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Recurrence matrices

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1 Introduction

The subject of this paper are *p*-recurrence matrices (or recurrence matrices for short) over a fixed ground field \mathbf{K} . A recurrence matrix is an element of the product

$$\prod_{l=0}^{\infty} \mathbf{K}^{p^l \times q^l}$$

(where $\mathbf{K}^{p^l \times q^l}$ denotes the vector-space of $p^l \times q^l$ matrices with coefficients in \mathbf{K}) satisfying finiteness conditions which are suitable for computations. Concretely, a recurrence matrix can be described by a finite amount of elements in \mathbf{K} , the set $\text{Rec}_{p \times q}(\mathbf{K})$ of all recurrence matrices in $\prod_{l=0}^{\infty} \mathbf{K}^{p^l \times q^l}$ is a vector space and the obvious product AB of two recurrence matrices $A \in \text{Rec}_{p \times r}(\mathbf{K}), B \in \text{Rec}_{r \times q}(\mathbf{K})$ is again a recurrence matrix. The subset $\text{Rec}_{p \times p}(\mathbf{K}) \subset \prod_{l=0}^{\infty} \mathbf{K}^{p^l \times q^l}$ of recurrence matrices is thus an algebra. We show how to do computations in this algebra and describe a few features of it.

The set of all invertible elements in the algebra $\text{Rec}_{p \times p}(\mathbf{K})$ forms a group $\text{GL}_{p\text{-rec}}(\mathbf{K})$ containing interesting subgroups. Indeed, a recurrence matrix $\text{GL}_{p\text{-rec}}(\mathbf{K})$ is closely linked to a finite-dimensional matrix-representation of the free monoid on p^2 elements. Recurrence matrices for which the image of this representation is a finite monoid are in bijection with (suitably defined) “automatic functions” associated to finite-state automata. This implies easily that $\text{GL}_{p\text{-rec}}(\mathbf{K})$ contains all *p*-automatic (in the Russian sense) or “*p*-self-similar groups” (formed by bijective finite-state transducers acting by automorphisms on the plane rooted *p*-regular tree).

In particular, the group $\text{GL}_{2\text{-rec}}(\mathbf{K})$ contains a famous group of Grigorchuk, see [6] (and all similarly defined groups) in a natural way.

Part of the present paper is contained in a condensed form in [2] whose main result was the initial motivation for developing the theory.

2 The category $K^{\mathcal{M}}$

Given two natural integers $p, q \in \mathbb{N}$ and a natural integer $l \in \mathbb{N}$, we define $\mathcal{M}_{p \times q}^l$ as the set of all $p^l q^l$ pairs of words (U, W) of common length l with

$U = u_1 \dots u_l \in \{0, \dots, p-1\}^l$ and $W = w_1 \dots w_l \in \{0, \dots, q-1\}^l$. The set $\mathcal{M}_{p \times q}^0$ contains by convention only (\emptyset, \emptyset) . We denote by $\mathcal{M}_{p \times q} = \bigcup_{l \in \mathbb{N}} \mathcal{M}_{p \times q}^l$ the union of all finite sets $\mathcal{M}_{p \times q}^l$. The common length $l = l(U) = l(W) \in \mathbb{N}$ is the *length* $l(U, W)$ of a *word* $(U, W) \in \mathcal{M}_{p \times q}^l \subset \mathcal{M}_{p \times q}$. We write $\mathcal{M}_{p \times q}^{\leq l}$ for the obvious set of all $1 + pq + \dots + p^l q^l = \frac{(pq)^{l+1} - 1}{pq - 1}$ words of length at most l in $\mathcal{M}_{p \times q}$.

The concatenation $(U, W)(U', W') = (UU', WW')$ turns $\mathcal{M}_{p \times q} = (\mathcal{M}_{p \times q}^1)^*$ into a monoid freely generated by the pq words of length 1 in $\mathcal{M}_{p \times q}^1$.

Equivalently, $\mathcal{M}_{p \times q}$ can be described as the submonoid of the product-monoid $\mathcal{M}_p \times \mathcal{M}_q$ (where \mathcal{M}_r stands for the free monoid on r generators) whose elements (U, W) are all pairs $U \in \mathcal{M}_p, W \in \mathcal{M}_q$ of the same length.

Remark 2.1. *Since the free cyclic monoid $\{0\}^*$ contains a unique word of length l for every $l \in \mathbb{N}$, we have an obvious isomorphism $\mathcal{M}_{p \times 1} \sim \mathcal{M}_p = \{0, \dots, p-1\}^*$. Moreover, $\mathcal{M}_{p \times q}$ is reduced to the empty word (\emptyset, \emptyset) if $pq = 0$.*

Remark 2.2. *Many constructions involving the monoid $\mathcal{M}_{p \times q}$ have generalizations to $\mathcal{M}_{p_1 \times p_2 \times \dots \times p_k}$ defined in the obvious way. In particular, using Remark 2.1, we write often simply \mathcal{M}_p instead of $\mathcal{M}_{p \times 1}$ or $\mathcal{M}_{1 \times p}$.*

In the sequel, we consider a fixed commutative field \mathbf{K} (many results continue to hold for commutative rings). We denote by $\mathbf{K}^{\mathcal{M}_{p \times q}}$ the vector space of all functions from $\mathcal{M}_{p \times q}$ into \mathbf{K} . We denote by $A[\mathcal{S}]$ the restriction of $A \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ to a subset $\mathcal{S} \subset \mathcal{M}_{p \times q}$. In particular, we write $A[U, W]$ for the evaluation of A on a word $(U, W) \in \mathcal{M}_{p \times q}$. For $A \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ and $l \in \mathbb{N}$ we consider the restriction $A[\mathcal{M}_{p \times q}^l]$ of A to $\mathcal{M}_{p \times q}^l$ as a matrix of size $p^l \times q^l$ with coefficients $A[U, W]$ indexed by all words $(U, W) = (u_1 \dots u_l, w_1 \dots w_l) \in \mathcal{M}_{p \times q}^l$.

For $A \in \mathbf{K}^{\mathcal{M}_{p \times r}}$ and $B \in \mathbf{K}^{\mathcal{M}_{r \times q}}$, we define the *matrix product* or *product* $AB \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ of A and B by

$$(AB)[U, W] = \sum_{V \in \{0, \dots, r-1\}^l} A[U, V]B[V, W]$$

where $(U, W) \in \mathcal{M}_{p \times q}^l$ is a word of length l . The matrix product is obviously bilinear and associative. We get thus a category $\mathbf{K}^{\mathcal{M}}$ as follows: An *object* of $\mathbf{K}^{\mathcal{M}}$ is a vector space of the form $\mathbf{K}^{\mathcal{M}_p} = \mathbf{K}^{\mathcal{M}_{p \times 1}}$ with $p \in \mathbb{N}$. A *morphism* (or *arrow*) is given by an element A of the vector space $\mathbf{K}^{\mathcal{M}_{q \times p}}$ and defines a linear application from $\mathbf{K}^{\mathcal{M}_p}$ to $\mathbf{K}^{\mathcal{M}_q}$ by matrix-multiplication.

The matrix-product turns the set $\mathbf{K}^{\mathcal{M}_{p \times p}}$ of endomorphisms of an object $\mathbf{K}^{\mathcal{M}_p}$ into an algebra.

Remark 2.3. *The category $\mathbf{K}^{\mathcal{M}}$ has also the following slightly different realization: Associate to an object $\mathbf{K}^{\mathcal{M}_p}$ corresponding to the natural integer*

$p \in \mathbb{N}$ the \mathbb{N} -graded vector space $\mathcal{F}S_p = \bigoplus_{l=0}^{\infty} \mathbf{K}^{p^l}$, identified with the subspace of $\mathbf{K}^{\mathcal{M}_p}$ of all functions with finite support. Morphisms are linear applications $\mathcal{F}S_p \rightarrow \mathcal{F}S_q$ preserving the grading (and are given by a product $\prod_{l=0}^{\infty} \mathbf{K}^{q^l \times p^l}$ of linear maps $\mathbf{K}^{p^l} \rightarrow \mathbf{K}^{q^l}$).

Remark 2.4. The category $\mathbf{K}^{\mathcal{M}}$, realized as in Remark 2.3 can be embedded as a full subcategory into a larger category with objects given by graded vector spaces $\mathcal{F}S_{(d_0, d_1, \dots)} = \bigoplus_{l=0}^{\infty} \mathbf{K}^{d_l}$ indexed by arbitrary sequences $d = (d_0, d_1, d_2, \dots) \in \mathbb{N}^{\mathbb{N}}$. Morphisms from $\mathcal{F}S_{(d_0, d_1, \dots)}$ to $\mathcal{F}S_{(e_0, e_1, \dots)}$ are linear applications preserving the grading and correspond to elements in the direct product of matrices $\prod_{l=0}^{\infty} \mathbf{K}^{e_l \times d_l}$. The category $\mathbf{K}^{\mathcal{M}}$ corresponds to the full subcategory with objects indexed by geometric progressions.

Remark 2.5. The vector-spaces $\mathcal{F}S_{p \times q} \subset \mathbf{K}^{\mathcal{M}_{p \times q}}$ can be identified with the vector spaces $\mathbf{K}[X_{0,0}, \dots, X_{p-1, q-1}] \subset \mathbf{K}[[X_{0,0}, \dots, X_{p-1, q-1}]]$ of polynomials and formal power-series in pq non-commuting variables $X_{u,w}, 0 \leq u < p, 0 \leq w < q$. Moreover, the vector-space $\mathbf{K}^{\mathcal{M}_{p \times q}}$ is also the algebraic dual of $\mathcal{F}S_{p \times q}$.

Remark 2.6. A vector $X \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ can be given as a projective limit by considering the projection

$$\mathbf{K}^{\mathcal{M}_{p \times q}^{\leq l+1}} \longrightarrow \mathbf{K}^{\mathcal{M}_{p \times q}^{\leq l}}$$

obtained by restricting the function $X[\mathcal{M}_{p \times q}^{\leq l+1}]$ to the subset $\mathcal{M}_{p \times q}^{\leq l} \subset \mathcal{M}_{p \times q}^{\leq l+1}$.

Remark 2.7. The following analogue of the tensor product yields a natural functor of the category $\mathbf{K}^{\mathcal{M}}$: For $A \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ and $B \in \mathbf{K}^{\mathcal{M}_{p' \times q'}}$, define $A \otimes B \in \mathbf{K}^{\mathcal{M}_{pp' \times qq'}}$ in the obvious way by considering tensor products $(A \otimes B)[\mathcal{M}_{pp' \times qq'}^l] = A[\mathcal{M}_{p \times q}^l] \otimes B[\mathcal{M}_{p' \times q'}^l]$ of the graded parts. This “tensor product” is bilinear and natural with respect to most constructions of this paper. The main difference with the usual tensor product is however the fact that $A \otimes B$ can be zero even if A and B are both non-zero: Consider A and B such that $A[U, W] = 0$ if (U, W) is of even length and $B[U', W'] = 0$ if (U', W') is of odd length.

3 The category $\mathbf{Rec}(\mathbf{K})$

A word $(S, T) \in \mathcal{M}_{p \times q}$ defines an endomorphism $\rho(S, T)$ of the vector space $\mathbf{K}^{\mathcal{M}_{p \times q}}$ by setting

$$(\rho(S, T)A)[U, W] = A[US, WT]$$

for $A \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ and $(U, W) \in \mathcal{M}_{p \times q}$. The easy computation

$$\begin{aligned} \rho(S, T)(\rho(S', T')A)[U, W] &= \rho(S', T')A[US, WT] \\ &= A[USS', WTT'] = \rho(SS', TT')A[U, W] \end{aligned}$$

shows that this defines a morphism ρ from the monoid $\mathcal{M}_{p \times q}$ into the algebra $\text{End}(\mathbf{K}^{\mathcal{M}_{p \times q}})$ of all linear endomorphisms of $\mathbf{K}^{\mathcal{M}_{p \times q}}$.

Definition 3.1. We call the monoid $\rho(\mathcal{M}_{p \times q}) \subset \text{End}(\mathbf{K}^{\mathcal{M}_{p \times q}})$ the *shift-monoid*. An element $\rho(S, T) \in \rho(\mathcal{M}_{p \times q})$ is a *shift-map*.

The shift-monoid is isomorphic to the free monoid $\mathcal{M}_{p \times q}$.

Remark 3.2. The terminology is motivated by the special case $p = q = 1$: An element $A \in \mathbf{K}^{\mathcal{M}_{1 \times 1}}$ is completely described by the sequence $\alpha_0, \alpha_1, \dots \in \mathbf{K}^{\mathbb{N}}$ defined by the evaluation $\alpha_l = A[0^l, 0^l]$ on the unique word $(0^l, 0^l) \in \mathcal{M}_{1 \times 1}^l$ of length l . The generator $\rho(0, 0) \in \rho(\mathcal{M}_{1 \times 1})$ acts on A by the usual shift $(\alpha_0, \alpha_1, \alpha_2, \dots) \mapsto (\alpha_1, \alpha_2, \alpha_3, \dots)$ which erases the first element of the sequence $\alpha_0, \alpha_1, \dots$ corresponding to A .

Given subsets $\mathcal{S} \subset \mathcal{M}_{p \times q}$ and $\mathcal{A} \subset \mathbf{K}^{\mathcal{M}_{p \times q}}$, we write

$$\rho(\mathcal{S})\mathcal{A} = \{\rho(S, T)A \mid (S, T) \in \mathcal{S}, A \in \mathcal{A}\} \subset \mathbf{K}^{\mathcal{M}_{p \times q}}.$$

Since $\rho(\emptyset, \emptyset)$ acts as the identity, we have $\mathcal{A} \subset \rho(\mathcal{S})\mathcal{A}$ for any subset $\mathcal{S} \subset \mathcal{M}_{p \times q}$ containing the empty word (\emptyset, \emptyset) of length 0.

Definition 3.3. A subset $\mathcal{S} \subset \mathbf{K}^{\mathcal{M}_{p \times q}}$ is a *recursively closed set* if $\rho(\mathcal{M}_{p \times q})\mathcal{S} = \mathcal{S}$. The *recursive set-closure* of $\mathcal{S} \subset \mathbf{K}^{\mathcal{M}_{p \times q}}$ is given by $\rho(\mathcal{M}_{p \times q})\mathcal{S}$ and is the smallest recursively closed subset of $\mathbf{K}^{\mathcal{M}_{p \times q}}$ containing \mathcal{S} . The *recursive closure* $\overline{\mathcal{S}}^{rec}$ is the linear span of $\rho(\mathcal{M}_{p \times q})\mathcal{S}$ and is the smallest linear subspace of $\mathbf{K}^{\mathcal{M}_{p \times q}}$ which contains \mathcal{S} and is recursively closed.

A subspace \mathcal{A} is recursively closed if and only if $\mathcal{A} = \overline{\mathcal{A}}^{rec}$. Intersections and sums (unions) of recursively closed subspaces (subsets) in $\mathbf{K}^{\mathcal{M}_{p \times q}}$ are recursively closed (subsets).

Any recursively closed subspace $\mathcal{A} \subset \mathbf{K}^{\mathcal{M}_{p \times q}}$ is invariant under the shift-monoid and we call the restriction $\rho_{\mathcal{A}}(\mathcal{M}_{p \times q}) \in \text{End}(\mathcal{A})$ of $\rho(\mathcal{M}_{p \times q})$ to the invariant subspace \mathcal{A} the *shift-monoid* of \mathcal{A} .

Definition 3.4. Given an element $A \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ with recursive closure \overline{A}^{rec} , we call the dimension $\dim(\overline{A}^{rec}) \in \mathbb{N} \cup \{\infty\}$ the (recursive) *complexity* of A . A *recurrence matrix* is an element of the vector space

$$\text{Rec}_{p \times q}(\mathbf{K}) = \{A \in \mathbf{K}^{\mathcal{M}_{p \times q}} \mid \dim(\overline{A}^{rec}) < \infty\}$$

spanned by all elements of finite complexity.

It is easy to check that $\text{Rec}_{p \times q}(\mathbf{K})$ is a recursively closed subspace of $\mathbf{K}^{\mathcal{M}_{p \times q}}$. An element $A \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ is a recurrence matrix, if and only if the shift-monoid $\rho_{\overline{A}^{rec}}(\mathcal{M}_{p \times q})$ of \overline{A}^{rec} is a finite dimensional linear representation of the free monoid $\mathcal{M}_{p \times q}$.

Proposition 3.5. *We have*

$$\dim(\overline{AB}^{rec}) \leq \dim(\overline{A}^{rec}) \dim(\overline{B}^{rec})$$

for the matrix-product $AB \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ of $A \in \mathbf{K}^{\mathcal{M}_{p \times r}}$ and $B \in \mathbf{K}^{\mathcal{M}_{r \times q}}$.

Corollary 3.6. *The matrix-product $AB \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ of two recurrence matrices $A \in \text{Rec}_{p \times r}(\mathbf{K})$, $B \in \text{Rec}_{r \times q}(\mathbf{K})$, is a recurrence matrix.*

Corollary 3.6 suggests the following definition.

Definition 3.7. *The category $\text{Rec}(\mathbf{K})$ of recurrence matrices is the subcategory of $\mathbf{K}^{\mathcal{M}}$ containing only recurrence matrices as arrows. Its objects can be restricted to $\text{Rec}_{p \times 1}(\mathbf{K})$ or even to the recursively closed subspaces $\mathcal{F}S_p = \bigoplus_{l=0}^{\infty} \mathbf{K}^{p^l}$ of elements with finite support.*

Proof of Proposition 3.5 Given bases A_1, A_2, \dots of $\overline{A}^{rec} \subset \text{Rec}_{p \times r}$ and B_1, B_2, \dots of $\overline{B}^{rec} \subset \text{Rec}_{r \times q}$, the computation

$$\begin{aligned} (\rho(s, t)(A_i B_j))[U, W] &= (A_i B_j)[Us, Wt] \\ &= \sum_{v=0}^{r-1} \sum_{V \in \{0, \dots, r-1\}^l} A_i[Us, Vv] B_j[Vv, Wt] \\ &= \sum_{v=0}^{r-1} \sum_{V \in \{0, \dots, r-1\}^l} (\rho(s, v) A_i)[U, V] (\rho(v, t) B_j)[V, W] \\ &= \left(\sum_{v=0}^{r-1} (\rho(s, v) A_i) (\rho(v, t) B_j) \right) [U, W] \end{aligned}$$

(with $(U, W) \in \mathcal{M}_{p \times q}^l$) shows that $\mathcal{C} = \sum_{1 \leq i, j} \mathbf{K} A_i B_j$ is recursively closed in $\mathbf{K}^{\mathcal{M}_{p \times q}}$. The inclusion $\overline{AB}^{rec} \subset \mathcal{C}$ finishes the proof. \square

3.1 Other ring-structures on $\text{Rec}_{p \times q}(\mathbf{K})$

The vector-space $\mathbf{K}^{\mathcal{M}_{p \times q}}$ carries two natural ring-structures which are both inherited by $\text{Rec}_{p \times q}(\mathbf{K})$.

A first ring-structure on $\mathbf{K}^{\mathcal{M}_{p \times q}}$ originates in the usual commutative product of functions $(A \circ B)[U, W] = (A[U, W])(B[U, W])$.

A second product (non-commutative if $pq > 1$) is given by associating to $A \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ the formal power series

$$\sum_{(U, W) = (u_1 \dots u_n, w_1 \dots w_n) \in \mathcal{M}_{p \times q}} A[u_1 \dots u_n, w_1 \dots w_n] X_{u_1, w_1} \cdots X_{u_n, w_n}$$

in pq non-commuting variables $X_{u, w}$, $0 \leq u < p$, $0 \leq w < q$. Multiplication of non-commutative formal power-series endows $\mathbf{K}^{\mathcal{M}_{p \times q}}$ with the *convolution-product*

$$(A * B)[U, W] = \sum_{(U, W) = (U_1, W_1)(U_2, W_2)} A[U_1, W_1] B[U_2, W_2]$$

where the sum is over all $l + 1$ factorizations of a word $(U, W) \in \mathcal{M}_{p \times q}^l$.

The following result shows that both ring-structures restrict to $\text{Rec}_{p \times q}(\mathbf{K})$.

Proposition 3.8. (i) $\text{Rec}_{p \times q}(\mathbf{K})$ is a commutative ring for the ordinary product of functions $(A \circ B)[U, W] = A[U, W]B[U, W]$. More precisely, we have $\overline{A \circ B}^{rec} \subset \sum \mathbf{K}A_i \circ B_j$ where $\overline{A}^{rec} = \sum \mathbf{K}A_i$ and $\overline{B}^{rec} = \sum \mathbf{K}B_j$.

(ii) $\text{Rec}_{p \times q}(\mathbf{K})$ is a ring for the convolution-product $(A * B)[U, W] = \sum_{(U, W) = (U_1, W_1)(U_2, W_2)} A[U_1, W_1]B[U_2, W_2]$. More precisely, we have $\overline{A * B}^{rec} \subset \sum \mathbf{K}A_i + \sum \mathbf{K}A * B_j$ where $\overline{A}^{rec} = \sum \mathbf{K}A_i$ and $\overline{B}^{rec} = \sum \mathbf{K}B_j$.

Proof Assertion (i) follows from Corollary 3.6 and the existence of a diagonal embedding of $\mathbf{K}^{\mathcal{M}_{p \times q}}$ into $\mathbf{K}^{\mathcal{M}_{(pq) \times (pq)}}$ preserving the complexity.

The computation

$$\begin{aligned} (\rho(s, t)(A * B))[U, W] &= (A * B)[Us, Wt] \\ &= \sum_{(U, W) = (U_1, W_1)(U_2, W_2)} A[U_1, W_1]B[U_2s, W_2t] + A[Us, Wt]B[\emptyset, \emptyset] \\ &= A * (\rho(s, t)B)[U, W] + \rho(s, t)A[U, W]B[\emptyset, \emptyset] \end{aligned}$$

shows the identity

$$\rho(s, t)(A * B) = A * (\rho(s, t)B) + B[\emptyset, \emptyset]\rho(s, t)A$$

and implies assertion (ii). \square

3.2 Convergent elements in $\mathbf{K}^{\mathcal{M}_{p \times q}}$ and $\text{Rec}_{p \times q}(\mathbf{K})$

Using the bijection

$$s_1 \dots s_n \longmapsto \sum_{j=1}^n s_j p^{j-1}$$

between $\{0, \dots, p-1\}^n$ and $\{0, \dots, p^n-1\}$, an infinite matrix \tilde{A} with coefficients $\tilde{A}_{s,t}, 0 \leq s, t \in \mathbb{N}$ gives rise to an element $A \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ by setting

$$A[s_1 \dots s_n, t_1 \dots t_n] = \tilde{A}_{s,t} \text{ where } s = \sum_{j=1}^n s_j p^{j-1}, t = \sum_{j=1}^n t_j q^{j-1}.$$

We call such an element A *convergent* with limit (the infinite matrix) \tilde{A} . If $p = 1$ or $q = 1$, the limit matrix \tilde{A} degenerates to a row or column-vector and it degenerates to a single coefficient $\tilde{A}_{0,0}$ if $pq = 0$. Obviously, an element $A \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ is convergent if and only if $\rho(0,0)A = A$. The vector space spanned by all converging elements in $\mathbf{K}^{\mathcal{M}_{p \times p}}$ (respectively in $\text{Rec}_{p \times p}(\mathbf{K})$) is not preserved under matrix-products (except if $p \leq 1$) but contains a few subalgebras given for instance by converging matrices with limit an infinite lower (or upper) triangular matrix.

