

On the filtering problem for stationary random \mathbb{Z}^2 -fields

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Abstract

It is shown that whenever a stationary random field $(Z_{n,m})_{n,m \in \mathbb{Z}}$ is given by a Borel function $f : \mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}$ of two stationary processes $(X_n)_{n \in \mathbb{Z}}$ and $(Y_m)_{m \in \mathbb{Z}}$, i.e. $(Z_{n,m}) = (f((X_{n+k})_{k \in \mathbb{Z}}, (Y_{m+\ell})_{\ell \in \mathbb{Z}}))$, then under some mild first coordinate univalence assumption on f , the process $(X_n)_{n \in \mathbb{Z}}$ is measurable with respect to $(Z_{n,m})_{n,m \in \mathbb{Z}}$ whenever the process $(Y_m)_{m \in \mathbb{Z}}$ is ergodic.

The notion of universal filtering property of an ergodic stationary process is introduced and then using ergodic theory methods it is shown that an ergodic stationary process has this property if and only if the centralizer of the dynamical system canonically associated to the process does not contain a non-trivial compact subgroup.

Introduction

In the seminal paper [7], H. Furstenberg introduced the notion of *absolute independence* (or *disjointness*) of two stationary random processes. Among other purposes this notion was used to study the following filtering problem: given two stationary processes $(X_n)_{n \in \mathbb{Z}}$ and $(Y_n)_{n \in \mathbb{Z}}$ and assuming stationarity of their joint distribution $(X_n, Y_n)_{n \in \mathbb{Z}}$, is it true that the process $(X_n)_{n \in \mathbb{Z}}$ is measurable with respect to the (stationary) process $(Z_n)_{n \in \mathbb{Z}}$, where $Z_n = X_n + Y_n$? If the answer is “yes” then we can interpret this result as a possibility to filter, from the output process $(Z_n)_{n \in \mathbb{Z}}$ the signal $(X_n)_{n \in \mathbb{Z}}$ that has been transmitted through a channel with noise $(Y_n)_{n \in \mathbb{Z}}$. Thus the filtering problem is to represent the process $(X_n)_{n \in \mathbb{Z}}$ as a function (“algorithm”) of the received process $(Z_n)_{n \in \mathbb{Z}}$. Furstenberg has shown that the answer to the filtering problem is positive provided that the signal and the noise are absolutely independent L^1 -processes. It is not known whether the absolute independence of the processes,

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without integrability assumption, is enough to give a positive answer to the filtering problem.

Furstenberg noticed that the answer is positive without integrability assumption however under much stronger assumption, of so called *double disjointness* of the two processes. The idea of the double disjointness has then been exploited in [9] in which a more general version of the filtering problem:

$$(1) \quad Z_n = f(X_n, Y_n), \quad n \in \mathbb{Z}$$

($f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a Borel map) was considered. It turns out that under the double disjointness condition and some mild assumption of pointwise distributional univalence property of f one can filter the process $(X_n)_{n \in \mathbb{Z}}$ from $(Z_n)_{n \in \mathbb{Z}}$.

Being unable to solve the original Furstenberg's filtering problem, in this note we deal with a \mathbb{Z}^2 -variant of it. Assume that $\underline{X} := (X_n)_{n \in \mathbb{Z}}$ and $\underline{Y} := (Y_n)_{n \in \mathbb{Z}}$ are stationary independent sources. Suppose that $f : \mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}$ is Borel and let us define a random (stationary) \mathbb{Z}^2 -field

$$(2) \quad \underline{Z} = (Z_{n,m})_{n,m \in \mathbb{Z}} := (f((X_{n+k})_{k \in \mathbb{Z}}, (Y_{m+\ell})_{\ell \in \mathbb{Z}}))_{n,m \in \mathbb{Z}}.$$

The problem is now to understand if we can filter the process \underline{X} from the random \mathbb{Z}^2 -field \underline{Z} . It is clear that a certain type of univalence property of the function f will be necessary to solve this problem.

We pose this filtering problem under two forms:

1. Under what conditions on \underline{Y} and f is it possible to filter \underline{X} from \underline{Z} ?
2. What property of the process \underline{X} could ensure that the filtering is possible whatever is the process \underline{Y} (when the map f satisfies a basic necessary pointwise univalence condition)?

We will see in Section 1 that the answer to the first question is positive as soon as the map f satisfies a mild pointwise distributional univalence condition and the process \underline{Y} is ergodic. The condition on f is similar to the one in [9], but in that paper only functions depending on the zero coordinate are considered.

In the remaining part of the article, we study the second question. We define a notion of *universal filtered property* (UFP) for a measure preserving dynamical system. This UFP seems to be the right condition to be imposed on the dynamical system canonically associated to the process \underline{X} in order to have a positive answer to our universal filtering problem. We obtain an ergodic characterization: an ergodic measure preserving dynamical system has the UFP if and only if it does not have any non-trivial compact subgroup in its centralizer or, which is the same, it cannot be represented as a compact group extension of any of its proper factors.

From pure ergodic theory point of view we describe all invariant sub- σ -algebras of so called product \mathbb{Z}^2 -actions (considered e.g. in [4], [5]). Slightly surprisingly a full description of these factors is obtained via Veech's theorem

from [17] (see e.g. [11], [12], [14] for other situations in which factors of some systems were described with a use of Veech's theorem). The precise description of invariant sub- σ -algebras of product \mathbb{Z}^2 -actions is given in Proposition 5.

1 The simple solution of our first filtering problem

The random variables we consider are all defined on a standard probability Borel space $(\Omega, \mathcal{T}, \mathbb{P})$.

Proposition 1 *Let X be a random variable with values in a standard Borel space S , and $(Y_m)_{m \in \mathbb{Z}}$ be a stationary ergodic real process independent of X . Denote by ν the law of the process $(Y_m)_{m \in \mathbb{Z}}$.*

Let F be a Borel real function defined on $S \times \mathbb{R}^{\mathbb{Z}}$. Assume that there exists a Borel subset B of S such that $X(\omega) \in B$ for almost all ω and such that for all $x, x' \in B$

$$x \neq x' \implies F(x, \cdot)_*(\nu) \neq F(x', \cdot)_*(\nu) .$$

Then the random variable X is measurable with respect to the (complete) σ -algebra generated by the process $(F(X, (Y_{m+\ell})_{\ell \in \mathbb{Z}}))_{m \in \mathbb{Z}}$.

A similar result can be stated when the random variable X is itself a random process $(X_n)_{n \in \mathbb{Z}}$, giving an answer to the first question we asked in Introduction.

