate category between $\mathbf{K}^{\mathcal{M}}$ and $\mathbf{Rec}(\mathbf{K})$

The category $\mathbf{K}^{\mathcal{M}}$ contains many other, perhaps interesting, subcategories. An example is given by the set of all elements $A \in \mathbf{K}^{\mathcal{M}_{p\times q}}$ (for arbitrary $p, q \in \mathbb{N}$) such that

$$\dim \left(\overline{A}^{rec}[\mathcal{M}_{p\times q}^{l}] \right)$$

is bounded by a polynomial in l. Proposition 4.6 shows easily that this property is preserved by products. An example of such an element in $\mathbb{Q}^{\mathcal{M}_{2\times 2}}$ is perhaps given by the inverse element of the converging element with limit the infinite Hadamard matrix associated to the sequence of coefficients of $\prod_{k=0}^{\infty}(1 - x^{2^k})$.

Remark 28.2. *One can of course also consider suitable intermediate growth classes for definining other subcategories of* $\mathbf{K}^{\mathcal{M}}$.

29 Examples of a few Toeplitz matrices in $\mathbf{Rec}_{p\times p}(\mathbf{K})$

This and the next chapters present a few (hopefully) interesting recurrence examples of recurrence matrices, mainly elements of $\mathrm{GL}_{p-rec}(\mathbf{K})$ for a suitable integer $p \geq 2$ and K a subfield or subring of $\mathbb{C}$ or a finite field. A few examples contain parameters which can be choosen in suitable, easily specified rings or fields.

Most of the computations are straightforward but tedious and omitted.

29.1

Consider the matrix $A(n)$ of square size $n \times n$ with coefficients $A(n)_{i,j}, 0 \leq i, j < n$ given by

$$A(n)_{i,j} = \begin{cases} 4 & i = j, \\ 3 & i < j, \\ 3 + 3(i-j) & i > j. \end{cases}$$

The infinite matrix $A(\infty)$ defines thus a converging element (still denoted) $A \in \mathrm{Rec}_{p\times p}(\mathbb{Z})$ for all $p \in \mathbb{N}$. We leave it to the reader to prove that $A = LU$ (where L is lower triangular unipotent and U is upper triangular) with $L, U \in \mathrm{GL}_{p-rec}(\mathbb{Q})$ for all $p \in \mathbb{N}$. An inspection of U proves

$$\det(A(n)) = \begin{cases} 1 & n \equiv 0 \pmod 3, \\ 4 & n \equiv 1 \pmod 3, \\ -2 & n \equiv 2 \pmod 3. \end{cases}$$

It follows that $A \in \mathrm{GL}_{p-rec}(\mathbb{Z})$ if $p \equiv 0 \pmod 3$.

Remark 29.1. *More generally, one can consider the $n \times n$ matrix $A(n)$ with coefficients*

$$A(n)_{i,j} = \begin{cases} x+1 & i = j, \\ x & i < j, \\ x + x(i-j) & i > j \end{cases}$$

for $0 \leq i, j < n$. It follows then for instance from [11] that the sequence $\det(A(1)), \det(A(2)), \det(A(3)), \ldots$ satisfies a linear recursion. More precisely we have

$$\det(A(n)) + (x-3)\det(A(n-1)) + (3-x)\det(A(n-2)) - \det(A(n-3)) = 0$$

which implies

$$\det(A(n)) = 1 - (-1)^n \sum_{k=0}^{n} (-1)^k \binom{2n-1-k}{k} x^{n-k} .$$

Particularly interesting are the evaluations $x \in \{1, 2, 3\}$ where the case $x = 3$ has been considered above.

29.2

A similar example is given by the $n \times n$ matrix $A(n)$ with coefficients

$$A(n)_{i,j} = \begin{cases} 1 & i = j, \\ 1 & i < j, \\ 1 + i - j & i > j. \end{cases}$$

Its characteristic polynomial is

$$t^n - \sum_{k=1}^{n} \binom{n-1+k}{n-k} t^{n-k}$$

and $A(n)$ is thus invertible over $\mathbb{Z}$ for all n since $\det(A(n)) = -(-1)^n$ for $n \geq 1$.

Setting $A[\mathcal{M}^l] = A(p^l)$ we get for all $p \in \mathbb{N}$ a converging element (still denoted) $A \in \mathrm{GL}_{p-rec}(\mathbb{Z})$ which satisfies $A = LU$ with $L \in \mathrm{GL}_{p-rec}(\mathbb{Z})$ converging lower triangular unipotent and $U \in \mathrm{GL}_{p-rec}(\mathbb{Z})$ converging upper triangular.

29.3

The square matrix $A(2n)$ of even size $2n \times 2n$ with coefficients $A(2n)_{i,j}, 0 \leq i, j < 2n$ given by $A(2n)_{i,j} = 1$ if $(i-j)^2 = 1$ and 0 otherwise has determinant $(-1)^n$. The inverse matrix $B(2n)$ of $A(2n)$ has coefficients $B(2n)_{i,j} = 0$ if $\min(i,j) \equiv 1 \pmod 2$ and $B(2n)_{i,j} = [x^{|i-j|}]\frac{x}{1+x^2}$ otherwise (for $0 \leq i, j < 2n$). For even $p \in 2\mathbb{N}$, this yields thus inverse matrices (still denoted) $A, B = A^{-1} \in \mathrm{Rec}_{p\times p}(\mathbb{Z})$ defined by $A[\emptyset, \emptyset] = B[\emptyset, \emptyset] = 1$ and $A[\mathcal{M}^l_{p\times p}] = A(p^l)$, $B[\mathcal{M}^l_{p\times p}] = B(p^l)$.

29.3.1

Over $\mathbb{F}_2$ there is a very similar $2-$recursive example with coefficients $A_{i,j} = 1$ except for $i = j$.

29.4

Define a symmetric $n \times n$ Toeplitz matrix $A(n)$ by $A(n)_{i,j} = \alpha_{|i-j|}, 0 \leq i, j$ where

$$\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = -\frac{1}{2}, \alpha_3 = \frac{5}{4}, \alpha_4 = -\frac{9}{8}, \ldots, \alpha_n = -\frac{1}{2}\alpha_{n-1} + \alpha_{n-2}, \ldots$$

are the coefficients of the rational series $\sum_{n=1}^{\infty} \alpha_n x^n = \frac{x}{1+x/2-x^2}$. For $n \geq 4$, the coefficients $B(n)_{i,j}, 0 \leq i, j < n$ of the inverse matrix $B(n) = A(n)^{-1}$

are then given by

$$\begin{cases} 1 & \text{if } i=j\in\{0,n-1\},\\ 5/4 & \text{if } i=j\in\{1,n-2\},\\ 9/4 & \text{if } i=j\in\{2,3,\dots,n-3\},\\ 1/2 & \text{if } (i,j)\in\{(0,1),(1,0),(n-2,n-1),(n-1,n-2)\},\\ -1 & \text{if } |i-j|=2,\\ 0 & \text{otherwise.}\end{cases}$$

Choosing an odd prime $\wp$ and a natural number $p\in\mathbb{N}$, we get thus an element (still denoted) $A\in \mathrm{GL}_{p-rec}(\mathbb{F}_\wp)$ defined by $A[\mathcal{M}^l]=A(p^l) \pmod{}F_\wp$.