Remark 3.9. *The vector-space of convergent elements contains a unique maximal subspace which is recursively closed. This subspace is the set of all convergent elements $A \in \text{Rec}_{p \times q}(\mathbf{K})$ such that $\overline{A}^{\text{rec}}$ consists only of convergent elements.*

3.3 The quotient category $\text{Rec}(\mathbf{K})/\mathcal{FS}$ modulo elements of finite support

We call a quotient \mathcal{A}/\mathcal{B} with vector-spaces $\mathcal{B} \subset \mathcal{A} \subset \text{Rec}_{p \times q}(\mathbf{K})$ a *quotient-space of recurrence matrices* if \mathcal{A} and \mathcal{B} are both recursively closed.

An important example is given by $\mathcal{A} = \text{Rec}_{p \times q}(\mathbf{K})$ and $\mathcal{B} = \mathcal{FS}_{p \times q}$ the vector space of all functions $B \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ with finite support. The vector space $\mathcal{FS}_{p \times q}$ is recursively closed since it consists of all elements A such that the shift map $\rho(\mathcal{M}_{p \times q})$ acts in a nilpotent way on $\overline{A}^{\text{rec}}$ (more precisely, for such an element A there exists a natural integer N such that $\rho(U, W)A = 0$ if $(U, W) \in \mathcal{M}_{p \times q}$ is of length $\geq N$). The elements of $\mathcal{FS}_{p \times q}$ can be considered as “trivial” in $\text{Rec}_{p \times q}(\mathbf{K})$. Since the subset $\mathcal{FS} = \cup_{p, q \in \mathbb{N}} \mathcal{FS}_{p \times q} \subset \cup_{p, q \in \mathbb{N}} \text{Rec}_{p \times q}(\mathbf{K})$ is in an obvious sense a “two-sided ideal” for the matrix-product (whenever defined), we get a functor from $\text{Rec}(\mathbf{K})$ onto the “quotient-category” $\text{Rec}(\mathbf{K})/\mathcal{FS}$.

Remark 3.10. *The subspace $\mathcal{FS}_{p \times q} = \text{Rec}_{p \times q}(\mathbf{K}) \cap \mathcal{FS}$ of all elements with finite support in $\text{Rec}_{p \times q}(\mathbf{K})$ is also an ideal for the commutative product of functions. Although $\mathcal{FS}_{p \times q}$ is also closed for the convolution-product (and corresponds to polynomials in non-commutative variables) it is not an ideal in the convolution-ring since it contains the convolutional identity Id_* given by $\text{Id}_*[\emptyset, \emptyset] = 1$ and $\text{Id}_*[U, W] = 0$ for $(U, W) \in \mathcal{M}_{p \times q} \setminus (\emptyset, \emptyset)$.*

4 Presentations

The aim of this section is to give finite descriptions of recurrence matrices in $\text{Rec}_{p \times q}(\mathbf{K})$.

Let $\mathcal{A} = \bigoplus_{j=1}^a \mathbf{K}A_j \subset \text{Rec}_{p \times q}(\mathbf{K})$ be a finite-dimensional recursively closed vector space with basis A_1, \dots, A_a . The isomorphism $\mathbf{K}^a \rightarrow \mathcal{A}$ defined by $(\alpha_1, \dots, \alpha_a) \mapsto \sum_{j=1}^a \alpha_j A_j$ realizes the shift-monoid $\rho_{\mathcal{A}}(\mathcal{M}_{p \times q}) \subset \mathbf{K}^{a \times a}$ of \mathcal{A} as a matrix monoid in $\text{End}(\mathbf{K}^a)$. The generators $\rho_{\mathcal{A}}(s, t) \in \mathbf{K}^{a \times a}$, $0 \leq s < p, 0 \leq t < q$ (with respect to the fixed basis A_1, \dots, A_a of \mathcal{A}) are called *shift-matrices*. Their coefficients $\rho_{\mathcal{A}}(s, t)_{j, k}$ (for $1 \leq j, k \leq a$) are given by

$$\rho(s, t)A_k = \sum_{j=1}^a \rho_{\mathcal{A}}(s, t)_{j, k} A_j .$$

More generally, we can define shift-matrices $\rho_{\mathcal{A}}(s, t) \in \mathbf{K}^{d \times d}$ with respect to any finite (not necessarily linearly independent) generating set A_1, \dots, A_d

of a recursively closed finite-dimensional vector-space $\mathcal{A} \subset \text{Rec}_{p \times q}(\mathbf{K})$ by requiring the identity $\rho(s, t)A_k = \sum_{j=1}^d \rho_{\mathcal{A}}(s, t)_{j,k} A_j$. These equations define the matrices $\rho_{\mathcal{A}}(s, t)$ up to linear applications $\mathbf{K}^d \rightarrow K$ where $K = \{(\alpha_1, \dots, \alpha_d) \in \mathbf{K}^d \mid \sum_{j=1}^d \alpha_j A_j = 0\}$ is the subspace of relations among the generators $A_1, \dots, A_d \in \mathcal{A}$.

Proposition 4.1. *Let $\mathcal{A} = \sum_{j=1}^d \mathbf{K}A_j \subset \text{Rec}_{p \times q}(\mathbf{K})$ be a recursively closed vector-space with shift-matrices $\rho_{\mathcal{A}}(s, t) \in \mathbf{K}^{d \times d}$ with respect to the (not necessarily free) generating set A_1, \dots, A_d . We have*

$$\begin{pmatrix} A_1[s_1 \dots s_n, t_1 \dots, t_n] \\ \vdots \\ A_d[s_1 \dots s_n, t_1 \dots, t_n] \end{pmatrix} = \rho_{\mathcal{A}}(s_n, t_n)^t \cdots \rho_{\mathcal{A}}(s_1, t_1)^t \begin{pmatrix} A_1[\emptyset, \emptyset] \\ \vdots \\ A_d[\emptyset, \emptyset] \end{pmatrix}.$$

Proof For $(U, W) = (s_1 \dots s_n, t_1 \dots t_n) \in \mathcal{M}_{p \times q}^n$, the formula

$$A[U, W] = (\rho_{\mathcal{A}}(U, W)A)[\emptyset, \emptyset] = (\rho_{\mathcal{A}}(s_1, t_1) \cdots \rho_{\mathcal{A}}(s_n, t_n)A)[\emptyset, \emptyset]$$

implies the result by duality.

A second proof is given by the computation

$$\begin{aligned} \begin{pmatrix} A_1[s_1 \dots s_n, t_1 \dots, t_n] \\ \vdots \\ A_d[s_1 \dots s_n, t_1 \dots, t_n] \end{pmatrix} &= \begin{pmatrix} \rho_{\mathcal{A}}(s_n, t_n)A_1[s_1 \dots s_{n-1}, t_1 \dots, t_{n-1}] \\ \vdots \\ \rho_{\mathcal{A}}(s_n, t_n)A_d[s_1 \dots s_{n-1}, t_1 \dots, t_{n-1}] \end{pmatrix} \\ &= \rho_{\mathcal{A}}(s_n, t_n)^t \begin{pmatrix} A_1[s_1 \dots s_{n-1}, t_1 \dots, t_{n-1}] \\ \vdots \\ A_d[s_1 \dots s_{n-1}, t_1 \dots, t_{n-1}] \end{pmatrix} \end{aligned}$$

and induction on n . □

Definition 4.2. *A presentation \mathcal{P} of complexity d is given by the following data:*

a vector $(\alpha_1, \dots, \alpha_d) \in \mathbf{K}^d$ of d initial values,

pq shift-matrices $\rho_{\mathcal{P}}(s, t) \in \mathbf{K}^{d \times d}$ of size $d \times d$ with coefficients $\rho_{\mathcal{P}}(s, t)_{k,j}, 1 \leq k, j \leq d$.

Proposition 4.3. *A presentation \mathcal{P} of complexity d as above defines a unique set $A_1, \dots, A_d \in \text{Rec}_{p \times q}$ of d recurrence matrices such that $A_k[\emptyset, \emptyset] = \alpha_k$ and $\rho(s, t)A_k = \sum_{j=1}^d \rho_{\mathcal{P}}(s, t)_{j,k} A_j$ for $1 \leq k \leq d$ and for all $(s, t) \in \mathcal{M}_{p \times q}^1$.*

Proof For $(U, W) = (s_1 \dots s_n, t_1 \dots t_n) \in \mathcal{M}_{p \times q}^n$, we define the evaluations $A_1[U, W], \dots, A_d[U, W] \in \mathbf{K}$ by

$$\begin{pmatrix} A_1[U, W] \\ \vdots \\ A_d[U, W] \end{pmatrix} = \rho_{\mathcal{P}}(s_n, t_n)^t \cdots \rho_{\mathcal{P}}(s_1, t_1)^t \begin{pmatrix} A_1[\emptyset, \emptyset] \\ \vdots \\ A_d[\emptyset, \emptyset] \end{pmatrix}.$$

The result follows from Proposition 4.1. \square

In the sequel, we will often drop the subscript \mathcal{P} for shift-matrices of a presentation. Thus, we identify (abusively) shift-matrices with the corresponding shift-maps, restricted to the subspace defined by the presentation.

A presentation \mathcal{P} is *reduced* if the elements $A_1, \dots, A_d \in \text{Rec}_{p \times q}(\mathbf{K})$ defined by \mathcal{P} are linearly independent. We say that a presentation \mathcal{P} *presents* (or is a presentation of) the recurrence matrix $A = A_1 \in \text{Rec}_{p \times q}(\mathbf{K})$. The empty presentation of complexity 0 presents by convention the zero-element of $\text{Rec}_{p \times q}(\mathbf{K})$. A presentation of $A \in \text{Rec}_{p \times q}(\mathbf{K})$ with complexity $a = \dim(\overline{A}^{rec})$ is *minimal*. Every recurrence matrix $A \in \text{Rec}_{p \times q}(\mathbf{K})$ has a minimal presentation \mathcal{P} : Complete $0 \neq A \in \text{Rec}_{p \times q}(\mathbf{K})$ to a basis $A_1 = A, \dots, A_a$ of its recursive closure \overline{A}^{rec} . For $1 \leq k \leq a$, set $\alpha_k = A_k[\emptyset, \emptyset] \in \mathbf{K}$ and consider the shift matrices $\rho(s, t)_{\mathcal{P}} \in \text{End}(\mathbf{K}^a)$ uniquely defined by

$$\rho(s, t)A_k = \sum_{j=1}^a \rho_{\mathcal{P}}(s, t)_{j,k} A_j .$$

In the sequel, a presentation denotes often a finite set of recurrence matrices $A_1, \dots, A_d \in \text{Rec}_{p \times q}(\mathbf{K})$ spanning a recursively closed subspace $\sum_{k=1}^d \mathbf{K}A_k$ together with pq suitable shift matrices in $\text{End}(\mathbf{K}^d)$ (which are sometimes omitted if they are obvious). We denote by $(A_1, \dots, A_d)[\emptyset, \emptyset] \in \mathbf{K}^d$ the corresponding initial values.

Proposition 4.4. *Given a presentation \mathcal{P} of d recurrence matrices $A_1, \dots, A_d \in \text{Rec}_{p \times q}(\mathbf{K})$ spanning a recursively closed subspace, a presentation \mathcal{P}^t of the transposed recurrence matrices $A_1^t, \dots, A_d^t \in \text{Rec}_{q \times p}(\mathbf{K})$ is given by the same initial data $(A_1^t, \dots, A_d^t)[\emptyset, \emptyset] = (A_1, \dots, A_d)[\emptyset, \emptyset]$ and the same shift matrices $\tilde{\rho}(t, s) = \rho(s, t)$, up to “transposition” of their labels.*

Proof Use an easy induction on l for the restricted functions $A_1[\mathcal{M}_{p \times q}^l], \dots, A_d[\mathcal{M}_{p \times q}^l]$.

\square

Remark 4.5. *Shift-matrices of transposed recurrence matrices are identical to the original shift-matrices (and are not to be transposed), only their labels are rearranged!*

Remark 4.6. *Endowing $\mathcal{M}_{p \times q}$ with a complete order, the recursive closure \overline{A}^{rec} of an element $A \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ has a unique basis of the form $\rho(U_1, W_1)A, \rho(U_2, W_2)A, \dots$ where the word $(U_j, W_j) \in \mathcal{M}_{p \times q}$ (if it exists) is inductively defined as the smallest element such that $\rho(U_1, W_1)A, \dots, \rho(U_j, W_j)A$ are linearly independent.*

5 Topologies and a metric

For \mathbf{K} a topological field, the vector space $\mathbf{K}^{\mathcal{M}_{p \times q}}$ carries two natural “obvious” topologies.

The first one is the product topology on $\prod_{l=0}^{\infty} \mathbf{K}^{\mathcal{M}_{p \times q}^l}$. It is defined as the coarsest topology for which all projections $A \mapsto A[\mathcal{M}_{p \times q}^l]$ are continuous. Its open subsets are generated by $\mathcal{O}_{\leq m} \times \prod_{l > m} \mathbf{K}^{\mathcal{M}_{p \times q}^l}$ with $\mathcal{O}_{\leq m} \subset \mathbf{K}^{\mathcal{M}_{p \times q}^{\leq m}}$ open for the natural topology on the finite-dimensional vector space $\mathbf{K}^{\mathcal{M}_{p \times q}^{\leq m}}$.

The second topology is the strong topology or box topology. Its open subsets are generated by $\prod_{l=0}^{\infty} \mathcal{O}_l$, with $\mathcal{O}_l \subset \mathbf{K}^{\mathcal{M}_{p \times q}^l}$ open for all $l \in \mathbb{N}$.

The restrictions to $\text{Rec}_{p \times q}(\mathbf{K})$ of both topologies are not very interesting: The product topology is very coarse, the strong topology is too fine: it gives in general rise to the discrete topology on the quotient $\text{Rec}_{p \times q}(\mathbf{K})/\mathcal{FS}_{p \times q}$ considered in 3.3.

For a recurrence matrix $A \in \text{Rec}_{p \times q}(\mathbb{C})$ defined over \mathbb{C} we set

$$\| A \|_{\infty} = \limsup_{l \rightarrow \infty} \sup_{(U,W) \in \mathcal{M}_{p \times q}^l} |A[U, W]|^{1/l}.$$

Remark 5.1. $\| A \|_{\infty}$ equals the supremum over $\mathcal{M}_{p \times q}$ of the numbers $\sigma(S, T)^{1/l}$ where $(S, T) \in \mathcal{M}_{p \times q}^l$ has length l and where $\sigma(S, T)$ is the spectral radius (largest modulus among eigenvalues) of the shift matrix $\rho_{\overline{A}^{\text{rec}}}(S, T)$ with respect to a minimal presentation of A . We have thus the inequality $\sigma_{s,t} \leq \| A \|_{\infty}$ (which is in general strict) where $\sigma_{s,t}$ are the spectral radii of shift-matrices $\rho(s, t) \in \mathbb{C}^{d \times d}$ associated to a minimal presentation of A .

Proposition 5.2. The application $A \mapsto \| A \|_{\infty}$ defines a metric on $\mathbb{C}^* \setminus \text{Rec}_{p \times q}(\mathbb{C})/\mathcal{FS}_{p \times q}$ such that

$$\| A + \lambda B \|_{\infty} = \sup(\| A \|_{\infty}, \| B \|_{\infty})$$

for $\lambda \in \mathbf{K} \setminus \{0\}$, $A, B \in \text{Rec}_{p \times q}(\mathbb{C})$ and

$$\| AB \|_{\infty} \leq r \| A \|_{\infty} \| B \|_{\infty}$$

for $A \in \text{Rec}_{p \times r}(\mathbb{C})$, $B \in \text{Rec}_{r \times q}(\mathbb{C})$.

The proof is left to the reader.

Remark 5.3. The set of all elements $A \in \mathbb{C}^{\mathcal{M}_{p \times q}}$, $p, q \in \mathbb{N}$, such that $\| A \|_{\infty} < \infty$ form a subcategory containing $\text{Rec}(\mathbb{C})$ of $\mathbb{C}^{\mathcal{M}}$.

The vector space $\text{Rec}_{p \times q}(\mathbb{C})$ can be normed by

$$\| A \| = \sum_{l=0}^{\infty} \frac{\| A[\mathcal{M}_{p \times q}^l] \|_{\infty}}{l!}$$

where $\| A[\mathcal{M}_{p \times q}^l] \|_{\infty}$ denotes the largest absolute value of all coefficients $A[U, W]$, $(U, W) \in \mathcal{M}_{p \times q}^l$. However, matrix-multiplication is not continuous for this norm.

This norm has different obvious variations:

The factorials $l!$ of the denominators can be replaced by any other sequence s_0, s_1, \dots of strictly positive numbers such that $\lim_{n \rightarrow \infty} \lambda^n / s_n = 0$ for all $\lambda > 0$.

The sup-norm $\|A[\mathcal{M}_{p \times q}^l]\|_\infty$ can be replaced by many other “reasonable” norms (like l_1 or l_2 norms) on $\mathbb{C}^{\mathcal{M}_{p \times q}^l}$.

It would be interesting to find a norm on the algebra $\text{Rec}_{p \times p}(\mathbb{C})$ for which matrix products are continuous.

6 Criteria for non-recurrence matrices

This short section lists a few easy properties (which have sometimes obvious generalizations, eg by replacing \mathbb{Q} with a number field) of recurrence matrices and non-recurrence matrices in $\mathbf{K}^{\mathcal{M}_{p \times q}}$. Proofs are straightforward and left to the reader.

Proposition 6.1. *An element $A \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ is not in $\text{Rec}_{p \times q}(\mathbf{K})$ if and only if there exist two sequences $(S_i, T_i), (U_j, W_j) \in (\mathcal{M}_{p \times q})^{\mathbb{N}}$ such that for all $n \in \mathbb{N}$, the $n \times n$ matrix with coefficients $c_{i,j} = \rho(S_i, T_i)A[U_j, W_j] = A[U_j S_i, W_j T_i], 1 \leq i, j \leq n$, has non-zero determinant.*

Proposition 6.2. *Consider $A \in \text{Rec}_{p \times q}(\mathbb{C})$. Then there exists a constant $C \geq 0$ such that $|A[U, W]| \leq C^{n+1}$ for all $(U, W) \in \mathcal{M}_{p \times q}^n$.*

Remark 6.3. *Associate to $A \in \mathbb{C}^{\mathcal{M}_p}$ the formal power series*

$$f_A = A[\emptyset] + \sum_{n=1}^{\infty} \sum_{0 \leq u_1, \dots, u_n < p} A[u_1 \dots u_n] Z_{u_1} \cdots Z_{u_n}$$

in p commuting variables Z_0, \dots, Z_{p-1} . Proposition 6.2 shows that f_A defines a holomorphic function in a neighbourhood of $(0, \dots, 0) \in \mathbb{C}^p$ if $A \in \text{Rec}_p(\mathbb{C})$.

Proposition 6.4. *The set of values $\{A[U, W] \mid (U, W) \in \mathcal{M}_{p \times q}\}$ of a recurrence matrix $A \in \text{Rec}_{p \times q}(\mathbf{K})$ is contained in a subring $\tilde{\mathbf{K}} \subset \mathbf{K}$ which is finitely generated (as a ring).*

Proposition 6.5. *For $A \in \text{Rec}_{p \times q}(\mathbb{Q})$, there exists a natural integer N such that $N^{n+1}A[U, W] \in \mathbb{Z}$ for all $(U, W) \in \mathcal{M}_{p \times q}^n$.*

7 Saturation

The aim of this section is to give finite algorithms for computations in the category $\text{Rec}(\mathbf{K})$. More precisely, given presentations of two suitable recurrence matrices A, B , we show how to compute a presentation of the sum

$A + B$, or of the matrix-product AB (whenever defined) of A and B in a finite number of steps.

For a vector space $\mathcal{A} \subset \mathbf{K}^{\mathcal{M}_{p \times q}}$ we denote by $\mathcal{A}[\mathcal{M}_{p \times q}^{\leq l}] \subset \mathbf{K}^{\mathcal{M}_{p \times q}^{\leq l}}$ the image of \mathcal{A} under the projection $A \mapsto A[\mathcal{M}_{p \times q}^{\leq l}]$ associating to $A \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ its restriction $A[\mathcal{M}_{p \times q}^{\leq l}] \in \mathbf{K}^{\mathcal{M}_{p \times q}^{\leq l}}$ to the subset of all $\frac{(pq)^{l+1}-1}{pq-1}$ words of length at most l in $\mathcal{M}_{p \times q}$.

Definition 7.1. *The saturation level of a vector space $\mathcal{A} \subset \mathbf{K}^{\mathcal{M}_{p \times q}}$ is the smallest natural integer $N \geq 0$, if it exists, for which the obvious projection*

$$\mathcal{A}[\mathcal{M}_{p \times q}^{\leq N+1}] \longrightarrow \mathcal{A}[\mathcal{M}_{p \times q}^{\leq N}]$$

is an isomorphism. A vector-space \mathcal{A} has saturation level ∞ if such an integer N does not exist. The saturation level of $A \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ is the saturation level of its recursive closure \overline{A}^{rec} .

Proposition 7.2. *Let $\mathcal{A} \subset \mathbf{K}^{\mathcal{M}_{p \times q}}$ be a recursively closed vector space of finite saturation level N . Then \mathcal{A} and $\mathcal{A}[\mathcal{M}_{p \times q}^{\leq N}]$ are isomorphic.*

In particular, \mathcal{A} is of finite dimension and contained in $\text{Rec}_{p \times q}(\mathbf{K})$.

Corollary 7.3. *A finite set $A_1, \dots, A_d \subset \text{Rec}_{p \times q}(\mathbf{K})$ spanning a recursively closed subspace $\mathcal{A} = \sum_{j=1}^d \mathbf{K}A_j \subset \text{Rec}_{p \times q}(\mathbf{K})$ is linearly independent if and only if $A_1[\mathcal{M}_{p \times q}^{\leq N}], \dots, A_d[\mathcal{M}_{p \times q}^{\leq N}] \in \mathbf{K}^{\mathcal{M}_{p \times q}^{\leq N}}$ are linearly independent where $N \leq \dim(\mathcal{A}) - 1 < d$ denotes the saturation level of \mathcal{A} .*

Proof of Proposition 7.2 We denote by $K_l = \{A \in \mathcal{A} \mid A[U, W] = 0 \text{ for all } (U, W) \in \mathcal{M}_{p \times q}^{\leq l}\} \subset \mathcal{A}$ the kernel of the natural projection $\mathcal{A} \longrightarrow \mathcal{A}[\mathcal{M}_{p \times q}^{\leq l}]$. We have

$$K_{N+1} = \{A \in K_N \mid \rho(\mathcal{M}_{p \times q}^1)A \subset K_N\} = K_N$$

which shows the equality $\rho(\mathcal{M}_{p \times q})K_N = K_N$. For $A \in K_N \subset K_0$ we have thus $A[U, W] = \rho(U, W)A[\emptyset, \emptyset] = 0$ for all $(U, W) \in \mathcal{M}_{p \times q}$ which implies $A = 0$ and shows the result. \square

Corollary 7.3 follows immediately.