Proposition 2 *Let $\underline{X} = (X_n)_{n \in \mathbb{Z}}$ be a real random process, and $(Y_m)_{m \in \mathbb{Z}}$ be a stationary ergodic real process independent of \underline{X} . Denote by ν the law of the process $(Y_m)_{m \in \mathbb{Z}}$.*

Let F be a Borel real function defined on $\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}$. Assume that there exists a Borel subset B of $\mathbb{R}^{\mathbb{Z}}$ such that $\underline{X}(\omega) \in B$ for almost all ω and such that for all $\underline{x}, \underline{x}' \in B$

$$\underline{x} \neq \underline{x}' \implies \exists n \in \mathbb{Z}, F((x_{n+k})_{k \in \mathbb{Z}}, \cdot)_*(\nu) \neq F((x'_{n+k})_{k \in \mathbb{Z}}, \cdot)_*(\nu) .$$

Then the process \underline{X} is measurable with respect to the (complete) σ -algebra generated by the random field $(F((X_{n+k})_{k \in \mathbb{Z}}, (Y_{m+\ell})_{\ell \in \mathbb{Z}}))_{n, m \in \mathbb{Z}}$.

Establishing a link with the original filtering problem, we note here that the condition imposed on F in this proposition is satisfied by the map $F : ((x_n), (y_m)) \mapsto x_0 + y_0$.

In the proofs of these propositions, we consider random probability measures. Let us denote by \mathcal{P} the set of Borel probabilities on the real line, equipped with the vague topology. It is a standard Borel space. The measurability of a map taken its values in \mathcal{P} can be characterized in various ways. If (Ω, \mathcal{C}) is a measurable space and ρ is a map from Ω into \mathcal{P} , there is equivalence between the following properties:

- (i) the map ρ is measurable;
- (ii) for all $a \in \mathbb{R}$, $\rho((-\infty, a])$ is a measurable function on Ω ;
- (iii) for all $t \in \mathbb{R}$, $\hat{\rho}(t) := \int e^{itx} d\rho(x)$ is a measurable function on Ω ;
- (iv) for all $t \in \mathbb{Q}$, $\hat{\rho}(t) := \int e^{itx} d\rho(x)$ is a measurable function on Ω .

Indeed, we have (i) \Rightarrow (ii) by approximation of characteristic functions of intervals by continuous functions, (ii) \Rightarrow (i) by approximation of continuous functions by linear combinations of characteristic functions of intervals, (iii) \Rightarrow (ii) by the Fourier inversion formula for measures, and (iii) \Leftrightarrow (iv) by continuity of the Fourier transform $\hat{\rho}$.

Proof of Proposition 1

This proof is based on two claims.

Claim 1. The random measure $F(X, \cdot)_*(\nu)$ is measurable with respect to the (complete) σ -algebra generated by the process $(F(X, (Y_{m+\ell})_{\ell \in \mathbb{Z}}))_{m \in \mathbb{Z}}$.

Claim 2. If a Borel function g , from \mathbb{R} into itself, is one to one on a Borel subset B containing the range of a real random variable W , then W is measurable with respect to the σ -algebra generated by $g(W)$.

The stationarity of the process (Y_m) and its independence from X , imply that, for each Borel function b on $S \times \mathbb{R}^{\mathbb{Z}}$, the process $(b(X, (Y_{m+\ell})_{\ell \in \mathbb{Z}}))_{m \in \mathbb{Z}}$ is stationary. Hence the ergodic theorem ensures that for each bounded Borel function b on $S \times \mathbb{R}^{\mathbb{Z}}$, the sequence

$$(3) \quad \left(\frac{1}{M} \sum_{m=0}^{M-1} b(X, (Y_{m+\ell})_{\ell \in \mathbb{Z}}) \right)_{M > 0}$$

converges almost everywhere and in the mean when M goes to $+\infty$.

If the function b is of the form $b(x, \underline{y}) = b_1(x) \times b_2(\underline{y})$ then the limit of the sequence (3) is (almost everywhere) $\bar{b}_1(X) \times \int_{\mathbb{R}^{\mathbb{Z}}} b_2(\underline{y}) d\nu(\underline{y})$, since we have assumed that the process (Y_m) is ergodic.

By density of the linear combinations of such “decomposed” functions we deduce that, for all bounded measurable functions b , the almost everywhere limit of the sequence (3) is

$$\int_{\mathbb{R}^{\mathbb{Z}}} b(X, \underline{y}) d\nu(\underline{y}).$$

This can be applied to the functions b of the form

$$b(x, \underline{y}) = \exp(itF(x, \underline{y}))$$

and we obtain that, almost everywhere, for all rational numbers t ,

$$\lim_{M \rightarrow \infty} \frac{1}{M} \sum_{m=0}^{M-1} \exp(itF(X, (Y_{m+\ell})_{\ell \in \mathbb{Z}})) = \int_{\mathbb{R}^{\mathbb{Z}}} \exp(itF(X, \underline{y})) d\nu(\underline{y}) = \widehat{\rho_X}(t),$$

where $\rho_X := F(X, \cdot)_*(\nu)$. We see that the Fourier transform of the random probability ρ_X depends measurably on the process $(F(X, (Y_{m+\ell})_{\ell \in \mathbb{Z}}))_{m \in \mathbb{Z}}$, and this completes the proof of our first claim.

We recall the fact that a real 1-1, Borel map defined on a Borel subset B of a standard Borel space sends the Borel subsets of B into Borel subsets of \mathbb{R} (see the Measurable Image Theorem in Appendix). Hence, if g , W and B are given like in Claim 2 and if A is a Borel subset of \mathbb{R} , then

$$W^{-1}(A) = W^{-1}(A \cap B) = (g \circ W)^{-1}(g(A \cap B))$$

and (from the recalled fact) $g(A \cap B)$ is Borel. From that we see that the σ -algebra generated by W is included in the σ -algebra generated by $g(W)$. Hence the second claim follows.

The second claim still remains true for a map g defined from a standard Borel space into another one. In order to prove Proposition 1, we consider the map $g(x) = F(x, \cdot)_*(\nu)$ from S into \mathcal{P} . Starting from the assumption of Proposition 1, we can modify X on a zero measure set, in order to have its range included in B . Claim 2 tells us that X is measurable with respect to $g(X)$ and Claim 1 tells us that $g(X)$ is measurable with respect to process $(F(X, (Y_{m+\ell})_{\ell \in \mathbb{Z}}))_{m \in \mathbb{Z}}$. \square

Proof of Proposition 2

The proof of this proposition follows the same line as the preceding one.