Remark 29.2. *We have The matrices $B(n)=A(n)^{-1}$ are converging for $n\to\infty$ in the sense that the all coefficients with fixed indices are ultimately constant. The limit-matrix $B(\infty)$ has an $LU-$decomposition given by*

$$LL^t=\begin{pmatrix} 1 & 1/2 & -1 & 0 & 0 & 0 & 0 & \cdots\\ 1/2 & 5/4 & 0 & -1 & 0 & 0 & 0 & \\ -1 & 0 & 9/4 & 0 & -1 & 0 & 0 & \\ 0 & -1 & 0 & 9/4 & 0 & -1 & 0 & \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \end{pmatrix}$$

for

$$L=\begin{pmatrix} 1 & & & & \\ 1/2 & 1 & & & \\ -1 & 1/2 & 1 & & \\ 0 & -1 & 1/2 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

the lower triangular Toeplitz matrix defined by $L_{i,i}=1, L_{i+1,i}=1/2, L_{i+2,i}=-1$ and $L_{i,j}=0$ otherwise. The reduction of L modulo an odd prime $\wp$ defines a converging element $L\in GL_{p-rec}(\mathbb{F}_\wp)$. Proposition 11.2 shows however that $L\in Rec_{p\times p}(\mathbb{Q})$ is not invertible in $Rec_{p\times p}(\mathbb{Q})$.

Let us also mention a curious experimental fact concerning the characteristic polynomial of $-A(2n)$.

A symmetric Toeplitz matrix of size $n\times n$ preserves the eigenspaces of the involution

$$\iota:(x_0,\dots,x_{n-1})\longmapsto(x_{n-1},x_{n-2},\dots,x_1,x_0)$$

and its characteristic polynomial χ factorizes thus as $\chi=\chi_+\chi_-$ where χ_+ of degree $\lceil n/2\rceil$ corresponds to the trivial eigenspace (formed by eigenvectors $(x_0,\dots,x_{n-1})$ of eigenvalue 1 satisfying $x_i=x_{n-1-i}$ for $i=0,\dots,\lfloor n/2\rfloor$) and where χ_- of degree $\lfloor n/2\rfloor$ corresponds to the eigenspace of eigenvalue -1 formed by all vectors $(x_0,\dots,x_{n-1})$ such that $x_i=-x_{n-1-i}$ for $i=0,\dots,\lfloor n/2\rfloor$.

For the characteristic polynomial of $-A(2n)$ the corresponding factorisation seems to be of the form

$$\det(t\ Id + A(2n)) = (t^n + \sum_{k=0}^{n-1} \gamma_{n,k} t^k)(t^n - \sum_{k=0}^{n-1} \gamma_{n,k} t^k)$$

with $\gamma_{n,0}, \ldots, \gamma_{n,n-1}$ strictly positive rational numbers. In particular, the bilinear product defined by $A(2n)$ endows seemingly the trivial eigenspace (associated to the eigenvalue 1) of ι with an Euclidean scalar product. Normalized in order to have leading term $4^{n-1}t^n$, the first few divisors $\tilde{\chi}_+(n)$ of $\det(t\ Id + A(2n))$ associated to the trivial eigenspace for ι are given by

$$\begin{aligned}
&\tilde{\chi}_+(1) = t + 1\\
&\tilde{\chi}_+(2) = 4t^2 + 9t + 4\\
&\tilde{\chi}_+(3) = 16t^3 + 65t^2 + 72t + 16\\
&\tilde{\chi}_+(4) = 64t^4 + 441t^3 + 844t^2 + 432t + 64\\
&\tilde{\chi}_+(5) = 256t^5 + 2929t^4 + 8208t^3 + 7008t^2 + 2304t + 256
\end{aligned}$$

and satisfy the recurrence relation

$$\tilde{\chi}_+(n) - (13t + 4)\tilde{\chi}_+(n-1) + 4t(13t + 4)\tilde{\chi}_+(n-2) - 64t^3\tilde{\chi}_+(n-3) = 0$$

(the same recurrence relation is also satisfied by the corresponding complementary divisors, or equivalently, by the geometric progression $1, (4t), (4t)^2, (4t)^3, \ldots$). Equivalently, for $k \geq 1$, the coefficient t^{n-k} of $\chi_+(n)$ is given by the coefficient of x^n in the rational series $4^{k-1}(x/(1 - 9x + 16x^2))^k$.

29.4.1

A similar example (defining also elements in $\mathrm{GL}_{p-rec}(\mathbb{F}_\wp)$ for $\wp$ an odd prime) is obtained by considering $A(n)_{i,j} = f_{|i-j|}, 0 \leq i, j$ where

$$f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, \ldots, f_n = f_{n-1} + f_{n-2}$$

defined by $\sum_{n=1}^{\infty} f_n x^n = \frac{x}{1-x-x^2}$ is the Fibonacci sequence. We have then $\det(A(n)) = -(-1)^{n-2}$ for $n \geq 2$.

29.4.2

Another similar example (satisfying $\det(A(n)) = -(-1)^n(\lambda^2 - 1)^{n-1}$ for $n \geq 1$ and giving non-trivial elements in $\mathrm{GL}_{p-rec}(\mathbb{F}_\wp)$ for $\wp$ a suitable prime depending on $\lambda \in \mathbb{Q}$) is given by $A(n)_{i,j} = \lambda^{|i-j|}$.

29.5

Given an even integer n such that $n \equiv 0 \pmod 4$, consider the symmetric Toeplitz matrix $A(n)$ of square size $n \times n$ with coefficients given by

$$A(n)_{i,j} = \begin{cases} (2-n)/2 + a & \text{if } i = j \\ -n/2 + |i-j| + a & \text{if } i \neq j \end{cases}$$

The matrix $A(n)$ is then invertible with inverse matrix having coefficients $\left(A(n)^{-1}\right)_{i,j}, 0 \leq i, j < n$ given by

$$\begin{cases} 1 - a & \text{if } i = j, \\ -(-1)^{(i-j)/2} a & \text{if } i \neq j \text{ and } i \equiv j \pmod 2, \\ -(-1)^{\lfloor |i-j|/2 \rfloor} + (-1)^{(i+j-1)/2} a & \text{if } i \not\equiv j \pmod 2. \end{cases}$$

For $p \geq 2$ an even natural number, one gets thus an element (still denoted) $A \in \mathrm{GL}_{p-rec}(\mathbb{C})$ (respectively $A \in \mathrm{GL}_{p-rec}(\mathbb{Z})$) by choosing $a \in \mathbb{C}$ (respectively $a \in \mathbb{Z}$) and setting $A[\mathcal{M}^l] = A(p^l)$ if $l \geq 2$ (after appropriate choices for $A[\mathcal{M}^0], A[\mathcal{M}^1]$).