Remark 7.4. *The proof of Proposition 7.2 shows the inequality $N + 1 \leq \dim(\overline{A}^{rec})$ for the saturation level N of $A \in \text{Rec}_{p \times q}(\mathbf{K})$. Equality is achieved eg. for the recurrence matrix defined by $A[U, W] = 1$ if $(U, W) \in \mathcal{M}_{p \times q}^N$ and $A[U, W] = 0$ otherwise.*

7.1 Algorithmic aspects

It is easy to extract a minimal presentation from a presentation \mathcal{P} of a recurrence matrix $A \in \text{Rec}_{p \times q}(\mathbf{K})$: Compute the saturation level N of the

recursively closed space $\mathcal{A} = \sum_{j=1}^d \mathbf{K}A_j$ defined by the presentation \mathcal{P} . This allows to detect linear dependencies among A_1, \dots, A_d .

Using Proposition 4.1, these computations can be done in polynomial time with respect to the complexity d of the presentation \mathcal{P} for A .

Adding two recurrence matrices A, B in $\text{Rec}_{p \times q}(\mathbf{K})$ is now easy: given presentations A_1, \dots, A_a and B_1, \dots, B_b of A and B , one can write down a presentation $C_1 = A_1 + B_1, C_2 = A_2, \dots, C_a = A_a, C_{a+1} = B_1, C_{a+2} = B_2, \dots, C_{a+b} = B_b$ of $A + B$.

Similarly, a presentation A_1, \dots, A_a and B_1, \dots, B_b of $A \in \text{Rec}_{p \times r}(\mathbf{K})$ and $B \in \text{Rec}_{r \times q}(\mathbf{K})$ gives rise to a presentation $C_1 = A_1 B_1, \dots, C_{ab} = A_a B_b$ of $C = C_1 = A_1 B_1$ (computing the corresponding shift-matrices is more involved and can be done in $\mathbf{K}^{\mathcal{M}_{p \times q}^{\leq N+1}}$ where N denotes the saturation level of $\sum \mathbf{K}A_i B_j$).

8 Diagonal and lower triangular subalgebras in $\mathbf{K}^{\mathcal{M}_{p \times p}}$ and $\text{Rec}_{p \times p}(\mathbf{K})$

This section describes a maximal commutative subalgebra formed by diagonal elements, the center (which is contained as a subalgebra in the diagonal algebra) and the lower triangular subalgebra of the algebras $\mathbf{K}^{\mathcal{M}_{p \times p}}$ and $\text{Rec}_{p \times p}(\mathbf{K})$.

8.1 The diagonal subalgebra and the center

An element $A \in \mathbf{K}^{\mathcal{M}_{p \times p}}$ is *diagonal* if $A[U, W] = 0$ for all $(U, W) \in \mathcal{M}_{p \times p}$ such that $U \neq W$. We denote by $\mathcal{D}_p(\mathbf{K}) \subset \mathbf{K}^{\mathcal{M}_{p \times p}}$ the vector space of all diagonal elements. It is easy to show that $\mathcal{D}_p(\mathbf{K})$ is a commutative algebra which is isomorphic to the function ring underlying $\mathbf{K}^{\mathcal{M}_p}$.

The center of the algebra $\mathbf{K}^{\mathcal{M}_{p \times p}}$ is the subalgebra $\mathcal{C}_p(\mathbf{K}) \subset \mathcal{D}_p(\mathbf{K})$ formed by all diagonal matrices A with diagonal coefficients $A[U, U]$ depending only on the length l of $(U, U) \in \mathcal{M}_{p \times p}^l$. For $p \geq 1$, the map

$$\mathcal{C}_p(\mathbf{K}) \ni C \longmapsto (C[\emptyset, \emptyset], C[0, 0], C[0^2, 0^2], \dots)$$

defines an isomorphism between $\mathcal{C}_p(\mathbf{K})$ and the algebra $\mathbf{K}^{\mathcal{M}_{1 \times 1}}$ corresponding to all sequences $\mathbb{N} \rightarrow \mathbf{K}$, endowed with the coefficient-wise (or Hadamard) product.

We denote by $\mathcal{D}_{p\text{-rec}}(\mathbf{K}) = \mathcal{D}_p(\mathbf{K}) \cap \text{Rec}_{p \times p}(\mathbf{K})$ and by $\mathcal{C}_{p\text{-rec}}(\mathbf{K}) = \mathcal{C}_p(\mathbf{K}) \cap \text{Rec}_{p \times p}(\mathbf{K})$ the subalgebras of $\mathcal{D}_p(\mathbf{K})$ and $\mathcal{C}_p(\mathbf{K})$ formed by all recurrence matrices. Associating to $C \in \mathcal{C}_{p\text{-rec}}(\mathbf{K})$ the generating series $\sum_{l=0}^{\infty} C[0^l, 0^l]z^l$ (where $0^0 = \emptyset$) yields an isomorphism between the algebra $\mathcal{C}_{p\text{-rec}}(\mathbf{K})$ and the algebra $\text{Rec}_{1 \times 1}(\mathbf{K})$ corresponding to the vector-space of rational functions in one variable without pole at the origin. The product is

the coefficientwise product $(\sum_{l=0}^{\infty} \alpha_l z^l) \cdot (\sum_{l=0}^{\infty} \beta_l z^l) = (\sum_{l=0}^{\infty} \alpha_l \beta_l z^l)$ of the corresponding series-expansions.

Diagonal and central recurrence matrices in $\text{Rec}_{p \times p}(\mathbf{K})$ can be characterized by the following result.

Proposition 8.1. (i) A recurrence matrix $A \in \text{Rec}_{p \times p}(\mathbf{K})$ is diagonal if and only if it can be given by a presentation with shift-matrices $\rho(s, t) = 0$ whenever $s \neq t, 0 \leq s, t < p$.

(ii) A diagonal recurrence matrix $A \in \text{Rec}_{p \times p}(\mathbf{K})$ is central if and only if it can be given by a presentation with shift-matrices $\rho(s, t) = 0$ whenever $s \neq t, 0 \leq s, t < p$ and $\rho(s, s) = \rho(0, 0)$ for all $s, 0 \leq s < p$.

We leave the easy proof to the reader.

Remark 8.2. The vector spaces $\mathcal{D}_p, \mathcal{D}_{p\text{-rec}}, \mathcal{C}_p, \mathcal{C}_{p\text{-rec}}$ are also (non-commutative for \mathcal{D}_p and $\mathcal{D}_{p\text{-rec}}$, if $p > 1$) algebras for the convolution-product.

8.2 Lower triangular subalgebras

Lower (or upper) triangular elements in $\mathbf{K}^{\mathcal{M}_{p \times p}}$ can be defined using the bijection

$$(u_1 \dots u_l, w_1 \dots w_l) \mapsto \left(\sum_{j=1}^l u_j p^{j-1}, \sum_{j=1}^l w_j p^{j-1} \right)$$

between $\mathcal{M}_{p \times p}^l$ and $\{0, \dots, p^l - 1\} \times \{0, \dots, p^l - 1\}$. More precisely, $A \in \mathbf{K}^{\mathcal{M}_{p \times p}}$ is *lower triangular* if for all $l \in \mathbb{N}$ the equality $A[U, W] = 0$ holds for $(U, W) = (u_1 \dots u_l, w_1 \dots w_l) \in \mathcal{M}_{p \times p}$ such that $\sum_{j=1}^l u_j p^{j-1} < \sum_{j=1}^l w_j p^{j-1}$.

Similarly, an element $A \in \mathbf{K}^{\mathcal{M}_{p \times p}}$ is *upper triangular* if A^t (defined by $A^t[U, W] = A[W, U]$) is lower triangular.

We denote by $\mathcal{L}_p(\mathbf{K}) \subset \mathbf{K}^{\mathcal{M}_{p \times p}}$ the vector-space of all lower triangular elements in $\mathbf{K}^{\mathcal{M}_{p \times p}}$. It is easy to check that $\mathcal{L}_p(\mathbf{K})$ is closed under the matrix-product and is thus an algebra. The subset of all convergent elements in $\mathcal{L}_p(\mathbf{K})$ is also closed under the matrix-product and forms a subalgebra.

A lower triangular matrix $A \in \mathcal{L}_p(\mathbf{K})$ is *unipotent* if $A[U, U] = 1$ for all “diagonal words” $(U, U) \in \mathcal{M}_{p \times p}$ and *strictly lower triangular* if $A[U, U] = 0$ for all diagonal words $(U, U) \in \mathcal{M}_{p \times p}$. The subset $\mathcal{N}_p(\mathbf{K})$ of all lower strictly-triangular matrices is a two-sided ideal in $\mathcal{L}_p(\mathbf{K})$. The associated quotient $\mathcal{L}_p(\mathbf{K})/\mathcal{N}_p(\mathbf{K})$ is isomorphic to the commutative algebra $\mathcal{D}_p(\mathbf{K})$ of diagonal elements. Unipotent lower triangular matrices form a multiplicative subgroup and correspond to the (multiplicative) identity of the quotient algebra $\mathcal{L}_p(\mathbf{K})/\mathcal{N}_p(\mathbf{K})$.

The following proposition is useful for recognizing triangular recurrence matrices:

Proposition 8.3. A recurrence matrix $A \in \text{End}_{p\text{-rec}}(\mathbf{K})$ is triangular if and only if it admits a presentation of the form $A = A_1, \dots, A_k, A_{k+1}, \dots, A_a$

such that $\rho(s, s)A_1, \dots, \rho(s, s)A_k \in \sum_{j=1}^k A_j$ for $0 \leq s < p$ and $\rho(s, t)A_1, \dots, \rho(s, t)A_k = 0$ for $0 \leq s < t < p$.

Such a recurrence matrix A is unipotent if and only if it admits a pre-representation as above which satisfies moreover $A_1[\emptyset, \emptyset] = 1, A_2[\emptyset, \emptyset] = \dots = A_k[\emptyset, \emptyset] = 0, \rho(s, s)A_1$ is in the affine space $A_1 + \sum_{j=2}^k \mathbf{K}A_j$ for $0 \leq s < p$, and $\rho(s, s)A_h \in \sum_{j=2}^k \mathbf{K}A_j$ for $2 \leq h \leq k$ and $0 \leq s < p$.

Proof These conditions are clearly sufficient since they imply by an easy induction on l that all matrices $A_1[\mathcal{M}_{p \times p}^l], \dots, A_k[\mathcal{M}_{p \times p}^l]$ are lower triangular. In the unipotent case they imply that $A_1[\mathcal{M}_{p \times p}^l]$ is unipotent and that $A_2[\mathcal{M}_{p \times p}^l], \dots, A_k[\mathcal{M}_{p \times p}^l]$ are strictly lower triangular for all $l \in \mathbb{N}$.

In order to see that they are also necessary, consider a basis $A_1 = A, A_k$ of the vector space $\mathcal{T} \subset \overline{A}^{rec}$ spanned by all triangular recurrence matrices of the form $\rho(S, S)A$, with $(S, S) \in \mathcal{M}_{p \times p}$. In the unipotent case, all diagonal coefficients of an element $X \in \mathcal{T}$ are equal and we can consider a basis A_2, \dots, A_k spanning the subspace of all strictly inferior triangular matrices in \mathcal{T} . \square

9 Elements of Toeplitz type

In this section we identify the set \mathcal{M}_p^l with $\{0, \dots, p^l - 1\}$ using the bijection

$$u_1 \dots u_l \mapsto \sum_{j=1}^l u_j p^{j-1}.$$

Analogously, we identify $\mathcal{M}_{p \times p}$ in the obvious way with $\{0, \dots, p^l - 1\} \times \{0, \dots, p^l - 1\}$. This identification yields a natural isomorphism between $\mathbf{K}^{\mathcal{M}_p^l}$ and vectors $(\alpha_0^l, \dots, \alpha_{p^l-1}^l) \in \mathbf{K}^{p^l}$, respectively between $\mathbf{K}^{\mathcal{M}_{p \times p}^l}$ and matrices with coefficients $(\alpha_{i,j}^l)_{0 \leq i, j < p^l}$.

9.1 Toeplitz matrices

A (finite or infinite) matrix T of square-order $n \in \mathbb{N} \cup \{\infty\}$ with coefficients $t_{u,w}, 0 \leq u, w < n$ is a *Toeplitz matrix* if $t_{u,w}$ depends only on the difference $u - w$ of its indices. A Toeplitz matrix is thus described by a (finite or infinite) sequence $\dots, \alpha_{-2}, \alpha_{-1}, \alpha_0, \alpha_1, \dots$ defined by $\alpha_{u-w} = t_{u,w}$.

We call a matrix algebra $\mathcal{A} \subset M_{n \times n}(\mathbf{K})$ of $n \times n$ square matrices (with $n \in \mathbb{N} \cup \{\infty\}$ finite or infinite) an algebra of *Toeplitz type* if all elements of \mathcal{A} are Toeplitz matrices. For finite n , the typical example is the n -dimensional

commutative algebra $\text{Toep}_\rho(n) = \sum_{j=0}^{n-1} \mathbf{K}T_\rho^j$ generated by the $n \times n$ matrix

$$T_\rho = \begin{pmatrix} 0 & \dots & 0 & \rho \\ 1 & 0 & & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & & \ddots & \\ 0 & \dots & 1 & 0 \end{pmatrix}$$

with minimal polynomial $T_\rho^n - \rho$ and yielding a cyclic permutation for $\rho = 1$.

An example of an element in $\text{Toep}_1(2^n)$ is the matrix with coefficients $t_{u,w} = \gamma_{v_2(u-w)}$, $0 \leq u, w < 2^n$ depending only on the highest power $2^{v_2(u-w)} \in \{1, 2, 4, 8, 16, \dots\} \cup \infty$ dividing $u - w$. If such a matrix

$$\begin{pmatrix} \gamma_\infty & \gamma_0 & \gamma_1 & \gamma_0 & \gamma_2 & \gamma_0 & \dots \\ \gamma_0 & \gamma_\infty & \gamma_0 & \gamma_1 & \gamma_0 & \gamma_2 & \\ & & \ddots & & & & \end{pmatrix}$$

is invertible, then its inverse is of the same type.

In the sequel, we will mainly consider the algebra $\text{Toep}(n) = \text{Toep}_0(n)$ of all lower triangular Toeplitz matrices having finite or infinite square order $n \in \mathbb{N} \cup \{\infty\}$. We call $\text{Toep}(n)$ the *Toeplitz algebra* of order n .

The proof of the following well-known result is easy and left to the reader.

Proposition 9.1. *For finite n , the algebra $\text{Toep}(n)$ is isomorphic to the ring $\mathbf{K}[x] \pmod{x^n}$ of polynomials modulo x^n and $\text{Toep}(\infty)$ is isomorphic to the ring $\mathbf{K}[[x]]$ of formal power series.*

In both cases, the isomorphism is given by considering the generating series $\sum_{j=0}^{n-1} t_{j,0}x^j$ associated to the first column of a lower triangular Toeplitz matrix $t_{i,j}$, $0 \leq i, j$.

An element $A \in \mathbf{K}^{\mathcal{M}_{p \times p}}$ is of *Toeplitz type* if all matrices $A[\mathcal{M}_{p \times p}^l]$ are Toeplitz matrices. *Algebras of Toeplitz type* in $\mathbf{K}^{\mathcal{M}_{p \times p}}$ or $\text{Rec}_{p \times p}(\mathbf{K})$ are defined in the obvious way as containing only elements of Toeplitz type.

The algebra $\mathcal{T}_p(\mathbf{K})$ of lower triangular elements of Toeplitz type in $\mathbf{K}^{\mathcal{M}_{p \times p}}$ will be studied below.

Shift maps $\rho(s, t)$ preserve the vector space of (recurrence) matrices of Toeplitz type in $\mathbf{K}^{\mathcal{M}_{p \times p}}$. The recursive closure \overline{T}^{rec} of an element of Toeplitz type in $\mathbf{K}^{\mathcal{M}_{p \times p}}$ contains thus only elements of Toeplitz type. We have the following result:

Proposition 9.2. *All recurrence matrices $T_1, \dots, T_d \in \text{Rec}_{p \times p}(\mathbf{K})$ of a reduced presentation are of Toeplitz type if and only if the shift matrices $\rho(s, t)$ satisfy the following two conditions:*

- (1) *The shift matrices $\rho(s, t)$, $0 \leq s, t < p$ depend only on $s - t$.*

(2) Assuming condition (1) and writing (somewhat abusively) $\rho(s-t)$ for a shift matrix $\rho(s,t)$ we have the identities

$$\rho(s+p)\rho(t) = \rho(s)\rho(t+1)$$

for $1-p \leq s < 0$ and $1-p \leq t < p-1$.

For $p=2$, the conditions of Proposition 9.2 boil down to the identities $\rho(0,0) = \rho(1,1)$ and

$$\begin{aligned}\rho(1,0)\rho(0,0) &= \rho(0,1)\rho(1,0), \\ \rho(1,0)\rho(0,1) &= \rho(0,1)\rho(0,0).\end{aligned}$$

Proof of Proposition 9.2 We show by induction on l that all matrices $T_i[\mathcal{M}_{p \times p}^l]$ are Toeplitz matrices.

For $l=0$, there is nothing to do. Condition (1) implies that $T_1[\mathcal{M}_{p \times p}^1], \dots, T_d[\mathcal{M}_{p \times p}^1]$ are Toeplitz matrices. Setting $T = T_i$ and denoting $(\rho(s,t)T)[\mathcal{M}_{p \times p}^l]$ by $\rho(s-t)T$ we get

$$T[\mathcal{M}_{p \times p}^{l+1}] = \begin{pmatrix} \rho(0)T & \rho(-1)T & \rho(-2)T & \cdots & \rho(1-p)T \\ \rho(1)T & \rho(0)T & \rho(-1)T & \cdots & \rho(2-p)T \\ \vdots & & \ddots & & \vdots \\ \rho(p-2)T & \rho(p-3)T & \rho(p-4)T & \cdots & \rho(1)T \\ \rho(p-1)T & \rho(p-2)T & \rho(p-3)T & \cdots & \rho(0)T \end{pmatrix}$$

where all matrices $\rho(1-p)T, \dots, \rho(p-1)T$ are Toeplitz matrices by induction. We have to show that two horizontally adjacent blocks $(\rho(t+1)T \mid \rho(t)T)$, $1-p \leq t < p-1$ define a $p^l \times (2 \cdot p^l)$ matrix of ‘‘Toeplitz-type’’ (the case of vertically adjacent blocks gives rise to the same conditions and is left to the reader). We have

$$(\rho(t+1)T \mid \rho(t)T) = \left(\begin{array}{cccc|cccc} \cdots & \rho(1-p)\rho(t+1)T & & & \rho(0)\rho(t)T & & \cdots & \\ \cdots & \rho(2-p)\rho(t+1)T & & & \rho(1)\rho(t)T & & \cdots & \\ & & \vdots & & \rho(2)\rho(t)T & & \cdots & \\ & & \vdots & & \vdots & & & \\ \cdots & \rho(-1)\rho(t+1)T & & & \vdots & & & \\ \cdots & \rho(0)\rho(t+1)T & & & \rho(p-1)\rho(t)T & & \cdots & \end{array} \right)$$

(the vertical line separates the matrix into two square blocks) where $\rho(\alpha)\rho(\beta)T = (\rho(\alpha)\rho(\beta)T)[\mathcal{M}_{p \times p}^{l-1}]$. If

$$\rho(s-p)\rho(t+1)T = \rho(s)\rho(t)T, \quad 1 \leq s < p-1,$$

the matrix $T = T_i[\mathcal{M}_{p \times p}^{l+1}]$ is of Toeplitz type.

The opposite direction is easy and left to the reader. \square

9.2 The algebra $\mathcal{T}_p(\mathbf{K}) \subset \mathbf{K}^{\mathcal{M}_{p \times p}}$ formed by lower triangular elements of Toeplitz type

We denote by $\mathcal{T}_p(\mathbf{K})$ the algebra given by all lower triangular elements of Toeplitz type in $\mathbf{K}^{\mathcal{M}_{p \times p}}$. This algebra is isomorphic to $\prod_{l=0}^{\infty} (\mathbf{K}[x] \pmod{x^{p^l}})$ by Proposition 9.1. The subalgebra of converging elements in $\mathcal{T}_p(\mathbf{K})$ can thus be identified with the algebra $\mathbf{K}[[x]]$ of formal power series. We denote by $\mathcal{T}_{p-rec}(\mathbf{K}) = \mathcal{T}_p(\mathbf{K}) \cap \text{Rec}_{p \times p}(\mathbf{K})$ the subalgebra of recurrence matrices.

Using the sequence of vectors

$$A[\mathcal{M}_p^l] = (\alpha_0^l, \dots, \alpha_{p^l-1}^l) \in \mathbf{K}^{p^l}$$

defined by $A \in \mathbf{K}^{\mathcal{M}_p}$, we associate to $A \in \mathbf{K}^{\mathcal{M}_p}$, the lower triangular element $L_A \in \mathbf{K}^{\mathcal{M}_{p \times p}}$ of Toeplitz type given by

$$L_A[\mathcal{M}_{p \times p}^l] = \begin{pmatrix} \alpha_0^l & & & & \\ \alpha_1^l & \alpha_0^l & & & \\ \alpha_2^l & \alpha_1^l & \alpha_0^l & & \\ \vdots & & & \ddots & \\ \alpha_{p^l-1}^l & \dots & & \alpha_1^l & \alpha_0^l \end{pmatrix}.$$

The map $A \mapsto L_A$ is clearly an isomorphism of vector spaces between $\mathbf{K}^{\mathcal{M}_p}$ and the algebra $\mathcal{T}_p(\mathbf{K})$.

Proposition 9.3. *The application $A \mapsto L_A$ with L_A satisfies*

$$\dim(\overline{A}^{rec}) \leq \dim(\overline{L_A}^{rec}) \leq 2 \dim(\overline{A}^{rec})$$

and induces an isomorphism of vector spaces between $\text{Rec}_p(\mathbf{K})$ and the algebra $\mathcal{T}_{p-rec}(\mathbf{K}) \subset \text{Rec}_{p \times p}(\mathbf{K})$.

Remark 9.4. *The inequality $\dim(\overline{A}^{rec}) \leq \dim(\overline{L_A}^{rec})$ is in general strict as illustrated by the example $A \in \mathbb{Q}^{\mathcal{M}_{2 \times 2}}$ with coefficients $A[U] = 1$ for all $U \in \mathcal{M}_2$. The recurrence vector $A \in \text{Rec}_2(\mathbb{Q})$ is thus of complexity 1 while L_A has complexity 2 since $\rho(0,0)L_A = L_A$ and $\rho(1,0)L_A$ (with coefficients $(\rho(1,0)L_A)[U,W] = 1$ for all $(U,W) \in \mathcal{M}_{2 \times 2}$) are linearly independent.*

Proof of Proposition 9.3 The inequality $\dim(\overline{A}^{rec}) \leq \dim(\overline{L_A}^{rec})$ follows easily from the observation that the first column of $L_A[\mathcal{M}_{p \times p}^l]$ is given by $A[\mathcal{M}_p^l]$.

For $A \in \mathbf{K}^{\mathcal{M}_p}$, we denote by $U_A \in \mathbf{K}^{\mathcal{M}_{p \times p}}$ the strictly upper triangular

matrix of Toeplitz type defined by

$$U_A[\mathcal{M}_{p \times p}^l] = \begin{pmatrix} 0 & \alpha_{p^{l-1}}^l & \cdots & \alpha_3^l & \alpha_2^l & \alpha_1^l \\ & 0 & \alpha_{p^{l-1}}^l & \cdots & \alpha_3^l & \alpha_2^l \\ & & \ddots & & & \alpha_3^l \\ & & & \ddots & & \\ & & & & 0 & \alpha_{p^{l-1}}^l \\ & & & & & 0 \end{pmatrix}$$

where $A[\mathcal{M}_p^l] = (\alpha_0^l, \alpha_1^l, \dots, \alpha_{p^{l-1}}^l)$.