From Claim 1, applied to the random variable \underline{X} instead of X , we deduce that for any $n \in \mathbb{Z}$, the random measure $F((X_{n+k})_{k \in \mathbb{Z}}, \cdot)_*(\nu)$ is measurable with respect to the process $(F((X_{n+k})_{k \in \mathbb{Z}}, (Y_{m+\ell})_{\ell \in \mathbb{Z}}))_{m \in \mathbb{Z}}$. This implies that the random sequence of measures $(F((X_{n+k})_{k \in \mathbb{Z}}, \cdot)_*(\nu))_{n \in \mathbb{Z}}$ is measurable with respect to the random field $(F((X_{n+k})_{k \in \mathbb{Z}}, (Y_{m+\ell})_{\ell \in \mathbb{Z}}))_{n, m \in \mathbb{Z}}$.

On the other hand we can apply a result similar to Claim 2, for the map g defined from $\mathbb{R}^{\mathbb{Z}}$ into $\mathcal{P}^{\mathbb{Z}}$ by

$$g(\underline{x}) = (F((x_{n+k})_{k \in \mathbb{Z}}, \cdot)_*(\nu))_{n \in \mathbb{Z}} ,$$

and from the univalence assumption of Proposition 2 we deduce that the process \underline{X} is measurable with respect to the σ -algebra generated by the random sequence of measures $(F((X_{n+k})_{k \in \mathbb{Z}}, \cdot)_*(\nu))_{n \in \mathbb{Z}}$. \square

2 Universally filtered automorphisms

We want now to study the filtering phenomenon as a property of the dynamical system associated to the stationary process \underline{X} . Recall that the classical canonical construction on the space of trajectories associates to the stationary process a probability measure preserving dynamical system $(\Omega, \mathcal{B}, \mu, T)$ and a measurable map h on Ω such that the process $(h \circ T^n)_{n \in \mathbb{Z}}$ has the same law as

\underline{X} and generates the σ -algebra \mathcal{B} . In terms of dynamical systems, our filtering problem becomes the following.

Given two invertible dynamical systems $(\Omega, \mathcal{B}, \mu, T)$ and $(\Omega', \mathcal{C}, \nu, S)$, and a $(\mathcal{B} \otimes \mathcal{C})$ -measurable map $F : \Omega \times \Omega' \rightarrow \mathbb{R}$ satisfying some univalence property, does the σ -algebra \mathcal{A} generated by the random field

$$(4) \quad ((\omega, \omega') \mapsto F(T^n \omega, S^m \omega'))_{n,m \in \mathbb{Z}}$$

contain the whole σ -algebra of the first coordinate, that is to say the algebra $\mathcal{B} \otimes \{\emptyset, \Omega'\}$?

Note that this algebra \mathcal{A} is the smallest σ -algebra of subsets of $\Omega \times \Omega'$ making the map F measurable and invariant under the \mathbb{Z}^2 -measure preserving action \underline{T} defined on $(\Omega \times \Omega', \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$ by

$$T_{n,m}(\omega, \omega') = (T^n \omega, S^m \omega') \text{ for each } (n, m) \in \mathbb{Z}^2.$$

This particular type of \mathbb{Z}^2 -action is called a *product* \mathbb{Z}^2 -action.

The univalence condition that we impose on the map F is the following.

Definition 1 Let $(\Omega, \mathcal{B}, \mu)$ and $(\Omega', \mathcal{C}, \nu)$ be two probability spaces, and T be an automorphism of the first one. Let F be a measurable map from $\Omega \times \Omega'$ into \mathbb{R} . This map is said to have the *T-pointwise univalence property* if there exists a set of full probability $\Omega_0 \subset \Omega$ such that, for any distinct $\omega_1, \omega_2 \in \Omega_0$, there exists $n \in \mathbb{Z}$ such that the maps $F(T^n \omega_1, \cdot)$ and $F(T^n \omega_2, \cdot)$ are not ν -almost everywhere equal.

This univalence condition is natural for our problem because if we consider the measurable equivalence relation \mathcal{R} on Ω defined by

$$\omega_1 \mathcal{R} \omega_2 \iff \forall n, F(T^n \omega_1, \cdot) = F(T^n \omega_2, \cdot) \nu - \text{a.e.},$$

then the σ -algebra generated by the random field (4) is contained in $(\mathcal{B}/\mathcal{R}) \otimes \mathcal{C}$.

Definition 2 Let T be an automorphism of the probability space $(\Omega, \mathcal{B}, \mu)$. We say that the dynamical system $(\Omega, \mathcal{B}, \mu, T)$ is *universally filtered* if the following holds: for any standard probability space $(\Omega', \mathcal{C}, \nu)$, for any measurable map $F : \Omega \times \Omega' \rightarrow \mathbb{R}$ with the T -pointwise univalence property and for any ergodic automorphism S of $(\Omega', \mathcal{C}, \nu)$, any σ -algebra invariant under the product \mathbb{Z}^2 -action generated by T and S , and making F measurable contains $\mathcal{B} \otimes \{\emptyset, \Omega'\}$ (modulo $\mu \otimes \nu$).

Recall that by the *centralizer* $C(T)$ of an automorphism T we mean the group of all automorphisms of $(\Omega, \mathcal{B}, \mu)$ that commute with T . Each member S of $C(T)$ determines a unitary operator $U_S : f \mapsto f \circ S$ on the space $L^2(\Omega, \mu)$. The set $\{U_S : S \in C(T)\}$ is closed in the strong topology of the group $U(L^2(\Omega, \mu))$ of unitary operators of L^2 . This gives to $C(T)$ a structure of a Polish group.

The following result completely characterizes universally filtered automorphisms.

Theorem 1 *An ergodic dynamical system $(\Omega, \mathcal{B}, \mu, T)$ is universally filtered if and only if the centralizer $C(T)$ of T has no nontrivial compact subgroup.*

We postpone the proof of this theorem to Section 4, because we need more ergodic theory tools. The “if” part of the proof of this theorem relies on a precise description of factors of product \mathbb{Z}^2 -action. This description, given in Proposition 5 is obtained via Veech’s theorem on graph joinings (see Section 3). For the “only if” part, we show that if $C(T)$ has a nontrivial compact subgroup then the property of filtering is not satisfied with $S = T$ and F appropriately chosen.

3 Factors of product \mathbb{Z}^2 -actions

Assume that T and S are automorphisms of standard probability spaces $(\Omega, \mathcal{B}, \mu)$ and $(\Omega', \mathcal{C}, \nu)$ respectively. The aim of this section is to give a precise description of factors of the product \mathbb{Z}^2 -action \underline{T} defined on $(\Omega \times \Omega', \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$ by

$$(5) \quad T_{n,m}(\omega, \omega') = (T^n \omega, S^m \omega') \text{ for each } (n, m) \in \mathbb{Z}^2.$$

Notice that \mathbb{Z}^2 -product action \underline{T} is ergodic if and only if T and S are ergodic. We intend to describe all sub- σ -algebras $\mathcal{A} \subset \mathcal{B} \otimes \mathcal{C}$ that are invariant under \underline{T} whenever \underline{T} is ergodic.