Remark 29.3. *The matrix $A(n)$ is singular if $n \equiv 2 \pmod 4$ since it contains then the vector $(1, -1, 1, -1, \ldots, 1, -1)$ in its kernel.*

29.5.1

There are many possible variations on the above example. One can for instance consider the symmetric Toeplitz matrix $A(n)$ with coefficients

$$A(n)_{i,j} = \begin{cases} a + 1 & \text{if } i = j, \\ a + |i-j| & \text{if } i \neq j \end{cases}.$$

for $0 \leq i, j < n$. For $n \equiv 0 \pmod 4$, the inverse matrix $A(n)^{-1}$ has coefficients $(A(n)^{-1})_{i,j}, 0 \leq i, j < n$ given by

$$\begin{cases} (2-n)/2 - a & \text{if } i = j, \\ -(-1)^{(i-j)/2}(n/2 + a) & \text{if } i \neq j \text{ and } i \equiv j \pmod 2, \\ (-1)^{\lfloor |i-j|/2 + j \rfloor}(n/2 + a - (-1)^j) & \text{if } i \not\equiv j \pmod 2 \text{ and } i > j, \\ (-1)^{\lfloor |i-j|/2 + i \rfloor}(n/2 + a - (-1)^i) & \text{if } i \not\equiv j \pmod 2 \text{ and } i < j. \end{cases}$$

For p even and $a \in \mathbb{Z}$, we get thus an element (still denoted) $A \in \mathrm{GL}_{p-rec}(\mathbb{Z})$ by setting $A[\mathcal{M}^l] = A(p^l)$ for $l \geq 2$ (and by defining $A[\mathcal{M}^0], A[\mathcal{M}^1]$ suitably).

29.5.2

Similarly, consider a natural odd integer n and define $\epsilon(n) \in \{\pm 1\}$ such that $n \equiv \epsilon(n) \pmod 4$. Let $A(n)$ denote the $n \times n$ matrix with coefficients given

by

$$(A(n))_{i,j} = \begin{cases} \dfrac{2-\epsilon(n)-n}{2} + a & \text{if } i = j \\ \dfrac{-\epsilon(n)-n}{2} + |i-j| + a & \text{if } i \neq j \end{cases}$$

For invertible a, the matrix $A(n)$ is then invertible; If $n \equiv 1 \pmod 4$ the inverse matrix has coefficients $\left(A(n)^{-1}\right)_{i,j}, 0 \leq i,j < n$ given by

$$\begin{cases} 1/a & \text{if } i = j \equiv 0 \pmod 2, \\ 2 & \text{if } i = j \equiv 1 \pmod 2, \\ (-1)^{(i-j)/2}(1-a)/a & \text{if } i \neq j \text{ and } i \equiv j \equiv 0 \pmod 2, \\ (-1)^{(i-j)/2} & \text{if } i \neq j \text{ and } i \equiv j \equiv 1 \pmod 2, \\ -(-1)^{\lfloor |i-j|/2 \rfloor} & \text{if } i \not\equiv j \pmod 2. \end{cases}$$

If $n \equiv 3 \pmod 4$ the inverse matrix has coefficients $\left(A(n)^{-1}\right)_{i,j}, 0 \leq i,j < n$ given by

$$\begin{cases} 0 & \text{if } i = j \equiv 0 \pmod 2, \\ 2 + 1/a & \text{if } i = j \equiv 1 \pmod 2, \\ -(-1)^{(i-j)/2} & \text{if } i \neq j \text{ and } i \equiv j \equiv 0 \pmod 2, \\ (-1)^{(i-j)/2}(a+1)/a & \text{if } i \neq j \text{ and } i \equiv j \equiv 1 \pmod 2, \\ -(-1)^{\lfloor |i-j|/2 \rfloor} & \text{if } i \not\equiv j \pmod 2. \end{cases}$$

For $p \geq 3$ an odd natural integer and $a \in \mathbb{C}^*$, one gets thus an element $A \in \mathrm{GL}_{p-rec}(\mathbb{C})$ by setting $A[\mathcal{M}^l] = A(p^l)$. The inverse element A^{-1} is converging if $p \equiv 1 \pmod 4$. Moreover, $A \in \mathrm{GL}_{p-rec}(\mathbb{Z})$ for $a = 1$ and $a = -1$.

29.6

Another nice example is given by considering the matrix $A(n)$ of square size $n \times n$ with coefficients

$$(A(n))_{i,j} = \begin{cases} (-1)^{(j-i)/2} & \text{if } i \leq j \text{ and } i \equiv j \pmod 2, \\ 0 & \text{if } i \leq j \text{ and } i \not\equiv j \pmod 2, \\ (-1)^{i-j} & \text{if } i \geq j. \end{cases}$$

There are nice formulae for $A(n)^{-1}$ (having all its coefficients in the finite set $\{0,1,2,3,4\}$) which the reader can easily write down inspecting the matrices $A(6)^{-1}$ and $A(7)^{-1}$:

$$\begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 1 \\ 1 & 2 & 3 & 4 & 4 & 2 \\ 0 & 1 & 2 & 3 & 4 & 2 \\ 0 & 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 2 & 1 \\ 1 & 2 & 3 & 4 & 4 & 4 & 2 \\ 0 & 1 & 2 & 3 & 4 & 4 & 2 \\ 0 & 0 & 1 & 2 & 3 & 4 & 2 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Setting $A[\mathcal{M}^l] = A(p^l)$ for $p \geq 2$ yields thus an element (still denoted) $A \in \mathrm{GL}_{p-rec}(\mathbb{Z})$. Moreover, A and A^{-1} have very simple LU decompositions in $\mathrm{Rec}_{p\times p}(\mathbb{Z})$ with L lower triangular and U upper triangular both unipotent.

29.7

Consider the $n \times n$ Toeplitz matrix

$$T(n) = \begin{pmatrix} 1 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & \dots \\ -2 & 1 & -1 & 1 & 0 & -1 & 1 & 0 & \\ 2 & -2 & 1 & -1 & 1 & 0 & & & \\ 0 & 2 & -2 & 1 & -1 & & & & \\ -2 & 0 & 2 & & & & & & \\ \vdots & & & & & & & & \end{pmatrix}$$

with coefficients $T(n)_{i,i} = 1, T(n)_{i,j} \equiv i-j \pmod 3$ with values in $\{-1, 0, 1\}$ for $i < j$ and with values in $\{-2, 0, 2\}$ for $i > j$.