We consider now the vector spaces $\mathcal{L}_{\overline{A}^{rec}} = \{L_A \mid A \in \overline{A}^{rec}\}$ and $\mathcal{U}_{\overline{A}^{rec}} = \{U_A \mid A \in \overline{A}^{rec}\}$ spanned by all lower triangular, respectively strictly upper triangular elements of Toeplitz type in $\mathbf{K}^{\mathcal{M}_{p \times p}}$.

It is now straightforward to check that $\mathcal{L}_{\overline{A}^{rec}} \oplus \mathcal{U}_{\overline{A}^{rec}} \subset \mathbf{K}^{\mathcal{M}_{p \times p}}$ is recursively closed. This shows the inequalities

$$\dim(\overline{L_A}^{rec}) \leq \dim(\mathcal{L}_{\overline{A}^{rec}} \oplus \mathcal{U}_{\overline{A}^{rec}}) \leq 2 \dim(\overline{A}^{rec}).$$

The last part of Proposition 9.3 follows from the fact that $A \mapsto L_A$ defines an isomorphism vector-spaces between $\mathbf{K}^{\mathcal{M}_p}$ and $\mathcal{T}_p(\mathbf{K})$. \square

9.3 The polynomial ring-structure on $\mathbf{K}^{\mathcal{M}_p}$ and $\text{Rec}_p(\mathbf{K})$

The isomorphisms of vector spaces $\mathbf{K}^{\mathcal{M}_p} \sim \mathcal{T}_p \sim \prod_{l=0}^{\infty} (\mathbf{K}[x] \text{ mod } x^{p^l})$ given by the map $A \mapsto L_A$ considered above and Proposition 9.1 endow $\mathbf{K}^{\mathcal{M}_p}$ with the *polynomial product*. More precisely, we consider the map given by

$$A \mapsto \psi(A) = \prod_{l=0}^{\infty} \psi_l(A) \in \prod_{l=0}^{\infty} (\mathbf{K}[x] \text{ (mod } x^{p^l}))$$

where $\psi_l(A) = \sum_{k=0}^{p^l-1} \alpha_k^l x^k$ if $A[\mathcal{M}_p^l] = (\alpha_0^l, \dots, \alpha_{p^{l-1}}^l)$ for $A \in \mathbf{K}^{\mathcal{M}_p}$. We have then $\psi(C) = \psi(A)\psi(B) \in \prod_{l=0}^{\infty} (\mathbf{K}[x] \text{ mod } x^{p^l})$ if and only if $\psi_l(C) \equiv \psi_l(A)\psi_l(B) \text{ (mod } x^{p^l})$ for all $l \in \mathbb{N}$ or equivalently if and only if $L_C = L_A L_B \in \mathbf{K}^{\mathcal{M}_{p \times p}}$.

In particular, if A and B are convergent and correspond to formal power series g_A, g_B , then the polynomial product $\psi(C) = \psi(A)\psi(B)$ corresponds to a convergent element $C \in \mathbf{K}^{\mathcal{M}_p}$ with associated formal power series $g_C = g_A g_B$ defined as the usual product of the formal power series g_A and g_B .

Using the isomorphism $A \mapsto L_A$ of the previous section, we identify the polynomial product algebra $\mathbf{K}^{\mathcal{M}_p}$ with the subalgebra $\mathcal{T}_p(\mathbf{K}) \subset \mathbf{K}^{\mathcal{M}_{p \times p}}$ of lower triangular matrices of Toeplitz type. Similarly, we identify $\text{Rec}_p(\mathbf{K})$ with $\mathcal{T}_{p-rec}(\mathbf{K}) \subset \text{Rec}_{p \times p}(\mathbf{K})$.

We denote by $\mathcal{IT}_p \subset \mathcal{T}_p(\mathbf{K})$ the subspace corresponding to all elements $A \in \mathbf{K}^{\mathcal{M}_p}$ such that $\rho(s)A$ is of finite support for $s = 0, 1, \dots, p-2$ and $\rho(p-1)A \in \mathcal{IT}_p$ (where we identify $\mathbf{K}^{\mathcal{M}_p}$ and $\mathcal{T}_p(\mathbf{K})$ using the map $A \mapsto L_A$). Although this definition is recursive, it makes sense: An element of \mathcal{IT}_p corresponds to sequences P_0, P_1, \dots of polynomials where P_l is of degree $\leq p^l - 1$ and almost all polynomials P_l satisfy $P_l \equiv 0 \pmod{x^{(p-1)(p^{l-1} + p^{l-2} + \dots + p^{l-a})}}$ for an arbitrary fixed natural integer $a \in \mathbb{N}$.

It is easy to check that \mathcal{IT}_p is an ideal of the algebra $\mathcal{T}_p(\mathbf{K})$. We denote by $\tilde{\mathcal{T}}_p(\mathbf{K}) = \mathcal{T}_p(\mathbf{K})/\mathcal{IT}_p$ and $\tilde{\mathcal{T}}_{p-rec}(\mathbf{K}) = \mathcal{T}_{p-rec}(\mathbf{K})/\mathcal{IT}_{p-rec}$ (where $\mathcal{IT}_{p-rec} = \mathcal{IT}_p \cap \mathcal{T}_{p-rec}(\mathbf{K})$) the obvious quotient algebras. It is also easy to check that the differential operator $\frac{d}{dx}$ (acting in the obvious way on polynomial sequences associated to $A \in \mathbf{K}^{\mathcal{M}_p}$) is well-defined on the quotient algebra $\tilde{\mathbf{K}}^{\mathcal{M}_p}(\mathbf{K})$.

Proposition 9.5. *The algebra $\tilde{\mathcal{T}}_{p-rec}(\mathbf{K})$ is a differential subalgebra of the differential algebra $\tilde{\mathcal{T}}_{p-rec}(\mathbf{K})$.*

In particular, converging recurrence vectors correspond to a differential subring of the differential ring $(\mathbf{K}[[x]], \frac{d}{dx})$ of formal power-series.

Proof Consider the factorization $\frac{d}{dx} = \frac{1}{x} \left(x \frac{d}{dx} \right)$ of the differential operator $\frac{d}{dx}$ into the differential operator $\left(x \frac{d}{dx} \right)$, followed by multiplication by x^{-1} . Given an element $A \in \mathbf{K}^{\mathcal{M}_p}$ (identified in the obvious way with the corresponding sequence P_l of polynomials), the differential operator $\psi(A) \mapsto \left(x \frac{d}{dx} \right) \psi(A)$ corresponds to Hadamard multiplication in $\mathbf{K}^{\mathcal{M}_p}$ with the convergent element N of complexity 2 defined by $N[\mathcal{M}_p^l] = (0, 1, 2, \dots, p^l - 1)$. The result $B \in \mathbf{K}^{\mathcal{M}_p}$ yields a strictly lower triangular element $L_B \in \mathbf{K}^{\mathcal{M}_p \times p}$ of Toeplitz type. Consider L_B as an element of ring $\prod_{l=0}^{\infty} \left(\mathbf{K}[x] \pmod{x^{p^l}} \right)$. Multiplying L_B by x^{-1} corresponds to application of the “shift element” S_1 as defined in the proof of Proposition 9.3 (of complexity 2) to the row vector B . This shows the inequality

$$\dim\left(\overline{\frac{d}{dx} A}^{rec}\right) \leq 4 \dim(\overline{A}^{rec})$$

for $A \in \text{Rec}_p(\mathbf{K})$ and implies the first part.

The second part follows from the obvious observation that the quotient map $\mathcal{T}_p(\mathbf{K}) \mapsto \tilde{\mathcal{T}}_p(\mathbf{K})$ restricts to an injection on the subalgebra of converging elements in $\mathcal{T}_p(\mathbf{K})$. \square

Remark 9.6. *Over the field $\mathbf{K} = \mathbb{C}$ of complex numbers, the formal power series g_A associated to a converging recurrence vector $A \in \text{Rec}_p(\mathbb{C})$ defines a holomorphic function in the open unit disc.*

10 Elements of Hankel type

A (finite or infinite) matrix H with coefficients $h_{s,t}$, $0 \leq s, t$ is a *Hankel matrix* if $h_{s,t} = \alpha_{s+t}$ for some sequence $\alpha_0, \alpha_1, \alpha_2, \dots$. An element $H \in \mathbf{K}_{\mathcal{M}_{p \times p}}$ is of *Hankel type* if for all $l \in \mathbb{N}$, the matrix $H[\mathcal{M}_{p \times p}^l]$ is a Hankel matrix (for the usual total order on rows and columns induced by $(u_1 \dots u_l) \mapsto \sum_{j=1}^l u_j p^{j-1}$).

Consider the involutive element $S \in \text{Rec}_{p \times p}(\mathbf{K})$ where the coefficients of $S[\mathcal{M}_{p \times p}^l]$ are 1 on the antidiagonal and zero elsewhere. More precisely, $S[u_1 \dots u_n, w_1 \dots w_n] = 1$ if $w_1 = p - 1 - u_1, \dots, w_n = p - 1 - u_n$ and $S[U, W] = 0$ otherwise. (In particular, S is a recurrence matrix of Hankel type with complexity 1.) The following result reduces the study of Hankel matrices to the study of Toeplitz matrices.

Proposition 10.1. *(Left- or right-)multiplication by the involution $S \in \text{Rec}_{p \times p}(\mathbf{K})$ defines a bijection preserving complexities between elements of Hankel type and elements of Toeplitz type in $\mathbf{K}^{\mathcal{M}_{p \times p}}$. In particular, the maps $A \mapsto SA$ and $A \mapsto AS$ yield bijections between recurrence matrices of Hankel type and recurrence matrices of Toeplitz type in $\text{Rec}_{p \times p}(\mathbf{K})$.*

The easy proof is left to the reader.

Elements of Hankel type satisfy $H[U, W] = H[W, U]$ for all $(U, W) \in \mathcal{M}_{p \times p}$ and are thus examples of symmetric elements in $\mathbf{K}^{\mathcal{M}_{p \times p}}$.

Let $H \in \mathbf{K}^{\mathcal{M}_{p \times p}}$ be an element of Hankel type. Since $\rho(s, t)H$, $0 \leq s, t < r$ is of Hankel type, the recursive closure $\overline{H}^{\text{rec}}$ contains only elements of Hankel type. The following result is the exact analogue of Proposition 9.2:

Proposition 10.2. *All recurrence matrices $H_1, \dots, H_d \in \text{Rec}_{p \times p}(\mathbf{K})$ of a reduced presentation are of Hankel type if and only if the shift matrices $\rho(s, t)$ satisfy the following two conditions:*

- (1) *The shift matrices $\rho(s, t)$, $0 \leq s, t < p$ depend only on $s + t$.*
- (2) *Assuming condition (1) and writing $\rho(s + t)$ for a shift matrix $\rho(s, t)$, we have the identities*

$$\rho(s + p)\rho(t) = \rho(s)\rho(t + 1)$$

for $0 \leq s < p - 2$ and $0 \leq t < 2p - 3$.

For $p = 2$, Proposition 10.2 boils down to the identities $\rho(1, 0) = \rho(0, 1)$ and

$$\begin{aligned} \rho(0, 0)\rho(1, 1) &= \rho(1, 1)\rho(0, 1), \\ \rho(1, 1)\rho(0, 0) &= \rho(0, 0)\rho(0, 1) \end{aligned}$$

for shift-matrices of a presentation containing only recurrence matrices of Hankel type.

Proof of Proposition 10.2 We show by induction on l that all matrices $H_i[\mathcal{M}_{p \times p}^l]$ are Hankel matrices.

For $l = 0$, there is nothing to do. Condition (1) implies that $H_1[\mathcal{M}_{p \times p}^1], \dots, H_d[\mathcal{M}_{p \times p}^1]$ are Hankel matrices. Setting $H = H_i$ and denoting $(\rho(s, t)H)[\mathcal{M}_{p \times p}^l]$ by $\rho(s + t)H$ we get

$$H[\mathcal{M}_{p \times p}^{l+1}] = \begin{pmatrix} \rho(0)H & \rho(1)H & \rho(2)H & \cdots & \rho(p-1)H \\ \rho(1)H & \rho(2)H & \rho(3)H & \cdots & \rho(p)H \\ \vdots & & & & \vdots \\ \rho(p-1)H & \rho(p)H & \rho(p+1)H & \cdots & \rho(2p-2)H \end{pmatrix}$$

where all matrices $\rho(0)H, \dots, \rho(2p-2)H$ are Hankel matrices by induction. We have to show that two horizontally adjacent blocks $(\rho(t)H \mid \rho(t+1)H)$, $0 \leq t < 2p-2$ define a $p^l \times (2 \cdot p^l)$ matrix of ‘‘Hankel-type’’ (the argument for vertically adjacent blocks gives rise to the same conditions and is left to the reader). We have

$$(\rho(t)H \mid \rho(t+1)H) = \left(\begin{array}{ccc|ccc} \cdots & \rho(p-1)\rho(t)H & & \rho(0)\rho(t+1)H & \cdots & \\ \cdots & \rho(p)\rho(t)H & & \rho(1)\rho(t+1)H & \cdots & \\ \cdots & \rho(p+1)\rho(t)H & & \vdots & & \\ & \vdots & & \vdots & & \\ & \vdots & & \rho(p-2)\rho(t+1)H & \cdots & \\ \cdots & \rho(2p-2)\rho(t)H & & \rho(p-1)\rho(t+1)H & \cdots & \end{array} \right)$$

(the vertical line separates the matrix into two square blocks) using the shorthand notation $\rho(\alpha)\rho(\beta)H = (\rho(\alpha)\rho(\beta)H)[\mathcal{M}_{p \times p}^{l-1}]$. Such a matrix is of Hankel-type if

$$\rho(s+p)\rho(t)H = \rho(s)\rho(t+1)H \text{ for } 0 \leq s < p-2.$$

The opposite direction is left to the reader. \square

11 Groups of recurrence matrices

The (multiplicative) identity Id of the algebra $\mathbf{K}^{\mathcal{M}_{p \times p}}$ satisfies $\rho(s, s)Id = Id$, $0 \leq s < p$ and $\rho(s, t)Id = 0$, $0 \leq s \neq t < p$, and is thus an element of complexity 1 in $\text{Rec}_{p \times p}(\mathbf{K})$.

Definition 11.1. We denote by $GL_{p\text{-rec}}(\mathbf{K})$ the general linear group of all recurrence matrices in $\text{Rec}_{p \times p}(\mathbf{K})$ which are invertible (for the matrix-product) in $\text{Rec}_{p \times p}(\mathbf{K})$.

A group of recurrence matrices is a subgroup of $GL_{p\text{-rec}}(\mathbf{K})$ for some $p \in \mathbb{N}$.

Remark 11.2. An element $X \in \text{Rec}_{p \times p}(\mathbf{K})$ having a multiplicative inverse $X^{-1} \in \mathbf{K}^{\mathcal{M}_{p \times p}}$ is not necessarily in $\text{GL}_{p\text{-rec}}(\mathbf{K})$ since X^{-1} has in general infinite complexity. A simple example is the “diagonal” matrix $X \in \mathbb{Q}^{\mathcal{M}_{p \times p}}$ which is in the center of the algebra $\mathbb{Q}^{\mathcal{M}_{p \times p}}$ and has diagonal coefficients $X[U, U] = n + 1 \in \{1, 2, \dots\}$ depending only on the length n of $(U, U) \in \mathcal{M}_{p \times q}^n$ (and off-diagonal coefficients $X[U, W] = 0$ for $U \neq W$ with $(U, W) \in \mathcal{M}_{p \times p}$). We have $\rho(s, t)X = 0$ for $0 \leq s \neq t < p$ and $\rho(s, s)X = X + \text{Id}$ where $\text{Id} \in \text{Rec}_{p \times p}(\mathbb{Q})$ denotes the multiplicative identity. The element X has thus complexity 2 and is in $\text{Rec}_{p \times p}(\mathbf{K})$. Since the denominators of X^{-1} involve all primes of \mathbb{Z} , it cannot be a recurrence matrix by Proposition 6.5. A second proof of this fact is given by the observation that the infinite matrix H with coefficients $c_{i,j} = (\rho(0^i, 0^i)X^{-1})[0^j, 0^j] = \frac{1}{i+j+1}, 0 \leq i, j$ (using the convention $0^0 = \emptyset$) is the Hilbert matrix

$$\begin{pmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The finite submatrix formed by the first n rows and columns of H is non-singular for all n . This implies that X^{-1} is of infinite complexity.

Another simple example for $p \geq 2$ is the converging lower triangular unipotent recurrence matrix $L \in \text{Rec}_{p \times p}(\mathbb{Q})$ of Toeplitz type with associated generating series $1 - 2x$. Its coefficients are 1 on the diagonal, -2 on the subdiagonal and 0 everywhere else. Its inverse $L^{-1} \in \mathbb{Q}^{\mathcal{M}_{p \times p}}$ is lower triangular of Toeplitz type with first row given by successive powers of 2 and corresponds to the generating series $\frac{1}{1-2x}$. Proposition 6.2 shows that L^{-1} cannot be a recurrence matrix (see also Remark 9.6). However, the element L^{-1} is a recurrence matrix over a field of positive characteristic.

Remark 11.3. For \mathbf{K} a topological field, the group of invertible elements in $\mathbf{K}^{\mathcal{M}_{p \times p}}$ is a topological group for both topologies considered in §5. The subgroup $\text{GL}_{p\text{-rec}}(\mathbf{K})$ is thus also a topological group (with discrete topology for the second topology described in §5).

Does $\text{GL}_{p\text{-rec}}(\mathbf{K})$ admit other, more interesting topologies? Is the map $X \mapsto X^{-1}$ of invertible elements in the quotient algebra $\text{Rec}_{p \times p}(\mathbf{K})/\mathcal{FS}_{p \times p}$ continuous for the metric $\|X\|_{\infty}$ described in §5?

11.1 A few homomorphisms and characters

Given an element $A \in \text{GL}_{p\text{-rec}}(\mathbf{K})$, the projection

$$A \longmapsto A[\mathcal{M}_{p \times p}^l] \in \text{GL}(\mathbf{K}^{p^l})$$

yields a homomorphism of groups. All these homomorphisms are surjective and have sections. In particular, $\mathrm{GL}_{p\text{-rec}}(\mathbf{K})$ contains (an isomorphic image of) every finite-dimensional matrix-group over \mathbf{K} for $p \geq 2$.

The *determinant* is the homomorphism $\mathrm{GL}_{p\text{-rec}}(\mathbf{K}) \longrightarrow \prod_{l=0}^{\infty} \mathbf{K}^*$ defined by

$$A \longmapsto \det(A) = (\det(A[\mathcal{M}_{p \times p}^0]), \det(A[\mathcal{M}_{p \times p}^1]), \det(A[\mathcal{M}_{p \times p}^2]), \dots) \in \prod_{l=0}^{\infty} \mathbf{K}^* .$$

Its image is in general not surjective: For a finite (countable) field \mathbf{K} there exist only finitely (countably) many elements of complexity $\leq d$ in $\mathrm{End}_{p\text{-rec}}(\mathbf{K})$ and the group $\mathrm{GL}_{p\text{-rec}}(\mathbf{K})$ is thus countable. If \mathbf{K}^* contains at least two elements, the subgroup $\det(\mathrm{GL}_{p\text{-rec}}(\mathbf{K}))$ is thus a proper subgroup of the uncountable group $\prod_{l=0}^{\infty} \mathbf{K}^*$.

Remark 11.4. *It would probably be interesting to have some knowledge of the subgroup*

$$\det(\mathrm{GL}_{p\text{-rec}}(\mathbf{K})) \subset \prod_{l=0}^{\infty} \mathbf{K}^* .$$

An obvious restriction is for instance given by the trivial remark that all denominators of $\det(A)$ involve only a finite number of prime-factors (see Proposition 6.5) for $A \in \mathrm{GL}_{p\text{-rec}}(\mathbb{Q})$.

Setting $\tau_n(A) = \sum_{U \in \mathcal{M}_p} A[U, U]$ we get the *trace-map* $A \longmapsto \mathrm{tr}(A) = (\tau_0(A), \tau_1(A), \dots, \dots)$ defining a linear application

$$\mathrm{tr} : \mathrm{Rec}_{p \times p}(\mathbf{K}) \longrightarrow \mathrm{Rec}_{1 \times 1}(\mathbf{K}) .$$

The trace map satisfies the identity $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and is thus constant on conjugacy classes under the action of $\mathrm{GL}_{p\text{-rec}}(\mathbf{K})$. The trace $\mathrm{tr}(A) \in \mathrm{Rec}_{1 \times 1}(\mathbf{K})$ of a recurrence matrix $A \in \mathrm{Rec}_{p \times p}(\mathbf{K})$ is easy to compute from a presentation $(A_1, \dots, A_a)[\emptyset, \emptyset], \rho_{\overline{A}^{\mathrm{rec}}}(s, t)$, $0 \leq s, t < p$: It admits the presentation $(\mathrm{tr}(A_1), \dots, \mathrm{tr}(A_a))[\emptyset, \emptyset] = (A_1, \dots, A_a)[\emptyset, \emptyset]$ with restriction matrix $\sum_{s=0}^{p-1} \rho_{\overline{A}^{\mathrm{rec}}}(s, s)$. The associated generating series

$$\sum_{n=0}^{\infty} \tau_n(A) t^n \in \mathbf{K}[[t]]$$

is always rational.

Remark 11.5. *The properties of the traces $\mathrm{tr}(A^0), \mathrm{tr}(A^1), \mathrm{tr}(A^2), \mathrm{tr}(A^3), \dots$ are essentially shared by the coefficients of x^{p^l-k} of the characteristic polynomial of $A[\mathcal{M}_{p \times p}^l]$.*

This suggests the following question: What can be said of the spectra of the matrices $A[\mathcal{M}_{p \times p}^l], l = 0, 1, 2, \dots$? Is there sometimes (perhaps after a

suitable normalization) a “spectral limit”, eg. a nice spectral measure, etc? (The answer to the last question is trivially yes for recurrence matrices of complexity 1 since they are tensor-powers, up to a scalar factor.)

Recall that a group Γ is *residually finite* if for every element $\gamma \in \Gamma$ different from the identity there exists a homomorphism $\pi : \Gamma \rightarrow F$ into a finite group F such that $\pi(\gamma) \neq 1 \in F$.

Proposition 11.6. (i) For \mathbf{K} a finite field, the group $GL_{p\text{-rec}}(\mathbf{K})$ is residually finite.

(ii) For \mathbf{K} an arbitrary field, a finitely generated group $\Gamma \subset GL_{p\text{-rec}}(\mathbf{K})$ is residually finite.

Proof. Given a finite field \mathbf{K} and an element $1 \neq A \in GL_{p\text{-rec}}(\mathbf{K})$, there exists l such that $A[\mathcal{M}_{p \times p}^l] \neq 1 \in \text{Aut}(\mathbf{K}^{p^l})$ thus proving assertion (i).