3.1 Markov operators and sub- σ -algebras

Let $(\Omega, \mathcal{B}, \mu)$ and $(\Omega', \mathcal{C}, \nu)$ be two standard probability Borel spaces. The formula

$$(6) \quad \int_{\Omega \times \Omega'} f \otimes g \, d\rho = \int_{\Omega'} \Phi_\rho(f) \cdot g \, d\nu$$

establishes a 1-1 correspondence between measures ρ defined on $(\Omega \times \Omega', \mathcal{B} \otimes \mathcal{C})$ whose marginals ρ_Ω on Ω and $\rho_{\Omega'}$ on Ω' are μ and ν respectively and the set of Markov (called also *doubly stochastic*) operators Φ from $L^2(\Omega, \mathcal{B}, \mu)$ to $L^2(\Omega', \mathcal{C}, \nu)$, *i.e.* of positive operators Φ satisfying $\Phi(1) = 1 = \Phi^*(1)$. Note that $\|\Phi\| \leq 1$, so each Markov operator is a contraction.

Assume for a while that $\Omega = \Omega'$ and that $\Phi = \Phi_\rho : L^2(\Omega, \mathcal{B}, \mu) \rightarrow L^2(\Omega, \mathcal{B}, \mu)$. We then define

$$\mathcal{B}_0(\Phi) = \mathcal{B}_0(\rho) = \{B \in \mathcal{B} : \rho((B \times \Omega) \Delta (\Omega \times B)) = 0\}$$

and

$$\mathcal{B}_1(\Phi) = \mathcal{B}_1(\rho) = \{B \in \mathcal{B} : (\exists B' \in \mathcal{B} : \rho((B \times \Omega) \Delta (\Omega \times B')) = 0)\}.$$

It is easy to check that both $\mathcal{B}_0(\Phi)$ and $\mathcal{B}_1(\Phi)$ are sub- σ -algebras of \mathcal{B} . It is well-known (see e.g. [10], [14]) that

$$(7) \quad \begin{cases} L^2(\mathcal{B}_0(\Phi)) = \{f \in L^2(\Omega, \mathcal{B}, \mu) : \Phi(f) = f\}, \\ L^2(\mathcal{B}_1(\Phi)) = \{f \in L^2(\Omega, \mathcal{B}, \mu) : \|\Phi(f)\| = \|f\|\}. \end{cases}$$

If $\Psi : L^2(\Omega', \mathcal{C}, \nu) \rightarrow L^2(\Omega', \mathcal{C}, \nu)$ is another Markov operator then $\Phi \otimes \Psi$ is a Markov operator of the tensor product of the two L^2 -spaces, hence, after a natural identification, it is a Markov operator from $L^2(\Omega \times \Omega', \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$ to itself.

Lemma 1 *Under the above assumptions, $\mathcal{B}_0(\Phi \otimes \Psi) \subset \mathcal{B}_1(\Phi) \otimes \mathcal{B}_1(\Psi)$.*

The proof of this lemma uses a description of invariant vectors for the tensor product of two contractions of a Hilbert space. This description is classical in the case of unitary operators. Its extension to contractions, obtained by the theory of unitary dilations, is presented in the appendix (Lemma 8).

Proof.

Let $f \in L^2(\mathcal{B}_0(\Phi \otimes \Psi))$. In view of (7), $(\Phi \otimes \Psi)f = f$ and therefore by Lemma 8 we obtain that

$$f = \sum_i g_i \otimes h_i,$$

where g_i and h_i are eigenfunctions for Φ and Ψ respectively corresponding to eigenvalues each being the inverse of the other (obviously of modulus 1), hence $f \in \mathcal{B}_1(\Phi) \otimes \mathcal{B}_1(\Psi)$. \square

We will also need the following well-known fact.

Lemma 2 *Let $(\Omega, \mathcal{B}, \mu)$ and $(\Omega', \mathcal{C}, \nu)$ be two standard probability Borel spaces. Let $\mathcal{A}_1^{(i)} \subset \mathcal{B}$, $\mathcal{A}_2^{(i)} \subset \mathcal{C}$ be sub- σ -algebras for arbitrary i in some index set I . Then*

$$\bigcap_{i \in I} (\mathcal{A}_1^{(i)} \otimes \mathcal{A}_2^{(i)}) = \bigcap_{i \in I} \mathcal{A}_1^{(i)} \otimes \bigcap_{i \in I} \mathcal{A}_2^{(i)} \quad \text{modulo } \mu \otimes \nu.$$

Proof.

If $D \subset \Omega \times \Omega'$ and $\omega' \in \Omega'$, we denote $D_{\omega'} := \{\omega \in \Omega : (\omega, \omega') \in D\}$. Similarly, if $\omega \in \Omega$, we denote $D^\omega := \{\omega' \in \Omega' : (\omega, \omega') \in D\}$.

If \mathcal{A} is a sub- σ -algebra of \mathcal{B} , then

$$\{D \subset \Omega \times \Omega' : D \in \mathcal{B} \otimes \mathcal{C} \text{ and } \forall \omega' \in \Omega', D_{\omega'} \in \mathcal{A}\}$$

is a σ -algebra which contains all the rectangles $A \times C$ with $A \in \mathcal{A}$ and $C \in \mathcal{C}$. Hence

$$(8) \quad \mathcal{A} \otimes \mathcal{C} \subset \{D \in \mathcal{B} \otimes \mathcal{C} : \forall \omega' \in \Omega', D_{\omega'} \in \mathcal{A}\}.$$

Let us prove that in fact these two algebras coincide modulo the product measure $\mu \otimes \nu$. Let $D \in \mathcal{B} \otimes \mathcal{C}$ and $\epsilon > 0$. There exists $k \geq 1$, there exist $B_1, B_2, \dots, B_k \in \mathcal{B}$ and $C_1, C_2, \dots, C_k \in \mathcal{C}$ such that

$$(9) \quad \mu \otimes \nu \left(D \Delta \bigcup_{j=1}^k (B_j \times C_j) \right) < \epsilon .$$

We can choose C_1, C_2, \dots, C_k as a partition of Ω' . Using Fubini Theorem, we can write

$$\sum_{j=1}^k \int_{C_j} \mu(D_{\omega'} \Delta B_j) \, d\nu(\omega') < \epsilon .$$

We now suppose that all the sections $D_{\omega'}$ are in \mathcal{A} , so for each j there exists $A_j \in \mathcal{A}$ such that

$$\nu(C_j)\mu(A_j \Delta B_j) \leq \int_{C_j} \mu(D_{\omega'} \Delta B_j) \, d\nu(\omega') .$$

Thus

$$\sum_{j=1}^k \nu(C_j)\mu(A_j \Delta B_j) < \epsilon ,$$

or equivalently

$$\mu \otimes \nu \left(\bigcup_{j=1}^k (B_j \times C_j) \Delta \bigcup_{j=1}^k (A_j \times C_j) \right) < \epsilon .$$

In view of (9) we obtain that

$$\mu \otimes \nu \left(D \Delta \bigcup_{j=1}^k (A_j \times C_j) \right) < 2\epsilon .$$

This shows that, modulo $\mu \otimes \nu$, $D \in \mathcal{A} \otimes \mathcal{C}$.