For $n \equiv 0 \pmod 3$ we have $T(n) \in \mathrm{GL}_3(\mathbb{Z})$ with inverse matrix having coefficients $(T(n)^{-1})_{i,j}, 0 \leq i, j < n$ given by

$$(T(n)^{-1})_{i,j} = \begin{cases} -2 \cdot n/3 + 1 & \text{if } i = j \\ -2 \cdot n/3 + j - i & \text{if } i < j \\ -2 \cdot n/3 + 2(i-j) & \text{if } i > j \end{cases}$$

for $0 \leq i, j < n$.

For $p \equiv 0 \pmod 3$ we get thus a converging element (still denoted) $T \in \mathrm{GL}_{p-rec}(\mathbb{Z})$ by setting $T[\mathcal{M}^l] = T(p^l)$.

Remark 29.4. *More generally, one can consider the $n \times n$ Toeplitz matrix with coefficients $T(n)_{i,i} = a, T(n)_{i,j} \equiv i-j \pmod 3$ with values in $\{-1, 0, 1\}$ for $i < j$ and with values in $\{-2, 0, 2\}$ for $i > j$. The example above corresponds to $a = 1$. Other interesting values are $a = 0, a = 3$ and $a = -3$. For $a = 3$ for instance, the values of $3^{-n}\det(T(n))$ display (experimentally) the following interesting $12-$periodic pattern (which one can probably prove adapting the methods of [11])*

$$\begin{array}{ll} n \equiv 0, \pm 1 \pmod{12} & 1, \\ n \equiv \pm 2 \pmod{12} & 7/9, \\ n \equiv \pm 3 \pmod{12} & 5/9, \\ n \equiv \pm 4 \pmod{12} & 1/3, \\ n \equiv \pm 5, 6 \pmod{12} & 1/9. \end{array}$$

Remark 29.5. *Another variation on the above example is given by considering the matrix $\tilde{T}(n)$ with coefficients $T(n)_{i,i} = a + 1, T(n)_{i,j} = a + j - i$ if $i < j$ and $T(n)_{i,j} = a + 2i - 2j$ if $i > j$ (obtained by adding the constant*

$a + 2n/3$ to all coefficients of the inverse matrix $T(n)^{-1}$ defined above for $n \equiv 0 \pmod 3$).

A last variation is given by the matrix defined by $T(n)_{i,i} = x, T(n)_{i,j} = 1+j-i$ if $i < j$ and $T(n)_{i,j} = 1+2i-2j$ if $i > j$ (corresponding, up to addition of a constant to all diagonal terms to the evaluation $a = 1$ of the matrix $\tilde{T}(n)$ considered above). Interesting evaluations are: $x = 2$ (yielding seemingly unimodular matrices for $n \equiv 0 \pmod 3$, this can probably been proven using the ideas of [11]), $x = 4$ (yielding seemingly matrices of determinant $\det(T(n)) = 3^n$ if $n \equiv 0 \pmod 6$ and $\det(T(n)) = 3^{n-2}$ if $n \equiv 3 \pmod 6$) and $x = 5/2$ yielding seemingly matrices of determinant $\det(T(n)) = (3/2)^n$ if $n \equiv 0 \pmod 4$ and $\det(T(n)) = 3^{n-2}/2^n$ if $n \equiv 2 \pmod 4$).

29.8 Digression

This small digression "explains" formulae for the determinants of several examples treated above.

Consider the Toeplitz matrix $T(n)$ with coefficients $T(n)_{i,j}, 0 \le i, j < n$ given by $T(n)_{i,j} = a + b(j-i)$ if $j > i$, $T(n)_{i,i} = x$, $T(n)_{i,j} = \alpha + \beta(i-j)$ if $i > j$. We have thus for example

$$T(4) = \begin{pmatrix} x & a+b & a+2b & a+3b \\ \alpha+\beta & x & a+b & a+2b \\ \alpha+2\beta & \alpha+\beta & x & a+b \\ \alpha=3\beta & \alpha+2\beta & \alpha+\beta & x \end{pmatrix}.$$

The generating function $\sum_{n=0}^{\infty} \det(T(n)t^n$ of the associated determinants is then a rational function P/Q (the proof is essentially the same as in [11]) given by $P = \sum_{n=0}^{5} P_n t^n, Q = \sum_{n=0}^{6} Q_n t^n$ with

$$\begin{aligned}
P_0 &= 1 \\
P_1 &= 3(a+\alpha) + 2(b+\beta) - 5x \\
P_2 &= 3(a+\alpha)^2 + 2a\alpha + (b^2 + b\beta + \beta^2) + 3(a+\alpha)(b+\beta) + (ab+\alpha\beta) \\
&\quad -(12(a+\alpha) + 6(b+\beta))x + 10x^2 \\
P_3 &= (a+\alpha)^3 + 4(a^2\alpha + a\alpha^2) + (a+\alpha)^2(b+\beta) + ab(a+b) + \alpha\beta(\alpha+\beta) \\
&\quad +3a\alpha(b+\beta) + ab\beta + b\alpha\beta \\
&\quad -(9(a+\alpha)^2 + 6a\alpha + (b+\beta)^2 + 6(a+\alpha)(b+\beta) + 2(ab+\alpha\beta))x \\
&\quad +(18(a+\alpha) + 6(b+\beta))x^2 - 10x^3 \\
P_4 &= (a-x)(\alpha-x)\Big(2(a+\alpha)^2 + a\alpha + (a+\alpha)(b+\beta) + (ab+\alpha\beta) + b\beta \\
&\quad -(7(a+\alpha) + 2(b+\beta))x + 5x^2\Big) \\
P_5 &= (a-x)^2(\alpha-x)^2(a+\alpha-x)
\end{aligned}$$

and

$$
\begin{aligned}
Q_0 &= 1\\
Q_1 &= 3(a+\alpha)+2(b+\beta)-6x\\
Q_2 &= 3(a+\alpha)^2+3a\alpha+(b+\beta)^2+4(a+\alpha)(b+\beta)\\
&\quad -(15(a+\alpha)+8(b+\beta))x+15x^2,\\
Q_3 &= (a+\alpha)^3+6(a\alpha^2+a^2\alpha)+(a+\alpha)(b+\beta)^2+2(a+\alpha)^2(b+\beta)\\
&\quad +4a\alpha(b+\beta)-(12(a+\alpha)^2+12a\alpha+2(b+\beta)^2+12(a+\alpha)(b+\beta))x\\
&\quad +(30(a+\alpha)+12(b+\beta))x^2-20x^3\\
Q_4 &= (a-x)(\alpha-x)Q_2\\
Q_5 &= (a-x)^2(\alpha-x)^2Q_1\\
Q_6 &= (a-x)^3(\alpha-x)^3\ .
\end{aligned}
$$

29.9 An example given by a symmetric Toeplitz matrix

We associate to a formal power series $s=\sum_{j=0}^{\infty}s_jx^j\in\mathbf{K}[[x]]$ the symmetric Toeplitz matrix $T(n)$ of size $n\times n$ with coefficients $T(n)_{i,j}, 0\le i,j<n$ given by

$$T(n)_{i,j}=[x^{|i-j|}]s=s_{|i-j|}\ .$$

Recall that a finite integral square-matrix A is unimodular if $\det(A)\in\{\pm1\}$.