For proving assertion (ii), we remark that presentations for a finite set of generators $\gamma_1^{\pm 1}, \dots, \gamma_m^{\pm 1}$ generating $\Gamma \subset GL_{p\text{-rec}}(\mathbf{K})$ involve only finitely many elements $k_1, \dots, k_N \in \mathbf{K}$ (appearing in the initial data and as coefficients of the shift-matrices). The group Γ is thus defined over the finitely generated field extension \mathbf{F} containing k_1, \dots, k_N over the primary field \mathbf{k} (which is either finite of prime order or \mathbf{Q}) of \mathbf{K} . Given an element $\gamma \neq 1d$ in Γ , choose $l \in \mathbb{N}$ such that $\gamma[\mathcal{M}_{p \times p}^l] \neq 1d$. Given a transcendental basis t_1, \dots, t_f of \mathbf{F} there exists a maximal ideal $\mathcal{J} \subset \mathbf{F}[t_1, \dots, t_f]$ such that Γ is defined over \mathbf{A}/\mathcal{J} where $\mathbf{A} \subset \mathbf{F}$ is a suitable subring and $\gamma[\mathcal{M}_{p \times p}^l] \neq 1d \pmod{\mathcal{J}}$. The quotient field \mathbf{A}/\mathcal{J} is either finite or a number field in which case we can reduce modulo a suitable prime \wp of the number field in order to get a finite field. \square

Remark 11.7. It follows from Proposition 6.5 or from the proof of Proposition 11.6, that any finitely generated group $\Gamma \subset GL_{p\text{-rec}}(\mathbf{Q})$ (or more generally $\Gamma \subset GL_{p\text{-rec}}(\mathbf{K})$ where \mathbf{K} is a number field) can be reduced modulo \wp to a quotient-group $\Gamma_\wp \subset GL_{p\text{-rec}}(\mathbb{Z}/\wp\mathbb{Z})$ for almost all primes \wp (or prime-ideals \wp of the number field \mathbf{K}).

Remark 11.8. Considering the group $\left(\text{Rec}_{p \times p}(\mathbf{K}) / \mathcal{FS}_{p \times p} \right)^*$ of invertible elements in the quotient algebra $\text{Rec}_{p \times p}(\mathbf{K}) / \mathcal{FS}_{p \times p}$ (where $\mathcal{FS}_{p \times p} \subset \text{Rec}_{p \times p}(\mathbf{K})$ denotes the two-sided ideal of all finitely supported elements X (characterized by $X[U, W] = 0$ except for finitely many words $(U, W) \in \mathcal{M}_{p \times p}$) we get a group homomorphism

$$GL_{p\text{-rec}}(\mathbf{K}) \longrightarrow \left(\text{Rec}_{p \times p}(\mathbf{K}) / \mathcal{FS}_{p \times p} \right)^*$$

with kernel $\bigoplus_{n=0}^{\infty} GL(\mathbf{K}^{p^n})$.

Remark 11.9. *There are three projective versions of the group $GL_{p-rec}(\mathbf{K})$. One can either consider the quotient-group $GL_{p-rec}(\mathbf{K})/(\mathbf{K}^* Id)$ or the quotient-group $GL_{p-rec}(\mathbf{K})/(GL_{p-rec}(\mathbf{K}) \cap \mathcal{C}_{p-rec}(\mathbf{K}))$ where $\mathcal{C}_{p-rec}(\mathbf{K}) \subset Rec_{p \times p}(\mathbf{K})$ denotes the center of $Rec_{p \times p}(\mathbf{K})$. Finally, one can also consider equivalence classes by invertible central elements in $\mathbf{K}^{\mathcal{M}_{p \times p}}$ of all recurrence matrices $X \in Rec_{p \times p}(\mathbf{K})$ for which there exists a recurrence matrix $Y \in Rec_{p \times p}(\mathbf{K})$ such that XY is central and invertible in $\mathbf{K}^{\mathcal{M}_{p \times p}}$.*

The obvious homomorphism $GL_{p-rec}(\mathbf{K})$ into this last group is in general neither injective nor surjective as can be seen as follows: For $p \geq 2$, we denote by $Id \in Rec_{p \times p}(\mathbf{K})$ the identity recurrence matrix and by $J \in Rec_{p \times p}(\mathbf{K})$ the recurrence matrix of complexity 1 defined by $J[U, W] = 1$ for all $(U, W) \in \mathcal{M}_{p \times p}$. The computation

$$\begin{aligned} & \left(\alpha Id[\mathcal{M}_{p \times p}^l] + \beta J[\mathcal{M}_{p \times p}^l] \right) \left((\alpha + p^l \beta) Id[\mathcal{M}_{p \times p}^l] - \beta J[\mathcal{M}_{p \times p}^l] \right) \\ & = \alpha(\alpha + p^l \beta) Id[\mathcal{M}_{p \times p}^l] \end{aligned}$$

shows that $X \in \mathbf{P}GL_{p-rec}(V)$ for X defined by

$$X[\mathcal{M}_{p \times p}^l] = \alpha Id[\mathcal{M}_{p \times p}^l] + \beta J[\mathcal{M}_{p \times p}^l]$$

with $\alpha, \beta \in \mathbb{C}$ such that $\alpha(\alpha + p^l \beta) \neq 0$ for all $l \in \mathbb{N}$. Choosing $\alpha = \beta = 1$, the eigenvalues of $(Id + J)[\mathcal{M}_{p \times p}^l]$ are $1 + p^l$ and 1 with multiplicity $p^l - 1$. We have thus $\det((Id + J)[\mathcal{M}_{p \times p}^l]) = 1 + p^l$. Choosing an odd prime \wp such that $p \pmod{\wp}$ is a non-square in $\mathbb{Z}/\wp\mathbb{Z}$ and setting $l = (\wp - 1)/2$, we have $\det((Id + J)[\mathcal{M}_{p \times p}^l]) \equiv 0 \pmod{\wp}$. Since the number of such primes \wp is infinite, Proposition 6.5 (see also Remark 11.7) implies that $(Id + J) \notin GL_{p-rec}(\mathbb{Q})$.

12 Examples of groups of recurrence matrices

12.1 $GL_p(\mathbf{K}) \subset GL_{p-rec}(\mathbf{K})$

Any matrix g of size $p \times p$ with coefficients $g_{u,w}, 0 \leq u, w < p$ in \mathbf{K} gives rise to a recurrence matrix $\mu(g) = G \in Rec_{p \times p}(\mathbf{K})$ of complexity 1 by considering the n -th tensor-power $g \otimes g \otimes \cdots \otimes g$ and setting

$$G[u_1 \dots u_n, w_1 \dots w_n] = g_{u_1, w_1} g_{u_2, w_2} \cdots g_{u_n, w_n}.$$

Since $\mu(g)\mu(g') = \mu(gg')$ (and $\mu(Id(GL_p)) = Id(GL_{p-rec})$) we have an injective homomorphism $GL_p(\mathbf{K}) \mapsto GL_{p-rec}(\mathbf{K})$.

12.2 An infinite cyclic group related to the shift

The recurrence matrix $A = A_1 \in Rec_{2 \times 2}(\mathbf{K})$ presented by $(A_1, A_2)[\emptyset, \emptyset] = (1, 1)$ and shift-matrices

$$\rho(0, 0) = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad \rho(0, 1) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

$$\rho(1,1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \rho(1,1) = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}.$$

yields permutation matrices $A_1[\mathcal{M}_{2 \times 2}^l]$ associated to the cyclic permutation $(0 \ 1 \ 2 \ \dots \ 2^l - 1)$ defined by addition of 1 modulo 2^l . The first few matrices $A_1[\mathcal{M}_{2 \times 2}^l]$, $l = 0, 1, 2$ (using the bijection $s_1 \dots s_n \mapsto \sum_{j=1}^n s_j 2^{j-1}$ for rows and columns) are

$$\begin{pmatrix} 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and the matrices $A_2[\mathcal{M}_{2 \times 2}^l]$ have only zero coefficients except for a coefficient 1 in the upper-right corner.

The inverse of A_1 is given by A_1^t . The two converging elements $\rho(0,0)A_1, \rho(0,0)A_1^t \in \text{End}_{2\text{-rec}}(\mathbb{Q})$ correspond to the section of the shift $(x_0, x_1, \dots) \mapsto (0, x_0, x_1, \dots)$ and the shift $(x_0, x_1, \dots) \mapsto (x_1, x_2, \dots)$ on converging elements in $\mathbf{K}^{\mathcal{M}^2}$. This example has an obvious generalization to $\text{GL}_{p\text{-rec}}(\mathbf{K})$ for all $p \geq 2$.

12.3 Diagonal groups

The set $\mathcal{D}_{p\text{-rec}}(\mathbf{K}) \cap \text{GL}_{p\text{-rec}}(\mathbf{K})$ of all diagonal recurrence matrices in $\text{GL}_{p\text{-rec}}(\mathbf{K})$ forms a commutative subgroup containing the maximal central subgroup $\mathcal{C}^* = \mathcal{C}_{p\text{-rec}}(\mathbf{K}) \cap \text{GL}_{p\text{-rec}}(\mathbf{K})$ of $\text{GL}_{p\text{-rec}}(\mathbf{K})$. For \mathbf{K} of characteristic $\neq p$ the trace map establishes a bijection between \mathcal{C}^* and $\text{Rec}_{1 \times 1}(\mathbf{K})$. An element of \mathcal{C}^* (in characteristic $\neq p$) corresponds to an element $A \in \text{Rec}_{1 \times 1}(\mathbf{K})$ which is invertible in the algebra (or, equivalently, in the function-ring) $\text{Rec}_{1 \times 1}(\mathbf{K})$. Such an element is thus encoded by a rational function $\sum_{n=0}^{\infty} \alpha_n x^n \in \mathbf{K}(x)$ having only non-zero coefficients such that $\sum_{n=0}^{\infty} \frac{x^n}{\alpha_n}$ is also rational. Examples of such functions are eg. $\frac{1}{1-\lambda z}$ with $\lambda \neq 0$, functions having only non-zero, ultimately periodic coefficients $\alpha_0, \alpha_1, \dots$, and, more generally, generating functions of recurrence matrices in $\text{Rec}_{1 \times 1}(\mathbf{K})$ with values in a finite subset of $\mathbf{K} \setminus \{0\}$.

12.4 Lower (or upper) triangular groups

Lower (or upper) triangular elements in $\text{GL}_{p\text{-rec}}(\mathcal{K})$ form a group containing the subgroup consisting of all converging lower triangular elements (of Toeplitz type) in $\text{GL}_{p\text{-rec}}(\mathcal{K})$.

Consider the commutative subgroup \mathcal{T}^* of all converging lower triangular elements of Toeplitz type in $\text{GL}_{p\text{-rec}}(\mathcal{K})$. Converging elements of $\mathcal{T}_{p\text{-rec}}(\mathbf{K})$ form a differential subring of $\mathbf{K}[[x]]$, (see §8.2), and the logarithmic derivative $G \mapsto G'/G$ defines a group homomorphism from \mathcal{T}^* into an additive subgroup of $\mathcal{T}_{p \times p}(\mathbf{K})$. In the case $\mathbf{K} \subset \mathbb{C}$, elements of \mathcal{T}^* correspond to

some holomorphic functions on the open unit disc of \mathbb{C} which have no zeroes in the open unit disc, cf. Remark 9.6.

Examples of such elements in \mathcal{T}^* are given by rational functions involving only cyclotomic polynomials. The rational function $\frac{1}{1-t}$ for instance corresponds to the invertible recurrence matrix of Toeplitz-type with limit the unipotent Toeplitz matrix consisting only of 1's below the diagonal. A more exotic example is given by the element of \mathcal{T}^* associated to the power series $\prod_{n=0}^{\infty}(1 + it^{2^n}) \in \mathbb{C}[[t]]$ (where $i^2 = -1$ is a square root of -1).

If \mathbf{K} is contained in the algebraic closure of a finite field, the group \mathcal{T}^* contains the subring of all rational functions without zero or pole at the origin.

12.5 Orthogonal groups

A recurrence matrix $A \in \text{Rec}_{p \times p}(\mathbf{K})$ is *symmetric* if $A[U, W] = A[W, U]$ for all $(U, W) \in \mathcal{M}_{p \times p}$. A complex recurrence matrix $A \in \text{Rec}_{p \times p}(\mathbf{K})$ is *hermitian* if $A[U, W] = \overline{A[W, U]}$, for all $(U, W) \in \mathcal{M}_{p \times p}$, with \bar{x} denoting the complex conjugate of $x \in \mathbb{C}$. (More generally, one can define hermitian matrices over fields with a non-trivial involutive automorphism.) A real symmetric recurrence matrix is *positive definite* if all matrices $A[\mathcal{M}_{p \times p}^l]$ are positive definite.

A symmetric or hermitian matrix $A \in \text{Rec}_{p \times p}(\mathbf{K})$ is *non-singular* if $\det(A[\mathcal{M}_{p \times p}^l]) \in \mathbf{K}^*$ for all $l \in \mathbb{N}$ and *non-degenerate* if $A \in \text{GL}_{p-\text{rec}}(\mathbf{K})$. An example of a non-singular positive definite symmetric recurrence matrix is the diagonal recurrence matrix $A \in \text{Rec}_{p \times p}(\mathbb{Q})$ with diagonal coefficients $A[U, U] = l + 1$ for $(U, U) \in \mathcal{M}_{p \times p}^l$ of length l . An example of a positive definite non-degenerate symmetric recurrence matrix is the identity matrix $\text{Id} \in \text{Rec}_{p \times p}(\mathbb{Q})$.

Such a non-singular matrix $A \in \text{Rec}_{p \times p}(\mathbf{K})$ defines the *orthogonal group*

$$\text{O}(A) = \{B \in \text{GL}_{p-\text{rec}}(\mathbf{K}) \mid B^t A B = A\} \subset \text{GL}_{p-\text{rec}}(\mathbf{K})$$

of A in the symmetric case and the *unitary group*

$$\text{U}(A) = \{B \in \text{GL}_{p-\text{rec}}(\mathbf{K}) \mid \overline{B}^t A B = A\}$$

of A in the case of a hermitian recurrence matrix A . For \mathbf{K} a real field, one speaks also of *Lorenzian groups* if the symmetric matrix A is not positive definite.

The following obvious proposition relates presentations of $A, A^t \in \text{Rec}_{p \times p}(\mathbf{K})$ and \overline{A} (over $\mathbf{K} = \mathbb{C}$). Its easy proof is omitted.

Proposition 12.1. *The following assertions are equivalent:*

(i) $(A_1, \dots, A_a)[\emptyset, \emptyset] = (\alpha_1, \dots, \alpha_a) \in \mathbf{K}^a, \rho(s, t) \in \mathbf{K}^{a \times a}, 0 \leq s, t < r$ is a presentation of $A = A_1$.

- (ii) $(A_1, \dots, A_a)[\emptyset, \emptyset] = (\alpha_1, \dots, \alpha_a), \tilde{\rho}(s, t) \in \mathbf{K}^{a \times a}, 0 \leq s, t < p$ with $\tilde{\rho}(s, t) = \rho(t, s)$ is a presentation of $A^t = A_1$.
- (iii) (Over $\mathbf{K} = \mathbb{C}$) $(A_1, \dots, A_a)[\emptyset, \emptyset] = (\overline{\alpha_1}, \dots, \overline{\alpha_a}), \overline{\rho}(s, t) \in \mathbf{K}^{a \times a}, 0 \leq s, t < r$ is a presentation of $\overline{A} = A_1$ with \overline{X} denoting complex conjugation applied to all coefficients of X .

12.6 Symplectic groups

A recurrence matrix $A \in \text{Rec}_{p \times p}(\mathbf{K})$ is *antisymmetric* if $A[U, W] = -A[W, U]$ for all $(U, W) \in \mathcal{M}_{p \times p}$. An antisymmetric recurrence matrix $A \in \text{Rec}_{p \times p}(\mathbf{K})$ is *symplectic* if $\det(A[\mathcal{M}_{p \times p}^l]) \in \mathbf{K}^*$ for all $l \geq 1$. If \mathbf{K} is of characteristic 2, we require moreover $A[U, U] = 0$ for all $(U, U) \in \mathcal{M}_{p \times p}$. Symplectic recurrence matrices exist only for p even.

A symplectic recurrence matrix A is *non-degenerate* if $\tilde{A} \in \text{GL}_{p-\text{rec}}(\mathbf{K})$ where $\tilde{A}[\emptyset, \emptyset] = 1$ and $\tilde{A}[U, W] = A[U, W]$ if $(U, W) \neq [\emptyset, \emptyset]$.

For $A \in \text{Rec}_{p \times p}(\mathbf{K})$ a symplectic recurrence matrix, the associated *symplectic group* $\text{Sp}(A) \subset \text{GL}_{p-\text{rec}}(\mathbf{K})$ of recurrence matrices is defined as

$$\text{Sp}(A) = \{B \in \text{GL}_{p-\text{rec}}(\mathbf{K}) \mid B^t A B = A\}$$

where the value $B[\emptyset, \emptyset]$ can be neglected.

Proposition 12.1 has an obvious analogue describing antisymmetric recurrence matrices.

12.7 Groups generated by elements of bounded complexity

Denote by $\Gamma_{a,b} \subset \text{GL}_{p-\text{rec}}(\mathbf{K})$ the group generated by all elements $A \in \text{GL}_{p-\text{rec}}(\mathbf{K})$ of complexity $\leq a$ with inverse $B = A^{-1}$ of complexity $\leq b$. We have $\Gamma_{a,b} = \Gamma_{b,a}$ and

$$\Gamma_{a,1} \subset \Gamma_{a,2} \subset \dots$$

Moreover, the set of generators of $\Gamma_{a,b}$ (elements in $\text{GL}_{p-\text{rec}}(\mathbf{K})$ of complexity $\leq a$ with inverse of complexity $\leq b$) is a union of algebraic sets since they can be described by (a finite union of) polynomial equations.

Remark 12.2. *The group $\Gamma_{1,1} = \Gamma_{1,2} = \Gamma_{1,3} = \dots$ is isomorphic to $\mathbf{K}^* \times \text{GL}(\mathbf{K}^p)$.*

For \mathbf{K} a finite field, the group $\Gamma_{a,b}$ is finitely generated.

It would be interesting to have a better understanding of the groups $\Gamma_{a,b}$. Obviously, $\Gamma_{a,b} \subset \Gamma_{a',b'}$ if $a \leq a', b \leq b'$. In which cases do we have equality? The inclusion $\Gamma_{1,1} \subset \Gamma_{2,2}$ for instance is strict. What can be said about $\Gamma_{n,n} \subset \Gamma_{n+1,n+1}, n \geq 2$? (For $\mathbf{K} = \mathbb{Q}$, many of these inclusions must be strict: Consider a central diagonal recurrence matrix with diagonal coefficients $A[U, U] = \alpha_l, (U, U) \in \mathcal{M}_{p \times p}^l$ for $\alpha_0, \alpha_1, \alpha_2, \dots$ with values in $\mathbb{Q} \setminus \{0\}$ a periodic sequence of minimal period-length a prime \wp . Such an

element has complexity $(\wp - 1)$ and cannot be obtained as a product of recurrence matrices in $\text{Rec}_{p \times p}(\mathbb{Q})$ having lower complexities.)

Remark 12.3. *One can similarly consider the subalgebra $\mathcal{R}_a \subset \text{Rec}_{p \times p}(\mathbf{K})$ generated as an algebra by all recurrence matrices of complexity $\leq a$ in $\text{Rec}_{p \times p}(\mathbf{K})$. The subalgebra \mathcal{R}_a of $\text{Rec}_{p \times p}(\mathbf{K})$ is always recursively closed.*

The above example shows again that many inclusions $\mathcal{R}_a \subset \mathcal{R}_{a+1}$ are strict for $\text{Rec}_{p \times p}(\mathbb{Q})$.

13 Computing G^{-1} for $G \in \text{GL}_{p-\text{rec}}(\mathbf{K})$

The aim of this section is to discuss a few difficulties when computing inverses of recurrence matrices in $\text{GL}_{p-\text{rec}}(\mathbf{K})$.

Definition 13.1. *The depth of $A \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ is the smallest element $D \in \mathbb{N} \cup \{\infty\}$ such that $\rho(\mathcal{M}_{p \times q}^{\leq D})A$ spans $\overline{A}^{\text{rec}}$ (where $\mathcal{M}_{p \times q}^{\infty} = \mathcal{M}$).*

It is easy to check that A is a recurrence matrix if and only if A has finite depth $D < \infty$.

Proposition 13.2. *We have*

$$\dim \left(\sum_{(U,W) \in \mathcal{M}_{p \times q}^{\leq k}} \mathbf{K} \rho(U,W)A \right) < \dim \left(\sum_{(U,W) \in \mathcal{M}_{p \times q}^{\leq k+1}} \mathbf{K} \rho(U,W)A \right)$$

if k is smaller than the depth D of A .

Corollary 13.3. *The depth of any non-zero recurrence matrix A is smaller than its complexity $\dim(\overline{A}^{\text{rec}})$.*

Proof of Proposition 13.2 The equality

$$\sum_{(U,W) \in \mathcal{M}_{p \times q}^{\leq k}} \mathbf{K} \rho(U,W)A = \sum_{(U,W) \in \mathcal{M}_{p \times q}^{\leq k+1}} \mathbf{K} \rho(U,W)A$$

implies that these vector-spaces are recursively closed and coincide thus with $\overline{A}^{\text{rec}}$. \square

Given a presentation \mathcal{P} of $G \in \text{GL}_{p-\text{rec}}(\mathbf{K})$, there are two obvious methods for computing a presentation of its inverse G^{-1} . The first method analyzes the matrices

$$(G[\mathcal{M}_{p \times p}^0])^{-1}, (G[\mathcal{M}_{p \times p}^1])^{-1}, \dots, (G[\mathcal{M}_{p \times p}^a])^{-1}$$

with a huge enough in order to guess a presentation $\tilde{\mathcal{P}}$ of G^{-1} . This can be done if $a \geq 1 + D + N$ where D is the depth and N the saturation level of

G^{-1} . It is then straightforward to check if the presentation \tilde{P} is correct by computing the matrix-product of G and the recurrence matrix presented by \tilde{P} . The limitation of this method is the need of large matrix-computations (inverting the square-matrix $G[\mathcal{M}_{p \times p}^a]$ of order $p^a \times p^a$). Below, we will describe an algorithm based on this method.

The second method is to guess an upper bound b for the complexity of G^{-1} , to write down a generic presentation \tilde{P} of complexity b where the initial data and shift matrices involve a set of $d + p^2 d^2$ unknowns. Equating the matrix-product of G with the recurrence matrix presented by \tilde{P} to the identity yields polynomial equations. We will omit a detailed discussion of this second method since it seems to be even worse than the first one.

Remark 13.4. *There is an other important issue which we will not adress here. Is there a finite algorithm which is able to tell if an element $A \in \text{Rec}_{p \times p}(\mathbf{K})$ (given, say, by a minimal presentation) is or is not invertible in $\text{GL}_{p \times p}(\mathbf{K})$? The algorithm presented below will always succeed (with finite time and memory requirements) in computing an inverse of A for $A \in \text{GL}_{p\text{-rec}}(\mathbf{K})$. It will however fail to stop (or more likely, use up all memory on your computing device) if $A \notin \text{GL}_{p\text{-rec}}(\mathbf{K})$ is invertible in $\mathbf{K}^{\mathcal{M}_{p \times p}}$.*

In particular, it would be interesting to prove or disprove the existence of a sequence $G_1, G_2, G_3, \dots \in \text{GL}_{p\text{-rec}}(\mathbf{K})$ of common complexity $\dim(\overline{G_i}^{\text{rec}}) = d$ such that $\lim_{i \rightarrow \infty} \dim(\overline{G_i}^{\text{rec}}) = \infty$. The first open case is $d = 2$ (see also §12.7).