We have proved that, modulo the product measure

$$(10) \quad \mathcal{A} \otimes \mathcal{C} = \{ D \in \mathcal{B} \otimes \mathcal{C} : \forall \omega' \in \Omega', D_{\omega'} \in \mathcal{A} \} .$$

Let us now consider two families of sub- σ -algebras $\mathcal{A}_1^{(i)} \subset \mathcal{B}$ and $\mathcal{A}_2^{(i)} \subset \mathcal{C}$, $i \in I$. We denote $\mathcal{A}_j := \bigcap_{i \in I} \mathcal{A}_j^{(i)}$, $j = 1, 2$. The inclusion

$$\mathcal{A}_1 \otimes \mathcal{A}_2 \subset \bigcap_{i \in I} (\mathcal{A}_1^{(i)} \otimes \mathcal{A}_2^{(i)}) .$$

is obvious. Let us show that it is in fact an equality (mod. $\mu \otimes \nu$).

From (8), we deduce that

$$\bigcap_{i \in I} \mathcal{A}_1^{(i)} \otimes \mathcal{A}_2^{(i)} \subset \left\{ D \in \mathcal{B} \otimes \mathcal{C} : \forall i \in I, \forall \omega' \in \Omega', \forall \omega \in \Omega, D_{\omega'} \in \mathcal{A}_1^{(i)} \text{ and } D^\omega \in \mathcal{A}_2^{(i)} \right\} .$$

Hence

$$\bigcap_{i \in I} \mathcal{A}_1^{(i)} \otimes \mathcal{A}_2^{(i)} \subset \{D \in \mathcal{B} \otimes \mathcal{C} : \forall \omega' \in \Omega', \forall \omega \in \Omega, D_{\omega'} \in \mathcal{A}_1 \text{ and } D^\omega \in \mathcal{A}_2\} .$$

Using (10), we obtain that, modulo the product measure,

$$\bigcap_{i \in I} \mathcal{A}_1^{(i)} \otimes \mathcal{A}_2^{(i)} \subset (\mathcal{A}_1 \otimes \mathcal{C}) \cap (\mathcal{B} \otimes \mathcal{A}_2)$$

It remains to show that, modulo $\mu \otimes \nu$,

$$(\mathcal{A}_1 \otimes \mathcal{C}) \cap (\mathcal{B} \otimes \mathcal{A}_2) \subset \mathcal{A}_1 \otimes \mathcal{A}_2.$$

Fix orthonormal bases $\{a_i^{(1)}\}_{i \in \mathbb{N}}$ in $L^2(\mathcal{A}_1)$ and $\{a_j^{(2)}\}_{j \in \mathbb{N}}$ in $L^2(\mathcal{A}_2)$.

Let $\{b_k^{(1)}\}_{k \in \mathbb{N}}$, $\{b_k^{(2)}\}_{k \in \mathbb{N}}$ be any complements of these bases to orthonormal bases of $L^2(\mathcal{B})$ and $L^2(\mathcal{C})$ respectively.

Let now $g \in L^2((\mathcal{A}_1 \otimes \mathcal{C}) \cap (\mathcal{B} \otimes \mathcal{A}_2))$. In particular $g \in L^2(\mathcal{A}_1 \otimes \mathcal{C})$, so

$$g = \sum_{i,j} \alpha_{ij} \cdot a_i^{(1)} \otimes a_j^{(2)} + \sum_{i,k} \beta_{ik} \cdot a_i^{(1)} \otimes b_k^{(2)}.$$

But at the same time $g \in L^2(\mathcal{B} \otimes \mathcal{A}_2)$ hence

$$g = E(g|\mathcal{B} \otimes \mathcal{A}_2) = \sum_{i,j} \alpha_{ij} \cdot a_i^{(1)} \otimes a_j^{(2)} + \sum_{i,k} \beta_{ik} \cdot a_i^{(1)} \otimes E(b_k^{(2)}|\mathcal{A}_2),$$

and $E(b_k^{(2)}|\mathcal{A}_2) = 0$, thus $g = E(g|\mathcal{B} \otimes \mathcal{A}_2) \in L^2(\mathcal{A}_1 \otimes \mathcal{A}_2)$. \square

3.2 Self-joinings and factors of product \mathbb{Z}^2 -actions

Assume that $(\Omega, \mathcal{B}, \mu)$ is a standard probability Borel space. The group of all μ -preserving automorphisms is denoted by $\text{Aut}(\Omega, \mathcal{B}, \mu)$. Given $T \in \text{Aut}(\Omega, \mathcal{B}, \mu)$ by U_T we denote the corresponding unitary action of T : $f \mapsto f \circ T$, on $L^2(\Omega, \mathcal{B}, \mu)$. Then $\text{Aut}(\Omega, \mathcal{B}, \mu)$ viewed as a subset of $U(L^2(\Omega, \mathcal{B}, \mu))$ is closed and therefore $\text{Aut}(\Omega, \mathcal{B}, \mu)$ endowed with the strong operator topology, becomes a Polish space. We will assume now that on $(\Omega, \mathcal{B}, \mu)$ we have an action $\underline{T} = (T_\gamma)_{\gamma \in \Gamma}$ (*i.e.* we have a group homomorphism $\gamma \mapsto T_\gamma \in \text{Aut}(\Omega, \mathcal{B}, \mu)$) of a countable Abelian group Γ (only \mathbb{Z} or \mathbb{Z}^2 will appear as Γ in this note). By $C(\underline{T})$ we mean the subset of $\text{Aut}(\Omega, \mathcal{B}, \mu)$ of those elements that commute with all T_γ , $\gamma \in \Gamma$. By a *factor* of \underline{T} we mean any sub- σ -algebra $\mathcal{A} \subset \mathcal{B}$ which is invariant under each T_γ , $\gamma \in \Gamma$ (more precisely, a factor is the action of Γ on the quotient probability space $(\Omega/\mathcal{A}, \mathcal{A}, \mu|_{\mathcal{A}})$). We will now consider the set $J(\underline{T})$

of *self-joinings* of \underline{T} , i.e. the set of $T_\gamma \times T_\gamma$ -invariant measures on $(\Omega \times \Omega, \mathcal{B} \otimes \mathcal{B})$ with the two marginals equal to μ . As we have already noticed to such a measure ρ there corresponds a Markov operator $\Phi = \Phi_\rho$ and the $(T_\gamma \times T_\gamma)_{\gamma \in \Gamma}$ -invariance is equivalent to the equivariance property

$$\Phi \circ U_{T_\gamma} = U_{T_\gamma} \circ \Phi \text{ for each } \gamma \in \Gamma$$