Proposition 29.6. *Let $k\ge1$ be a natural integer. For $s=(1-x-x^2)/(1-x-x^2-x^k+x^{k+2})$ and $n\ge 2k+5$, the matrix $T(n)$ is unimodular. For $j=k+2,\dots,n-k-3$, the generating function $g_j=g_j(x)$ of coefficients for the $j-$th row (with rows indexed from 0 to $n-1$) of $T(n)^{-1}$ is given by*

$$g_j=x^{j-k-2}(1-x-x^2-x^k+x^{k+2})(1-x^2-x^k-x^{k+1}+x^{k+2})\ .$$

The corresponding generating series $g_0,\dots,g_{k+1}$ for the first $k+2$ rows are given by

$$g_j=(1-x-x^2-x^k+x^{k+2})\left(\sum_{k=0}^{j}x^k[x^k]\left(x^{j-k-2}(1-x^2-x^k-x^{k+1}+x^{k+2})\right)\right)$$

where

$$\left(\sum_{k=0}^{j}x^k[x^k]\left(x^{j-k-2}(1-x^2-x^k-x^{k+1}+x^{k+2})\right)\right)$$

denotes the non-singular part of the Laurent-polynomial

$$x^{j-k-2}-x^{j-k}-x^{j-2}-x^{j-1}+x^j.$$

The easy identity $g_{n-j}(x)=x^{n-1}g_j\left(\frac{1}{x}\right)$ determines now the last $k-2$ rows of $T(n)^{-1}$.

Corollary 29.7. *Choosing an integer $p \geq 2$ and a prime $\wp$, reduction* (mod $\wp$) *of the matrices $T(p^l)$ yields an element (still denoted) $T \in Rec_{p\times p}(\mathbb{F}_\wp)$ which is invertible, up to modifications of the first few matrices $T[\mathcal{M}^l]$ with $p^l < 2k+5$.*

Remark 29.8. *We have $P(1) = -1$ and $lim_{x\to\infty} P(x) = \infty$ for the polynomial $P(x) = 1 - x - x^2 - x^k - x^{k+2}$. The polynomial $P(x)$ has thus a real root $\rho > 1$. For $p \geq 2$, it follows thus from Proposition 11.2 that the element of $\prod_{l=0}^{\infty} \mathbb{Z}^{p^l\times p^l}$ defined by $T[\mathcal{M}^l] = T(p^l)$ is not in $Rec_{p\times p}(\mathbb{Z})$.*

Sketch of proof for proposition 29.6 The identity $g_{n-j}(x) = x^{n-1} g_j\left(\frac{1}{x}\right)$ for the generating functions of the rows of $T(n)^{-1}$ follows easily from the fact that $T(n)$ is a symmetric Toeplitz matrix.

The generating series $\tilde{f}_j = \tilde{f}_j(x)$ of the $j-$th row of $T(n)$ is associated to the coefficients $[x^{-j}]f, \ldots, [x^{-j+n-1}]f$ where

$$f = \frac{1-x-x^2}{1-x-x^2-x^k+x^{k+2}} + \frac{1-x^{-1}-x^{-2}}{1-x^{-1}-x^{-2}-x^{-k}+x^{-k-2}} - 1 \in \mathbb{Z}[[x, x^{-1}]] \,.$$

The proposition follows now from the identities $[x^0]\left(\tilde{f}_i\left(\frac{1}{x}\right) g_j(x)\right) = 1$ if $i = j$ and $[x^0]\left(\tilde{f}_i\left(\frac{1}{x}\right) g_j(x)\right) = 0$ otherwise. □

Remark 29.9. *Computations suggest that the sequence $d_n = \det(T(n))$ of determinants is given by $d_n = 1$ if $n \leq k$, $d_{k+1} = 0$ and $d_n = -1$ if $n > k+1$.*

A analogous example is given by the matrices associated to the function $s = -(1-x^2)/(1-x^2-x^k-x^{k+1}-x^{k+2})$. The sequence of determinants $d_n = \det(T(n))$ is seemingly given by $d_n = (-1)^n$ for $n \leq k$, $d_{k+1} = 0$ and $d_n = -(-1)^k$ for $n > k+1$.

Let us also mention the matrices $T(n)$ associated to $s = (1-ax-x^2)/(1-ax-x^2-x^k-(1-a)x^{k+1}+x^{k+2})$. Experimentally, the determinant $d_n = \det(T(n))$ seems to be given by $d_n = 1$ if $n \leq k$, $d_{k+1} = 0$ and $d_n = -(2a-1)^{n-k-2}$ if $n > k+1$.

Remark 29.10. *The formal "infinite inverse" matrix $T(\infty)^{-1}$ associated to $T(\infty)$ for $s = (1-x-x^2)/(1-x-x^2-x^k+x^{k+2})$ with rows given by g_j as in Proposition 29.6 for $n > j+k$ is also interesting. All submatrices defined by its first n rows and columns are unimodular and it defines thus a converging unimodular matrix in $Rec_{p\times p}(\mathbb{Z})$ which is not invertible (for $p > 1$) over $\mathbb{Z}$ but has an invertible reduction in $Rec_p(\mathbb{F}_\wp)$ for $\wp$ an arbitrary prime.*

Remark 29.11. *There are other similar examples, eg. by considering $s = (1-x^4)/(1-x^2+x^6)$, $s = (1+x^2-x^4)/(1-x^4+x^6)$, $s = (1-x^2-x^4)/(1-x^2-x^3-x^4+x^7)$ or $s = (1-2x+x^2+2x^3-x^4)/(1-2x+4x^3-x^4-2x^5+x^6)$. It would perhaps be interesting to classify all rational fractions giving rise to series $s \in \mathbb{Z}[[x]]$ such that the associated symmetric Toeplitz matrices $T(n)$ are unimodular for almost all $n \in \mathbb{N}$.*

29.10 Two more symmetric Toeplitz matrices

For $\epsilon \in \{1, -1\}$, consider the infinite symmetric Toeplitz matrix $T(n)$ with coefficients

$$T(n)_{i,j} = [x^{|i-j|}]\left(\epsilon + \frac{x}{1+x^2}\right),\ 0 \leq i, j$$

associated to the series $s = \epsilon + \frac{x}{1+x^2} = \epsilon + x - x^3 + x^5 - x^7 + \dots$.