13.1 An algorithm

Given a presentation \mathcal{P} of $G \in \text{GL}_{p\text{-rec}}(\mathbf{K})$, the following algorithm computes a presentation \tilde{P} of G^{-1} in a finite number of steps.

Step 1 Set $D = 0$.

Step 2 Compute the saturation level N of the (not necessarily recursively closed) vector space spanned by $\rho(\mathcal{M}_{p \times p}^{\leq D})G^{-1}$. (This needs inversion of the $p^{D+N+1} \times p^{D+N+1}$ matrix $G[\mathcal{M}_{p \times p}^{\leq D+N+1}]$ where $N < \frac{p^{D+1}-1}{p-1}$ is the saturation level of the vector space spanned by $\rho(\mathcal{M}_{p \times p}^{\leq D})G^{-1}$.)

Step 3 Supposing the depth D correct, use the saturation level N of step 2 for computing a presentation \tilde{P}_D using the finite-dimensional vector space spanned by $\left(\rho(\mathcal{M}_{p \times p}^{\leq D})G^{-1}\right) [\mathcal{M}_{p \times p}^{\leq N+1}]$.

Step 4 If the recurrence matrix \tilde{G} defined by the presentation \tilde{P}_D satisfies $G\tilde{G} = \text{Id}$, stop and print the presentation \tilde{P} . Otherwise, increment D by 1 and return to step 2.

Remark 13.5. *The expensive part (from a computational view) of the algorithm are steps 2 and 3 and involve computations with large matrices (for $p \geq 2$). A slight improvement is to merge step 2 and 3 and to do the com-*

putations for guessing the presentation $\tilde{\mathcal{P}}_D$ of step 3 at once during step 2.

One could avoid a lot of iterations by running step 2 simultaneously for D and $D+1$. The cost of this “improvement” is however an extra factor of p in the size of the involved matrices and should thus be avoided since step 4 is probably quite cheap (compared to step 2).

14 Lie algebras

The Lie bracket $[A, B] = AB - BA$ turns the algebra $\text{Rec}_{p \times p}(\mathbf{K})$ (or $\mathbf{K}^{\mathcal{M}_{p \times p}}$) into a Lie algebra. For \mathbf{K} a suitable complete topological field (say $\mathbf{K} = \mathbb{R}$ or $\mathbf{K} = \mathbb{C}$), the exponential function $X \mapsto \exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!}$ is well-defined and continuous for both topologies defined in §5. The exponential function does however not preserve the subspace $\text{Rec}_{p \times p}(\mathbf{K})$ of recurrence matrices and the associated Lie group

$$\{\exp(A) \in \mathbf{K}^{\mathcal{M}_{p \times p}} \mid A \in \text{Rec}_{p \times p}(\mathbf{K})\}$$

is thus only a group in $\mathbf{K}^{\mathcal{M}_{p \times p}}$.

The Lie algebra $\text{Rec}_{p \times p}(\mathbb{C})$ contains analogues of the classical Lie algebras of type A, B, C and D .

The analogue of type A is given by the vector space of recurrence matrices $X \in \text{Rec}_{p \times p}(\mathbb{C})$ such that $\text{tr}(X) = 0 \in \text{Rec}_{1 \times 1}$.

The B and D series are defined as the set of all recursive antisymmetric matrices of $\text{Rec}_{p \times p}(\mathbb{C}), p \geq 3$ odd, for the B series and of $\text{Rec}_{p \times p}(\mathbb{C}), p \geq 4$ even, for the D series.

The C series is defined only for even p and consists of all recursive matrices $X \in \text{Rec}_{p \times p}(\mathbb{C})$ such that $\Omega X = (\Omega X)^t$ where $\Omega \in \text{Rec}_{p \times p}(\mathbb{C})$ has complexity 2 and $\Omega[\mathcal{M}_{p \times p}^l]$ is of the form $\begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix}$ for $l \geq 1$ with Id and 0 denoting the identity matrix and the zero matrix of size $p^l/2 \times p^l/2$ (the value of $\Omega[\emptyset, \emptyset]$ is irrelevant).

A different and perhaps more natural way to define the C series is to consider triplets $A, B, C \in \text{Rec}_{p \times p}(\mathbf{K})$ (for all $p \in \mathbb{N}$) of recurrence matrices with $B = B^t$ and $C = C^t$ symmetric. The C series is then the Lie subalgebra of $\prod_{l=0}^{\infty} \mathbf{K}^{2p^l \times 2p^l}$ of matrix products given by

$$\begin{pmatrix} A[\mathcal{M}_{p \times p}^l] & B[\mathcal{M}_{p \times p}^l] \\ C[\mathcal{M}_{p \times p}^l] & -A^t[\mathcal{M}_{p \times p}^l] \end{pmatrix}.$$

I ignore if there are natural “recursive” analogues of (some of) the exceptional simple Lie algebras.

Remark 14.1. *Analogues of type B, C, D Lie algebras can also be defined using arbitrary non-singular symmetric or symplectic recurrence matrices in $\text{Rec}_{p \times p}(\mathbf{K})$.*

Remark 14.2. A Lie-algebra $\mathcal{L} \subset \text{Rec}_{p \times p}(\mathbf{K})$ is in general not a recursively closed subspace of $\text{Rec}_{p \times p}(\mathbf{K})$.

Remark 14.3. It would be interesting to understand the algebraic structure of Lie algebras in $\text{Rec}_{p \times p}(\mathbf{K})$. Given such a Lie algebra $\mathcal{L} \subset \text{Rec}_{p \times p}(\mathbf{K})$, the intersection $\mathcal{L} \cap \mathcal{FS}_{p \times p}$ with the vector-space $\mathcal{FS}_{p \times p}$ of all finitely supported elements defines an ideal in \mathcal{L} . The interesting object is thus probably the quotient Lie algebra $\mathcal{L}/(\mathcal{L} \cap \mathcal{FS}_{p \times p})$. What is the structure of this quotient algebra for the analogues in $\text{Rec}_{p \times p}(\mathbb{C})$ of the A – D series?

14.1 Lie algebras in the convolution ring $\mathbf{K}^{\mathcal{M}_p}$

Using the convolution-ring structure on $\mathbf{K}^{\mathcal{M}_{p \times q}}$ we get other, different Lie algebras. Since the convolution-structure depends only on the product pq , we restrict ourself to $\mathbf{K}^{\mathcal{M}_p}$ for simplicity.

The three obvious subalgebras in $\mathbf{K}^{\mathcal{M}_p}$ (corresponding to formal power series in p non-commuting variables) given by $\mathbf{K}^{\mathcal{M}_p}$, $\text{Rec}_p(\mathbf{K})$ and the vector space \mathcal{FS}_p of finitely supported elements (corresponding to polynomials in p non-commuting variables) give rise to three Lie-algebras with Lie bracket $[A, B] = A * B - B * A$ (where $*$ stands for the convolution product in $\mathbf{K}^{\mathcal{M}_p}$).

The Lie algebra resulting from \mathcal{FS}_p contains the free Lie algebra on p generators.

All these Lie algebras are in a certain sense graded: We have indeed, $[A, B][\emptyset, \emptyset] = 0$ for all $A, B \in \mathbf{K}^{\mathcal{M}_p}$ and $[A, B][U, W] = 0$ for all $(U, W) \in \mathcal{M}_p^{\leq \alpha + \beta - 1}$ if $A[U', W'] = 0$ for all $(U', W') \in \mathcal{M}_p^{\leq \alpha}$ and $B[U'', W''] = 0$ for all $(U'', W'') \in \mathcal{M}_p^{\leq \beta}$.

15 Integrality and lattices of recurrence vectors

A recurrence matrix $A \in \text{Rec}_{p \times q}(\mathbb{Q})$ is *integral* if all its coefficients $A[U, W]$ are integers. This notion can easily be generalized by considering coefficients in a subring of the algebraic integers $\mathcal{O}_{\mathbf{K}}$ of a number field \mathbf{K} .

We denote by $\text{Rec}_{p \times q}(\mathbb{Z})$ the \mathbb{Z} -module of all integral recurrence matrices in $\text{Rec}_{p \times q}(\mathbb{Q})$. Since products of integral matrices are integral, one gets a category $\mathbb{Z}^{\mathcal{M}}$ having only integral recurrence matrices as arrows and algebras $\text{Rec}_{p \times p}(\mathbb{Z})$ consisting of all integral recurrence matrices in $\text{Rec}_{p \times p}(\mathbb{Q})$.

Proposition 15.1. Every integral recurrence matrix $A \in \text{Rec}_{p \times q}(\mathbb{Z})$ can be written as $A = \sum_{j=1}^a \lambda_j A_j$ with $\lambda_j \in \mathbb{Z}$ and $A_1, \dots, A_a \in \text{Rec}_{p \times q}(\mathbb{Z})$ a basis of $\text{Rec}_{p \times q}(\mathbb{Z})$ with span \overline{A}^{rec} and a presentation given by $(A_1, \dots, A_a)[\emptyset, \emptyset] \in \mathbb{Z}^a$ and integral shift-matrices $\rho(s, t) \in \mathbb{Z}^{a \times a}$.

Proof The set $(\text{Rec}_{p \times q}(\mathbb{Z}) \cap \overline{A}^{rec}) \subset \overline{A}^{rec} \subset \text{Rec}_{p \times q}(\mathbb{Q})$ is a free \mathbb{Z} -module. Since it contains the recursive set-closure $\rho(\mathcal{M}_{p \times q})A$ spanning \overline{A}^{rec} , it

is of maximal rank $a = \dim(\overline{A}^{rec})$. A \mathbb{Z} -basis A_1, \dots, A_a of $(\text{Rec}_{p \times q}(\mathbb{Z}) \cap \overline{A}^{rec})$ has the required properties. \square

We call an integral recurrence matrix $A \in \text{Rec}_{p \times p}(\mathbb{Z})$ *unimodular* if $\det(A[\mathcal{M}_{p \times p}^l]) \in \{\pm 1\}$ for all $l \in \mathbb{Z}$. The set of all integral unimodular recurrence matrices in $\text{GL}_{p-rec}(\mathbb{Q})$ is the *unimodular subgroup of integral recurrence matrices* in $\text{Rec}_{p \times p}(\mathbb{Q})$.

Remark 15.2. *The \mathbb{Q} -vector space $\text{Rec}_{p \times q}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is in general a strict subspace of $\text{Rec}_{p \times q}(\mathbb{Q})$. For $q = p$, this vector space is thus a subalgebra of the algebra $\text{Rec}_{p \times p}(\mathbb{Q})$. However, given an arbitrary recurrence matrix $A \in \text{Rec}_{p \times p}(\mathbb{Q})$, there exists by Proposition 6.5 an integer $\alpha \geq 1$ and an integer $\lambda \geq 1$ with $\alpha H_\lambda A \in \text{Rec}_{p \times p}(\mathbb{Z})$ where $H_\lambda \in \mathcal{C}_{p-rec}(\mathbb{Q})$ is the integral diagonal matrix in the center $\mathcal{C}_{p-rec}(\mathbb{Q})$ of $\text{Rec}_{p \times p}(\mathbb{Q})$ with diagonal coefficients $H_\lambda[U, U] = \lambda^l$ for $(u, u) \in \mathcal{M}_{p \times p}^l$.*

Remark 15.3. *An integral unimodular recurrence matrix of $\text{Rec}_{p \times p}(\mathbb{Z})$ is in general not invertible. An example is given by the converging lower triangular recurrence matrix $A \in \text{Rec}_{2 \times 2}(\mathbb{Z})$ of Toeplitz type with generating series $1 - z - z^2 \in \mathbb{Z}[[z]]$. Its inverse in $\mathbb{Q}^{\mathcal{M}_{2 \times 2}}$ corresponds to the generating series $1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + \dots \in \mathbb{Z}[[z]]$ defining Fibonacci numbers and cannot be a recurrence matrix by Proposition 6.2.*

15.1 Finite-dimensional lattices

Definition 15.4. *A finite-dimensional lattice of $\text{Rec}_{p \times q}(\mathbb{C})$ is a free \mathbb{Z} -module of finite rank in $\text{Rec}_{p \times q}(\mathbb{C})$.*

A finite-dimensional lattice of $\text{Rec}_{p \times q}(\mathbb{C})$ is discrete for both topologies introduced in §5.

Since the multiplicative structure is irrelevant for lattices, it is enough to consider lattices in $\text{Rec}_p(\mathbb{C})$. A particularly beautiful set of lattices is given by lattices which are recursively closed sets, ie. given by the identity $\Lambda = \rho(\mathcal{M}_p)\Lambda$. An example of such a lattice is the subset $\text{Rec}_p(\mathbb{Z}) \cap \overline{A}^{rec} \subset \text{Rec}_p(\mathbb{Q})$ of all integral recurrence vectors in the recursive closure of a recurrence vector $A \in \text{Rec}_p(\mathbb{Q})$.

Given two recurrence vectors $A, B \in \mathbb{R}$, we define their *scalar-product* as the element

$$\langle A, B \rangle = A^t B \in \text{Rec}_1(\mathbb{R}).$$

More precisely, such a scalar-product can be represented by the generating series

$$\langle A, B \rangle_z = \sum_{l=0}^{\infty} z^l \sum_{U \in \mathcal{M}_p^l} A[U]B[U] \in \mathbb{R}[[z]]$$

and defines a rational function. Given a finite dimensional subspace $\mathcal{A} \subset \text{Rec}_p(\mathbb{R})$ there exists an open interval $(0, \alpha(\mathcal{A})) \subset \mathbb{R}$ such that an evaluation with $z \in (0, \alpha(\mathcal{A}))$ yields an Euclidean scalar-product on \mathcal{A} .

In particular, a suitable evaluation of the scalar-product turns a lattice Λ into an Euclidean lattice.

Remark 15.5. *It is of course also possible to define scalar-products using a symmetric positive definite recurrence matrix of $\text{Rec}_{p \times p}(\mathbb{R})$ (the case considered above corresponds to the identity).*

Hermitian products over \mathbb{C} are defined in an analogous way.

16 Abelian monoids

There is a natural notion of a monoid presented by generators and relations. A free monoid has no relations and an arbitrary finitely generated monoid \mathcal{Q} can always be given in the form

$$\mathcal{Q} = \langle \mathcal{G} : \mathcal{R} \rangle$$

where \mathcal{G} is a finite set of generators and $\mathcal{R} \subset \mathcal{G}^* \times \mathcal{G}^*$ a (perhaps infinite) set of relations of the form $L = R$ with $L, R \in \mathcal{G}^*$. The quotient monoid \mathcal{Q} of the free monoid \mathcal{G}^* by the relations \mathcal{R} is the set of equivalence classes of words in \mathcal{G}^* by the equivalence relation generated by $UL_iV \sim UR_iV$ for $U, V \in \mathcal{G}^*$ and $(L_i, R_i) \in \mathcal{R}$.

Any quotient monoid $\mathcal{Q} = \langle \mathcal{M}_{p \times q}^1 : \mathcal{R} \subset \mathcal{M}_{p \times q} \times \mathcal{M}_{p \times q} \rangle$ of the monoid $\mathcal{M}_{p \times q}$ defines a subspace $V_{\mathcal{Q}} \subset \mathbf{K}^{\mathcal{M}_{p \times q}}$ by setting

$$V_{\mathcal{Q}} = \{A \in \mathbf{K}^{\mathcal{M}_{p \times q}} \mid \rho(UL_iV)A = \rho(UR_iV)A, U, V \in \mathcal{M}_{p \times q}, (L_i, R_i) \in \mathcal{R}\}.$$

The space $V_{\mathcal{Q}}$ is by construction recursively closed, and the shift monoid $\rho_{\mathcal{Q}}(\mathcal{M}_{p \times q})$ acts on $V_{\mathcal{Q}}$ with a “kernel” generated by the relations \mathcal{R} .

In the case $p = q$, the subspace $V_{\mathcal{Q}}$ is in general not multiplicatively closed. We will however describe an important particular case where this happens.

For $p, q \in \mathbb{N}$ we consider the *free abelian* quotient monoid $\mathcal{A}b_{p \times q} \sim \mathbb{N}^{pq}$ generated by all pq elements of $\mathcal{M}_{p \times q}^1$ with relations $\mathcal{R} = \{(XY, YX) \mid X, Y \in \mathcal{M}_{p \times q}^1\}$. The corresponding subspace $V_{\mathcal{A}b_{p \times q}}$ consists of all functions $A \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ such that

$$A[u_1 \dots u_n, w_1 \dots w_n] = A[u_{\pi(1)} \dots u_{\pi(n)}, w_{\pi(1)} \dots w_{\pi(n)}]$$

for all $(u_1 \dots u_n, w_1 \dots w_n) \in \mathcal{M}_{p \times q}$ and all permutations π of $\{1, \dots, n\}$.

The vector space $V_{\mathcal{A}b_{p \times q}}$ can thus be identified with the vector space $\mathbf{K}[[Z_{0,0}, \dots, Z_{p-1,q-1}]]$ of formal power series in pq commuting variables $Z_{u,w}, 0 \leq u < p, 0 \leq w < q$.

We denote the vector space $V_{\mathcal{A}b_{p \times q}}$ by $\mathbf{K}^{\mathcal{A}b_{p \times q}}$.

For $A \in \mathbf{K}^{\mathcal{A}b_{p \times q}}$ we have

$$\rho(s, t)\rho(s', t')A = \rho(s', t')\rho(s, t)A$$

for all $0 \leq s, s' < p, 0 \leq t, t' < q$. The easy computation

$$\begin{aligned}
\rho(s, t)\rho(s', t')(AB) &= \rho(s, t) \left(\sum_{v'} (\rho(s', v')A)(\rho(v', t')B) \right) \\
&= \sum_{v, v'} (\rho(s, v)\rho(s', v')A)(\rho(v, t)\rho(v', t')B) \\
&= \sum_{v, v'} (\rho(s', v')\rho(s, v)A)(\rho(v', t')\rho(v, t)B) \\
&= \dots = \rho(s', t')\rho(s, t)(AB)
\end{aligned}$$

shows that $AB \in \mathbf{K}^{\mathcal{A}b_{p \times q}}$ for $A \in \mathbf{K}^{\mathcal{A}b_{p \times r}}, B \in \mathbf{K}^{\mathcal{A}b_{r \times q}}$. We get thus subcategories $\mathbf{K}^{\mathcal{A}b}$ of $\mathbf{K}^{\mathcal{M}}$ and $\text{Rec}(\mathbf{K}) \cap \mathbf{K}^{\mathcal{A}b}$ of $\text{Rec}(\mathbf{K})$.

These subcategories contain all elements of complexity 1. The algebra formed by all recurrence matrices in $\mathbf{K}^{\mathcal{A}b_{p \times p}}$ is thus not commutative if $p > 1$.

It would be interesting to have other examples of “natural” quotient monoids of $\mathcal{M}_{p \times q}, p, q \in \mathbf{N}$, giving rise to subcategories in $\mathbf{K}^{\mathcal{M}}$ and $\text{Rec}(\mathbf{K})$.

Remark 16.1. *The association*

$$A \longmapsto f(A) = A[\emptyset] + \sum_{n=1}^{\infty} \sum_{0 \leq u_1, \dots, u_n < p} A[u_1 \dots u_n] Z_{u_1} \dots Z_{u_n}$$

of the formal power series f_A in p commuting variables to a recurrence element in $\mathbb{C}^{\mathcal{A}b_p}$ is here natural (cf. Remark 6.3). Does this have interesting analytic consequences for f_A (which is holomorphic in a neighbourhood of $(0, \dots, 0)$ by Proposition 6.2)?

17 Birecursivity

The *palindromic involution*

$$\iota(s_1 s_2 \dots s_{n-1} s_n, t_1 \dots t_n) = (s_n s_{n-1} \dots s_2 s_1, t_n \dots t_1)$$

defines an involutive antiautomorphism of $\mathcal{M}_{p \times q}$ and we get the *palindromic automorphism* $\iota : \mathbf{K}^{\mathcal{M}_{p \times q}} \longrightarrow \mathbf{K}^{\mathcal{M}_{p \times q}}$ by considering the involutive automorphism defined by $(\iota X)[U, W] = X[\iota(U, W)]$ for $X \in \mathbf{K}^{\mathcal{M}_{p \times q}}$. Since we have the identity $(\iota X)(\iota Y) = \iota(XY), X \in \mathbf{K}^{\mathcal{M}_{p \times r}}, Y \in \mathbf{K}^{\mathcal{M}_{r \times q}}$, the palindromic automorphism defines an involutive automorphic functor of the category $\mathbf{K}^{\mathcal{M}}$.

In characteristic $\neq 2$, the palindromic automorphism ι endows the algebra $\mathbf{K}^{\mathcal{M}_{p \times q}}$ with a $\mathbb{Z}/2\mathbb{Z}$ -grading in the usual way by considering the decomposition $X = X_+ + X_-$ into its *even* and *odd* (palindromic) parts defined by

$$X_+ = \frac{X + \iota X}{2} \text{ and } X_- = \frac{X - \iota X}{2}$$

for $X \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ and we have the sign-rules

$$(XY)_+ = X_+Y_+ + X_-Y_-, (XY)_- = X_+Y_- + X_-Y_+$$

whenever the matrix product XY is defined. In particular, we get a subcategory $(\mathbf{K}^{\mathcal{M}})_+$ consisting of all even parts in $\mathbf{K}^{\mathcal{M}}$.

Conjugating the shift-monoid $\rho(\mathcal{M}_{p \times q})$ by ι , we get a second morphism $\lambda : \mathcal{M}_{p \times q} \longrightarrow \text{End}(\mathbf{K}^{\mathcal{M}_{p \times q}})$, called the *left-shift-monoid*. It is defined by

$$(\lambda(s_1 \dots s_n, t_1 \dots t_n)X)[U, W] = X[s_n \dots s_1 U, t_n \dots t_1 W].$$

The obvious commutation rule $\rho(S, T)\lambda(S', T') = \lambda(S', T')\rho(S, T)$ yields an action of the (direct) product-monoid $\mathcal{M}_{p \times q} \times \mathcal{M}_{p \times q}$ on $\mathbf{K}^{\mathcal{M}_{p \times q}}$. An element $X \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ is a *birecurrence matrix* if the linear span $\overline{X}^{\text{birec}}$ of the orbit $\lambda(\mathcal{M}_{p \times q})\rho(\mathcal{M}_{p \times q})X$ has finite dimension. The *birecursive complexity* of X is defined as $\dim(\overline{X}^{\text{birec}}) \in \mathbb{N} \cup \{\infty\}$.