(see e.g. [10], [16]). If $\mathcal{A} \subset \mathcal{B}$ is a factor then the measure $\rho_{\mathcal{A}}$ given by

$$\rho_{\mathcal{A}}(B_1 \times B_2) = \int_{\Omega} E(B_1|\mathcal{A}) \cdot E(B_2|\mathcal{A}) d\mu$$

for $B_1, B_2 \in \mathcal{B}$ is a self-joining of \underline{T} , called the relative product measure over \mathcal{A} . We have $\Phi_{\rho_{\mathcal{A}}} = E(\cdot|\mathcal{A})$.

Suppose now that the action \underline{T} is ergodic. We denote by $J^e(\underline{T})$ the set of ergodic self-joinings: these are $\rho \in J(\underline{T})$ for which the Γ -action $(T_\gamma \times T_\gamma)_{\gamma \in \Gamma}$ on $(\Omega \times \Omega, \rho)$ is ergodic. Then $J^e(\underline{T})$ is always non-empty and more than that – each self-joining admits a unique decomposition into ergodic joinings (see e.g. [10]). If $S \in C(\underline{T})$ then the measure ρ_S given by

$$\rho_S(B_1 \times B_2) = \mu(B_1 \cap S^{-1}(B_2))$$

for $B_1, B_2 \in \mathcal{B}$ is a self-joining of \underline{T} . Such a joining is called a *graph* self-joining. Graph self-joinings are always ergodic. It is also clear that for a graph self-joining ρ_S we have $\mathcal{B}_1(\rho_S) = \mathcal{B}$. In fact whenever a self-joining ρ has the property that $\mathcal{B}_1(\rho) = \mathcal{B}$ then $\rho = \rho_S$ for some $S \in C(\underline{T})$ (see e.g. [14]). Given $\mathcal{A} \subset \mathcal{B}$ a factor of \underline{T} let

$$(11) \quad \rho_{\mathcal{A}} = \int_{J^e(\underline{T})} \eta dP(\eta)$$

denote the ergodic decomposition of $\rho_{\mathcal{A}}$ (equivalently, $E(\cdot|\mathcal{A}) = \int_{J^e(\underline{T})} \Phi_\eta dP(\eta)$). Note that for $B \in \mathcal{B}$ we have $B \in \mathcal{A}$ if and only if $\rho_{\mathcal{A}}((B \times \Omega) \Delta (\Omega \times B)) = 0$. An easy convexity argument then shows that whenever a set $A \in \mathcal{A}$ then $\eta((A \times \Omega) \Delta (\Omega \times A)) = 0$ for P -a.e. η . Equivalently, if $f \in L^2(\mathcal{A})$ then $\Phi_\eta(f) = f$ for P -a.e. η . Therefore, if $\{f_n : n \geq 1\} \subset L^2(\mathcal{A})$ then there exists a set of full P -measure such that for η in this set we have $\Phi_\eta(f_n) = f_n$ for all $n \in \mathbb{N}$. Assume that $f_{n_k} \rightarrow f$ in L^2 and let $g \in L^2(\Omega, \mathcal{B}, \mu)$. We have

$$\begin{aligned} \langle f, g \rangle &= \lim_{k \rightarrow \infty} \langle f_{n_k}, g \rangle = \lim_{k \rightarrow \infty} \langle \Phi_\eta f_{n_k}, g \rangle = \\ &= \lim_{k \rightarrow \infty} \langle f_{n_k}, \Phi_\eta^* g \rangle = \langle f, \Phi_\eta^* g \rangle = \langle \Phi_\eta f, g \rangle, \end{aligned}$$

whence $\Phi_\eta(f) = f$. Considering a dense countable subset of L^2 , it follows that

$$(12) \quad \text{on a set of } \eta \text{ of full } P\text{-measure, } \Phi_\eta(f) = f \text{ for each } f \in L^2(\mathcal{A}).$$

The following important result holds (see [12], [17]).

Proposition 3 (Veech's theorem) *Assume that the action \underline{T} is ergodic and let $\mathcal{A} \subset \mathcal{B}$ be a factor. Assume that in the ergodic decomposition (11), P -a.e. ergodic component η is a graph joining. Then there exists a compact subgroup $G \subset C(\underline{T})$ such that*

$$(13) \quad \mathcal{A} = \{B \in \mathcal{B} : R(B) = B \text{ for each } R \in G\}.$$

□

The following lemma is also well-known but we include its proof for completeness.

Lemma 3 *Assume that \underline{T} is a product \mathbb{Z}^2 -action given by (5). Let ρ be a probability measure defined on $(\Omega \times \Omega', \mathcal{B} \otimes \mathcal{C})$ with marginals μ and ν respectively and which is invariant for \underline{T} . If one of the transformations T or S is ergodic then $\rho = \mu \otimes \nu$.*

Proof.

Assume that S is ergodic. Take $B \in \mathcal{B}$ with $\mu(B) > 0$ and let $C \in \mathcal{C}$. Define the measure ρ_2^B on (Ω', \mathcal{C}) in the following way:

$$(14) \quad \rho_2^B(C) = \frac{\rho(B \times C)}{\rho(B \times \Omega)} = \frac{\rho(B \times C)}{\mu(B)}.$$

The measure ρ_2^B is S -invariant and, since $\rho_2^B(C) \leq \nu(C)/\mu(B)$, it is absolutely continuous with respect to ν which is S -ergodic. Hence $\rho_2^B = \nu$. Now (14) gives $\rho(B \times C) = \mu(B)\nu(C)$. □

Assume that \underline{T} is an ergodic product \mathbb{Z}^2 -action given by (5). Denote by $\pi_{\Omega \times \Omega}$ (resp. $\pi_{\Omega' \times \Omega'}$) the map from $\Omega \times \Omega' \times \Omega \times \Omega'$ to $\Omega \times \Omega$ (resp. $\Omega' \times \Omega'$) given by

$$\pi_{\Omega \times \Omega}(\omega_1, \omega'_1, \omega_2, \omega'_2) = (\omega_1, \omega_2)$$

(resp. $\pi_{\Omega' \times \Omega'}(\omega_1, \omega'_1, \omega_2, \omega'_2) = (\omega'_1, \omega'_2)$). As an important consequence of Lemma 3 we obtain the fact that every ergodic self-joining of a product \mathbb{Z}^2 -action is a product of ergodic self-joinings. More precisely, we have the following

Proposition 4 *Assume that \underline{T} is an ergodic product \mathbb{Z}^2 -action given by (5). Assume that $\rho \in J^e(\underline{T})$. Then there exist $\rho_1 \in J^e(T)$ and $\rho_2 \in J^e(S)$ such that*

$$(\pi_{\Omega \times \Omega} \times \pi_{\Omega' \times \Omega'})_*(\rho) = \rho_1 \otimes \rho_2.$$

Proof.