For $n \equiv 0 \pmod 4$ we have $T(n) \in \mathrm{GL}_n(\mathbb{Z})$ with coefficients $(T(n)^{-1})_{i,j}, 0 \leq i, j$ of the inverse matrix given by

$$\begin{cases} -(n/2-1)\epsilon & \text{if } i = j, \\ (n/2 - |i-j|)(-\epsilon)^{i-j+1} & \text{otherwise.} \end{cases}$$

For even $p \in \mathbb{N}$, we get thus a converging element (still denoted) $T \in \mathrm{Rec}_{p\times p}(\mathbb{Z})$ by setting $T[\mathcal{M}^l] = T(p^l)$ for $l \geq 2$ (and defining $T[\mathcal{M}^0], T[\mathcal{M}^1]$ appropriately).

Remark 29.12. *For $\epsilon = 1$ one gets also an interesting larger family by adding a constant a to all coefficients of $T(n)$.*

30 The Baobab-example related to powers of 2

The infinite symmetric Hankel matrix

$$R = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & \dots \\ 1 & 0 & 1 & 0 & 0 & & & & \\ 0 & 1 & 0 & \dots & & & & & \\ \vdots & & & & & & & & \end{pmatrix}$$

with coefficients $R_{i,j} \in \{0, 1\}, 0 \leq i, j$ given by $R_{i,j} = 1$ if $(i + j + 1)$ is a power of 2 and $R_{i,j} = 0$ otherwise defines a converging element (still called) $R \in \mathrm{Rec}_{2\times 2}(\mathbb{Z})$ recursively presented by

$$R = (1, \begin{pmatrix} R & A \\ A & 0 \end{pmatrix}), A = (1, \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix})\ .$$

Shift matrices with respect to the basis (R, A) (having initial values $(R, A)[\emptyset, \emptyset] = (1, 1)$) of $\overline{R}^{rec}$ are given by

$$\rho(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \rho(0,1) = \rho(1,0) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \rho(1,1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The matrix R has an LU decomposition with $L = L_1 \in \mathrm{Rec}_2(\mathbf{K})$ the lower triangular matrix presented by $(L_1, L_2, L_3, L_4)[\emptyset, \emptyset] = (1, 1, 1, -1)$ and

shift matrices

$$\rho(0,0)=\begin{pmatrix}1&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\rho(0,1)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&1\end{pmatrix},$$

$$\rho(0,0)=\begin{pmatrix}0&0&0&0\\1&1&-1&1\\0&0&0&0\\0&0&0&0\end{pmatrix},\rho(0,1)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&1&1&-1\\0&0&0&0\end{pmatrix}.$$

The upper triangular matrix U is given by DL^t where D is the converging diagonal matrix with diagonal limit the $2-$periodic sequence $1,-1,1,-1,1,-1,\ldots$ recursively presented by $(S_1,S_2)[\emptyset]=(1,-1)$, $S_1=S_2=(S_1\ S_2)$.

The recurrence matrix L is invertible in $\mathrm{Rec}_2(\mathbb{Z})$ with inverse $M_1=M=L^{-1}$ presented by $(M_1,M_2,M_3,M_4)=(1,-1,1,1)$ and

$$\rho(0,0)=\begin{pmatrix}1&-1&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\rho(0,1)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&1\end{pmatrix},$$

$$\rho(0,0)=\begin{pmatrix}0&0&0&0\\1&1&-1&1\\0&0&0&0\\0&0&0&0\end{pmatrix},\rho(0,1)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&1&0\\0&0&0&0\end{pmatrix}.$$

Consider the converging diagonal matrix $D'=D'_1\in\mathrm{Rec}_{2\times 2}(\mathbb{Z})$ presented by $(D'_1,D'_2)[\emptyset,\emptyset]=(1,1)$ and

$$\rho(0,0)=\begin{pmatrix}1&-1\\0&0\end{pmatrix},\rho(0,1)=\rho(1,0)=\begin{pmatrix}0&0\\0&0\end{pmatrix},\rho(1,1)=\begin{pmatrix}0&0\\1&1\end{pmatrix}.$$

All diagonal coefficients of D'_1 (or D'_2) are in $\{\pm 1\}$ and D'_1 is thus an element of order 2 in $\mathrm{GL}_{2-rec}(\mathbb{Z})$. The $2-$automatic sequence formed by all diagonal coefficients of D'_1 starts as $1,1,-1,1,-1,-1,-1,1,\ldots$ and can be constructed in the following way: Given two words W_n,W'_n of length 2^n in the alphabet $\{\pm 1\}$ construct $W_{n+1}=W_nW'_n$ and $W'_{n+1}=(-W_n)W'_n$ by concatenating W_n (respectively $-W_n$) with W'_n. The diagonal sequence of D'_1 is the limit word W_∞ obtained from $W_0=W'_0=1$. The converging element W can thus be recursively presented by $(W,W')[\emptyset]=(1,1)$ and substition matrices $W=(W\ W')$, $W'=(-W\ W')$.

One can show the following result (see [3]):

Proposition 30.1. *(i) The limit of the converging recurrence matrix $D'LD'\in GL_{2-rec}(\mathbb{Z})$ is the infinite lower triangular matrix $\tilde{L}$ with coefficients in $\{0,1\}$ defined by $\tilde{L}_{i,j}\equiv\binom{2i+1}{i-j}\pmod 2$ for $0\le i,j$.*

(ii) The limit of the converging recurrence matrix $DD'MD'D \in GL_{2-rec}(\mathbb{Z})$ is the infinite lower triangular matrix $\tilde{L}$ with coefficients in $\{0,1\}$ defined by $\tilde{M}_{i,j} \equiv \binom{i+j}{2j} \pmod 2$ for $0 \leq i,j$.

Remark 30.2. *The recurrence matrices (still denoted) $\tilde{L}, \tilde{M} \in Rec_{2\times 2}(\mathbb{Z})$ defined by Proposition 30.1 display the following self-similar structure: Writing*

$$\tilde{L}[\mathcal{M}^l_{2\times 2}] = \begin{pmatrix} A & \\ B & A \end{pmatrix}, \tilde{M}[\mathcal{M}^l_{2\times 2}] = \begin{pmatrix} A' & \\ B' & A' \end{pmatrix}$$

for $l \geq 1$ (where $A = (\rho(0,0)\tilde{L})[\mathcal{M}^{l-1}_{2\times 2}] = (\rho(1,1)\tilde{L})[\mathcal{M}^{l-1}_{2\times 2}]$, etc) we have

$$\tilde{L}[\mathcal{M}^{l+1}_{2\times 2}] = \begin{pmatrix} A & & & \\ B & A & & \\ 0 & B & A & \\ B & A & B & A \end{pmatrix}, \tilde{M}[\mathcal{M}^{l+1}_{2\times 2}] = \begin{pmatrix} A' & & & \\ B' & A' & & \\ A' & B' & A' & \\ B' & 0 & B' & A' \end{pmatrix}.$$

*The name of this example is motivated by the word A**BA0BAB**ABA defining the matrix $\tilde{L}$. The matrices $\tilde{L} = A$ and $\tilde{M} = A'$ have recursive presentations given by*

$$A = (1, \begin{pmatrix} A & 0 \\ B & A \end{pmatrix}), B = (1, \begin{pmatrix} 0 & B \\ B & A \end{pmatrix})$$

and

$$A' = (1, \begin{pmatrix} A' & 0 \\ B' & A' \end{pmatrix}), B' = (1, \begin{pmatrix} A' & B' \\ B' & 0 \end{pmatrix}).$$

31 The Prouhet-Thue-Morse example

This example has been described in [2] and was "guiding principle" and main motivation.