Proposition 17.1. *We have*

$$\dim(\overline{A}^{\text{rec}}) \leq \dim(\overline{A}^{\text{birec}}) \leq (\dim(\overline{A}^{\text{rec}}))^2.$$

Corollary 17.2. (i) *The vector spaces of recurrence matrices and birecurrence matrices in $\mathbf{K}^{\mathcal{M}_{p \times q}}$ coincide.*

(ii) *The palindromic automorphism ι of $\mathbf{K}^{\mathcal{M}_{p \times q}}$ restricts to an involutive automorphism of $\text{Rec}_{p \times q}(\mathbf{K})$. The decomposition of a recurrence matrix A into its even and odd parts $A_+ = (A + \iota A)/2$, $A_- = (A - \iota A)/2$ holds thus in $\text{Rec}(\mathbf{K})$ for \mathbf{K} of characteristic $\neq 2$.*

(iii) *The palindromic automorphism yields an involutive automorphic functor of the category $\text{Rec}(\mathbf{K})$.*

Remark 17.3. *Defining the palindromic element $P_p^\iota \in \mathbf{K}^{\mathcal{M}_{p \times p}}$ by $P_p^\iota[s_1 \dots s_n, s_n \dots s_1] = 1$ and $P_p^\iota[s_1 \dots s_n, t_1 \dots t_n] = 0$ if $s_1 \dots s_n \neq t_n \dots t_1$, the palindromic automorphism of $\mathbf{K}^{\mathcal{M}_{p \times q}}$ or $\text{Rec}_{p \times q}$ is given by $X \longmapsto P_p^\iota X P_q^\iota$. In particular, the palindromic automorphism is an inner automorphism of the algebra $\mathbf{K}^{\mathcal{M}_{p \times p}}$.*

The palindromic element P_p^ι is however not a recurrence matrix if $p \geq 2$. Indeed, we have $(\rho(s_1 \dots s_n, t_1 \dots t_n)P_p^\iota)[u_1 \dots u_n, w_1 \dots w_n] = 1$ if $u_1 \dots u_n = t_n \dots t_1, w_1 \dots w_n = s_n \dots s_1$ and $(\rho(s_1 \dots s_n, t_1 \dots t_n)P_p^\iota)[u_1 \dots u_n, w_1 \dots w_n] = 0$ otherwise. This implies $\dim(\overline{P_p^\iota}^{\text{rec}}) \geq p^{2n}$ for all $n \in \mathbb{N}$ and shows that the palindromic automorphism of the algebra $\text{Rec}_{p \times p}(\mathbf{K})$ is not inner for $p \geq 2$. (For $p = 1$ the palindromic automorphisms P_1^ι is trivial and thus inner in the commutative algebra $\text{Rec}_{1 \times 1}(\mathbf{K})$.)

Let us finish this remark by adding that the algebras $\text{Rec}_{p \times p}(\mathbf{K})$ admit many more similar “exterior” automorphisms. An example is for instance given by the automorphism φ_k (for $k \geq 1$) which acts as ι on $A[\mathcal{M}_{p \times p}^{nk}]$ and as the identity on $A[\mathcal{M}_{p \times p}^n]$ if n is not divisible by k . It would perhaps

be interesting to understand the algebraic structure of the quotient group of outer automorphism obtained as the quotient of all automorphisms of the algebra $\text{Rec}_{p \times p}(\mathbf{K})$ by the normal subgroup $GL_{p-\text{rec}}(\mathbf{K}) / (GL_{p-\text{rec}}(\mathbf{K}) \cap \mathcal{C}_{p-\text{rec}})$ (where $\mathcal{C}_{p-\text{rec}}$ denotes the center of $\text{Rec}_{p \times p}(\mathbf{K})$) of inner automorphisms acting by conjugation.

Proof of Proposition 17.1 The result clearly holds for the 0 element in $\text{Rec}_{p \times q}(\mathbf{K})$.

Consider a non-zero recurrence matrix $A \in \text{Rec}_{p \times q}(\mathbf{K})$ with complexity $a \geq 1$ and recursive closure $\mathcal{A} = \overline{A}^{\text{rec}}$. There exists thus a word $(U, W) \in \mathcal{M}_{p \times q}$ such that $A[U, W] = (\rho(U, W)A)[\emptyset, \emptyset] \neq 0$. Using the recurrence matrix $B = \rho(U, W)A \in \mathcal{A}$ (satisfying $B[\emptyset, \emptyset] \neq 0$), we can construct a basis A_1, \dots, A_a of \mathcal{A} such that $A_i[\emptyset, \emptyset] = 1$ for $1 \leq i \leq a$. For $k \in \{1, \dots, a\}$, consider the basis $A_{k,1}, \dots, A_{k,a}$ of \mathcal{A} defined by $A_{k,1} = A_k$, $A_{k,k} = A_1 - A_k$ and $A_{k,i} = A_i - A_k$ if $i \neq 1, k$. We have thus a presentation $(A_{k,1}, \dots, A_{k,a})[\emptyset, \emptyset] = (1, 0, 0, \dots, 0)$ with shift-matrices $\rho_k(s, t) \in \mathbf{K}^{a \times a}$. Using Proposition 4.1 we have

$$\begin{pmatrix} A_{k,1}[s_1 \dots s_n, t_1 \dots t_n] \\ A_{k,2}[s_1 \dots s_n, t_1 \dots t_n] \\ \vdots \\ A_{k,a}[s_1 \dots s_n, t_1 \dots t_n] \end{pmatrix} = \rho_k(s_n, t_n)^t \cdots \rho_k(s_1, t_1)^t \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Transposing we get

$$\begin{pmatrix} \tilde{A}_{k,1}[s_n \dots s_1, t_n \dots t_1] \\ \tilde{A}_{k,2}[s_n \dots s_1, t_n \dots t_1] \\ \vdots \\ \tilde{A}_{k,a}[s_n \dots s_1, t_n \dots t_1] \end{pmatrix} = \rho_k(s_1, t_1) \cdots \rho_k(s_n, t_n) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where the elements $\tilde{A}_{k,1}, \dots, \tilde{A}_{k,a}$ span a recursively closed subspace $\tilde{\mathcal{A}}_k \subset \text{Rec}_{p \times p}(\mathbf{K})$. The element $A_{k,1}[s_1 \dots s_n, t_1 \dots t_n]$ corresponds to the coefficient $m_{1,1}$ of $M = \rho_k(s_n, t_n)^t \cdots \rho_k(s_1, t_1)^t \in \mathbf{K}^{a \times a}$ and coincides thus with the corresponding coefficient $\tilde{m}_{1,1}$ of the transposed matrix $M^t = \rho_k(s_1, t_1) \cdots \rho_k(s_n, t_n)$ encoding $\tilde{A}_{k,1}[s_n \dots s_1, t_n \dots t_1]$. This implies the identity

$$\tilde{A}_{k,1}[s_n \dots s_1, t_n \dots t_1] = A_{k,1}[s_1 \dots s_n, t_1 \dots t_n]$$

for all $(s_1 \dots s_n, t_1 \dots t_n) \in \mathcal{M}_{p \times q}$ and proves that $\tilde{\mathcal{A}}_k = \iota(A_k)$ is a recurrence matrix contained in $\tilde{\mathcal{A}}_k$.

The vector-space $\iota(\tilde{\mathcal{A}}_k) \subset \mathbf{K}^{\mathcal{M}_{p \times q}}$ containing A_k is thus invariant under the action of $\lambda(\mathcal{M}_{p \times q})$. Consider now the vector-space $\mathcal{B} = \sum_{k=1}^a \iota(\tilde{\mathcal{A}}_k) \subset \mathbf{K}^{\mathcal{M}_{p \times q}}$ of dimension $\leq a^2$. For $(S, T), (S', T') \in \mathcal{M}_{p \times q}$, the computation

$$\lambda(S, T)\rho(S', T')A \in \lambda(S, T)\left(\sum_{k=1}^a \mathbf{K}A_k\right) \subset \sum_{k=1}^a \iota(\tilde{\mathcal{A}}_k) = \mathcal{B}$$

shows the inclusion $\overline{A}^{birec} \subset \mathcal{B}$ which ends the proof. \square

Remark 17.4. *An example of an element $A \in \text{Rec}_p(\mathbb{Q})$ with complexity $n + 1$ and birecursive complexity roughly $n^2/2$ (for $p \geq 2$ and huge n) is given by $A[u_1 \dots u_n] = 2^{\sum_{j=1}^n u_j p^j}$ and $A[u_1 \dots u_l] = 0$ if $l \neq n$. The vector $A[\mathcal{M}_p^n]$ can thus be identified with the finite geometric progression*

$$(A[0^n], A[10^{n-1}], \dots, A[(p-1)^n]) = (1, 2, 2^2, 2^3, \dots, 2^{p^n-1}).$$

The vector space generated by $\cup_{j=0}^k (\lambda(\mathcal{M}_p^j) \rho(\mathcal{M}_p^{k-j} A)) [\mathcal{M}_p^{n-k}]$ is spanned by the $(k+1)$ geometric progressions

$$1, 2^\alpha, 2^{2\alpha}, 2^{3\alpha}, \dots, 2^{(p^{n-k}-1)\alpha}$$

for $\alpha = 1, p, p^2, \dots, p^k$ and its dimension equals the minimum of $k+1$ and p^{n-k} . The birecursive complexity of A is thus given by $\sum_{k=0}^n \min(k+1, p^{n-k})$ which amounts roughly to $n^2/2$ for huge n and $p > 1$.

18 Finite monoids and finite state automata

This section describes a connection between automatic functions and sequences (defined by finite state automata) and recurrence matrices with finite shift-monoid.

18.1 Finite monoids

Proposition 18.1. *Given a recurrence matrix $A \in \text{Rec}_{p \times q}(\mathbf{K})$, the following four assertions are equivalent:*

- (i) *The subset $\{A[U, W] \mid (U, W) \in \mathcal{M}_{p \times q}\} \subset \mathbf{K}$ of values of A is finite.*
- (ii) *For all $B \in \overline{A}^{rec}$, the subset $\{B[U, W] \mid (U, W) \in \mathcal{M}_{p \times q}\} \subset \mathbf{K}$ of values of B is finite.*
- (iii) *The recursive set-closure $\rho(\mathcal{M}_{p \times q})A$ of A is finite.*
- (iv) *The shift-monoid $\rho_{\overline{A}^{rec}}(\mathcal{M}_{p \times q}) \in \text{End}(\overline{A}^{rec})$ of A is finite.*

Proof of Proposition 18.1 For a recurrence matrix $A \in \text{Rec}_{p \times q}(\mathbf{K})$ satisfying assertion (i), consider the finite set $\mathcal{F} = \{A[U, W] \mid (U, W) \in \mathcal{M}_{p \times q}\} \subset \mathbf{K}$ containing all evaluations of A . The set \mathcal{F} contains thus also all evaluations $\{X[U, W] \mid (U, W) \in \mathcal{M}_{p \times q}\}$ for $X \in \rho(\mathcal{M}_{p \times q})A$ in the set-closure of A . Choose $(U_1, W_1), \dots, (U_a, W_a) \in \mathcal{M}_{p \times q}$ such that $\rho(U_1, W_1)A, \dots, \rho(U_a, W_a)A$ form a basis of \overline{A}^{rec} . An arbitrary element $B \in \overline{A}^{rec}$ can thus be written as a linear combination $B = \sum_{j=1}^a \beta_j \rho(U_j, W_j)A$ and the set $\{B[U, W] \mid (U, W) \in \mathcal{M}_{p \times q}\}$ of all its evaluations is a subset of the finite set $\sum_{j=1}^a \beta_j \mathcal{F}$ containing at most $1 + a \#\{\mathcal{F} \setminus \{0\}\} < \infty$ elements. This shows the equivalence of assertions (i) and (ii).

All elements in the set closure $\rho(\mathcal{M}_{p \times q})A \subset \overline{A}^{rec}$ of $A \in \text{Rec}_{p \times q}(\mathbf{K})$ are determined by their projection onto $\overline{A}^{rec}[\mathcal{M}_{p \times q}^{\leq N}]$ where N denotes the saturation level of \overline{A}^{rec} . Assuming (i), the elements of $\rho(\mathcal{M}_{p \times q})A$ are thus in bijection with a subset of the finite set $\mathcal{F}^{\mathcal{M}_{p \times q}^{\leq N}}$ where $\mathcal{F} = \{A[U, W] \mid (U, W) \in \mathcal{M}_{p \times q}\} \subset \mathbf{K}$ is the finite set of all evaluations of A . This proves that (i) implies (iii).

The shift-monoid $\rho_{\overline{A}^{rec}}(\mathcal{M}_{p \times q}) \subset \text{End}(\overline{A}^{rec})$ of A acts faithfully on the set-closure $\rho(\mathcal{M}_{p \times q})A$ of A . Finiteness of $\rho(\mathcal{M}_{p \times q})A$ implies thus (iv).

We have $\{A[U, W] \mid (U, W) \in \mathcal{M}_{p \times q}\} = \{(\rho(U, W)A)[\emptyset, \emptyset] \mid (U, W) \in \mathcal{M}_{p \times q}\}$ which shows that (iv) implies (i). \square

18.2 Automata

An *initial automaton* with *input alphabet* X and *output alphabet* Y is a directed graph \mathcal{A} with vertices or *states* \mathcal{V} such that

- \mathcal{V} contains a marked *initial state* v_* .
- The set of directed edges originating in a given state $v \in \mathcal{V}$ is bijectively labelled by the input alphabet X .
- The set \mathcal{V} of states is weighted by an *output function* $w : \mathcal{V} \rightarrow Y$.

An initial automaton \mathcal{A} is *finite state* if its input alphabet X and its set of states \mathcal{V} are both finite.

An initial automaton yields an application $\alpha : \mathcal{M}_X \rightarrow Y$ from the free monoid \mathcal{M}_X generated by the input alphabet X into Y . Indeed, an element $x_1 \dots x_n \in \mathcal{M}_X$ defines a unique path $\gamma \subset \mathcal{A}$ of length n with consecutive vertices $v_0 = v_*, v_1, v_2, \dots, v_n$ defined by the requirement that there is a directed edge labelled x_i from v_{i-1} to v_i . We set $\alpha(x_1 \dots x_n) = w(v_n)$. Every function $\alpha \in Y^{\mathcal{M}_X}$ can be constructed in this way by a suitable initial automaton which is generally infinite. A function $\alpha \in Y^{\mathcal{M}_X}$ is *automatic* if it is realizable by a finite state initial automaton. Automatic functions (with $Y \subset \mathbf{K}$ a subset of a field) which are converging elements of $\mathbf{K}^{\mathcal{M}_p}$ are called *p-automatic* or *automatic sequences*.

Proposition 18.2. *Automatic functions with $X = \mathcal{M}_p$ and $Y = \mathbf{K}$ are in bijection with elements of $\text{Rec}_p(\mathbf{K})$ having finite shift-monoids.*

Proof We show first that an automatic function $\alpha \in \mathbf{K}^{\mathcal{M}_p}$ is a recurrence matrix with finite shift-monoid. We choose an initial automaton $\mathcal{A} = (v_* \in \mathcal{V}, w \in \mathbf{K}^{\mathcal{V}})$ which has a finite number $a = \#\mathcal{V}$ of states and realizes α . Given a word $S = s_1 \dots s_n \in \mathcal{M}_p^n$, we get an application $\gamma_S : \mathcal{V} \rightarrow \mathcal{V}$ defined by $\gamma_S(v) = w$ if the path of length n starting at state $v \in \mathcal{V}$ and consisting of directed edges with labels s_1, \dots, s_n , ends at state w . Associate to a word S the initial finite state automaton $\mathcal{A}_S = (v_* \in \mathcal{V}, \tilde{w} = w \circ \gamma_S)$ with the

same set \mathcal{V} of states and the same initial state v_* but with weight-function $\tilde{w}(v) = w(\gamma_S(v)) \in \mathbf{K}$. We have then by construction $\alpha(US) = \alpha_S(U)$ where α_S is the automatic function of the finite state initial automaton \mathcal{A}_S .

Given a fixed finite state initial automaton \mathcal{A} , the automatic function α_S depends only of the function γ_S which belongs to the finite set $\mathcal{V}^{\mathcal{V}}$ of all a^a functions from \mathcal{V} to \mathcal{V} . The recursive set-closure

$$\rho(\mathcal{M}_p)\alpha = \{\alpha_S \mid S \in \mathcal{M}_p\}$$

of $\alpha \in \mathbf{K}^{\mathcal{M}_p}$ is thus finite. This proves that $\alpha \in \text{Rec}_p(\mathbf{K})$ and Proposition 18.1 implies that the shift-monoid of α is finite.

Consider now a recurrence matrix $A \in \mathbf{K}^{\mathcal{M}_p}$ with finite shift-monoid. We have to show that $U \mapsto A[U]$ (for $U \in \mathcal{M}_p$) is an automatic function. If A is identically zero, there is nothing to prove. Otherwise, complete A to a basis $A_1 = A, A_2, \dots, A_a$ of its recursive closure $\overline{A}^{\text{rec}}$ of dimension a and consider the associated presentation with shift-matrices $\rho_{\overline{A}^{\text{rec}}}(s) \in \mathbf{K}^{a \times a}$ and initial values $W = (A_1[\emptyset], \dots, A_a[\emptyset]) \in \mathbf{K}^a$. We get thus a matrix-realization $\mathcal{M}_A = \rho_{\overline{A}^{\text{rec}}}(\mathcal{M}_p) \subset \text{End}(\mathbf{K}^a)$ of the finite shift-monoid of A . Consider now the Cayley graph Γ of \mathcal{M}_A with respect to the generators $\rho_{\overline{A}^{\text{rec}}}(s) \in \rho_{\overline{A}^{\text{rec}}}(\mathcal{M}_p^1)$. The graph Γ is the finite graph with vertices indexed by all elements of \mathcal{M}_A . An oriented (or directed) edge labelled s joins X to Y if $Y = X\rho_{\overline{A}^{\text{rec}}}(s)$. Consider the graph Γ as an initial finite state automaton with initial state the identity $\rho_A(\emptyset)$ and weight function $w(X) = (X^t W^t)_1$ where $(X^t W^t)_1$ is the first coordinate of the row-vector $X^t W^t \in \mathbf{K}^a$. By construction, the automatic function $\alpha \in \mathbf{K}^{\mathcal{M}_p}$ associated to this initial automaton is given by $\alpha(U) = A_1[U]$ since we have

$$\begin{pmatrix} A_1[u_1 \dots u_n] \\ A_2[u_1 \dots u_n] \\ \vdots \\ A_a[u_1 \dots u_n] \end{pmatrix} = \rho_{\overline{A}^{\text{rec}}}(u_n)^t \cdots \rho_{\overline{A}^{\text{rec}}}(u_1)^t \begin{pmatrix} A_1[\emptyset] \\ A_2[\emptyset] \\ \vdots \\ A_a[\emptyset] \end{pmatrix}.$$

□

Remark 18.3. *When dealing with automatic functions, one often works with output alphabet a finite field \mathbf{K} . Shift-monoids of recurrence matrices are then always finite and \mathbf{K} -valued automatic functions on \mathcal{M}_p are in bijection with $\text{Rec}_p(\mathbf{K})$. Automatic functions form thus a ring with respect to the Hadamard product (Proposition 3.8, assertion (i)) or with respect to the polynomial product (see §9.3). Converging automatic functions (also called automatic sequences,) form a subring for the Hadamard product (see also [1], Corollary 5.4.5 and Theorem 12.2.6) and a differential subring (after identification with the corresponding generating series) for the polynomial product (see also [1], Theorem 16.4.1).*

19 The categories of transducers and finite-state transducers

An element $A \in Y^{\mathcal{M}_X}$ gives rise to a length-preserving application $\tau_A : \mathcal{M}_X \longrightarrow \mathcal{M}_Y$ defined by $\tau_A(\emptyset) = \emptyset$ and

$$\tau_A(u_1 \dots u_n) = A[u_1 u_2 \dots u_n] A[u_2 \dots u_n] A[u_3 \dots u_n] \dots A[u_{n-1} u_n] A[u_n].$$

The application τ_A , called the *transducer* defined by A , is in general not a morphism of monoids (except if $A[u_1 \dots u_n] = A[u_1]$ for all $u_1 \dots u_n \in \mathcal{M}_X$ of length ≥ 1) and the identity $\tau_A = \tau_B$ holds if and only if $A[U, W] = B[U, W]$ except perhaps for $(U, W) = (\emptyset, \emptyset)$.

Proposition 19.1. *A length-preserving function $\tau : \mathcal{M}_X \longrightarrow \mathcal{M}_Y$ is a transducer if and only if $\tau(uU)$ is of the form $*\tau(U)$ for all $u \in X$ and $U \in \mathcal{M}_X$.*

Corollary 19.2. *The composition $\tau_B \circ \tau_A : \mathcal{M}_X \longrightarrow \mathcal{M}_Z$ of two transducers $\tau_A : \mathcal{M}_X \longrightarrow \mathcal{M}_Y, \tau_B : \mathcal{M}_Y \longrightarrow \mathcal{M}_Z$ is a transducer.*

Proof of Proposition 19.1 Given a length-preserving function $\tau : \mathcal{M}_X \longrightarrow \mathcal{M}_Y$, set $A[\emptyset] = \alpha$ with $\alpha \in Y$ arbitrary and $A[uU] = y$ if $\tau(uU) = y\tau(U)$. The transducer τ_A associated to A coincides then with the function τ if and only if τ satisfies the conditions of Proposition 19.1. \square

The easy proof of Corollary 19.2 is left to the reader.

Corollary 19.2 allows to define the *category of transducers* by considering free monoids \mathcal{M}_p on $\{0, 1, \dots, p-1\}$ as objects with arrows from \mathcal{M}_p to \mathcal{M}_q given by transducers $\tau_A : \mathcal{M}_p \longrightarrow \mathcal{M}_q$ associated to $A \in \{0, \dots, q-1\}^{\mathcal{M}_p}$. The set of all transducers from \mathcal{M}_p to \mathcal{M}_q is in bijection with the set of functions $\{0, \dots, q-1\}^{\mathcal{M}_p \setminus \{\emptyset\}}$.

Given a transducer $\tau : \mathcal{M}_p \longrightarrow \mathcal{M}_q$, we define the *transducer-matrix* $M_\tau \in \mathbf{K}^{\mathcal{M}_q \times \mathcal{M}_p}$ of τ by $M_\tau[U, W] = 1$ if $\tau(W) = U$ and $M_\tau[U, W] = 0$ otherwise. The set of transducer-matrices is contained in $\mathbf{K}^{\mathcal{M}}$ for any field \mathbf{K} . Since $M_{\tau \circ \tau'} = M_\tau M_{\tau'}$, the application $\tau \longmapsto M_\tau$ which associates to a transducer its transducer-matrix is a faithful functor from the category of transducers into the category $\mathbf{K}^{\mathcal{M}}$.

A transducer $\tau_A \in \mathcal{M}_q^{\mathcal{M}_p}$ is *finite-state* if $A \in \{0, \dots, q-1\}^{\mathcal{M}_p}$ is automatic.