Put $\rho_1 = (\pi_{\Omega \times \Omega})_*(\rho)$ and $\rho_2 = (\pi_{\Omega' \times \Omega'})_*(\rho)$. Since for each $(n, m) \in \mathbb{Z}^2$ the measure ρ is invariant under the automorphism $T^m \times S^n \times T^m \times S^n$ and it is \underline{T} -ergodic, $\rho_1 \in J^e(T)$ and $\rho_2 \in J^e(S)$. The measure $(\pi_{\Omega \times \Omega} \times \pi_{\Omega' \times \Omega'})_*(\rho)$ is now $(T \times T)^n \times (S \times S)^m$ -invariant for each $(n, m) \in \mathbb{Z}^2$ and it is ergodic under the corresponding \mathbb{Z}^2 -action. The result follows directly from Lemma 3. \square

It is a routine verification that the Markov operator $\Phi_{\rho_1 \otimes \rho_2}$ can naturally be identified with the tensor product $\Phi_{\rho_1} \otimes \Phi_{\rho_2}$. Reminding that graph self-joinings are characterized by the fact that their corresponding Markov operators are unitary, and noticing that if the operator $\Phi_{\rho_1} \otimes \Phi_{\rho_2}$ is unitary, then each operator Φ_{ρ_j} is unitary, we see that a consequence of Proposition 4 is the following fact.

Corollary 1 *Assume that \underline{T} is a product \mathbb{Z}^2 -action given by (5), with T and S ergodic. If $R \in C(\underline{T})$ then there exist $R_1 \in C(T)$ and $R_2 \in C(S)$ such that $R = R_1 \times R_2$. \square*

We are now able to prove the main result of this section which is a full description of factors of an ergodic product \mathbb{Z}^2 -action \underline{T} given by (5). If $\mathcal{B}_1 \subset \mathcal{B}$ is a factor of T and $\mathcal{C}_1 \subset \mathcal{C}$ is a factor of S then the σ -algebra $\mathcal{B}_1 \otimes \mathcal{C}_1$ is a factor of \underline{T} . We call such factors *product factors*. Assume now that $\mathcal{A} \subset \mathcal{B} \otimes \mathcal{C}$ is a factor of \underline{T} . Then, in view of Lemma 2, there exists the smallest product factor $\tilde{\mathcal{B}} \otimes \tilde{\mathcal{C}}$ containing \mathcal{A} . Denote by \tilde{T} (\tilde{S}) the quotient action of T (of S) on $(\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{\mu})$ (on $(\tilde{\Omega}', \tilde{\mathcal{C}}, \tilde{\nu})$).

Proposition 5 *Assume that $\mathcal{A} \subset \mathcal{B} \otimes \mathcal{C}$ is a factor of an ergodic product \mathbb{Z}^2 -action \underline{T} given by (5) and let $\tilde{\mathcal{B}} \otimes \tilde{\mathcal{C}}$ be the smallest product factor containing \mathcal{A} . Then there exist a compact metric group G and two continuous 1-1 homomorphisms*

$$g \mapsto R_g \in C(\tilde{T}), \quad g \mapsto R'_g \in C(\tilde{S})$$

such that

$$\mathcal{A} = \{A \in \tilde{\mathcal{B}} \otimes \tilde{\mathcal{C}}; (R_g \times R'_g)(A) = A \text{ for all } g \in G\}.$$

Proof.

Denote by $\tilde{\underline{T}}$ the product \mathbb{Z}^2 -action $\{\tilde{T}^n \times \tilde{S}^m\}_{n,m \in \mathbb{Z}}$ on $\tilde{\mathcal{B}} \otimes \tilde{\mathcal{C}}$. We have

$$L^2(\mathcal{A}) \subset L^2(\tilde{\Omega} \times \tilde{\Omega}', \tilde{\mu} \otimes \tilde{\nu}),$$

i.e. after a natural identification

$$(15) \quad L^2(\mathcal{A}) \subset L^2(\tilde{\Omega}, \tilde{\mu}) \otimes L^2(\tilde{\Omega}', \tilde{\nu}).$$

By taking first the ergodic decomposition of $\rho_{\mathcal{A}}$ in $J(\tilde{T})$, applying Proposition 4 and passing to (15) we obtain

$$Pr_{L^2(\mathcal{A})} = \int \Phi_1 \otimes \Phi_2 dQ(\Phi_1 \otimes \Phi_2),$$

where $\Phi_i = \Phi_{\eta_i}$ for $i = 1, 2$ and $\eta_1 \in J^e(\tilde{\mathcal{B}})$ and $\eta_2 \in J^e(\tilde{\mathcal{C}})$. In view of (12) on a set of full Q -measure, for arbitrary $f \in L^2(\mathcal{A})$ we have $f \in L^2(\mathcal{B}_0(\Phi_1 \otimes \Phi_2))$. By Lemma 1, on a set of full Q -measure, $f \in L^2(\mathcal{B}_1(\Phi_1) \otimes \mathcal{B}_1(\Phi_2))$ for an arbitrary $f \in L^2(\mathcal{A})$. We know already that $\mathcal{B}_1(\Phi_1) \subset \tilde{\mathcal{B}}$ and $\mathcal{B}_1(\Phi_2) \subset \tilde{\mathcal{C}}$. Hence, by minimality of $\tilde{\mathcal{B}} \otimes \tilde{\mathcal{C}}$, we obtain $\mathcal{B}_1(\Phi_1) \otimes \mathcal{B}_1(\Phi_2) = \tilde{\mathcal{B}} \otimes \tilde{\mathcal{C}}$ on a set of full Q -measure of $\Phi_1 \otimes \Phi_2$. Thus (Q -a.e.) $\mathcal{B}_1(\Phi_{\eta_1}) = \tilde{\mathcal{B}}, \mathcal{B}_1(\Phi_{\eta_2}) = \tilde{\mathcal{C}}$, which means that η_1 and η_2 are graph self-joinings and therefore the self-joining η , where $\Phi_{\eta} = \Phi_1 \otimes \Phi_2$, is also a graph self-joining. In view of Veech's theorem there exists a compact group $\mathcal{G} \subset C(\tilde{T})$ such that

$$\mathcal{A} = \{A \in \tilde{\mathcal{B}} \otimes \tilde{\mathcal{C}}; R(A) = A \text{ for all } R \in \mathcal{G}\}.$$