We denote by $\tau(n)$ the integer-valued Prouhet-Thue-Morse sequence defined by $\tau(n) = \tau\left(\sum_{j=0}^{l} \epsilon_j 2^j\right) = \sum_{j=0}^{l} \epsilon_j$ for n a binary integer with binary digits $\epsilon_0, \ldots, \epsilon_l$. The infinite symmetric Hankel matrix of the sequence $i^{\tau(0)}, i^{\tau(1)}, i^{\tau(2)}, \ldots$ (where $i^2 = -1$) with generating series $\prod_{k=0}^{\infty}(1 + ix^{2^k})$ defines then a converging element (still denoted) $H \in \mathrm{Rec}_{2\times 2}(\mathbb{Z}[i])$ presented by $(H = H_1, H_2)[\emptyset, \emptyset] = (1, i)$ and shift matrices

$$\rho(0,0) = \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}, \ \rho(0,1) = \rho(1,0) = \begin{pmatrix} 0 & -i \\ 1 & 1+i \end{pmatrix}, \ \rho(1,1) = \begin{pmatrix} i & i \\ 0 & 0 \end{pmatrix}$$

The element H has an LU decomposition with $L \in \mathrm{Rec}_{2\times 2}(\mathbb{Z}[i])$ unipotent lower triangular presented by $(L = L_1, L_2, L_3, L_4)[\emptyset, \emptyset] = (1, i, 1, 0)$ and

shift matrices

$$\rho(0,0)=\begin{pmatrix}1&i&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad \rho(0,1)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&1\end{pmatrix},$$

$$\rho(1,0)=\begin{pmatrix}0&-i&-1+i&-i\\1&1+i&-i&1\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad \rho(1,1)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&1+i&1&i\\0&i&0&i\end{pmatrix}.$$

We have $U=DL^t\in \mathrm{Rec}_{2\times 2}(\mathbb{Z}[i])$ for the upper triangular matrix $U\in \mathrm{Rec}_{2\times 2}(\mathbb{Z}[i])$ where $D\in \mathrm{Rec}_{2\times 2}(\mathbb{Z}[i])$ is diagonal with diagonal coefficients given by presentation $(D=D_1,D_2,D_3)[\emptyset]=(1,1+i,1+i)$ and shift matrices

$$\rho(0,0)=\begin{pmatrix}1&0&0\\0&0&0\\0&1&1\end{pmatrix},\rho(0,1)=\rho(1,0)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},\rho(1,1)=\begin{pmatrix}0&2&0\\1&1&1\\0&-2&0\end{pmatrix}.$$

The inverse $M=L^{-1}$ has presentation $(M=M_1,M_2,M_3,M_4,M_5)[\emptyset,\emptyset]=(1,-i,1,-i,-1+i)$ and shift-matrices

$$\rho(0,0)=\begin{pmatrix}1&0&1&-i&-1\\0&0&0&0&0\\0&0&0&-i&1\\0&1&0&-i&0\\0&0&0&-i&1\end{pmatrix}\qquad \rho(0,1)=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&i&0&1&0\\0&i&0&1&0\\0&i&0&1&0\end{pmatrix}$$

$$\rho(1,0)=\begin{pmatrix}0&0&0&0&0\\1&0&-i&-i&i\\0&0&1-i&2&-1+i\\0&0&0&0&0\\0&1&0&1&0\end{pmatrix}\quad \rho(1,1)=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\1&0&1&i&-1\\0&0&0&0&0\\0&i&0&i&0\end{pmatrix}$$

The product $\prod_{k=0}^{\infty}(1+ix^{2^k})$ has thus a continuous fraction of Jacobi-type with coefficients forming converging $2-$recursive sequences of $\mathrm{Rec}_2(\mathbb{Z}[i])$.

Remark 31.1. *The somewhat similar Hankel matrices with generating series* $(1-x)\prod_{k=1}^{\infty}(1+ix^{2^k})$ *or* $(1+x)\prod_{k=1}^{\infty}(1+ix^{2^k})$ *have very similar properties.*

31.1 The Hankel matrix of $\frac{1+x}{1+i}\prod_{k=0}^{\infty}(1+ix^{2^k})$

Consider the Hankel matrix H associated to the sequence

$$\beta_1,\beta_2,\cdots=1,1+i,i,i,i,-1+i,-1,i,-1+i,-1,\ldots$$

defined by

$$\sum_{n=0}^{\infty} \beta_n x^n = \frac{1+x}{1+i} \prod_{k=0}^{\infty} \left(1 + ix^{2^k}\right)$$

where we drop the constant term $\beta_0 = \frac{1-i}{2}$.

Theorem 31.2. *The determinant $d(n) = \det(H(n))$ of the $n \times n$ Hankel matrix with coefficients $H_{i,j} = [x^{i+j+1}]\frac{1+x}{1+i}\prod_{k=0}^{\infty}\left(1+ix^{2^k}\right)$ is given by $d(n) = (-i)^{\lfloor n/2 \rfloor}$.*

The converging Hankel matrix $H \in \mathrm{Rec}_{2\times 2}(\mathbb{Z}[i])$ is presented by $(H = H_1, H_2)[\emptyset, \emptyset] = (1, 1+i)$ with the same shift-matrices

$$\rho(0,0) = \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}, \rho(0,1) = \rho(1,0) = \begin{pmatrix} 0 & -i \\ 1 & 1+i \end{pmatrix}, \rho(1,1) = \begin{pmatrix} i & i \\ 0 & 0 \end{pmatrix}$$

as for the previous example.

L of LU decomposition: $(L = L_1, L_2, L_3, L_4)[\emptyset, \emptyset] = (1, 1+i, 1, i)$ and the shift-matrices

$$\rho(0,0) = \begin{pmatrix} 1 & i & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \rho(0,1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\rho(1,0) = \begin{pmatrix} 0 & -i & -1+i & -i \\ 1 & 1+i & -i & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \rho(1,1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1+i & 1 & i \\ 0 & i & 0 & i \end{pmatrix}$$

are again as in the previous case.