Proposition 19.3. *A transducer-matrix $M_\tau \in \mathbf{K}^{\mathcal{M}_q \times \mathcal{M}_p}$ is a recurrence matrix if and only if τ is a finite-state transducer.*

Corollary 19.4. *Finite-state transducers form a subcategory in the category of transducers.*

Corollary 19.5. *The category of finite-state transducers can be realized as a subcategory of the category $\text{Rec}(\mathbf{K})$ of recurrence matrices over any field \mathbf{K} .*

Proof or Proposition 19.3 Consider a transducer $\tau = \tau_A : \mathcal{M}_p \longrightarrow \mathcal{M}_q$ associated to $A \in \{0, \dots, q-1\}^{\mathcal{M}_p} \subset \mathbb{Q}^{\mathcal{M}_p}$. For $T \in \mathcal{M}_p$, we denote by τ_T the finite state transducer and by M_T the transducer matrix associated to the function $\rho(T)A$ (defined in the usual way by $(\rho(T)A)[W] = A[WT]$). We have then

$$\tau(WT) = \tau_T(W)\tau(T)$$

where $\tau = \tau_\emptyset$. Moreover, the formula

$$(\rho(s, t)M_T)[U, W] = M_T[Us, Wt] = M[U s \tau(T), W t T]$$

shows the identities $\rho(s, t)M_T = M_{tT}$ if $\tau(tT) = s\tau(T)$ (or, equivalently, if $A[tT] = s$) and $\rho(s, t)M_T = 0$ otherwise, for all $(s, t) \in \mathcal{M}_{q \times p}^1$.

If A is automatic, then $A \in \text{Rec}_p(\mathbb{Q})$ by Proposition 18.2. Since A takes all its values in the finite set $\{0, \dots, q-1\}$, Proposition 18.1 implies finiteness of its recursive set-closure $\rho(\mathcal{M}_p)A = \{\rho(T)A \mid T \in \mathcal{M}_p\}$. This proves finiteness of the set $\mathcal{S} = \{M_T \mid T \in \mathcal{M}_p\} \subset \mathbf{K}^{\mathcal{M}_{q \times p}}$. The recursive set-closure $\rho(\mathcal{M}_{q \times p})M \subset \{0 \cup \mathcal{S}\}$ (with $0 \in \mathbf{K}^{\mathcal{M}_{q \times p}}$ denoting the zero recurrence matrix) is thus also finite and $M = M_\emptyset = M_\tau$ is a recurrence matrix.

In the other direction, we consider a transducer matrix $M = M_\tau \in \text{Rec}_{q \times p}(\mathbf{K})$ which is a recurrence matrix of a transducer $\tau = \tau_A : \mathcal{M}_p \longrightarrow \mathcal{M}_q$ associated to $A \in \{0, \dots, q-1\}^{\mathcal{M}_p}$. The recurrence matrix $\rho(S, T)M$ is then either 0 or the transducer matrix of the transducer $\tau_{\rho(T)A}$ associated to $\rho(T)A$. Finiteness of $\rho(\mathcal{M}_{q \times p})M$ (which follows from Proposition 18.1) implies finiteness of the recursive set-closure $\rho(\mathcal{M}_p)A$ and shows that A is a recurrence matrix with finite shift-monoid. The function A is thus automatic by Proposition 18.2 and $\tau = \tau_A$ is an automatic transducer. \square

Proposition 19.6. *A recurrence matrix $M \in \mathbf{K}^{\mathcal{M}_{q \times p}}$ is a transducer-matrix of a finite-state transducer if and only if it can be given by a presentation with initial values $(M = M_1, M_2, \dots, M_d)[\emptyset, \emptyset] = (1, 1, \dots, 1)$ and shift matrices $\rho(s, t) \in \{0, 1\}^{d \times d}$ with coefficients in $\{0, 1\}$ such that all column-sums of the p matrices $\sum_{s=0}^{q-1} \rho(s, t)$ (with fixed $t \in \{0, \dots, p-1\}$) are 1.*

Moreover, all finite-state transducers τ_1, \dots, τ_d defined by d transducer-matrices M_1, \dots, M_d as above are surjective if and only if all column-sums of the q matrices $\sum_{t=0}^{p-1} \rho(s, t)$ are strictly positive.

Remark 19.7. *There are $(qd)^{pd}$ presentations of complexity d defining d finite-state transducer-matrices in $\text{Rec}_{q \times p}$ as in Proposition 19.6. Indeed, these presentations are in bijection with matrices $\{0, 1\}^{qd \times pd}$ having all column-sums equal to 1 as can be seen by contemplating the matrix*

$$\begin{pmatrix} \rho(0, 0) & \rho(0, 1) & \dots & \rho(0, p-1) \\ \rho(1, 0) & \rho(1, 1) & \dots & \rho(1, p-1) \\ \vdots & & & \vdots \\ \rho(q-1, 0) & \rho(q-1, 1) & \dots & \rho(q-1, p-1) \end{pmatrix}$$

obtained by gluing all qp shift-matrices $\rho(s, t)$ into a $qd \times pd$ matrix. Since all pd columns of such a matrix can be chosen independently with qd possibilities, there are $(qd)^{pd}$ such matrices.

Proof of Proposition 19.6 Given a transducer-matrix $M \in \text{Rec}_{q \times p}(\mathbf{K})$ associated to $A \in \text{Rec}_p(\mathbb{Q})$, we have $\rho(s, t)M = 0$ if $A[t] \neq s$ and $\rho(s, t)M$ is the transducer matrix M_t associated to $\rho(t)A \in \text{Rec}_p(\mathbb{Q})$ if $A[t] = s$. The first part of Proposition 19.6 follows by considering the presentation defined by all distinct transducer matrices in $\{M_T \mid T \in \mathcal{M}_p\} = \rho(\mathcal{M}_{q \times p})M \setminus \{0\}$ where M_T is the transducer matrix associated to $\rho(T)A \in \text{Rec}_p(\mathbb{Q})$.

By induction on the length l of words in \mathcal{M}_p^l , the transducers associated to the transducer matrices M_T are all surjective if for each $s \in \{0, \dots, q-1\}$ there exists an integer $t = t_T, 0 \leq t < p$ such that $A[tT] = s$. The row corresponding to a transducer matrix M_T of the shift-matrix $\rho(s, t_T) \in \{0, A\}^{d \times d}$ contains thus at least one non-zero coefficient and this proves the last part of Proposition 19.6. \square

Remark 19.8. An element $A \in \mathbf{K}^{\mathcal{M}_{q \times p}}$ can be described by its initial value $A[\emptyset, \emptyset]$ and the $q \times p$ matrix

$$\begin{pmatrix} \rho(0, 0)A & \dots & \rho(0, p-1)A \\ \vdots & & \vdots \\ \rho(q-1, 0)A & \dots & \rho(q-1, p-1)A \end{pmatrix}$$

with coefficients in $\mathbf{K}^{\mathcal{M}_{q \times p}}$ encoding the action of the generators of the shift-monoid on A .

This description is particularly useful for transducer-matrices: Every row of the above matrix contains exactly one non-zero element which is again a transducer-matrix.

As an example we consider the two finite-state transducer-matrices $M_1, M_2 \in \{0, 1\}^{\mathcal{M}_{2 \times 3}}$ for which the above matrix is given by

$$\begin{pmatrix} M_1 & M_2 & 0 \\ 0 & 0 & M_1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & M_2 & M_1 \\ M_2 & 0 & 0 \end{pmatrix}.$$

They span a recursively closed subspace in $\text{Rec}_{2 \times 3}(\mathbb{Q})$ and are presented by $(M_1, M_2)[\emptyset, \emptyset] = (1, 1)$ and

$$\begin{aligned} \rho(0, 0) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \rho(0, 1) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \rho(0, 2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ \rho(1, 0) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \rho(1, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \rho(1, 2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Remark 19.9. Many authors define the transducer of a function $A \in Y^{\mathcal{M}_X}$ as the length-preserving application $\tilde{\tau}_A : \mathcal{M}_X \rightarrow \mathcal{M}_Y$ defined by $\tau_A(\emptyset) = \emptyset$ and

$$\tilde{\tau}_A(x_1 \dots x_n) = A[x_1]A[x_1x_2]A[x_1x_2x_3] \dots A[x_1 \dots x_n].$$

This definition yields an equivalent theory since we have $\tilde{\tau}_A = \iota \circ \tau_A \circ \iota$ with ι denoting the palindromic antimorphism $w_1 \dots w_n \mapsto w_n \dots w_1$ of \mathcal{M}_X or \mathcal{M}_Y . The corresponding transducer matrices M_{τ_A} and $M_{\tilde{\tau}_A}$ (for $A \in (\mathcal{M}_q^1)^{\mathcal{M}_p}$) are thus related by $M_{\tilde{\tau}} = P_q^\iota M_\tau P_p^\iota$ where $P_n^\iota \in \mathbf{K}^{\mathcal{M}_n \times \mathcal{M}_n}$ is the palindromic element of section 17. By Proposition 17.1, the “transducer matrix” of $\tilde{\tau}_A$ is thus a recurrence matrix if and only if the transducer matrix of τ_A is a recurrence matrix.

19.1 Transducers and regular rooted planar trees

Definition 19.10. *The p -regular rooted planar tree is the infinite tree \mathbf{T}_p with vertices in bijection with \mathcal{M}_p and directed (or oriented) edges labelled by $s \in \{0, \dots, p-1\}$ joining a vertex $S \in \mathcal{M}_p$ to the vertex sS . The vertex \emptyset corresponds to the root.*

Geometrically, a vertex $s_1 \dots s_l \in \mathcal{M}_p$ corresponds to the endpoint of the continuous path of length l starting at the root \emptyset which is obtained by concatenating oriented edges with labels $s_l, s_{l-1}, \dots, s_2, s_1$.

A transducer $\tau : \mathcal{M}_p \rightarrow \mathcal{M}_q$ induces an application $\tau : \mathbf{T}_p \rightarrow \mathbf{T}_q$ by its action on vertices. In particular, a bijective transducer $\tau : \mathcal{M}_p \rightarrow \mathcal{M}_p$ corresponds to an automorphism of the p -regular rooted tree \mathbf{T}_p .

20 Automatic groups

Proposition 20.1. *The inverse of a bijective transducer $\tau : \mathcal{M}_p \rightarrow \mathcal{M}_p$ is a transducer.*

Moreover, if a bijective transducer $\tau : \mathcal{M}_p \rightarrow \mathcal{M}_p$ is finite-state, then its inverse transducer τ^{-1} is also finite-state.

Proof Consider a presentation M_1, \dots, M_d as in Proposition 19.6 with shift-matrices $\rho(s, t) \in \{0, 1\}^{d \times d}$ of the associated transducer-matrix $M = M_\tau = M_1$. By Proposition 19.6, the sum of all coefficients of the matrix $\sum_{0 \leq s, t < p} \rho(s, t) \in \mathbb{Z}^{d \times d}$ equals pd . The second part of Proposition 19.6 shows thus that for fixed $s, 0 \leq s < p$, each matrix $\sum_{t=0}^p \rho(s, t)$ has all row-sums equal to 1. Proposition 19.6 can thus be applied to the transposed matrices M_1^t, \dots, M_d^t (given by the presentation $(M_1^t, \dots, M_d^t)[\emptyset, \emptyset] = (1, \dots, 1)$) and shift-matrices $\tilde{\rho}(s, t)$ defined by $\tilde{\rho}(s, t) = \rho(t, s)$ and shows that $M_1^t = M_1^{-1}, \dots, M_d^t = M_d^{-1}$ are transducer-matrices.

The second part is obvious. \square

The set of all bijective finite-state transducers from \mathcal{M}_p to \mathcal{M}_p forms a group which is isomorphic to the subgroup $\mathcal{TD}_p \subset \text{GL}_{p\text{-rec}}(\mathbf{K})$ formed by all invertible transducer-matrices in $\text{Rec}_{p \times p}(\mathbf{K})$. We call a subgroup of \mathcal{TD}_p a p -automatic group or automatic group for short. (The reader should be warned that the terminology “automatic” is also used for a different kind of

discrete groups, our automatic groups are automatic in the Russian sense. Sometimes, they are also called branched groups.)

Remark 20.2. *There are $(d^p p!)^d$ presentations of complexity d defining d bijective finite-state transducer-matrices $M_1, \dots, M_d \in \mathbb{Q}^{\mathcal{M}_{p \times p}}$ which have the properties of Proposition 19.6. Indeed, using the description of Remark 19.8, such a matrix M_i can be encoded by a “coloured” permutation matrix of order $p \times p$ with all coefficients 1 “coloured” (independently) by a colour in the set $\{M_1, \dots, M_d\}$. For each such matrix M_i , there are thus $p! d^p$ possibilities.*

20.1 An example: The first Grigorchuk group Γ

This fascinating group appeared first in [6]. A few interesting properties (see [4], page 211) of Γ are:

- Γ has no faithful finite-dimensional representation.
- Γ is not finitely presented.
- Γ is of intermediate growth.
- Γ contains every finite 2–group.

The group Γ is the subgroup generated by four bijections a, b, c, d of the set $\mathcal{S} = \bigcup_{l=0}^{\infty} \{\pm 1\}^l$. Since a, b, c, d preserve the subsets $\mathcal{S}_l = \{\pm 1\}^l$ we denote by a_l, b_l, c_l, d_l the restricted bijections induced by a, b, c, d on the finite subset \mathcal{S}_l . For $l = 0$, we have $\mathcal{S}_0 = \emptyset$ with trivial action of the permutations a_0, b_0, c_0, d_0 . For $l > 0$ we write $(\epsilon, x) = (y_1, y_2, \dots, y_l)$ with $\epsilon = y_1 \in \{\pm 1\}$ and $x = (y_2, y_3, \dots, y_l) \in \{\pm 1\}^{l-1}$. The action of a_l, b_l, c_l, d_l is then recursively defined by

$$\begin{aligned} a_l(1, x) &= (-1, x) & a_l(-1, x) &= (1, x) \\ b_l(1, x) &= (1, a_{l-1}(x)) & b_l(-1, x) &= (-1, c_{l-1}(x)) \\ c_l(1, x) &= (1, a_{l-1}(x)) & c_l(-1, x) &= (-1, d_{l-1}(x)) \\ d_l(1, x) &= (1, x) & d_l(-1, x) &= (-1, b_{l-1}(x)) \end{aligned}$$

(cf VIII.B, pages 217-218 of [4]).

The set \mathcal{S} corresponds to the vertices of the 2–regular rooted planar tree \mathbf{T}_2 and the four bijections a, b, c, d act as automorphisms on \mathbf{T}_2 . They correspond thus to transducers $\tau_a, \tau_b, \tau_c, \tau_d$. The associated transducer-matrices M_a, M_b, M_c, M_d can be encoded using Remark 19.8 by the four matrices

$$\begin{aligned} M_a &: \begin{pmatrix} 0 & \text{1d} \\ \text{1d} & 0 \end{pmatrix}, M_b : \begin{pmatrix} M_a & 0 \\ 0 & M_c \end{pmatrix}, \\ M_c &: \begin{pmatrix} M_a & 0 \\ 0 & M_d \end{pmatrix}, M_d : \begin{pmatrix} \text{Id} & 0 \\ 0 & M_b \end{pmatrix} \end{aligned}$$

with coefficients in $\mathbb{Q}^{\mathcal{M}_{2 \times 2}}$.

A presentation can be given by $(\text{Id}, M_a, M_b, M_c, M_d)[\emptyset, \emptyset] = (1, 1, 1, 1, 1)$ and shift-matrices

$$\rho(0,0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho(0,1) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\rho(1,0) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho(1,1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Remark 20.3. *The fact that the four generators a, b, c, d of the Grigorchuk group are of order 2 is equivalent to the identity $\rho(0,1) = \rho(1,0)$ of the shift-matrices described above.*

21 Asymptotic problems in $\text{Rec}_{p \times p}(\mathbf{K})$

This section introduces generating series counting dimensions associated to recurrence matrices. It contains mainly definitions and (open) problems.

Given a recurrence matrix $A \in \text{Rec}_{p \times p}(\mathbf{K})$ one can consider the generating series

$$1 + \dim(\overline{A}^{rec})t + \dots = \sum_{n=0}^{\infty} \dim(\overline{A}^{n rec})t^n$$

associated to the complexities of its powers. One has the inequality $\dim(\overline{A}^{n rec}) \leq (\dim(\overline{A}^{rec}))^n$ which implies convergency of the series for $\{t \in \mathbb{C} \mid |t| < 1/\dim(\overline{A}^{rec})\}$.

What can be said about the analytic properties of this generating series? In a few easy cases (nilpotent or of complexity 1 for instance), it is meromorphic in \mathbb{C} .

If $A \in \text{GL}_{p-rec}(\mathbf{K})$ is invertible in $\text{Rec}_{p \times p}(\mathbf{K})$, one can of course also consider the formal sum

$$\sum_{n=-\infty}^{\infty} \dim(\overline{A}^{n rec})t^n$$

encoding the complexities of all integral powers of A .

Remark 21.1. *There are many variations for the generating function(s) defined above. One can replace the complexity by the birecursive complexity*

of powers. One can also consider the generating function encoding the the dimension α_n of the recursive closure Id, A^1, \dots, A^n containing all powers $A^0 = Id, A^1, \dots, A^n$ etc.

Similarly, given a monoid or group generated by a finite set $\mathcal{G} \subset \text{Rec}_{p \times p}(\mathbf{K})$ of recurrence matrices, one can consider the generating series $\sum_{n=0}^{\infty} \dim(\mathcal{A}_n) t^n$ encoding the complexities of the recursive closures \mathcal{A}_n associated to all products of (at most) n elements in \mathcal{G} .

A different kind of generating series is given by considering for $A \in \text{Rec}_{p \times p}(\mathbb{C})$ the series

$$1 + \|A\|_{\infty}^{\infty} t + \dots = \sum_{n=0}^{\infty} \|A^n\|_{\infty}^{\infty} t^n.$$

The inequality $\|A^n\|_{\infty}^{\infty} \leq p^{2n-1} (\|A\|_{\infty}^{\infty})^n$ (which can be proven by considering a recursive matrix whose coefficients $A[U, W]$ depend only on the length l of $(U, W) \in \mathcal{M}_{p \times p}^l$) shows again convergency for $t \in \mathbb{C}$ small enough.

22 A generalization

Consider a ring \mathbf{R} of functions $\mathbb{N} \rightarrow \mathbf{K}$ with values in a commutative field \mathbf{K} . An element $A \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ is an \mathbf{R} -recurrence matrix (or simply a recurrence matrix if the underlying function ring \mathbf{R} is obvious) if there exists a finite number of elements $A_1 = A, A_2, \dots, A_a \in \mathbf{K}^{\mathcal{M}_{p \times q}}$ and pq shift-matrices $\rho_A(s, t) \in \mathbf{R}^{a \times a}$ with coefficients $\rho_A(s, t)_{j, k} \in \mathbf{R}$ (for $1 \leq j, k \leq a$) in the function ring \mathbf{R} such that

$$(\rho(s, t) A_k)[\mathcal{M}_{p \times q}^{l+1}] = \sum_{j=1}^a \rho_A(s, t)_{j, k}(l) A_j[\mathcal{M}_{p \times q}^l], \quad 1 \leq k \leq a,$$

for all $0 \leq s < p, 0 \leq t < q$.

We denote by $\mathbf{R} - \text{Rec}_{p \times q}(\mathbf{K})$ the vector-space of all \mathbf{R} -recurrence matrices in $\mathbf{K}^{\mathcal{M}_{p \times q}}$.

Proposition 3.5 and its proof can easily be modified in order to deal with \mathbf{R} -recurrence matrices and we get thus a category $\mathbf{R} - \text{Rec}(\mathbf{K})$ with \mathbf{R} -recurrence matrices as morphisms. Moreover, the vector-space $\mathbf{R} - \text{Rec}_{p \times q}(\mathbf{K})$ is invariant under the action of the shift-monoid if the function ring \mathbf{R} is preserved by the translation $x \mapsto x + 1$ of the argument (ie. if $\alpha \in \mathbf{R}$ implies that the function $x \mapsto \alpha(x + 1)$ is also in \mathbf{R}).

A few analogies and differences between \mathbf{R} -recurrence matrices and (ordinary) recurrence matrices are:

- Elements of $\mathbf{R} - \text{Rec}_{p \times q}(\mathbf{K})$ have finite descriptions involving a finite number of elements in the function ring \mathbf{R} .

- The notion of a presentation is in general more involved: Every presentation of an ordinary recurrence matrix A contains a basis of \overline{A}^{rec} and can thus be used to construct a minimal presentation (defining a basis of \overline{A}^{rec}). This is no longer true in general for \mathbf{R} -recurrence matrices since \mathbf{R} might contain non-zero elements which have no multiplicative inverse. One has thus to work with (not necessarily free) \mathbf{R} -modules when dealing with presentations.
- Proposition 4.1, slightly modified, remains valid. We have

$$\begin{pmatrix} A_1[U, W] \\ \vdots \\ A_d[U, W] \end{pmatrix} = \rho_{\mathcal{A}}(u_n, w_n)^t (n-1) \cdots \rho_{\mathcal{A}}(u_1, w_1)^t (0) \begin{pmatrix} A_1[\emptyset, \emptyset] \\ \vdots \\ A_d[\emptyset, \emptyset] \end{pmatrix}$$

where $(U, W) = (u_1 \dots u_n, w_1 \dots w_n) \in \mathcal{M}_{p \times q}^n$. The matrices $\rho_{\mathcal{A}}(s, t) \in \mathbf{R}^{d \times d}$ are the obvious shift-matrices of the subspace $\mathcal{A} = \sum \mathbf{K}A_j$ with respect to the generators A_1, \dots, A_d .

- An analogue of the saturation level is no longer available in general. This makes automated computations impossible in the general case.

Remark 22.1. *Non-existence of a saturation level does not necessarily imply the impossibility of proving a few identities among \mathbf{R} -recurrence matrices, since such an identity can perhaps be proven by induction on the word-length l , after restriction to $\mathcal{M}_{p \times q}^{\leq l}$.*

22.1 Examples

The case where the function ring \mathbf{R} consists of all constant functions $\mathbb{N} \rightarrow \mathbf{K}$ corresponds of course to the case of (ordinary) recurrence matrices studied in this paper.

The case where the function ring \mathbf{R} consists of ultimately periodic functions $\mathbb{N} \rightarrow \mathbf{K}$ yields again only ordinary recurrence matrices.

The first interesting new case is given by considering the ring $\mathbf{R} = \mathbf{K}[x]$ of all polynomial functions. It would be interesting to have finiteness results in this case: Given $A, B, C \in \mathbf{K}[x] - \text{Rec}_{p \times q}$ defined by presentations of complexity a, b, c (where the complexity is the minimal number of elements appearing in a finite presentation) with shift-matrices involving only polynomials of degree $\leq \alpha, \beta, \gamma$, can one give a bound $N_+ = N_+(a, b, c, \alpha, \beta, \gamma, p, q)$ such that the equality $(A + B)[\mathcal{M}_{p \times q}^{\leq N_+}] = C[\mathcal{M}_{p \times q}^{\leq N_+}]$ ensures the equality $A + B = C$ in $\mathbf{K}[x] - \text{Rec}_{p \times q}$? Similarly, given $A \in \mathbf{K}[x] - \text{Rec}_{p \times r}, B \in \mathbf{K}[x] - \text{Rec}_{r \times q}, C \in \mathbf{K}[x] - \text{Rec}_{p \times q}$ defined by presentations of complexity a, b, c with shift-matrices involving only polynomials of degree $\leq \alpha, \beta, \gamma$, can one give a bound $N_{\times} = N_{\times}(a, b, c, \alpha, \beta, \gamma, p, q, r)$ such that the equality $(AB)[\mathcal{M}_{p \times q}^{\leq N_{\times}}] = C[\mathcal{M}_{p \times q}^{\leq N_{\times}}]$ ensures the equality $AB = C$ in $\mathbf{K}[x] - \text{Rec}_{p \times q}$?

Are there natural and “interesting” examples of $\mathbf{K}[x]$ –recurrence matrices which are not (ordinary) recurrence matrices?

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