Inverse $M = L^{-1}$ is presented by $(M = M_1, M_2, M_3, M_4)[\emptyset, \emptyset] = (1, -1-i, 1, -1)$ with shift-matrices

$$\rho(0,0) = \begin{pmatrix} 1 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -i \end{pmatrix} \qquad \rho(0,1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 1 \end{pmatrix}$$

$$\rho(1,0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -i & -i \\ 0 & -1 & 1-i & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \rho(1,1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The diagonal matrix D such that $H = LDL^t$ is the converging matrix with $2-$periodic diagonal entries $1, -i, 1, -i, 1, -i, \ldots$.

Proof of theorem 31.2 The result follows at once from the form of the diagonal matrix D involved in $H = LDL^t$. □

Remark 31.3. *The converging Hankel matrix H presented by $(H_1 = H, H_2)[\emptyset, \emptyset] = (1, 1)$ with shift matrices*

$$\rho(0,0) = \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}, \rho(0,1) = \rho(1,0) = \begin{pmatrix} 0 & -i \\ 1 & 1+i \end{pmatrix}, \rho(1,1) = \begin{pmatrix} i & i \\ 0 & 0 \end{pmatrix}$$

as above is also in $GL_{2-rec}(\mathbb{Q}[i])$ with inverse R (of Hankel type) presented by $(R_1 = R, R_2, R_3)[\emptyset, \emptyset] = (1, \frac{1+i}{2}, \frac{-1-i}{2})$ having shift-matrices $\rho(0,0), \rho(0,1) = \rho(1,0), \rho(1,1)$ given by

$$\begin{pmatrix} 0 & -i & 1+i \\ 0 & -1+i & -2i \\ i & -1+i & 1-2i \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1+i \\ 1 & -1-i & 2-i \\ 0 & -1-i & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1+i & -1 \\ 0 & -2i & 1+i \\ 1 & 1-2i & 1+i \end{pmatrix}.$$

The Hankel matrix H has however no LU decomposition since (for instance) the submatrix consisting of its first 9 rows and columns is singular.

31.2 The Hankel matrix of $\frac{x^2-1}{1-i}\prod_{k=0}^{\infty}(1+ix^{2^k})$

Let H denote the infinite Hankel matrix associated to the sequence

$$\gamma_2, \gamma_3, \gamma_4, \cdots = 1, i, 0, 0, i, -1, -i, 1, i, -1, 0, 0, -1, \ldots$$

of coefficients of $\frac{x^2-1}{1-i}\prod_{k=0}^{\infty}(1+ix^{2^k})$ with $\gamma_0 = \frac{-1-i}{2}, \gamma_1 = \frac{1-i}{2}$ dropped.

The convergent Hankel matrix $H \in \mathrm{Rec}_{2\times 2}(\mathbb{Z}[i])$ is presented by $(H = H_1, H_2, H_3)[\emptyset, \emptyset] = (1, i, 0)$ with shift-matrices $\rho(0,0), \rho(0,1) = \rho(1,0), \rho(1,1)$ as follows

$$\begin{pmatrix} 1 & 0 & i \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1-i & 0 \\ 1 & 1+i & i \\ 0 & i & 0 \end{pmatrix}, \begin{pmatrix} 0 & i & -i \\ 0 & 0 & 0 \\ 1 & 0 & 1+i \end{pmatrix}.$$

The convergent lower triangular unipotent matrix L involved in the LU decomposition of H is presented by $(L = L_1, L_2, L_3)[\emptyset, \emptyset] = (1, i, 0)$ with shift-matrices

$$\rho(0,0) = \begin{pmatrix} 1 & 0 & i \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \rho(0,1) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\rho(1,0) = \begin{pmatrix} 0 & 1-i & 0 \\ 1 & 1+i & i \\ 0 & i & 0 \end{pmatrix} \quad \rho(1,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -i & 1 \end{pmatrix}$$

The inverse matrix $M = L^{-1}$ is presented by $(M = M_1, M_2, M_3, M_4)[\emptyset, \emptyset] =$

$(1,-i,1,0)$ with shift-matrices

$$\rho(0,0)=\begin{pmatrix}1&0&1&-i\\0&0&0&0\\0&0&0&0\\0&1&0&-i\end{pmatrix}\qquad \rho(0,1)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&i&0&1\end{pmatrix}$$

$$\rho(1,0)=\begin{pmatrix}0&0&0&0\\1&0&-i&-i\\0&-1&1-i&1\\0&0&0&0\end{pmatrix}\quad \rho(1,1)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&-i&1&0\\0&0&0&0\end{pmatrix}$$

The diagonal matrix has diagonal coefficients defining $D\in \mathrm{Rec}_2([i])$ presented by $(D=D_1,D_2,D_3)[\emptyset,\emptyset]=(1,1,i)$ with non-zero shift-matrices given by

$$\rho(0,0)=\begin{pmatrix}1&0&1\\0&0&0\\0&1&0\end{pmatrix},\ \rho(1,1)=\begin{pmatrix}0&0&-1\\1&1&1\\0&0&1\end{pmatrix}.$$

Remark 31.4. *The converging Hankel matrix of the sequence* $n\longmapsto(-1)^{\tau(n)}$ *with generating series* $\prod_{k=0}^{\infty}(1-x^{2^k})$ *seems to be a non-invertible element of* $Rec_{2\times 2}(\mathbb{Q})$. *The matrix* $H[\mathcal{M}^k]$ *(with coefficients* $(-1)^{\tau(i+j)}, 0\le i,j<2^k$*), presented by* $(H_1=H,H_2)[\emptyset,\emptyset]=(1,-1)$ *and shift-matrices*

$$\rho(0,0)=\begin{pmatrix}1&-1\\0&0\end{pmatrix},\rho(0,1)=\rho(1,0)=\begin{pmatrix}0&1\\1&0\end{pmatrix},\rho(1,1)=\begin{pmatrix}-1&-1\\0&0\end{pmatrix},$$

seems however to have a nice and interesting inverse matrix $(H[\mathcal{M}^k])^{-1}$ *given by the Hankel matrix associated to the sequence* $\frac{1}{2}\sigma(k),-\frac{1}{2}\sigma(k)$ *(with last term unused) where* $\sigma(k)$ *is of length* 2^k *and is recursively defined by* $\sigma(0)_1=2$, $\sigma(k+1)_1=\frac{1}{2}\sum_{i=1}^{2^k}\sigma(k)_i$, $\sigma(k+1)_{2i+1}=\sigma(k+1)_{2i}$ *for* $i>0$ *and* $\sigma(k+1)_{2i}=\sigma(k+1)_{2i-1}-\sigma(k)_i$.

The first sequences $\sigma(k)$ *are:*

$\sigma(0)=2$
$\sigma(1)=(1,-1)$
$\sigma(2)=(0,-1,-1,0)$
$\sigma(3)=(-1,-1,-1,0,0,1,1,1)$
$\sigma(4)=(0,1,1,2,2,3,3,3,3,3,3,2,2,1,1,0)$.

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