In order to conclude we only need to show the special form of \mathcal{G} . By Corollary 1 each $R \in \mathcal{G}$ is of the form $R_1 \times R_2$ and we need to show that if $Id \times R' \in \mathcal{G}$ then $R' = Id$. This is a direct consequence of the description of invariant vectors under the tensor product of two contractions (see Lemma 8 in Appendix) which guarantees that $\mathcal{B}_0(Id \otimes U_{R'}) \subset \tilde{\mathcal{B}} \otimes \mathcal{B}_0(U_{R'})$. \square

Remark 1 Proposition 5 tells us that the only possibility to find non-product factors for a product \mathbb{Z}^2 -action is to have isomorphic compact subgroups in the centralizers of the marginal \mathbb{Z} -actions. Therefore it may even happen that T and S are disjoint in the sense of Furstenberg (*i.e.* the only joining between T and S is product measure) but there are \underline{T} -factors different from product factors. For example if $T\omega = \omega + \alpha$ and $S\omega' = \omega' + \beta$ are two irrational rotations on the additive circle \mathbb{T} and $1, \alpha, \beta$ are rationally independent then T and S are disjoint but the corresponding product \mathbb{Z}^2 -action \underline{T} has a factor with 2-point fibers given as the σ -algebra of sets fixed by the group $\{Id \times Id, R_1 \times R_2\}$, where $R_1\omega = \omega + \frac{1}{2}$ and $R_2\omega' = \omega' + \frac{1}{2}$.

4 Proof of Theorem 1

Assume that T and S are ergodic automorphisms of standard probability Borel spaces $(\Omega, \mathcal{B}, \mu)$ and $(\Omega', \mathcal{C}, \nu)$ respectively.

We will use some basic facts about standard probability Borel spaces and their factors. These facts are described in the next three lemmas (for proofs see Appendix).

We say that the sub- σ -algebra \mathcal{B}' of \mathcal{B} is *determined by a factor* if there exist a standard Borel space (Ω'', \mathcal{D}) and a measurable map π from (Ω, \mathcal{B}) into (Ω'', \mathcal{D}) such that $\mathcal{B}' = \pi^{-1}(\mathcal{D})$.

Lemma 4 *A sub- σ -algebra \mathcal{B}' of \mathcal{B} is determined by a factor if and only if it is generated by a countable sub-family.*

Lemma 5 *If the sub- σ -algebra \mathcal{B}' is determined by a factor and separates the points of Ω , then $\mathcal{B}' = \mathcal{B}$.*

Lemma 6 *For any sub- σ -algebra \mathcal{B}'' of \mathcal{B} , there exists a sub- σ -algebra \mathcal{B}' determined by a factor and such that $\mathcal{B}' = \mathcal{B}''$ modulo μ .*

If \mathcal{A} is a T -invariant sub- σ -algebra of \mathcal{B} , we shall denote by $C(\mathcal{A})$ the centralizer of the factor determined by \mathcal{A} .

Proposition 6 *Let \underline{T} be the product \mathbb{Z}^2 -action given by (5). Assume that $F : \Omega \times \Omega' \rightarrow \mathbb{R}$ is measurable. Assume moreover that there exists a T -invariant subset $\Omega_0 \subset \Omega$ of full μ -measure such that for all $\omega_1, \omega_2 \in \Omega_0$ if $\omega_1 \neq \omega_2$ then there exists $n \in \mathbb{Z}$ such that the maps $F(T^n \omega_1, \cdot)$ and $F(T^n \omega_2, \cdot)$ are not ν -almost everywhere equal. Then (modulo $\mu \otimes \nu$) the smallest \underline{T} -factor $\mathcal{A} \subset \mathcal{B} \otimes \mathcal{C}$ making F measurable consists of sets fixed by a compact subgroup of the form $\{R_h \times R'_h : h \in G\}$, where $\{R_h : h \in G\} \subset C(\mathcal{B})$ while $\{R'_h : h \in G\} \subset C(\mathcal{A}_2)$ for some S -factor $\mathcal{A}_2 \subset \mathcal{C}$. In particular, if $C(T)$ has no nontrivial compact subgroups then $\mathcal{A} = \mathcal{B} \otimes \mathcal{A}_2$, that is T can be filtered out from the \underline{T} -factor given by F .*

Proof.

By Lemma 2 we know that there exists a smallest product σ -algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ containing \mathcal{A} , modulo the product measure $\mu \otimes \nu$. By Lemma 6, we can suppose that the sub- σ -algebra \mathcal{A}_1 is determined by a factor of (Ω, \mathcal{B}) . There exists a $\mathcal{A}_1 \otimes \mathcal{A}_2$ -measurable map F_1 such that $F = F_1 \mu \otimes \nu$ -a.e. For all $\omega' \in \Omega'$, the map $F_1(\cdot, \omega')$ is \mathcal{A}_1 -measurable.

By the Fubini theorem, for μ -almost all ω , $F_1(\omega, \cdot) = F(\omega, \cdot)$ ν -a.e. We know that there exists a set Ω_0 of full μ -measure such that

$$\omega_1, \omega_2 \in \Omega_0 \text{ and } \omega_1 \neq \omega_2 \implies \exists_{n \in \mathbb{Z}} F(T^n \omega_1, \cdot) \neq F(T^n \omega_2, \cdot) \text{ mod } \nu.$$

Hence there exists a set Ω_1 of full μ -measure such that

$$\omega_1, \omega_2 \in \Omega_1 \text{ and } \omega_1 \neq \omega_2 \implies \exists_{n \in \mathbb{Z}} F_1(T^n \omega_1, \cdot) \neq F_1(T^n \omega_2, \cdot) \text{ mod } \nu.$$

This implies that the σ -algebra \mathcal{A}_1 separates the points of Ω_1 . Thus by Lemma 5, we have $\mathcal{A}_1 = \mathcal{B}$.

According to Proposition 5 there exist compact subgroups $\{R_h\}_{h \in G} \subset C(\mathcal{B})$, $\{R'_h\}_{h \in G} \subset C(\mathcal{A}_2)$ such that, modulo $\mu \otimes \nu$, \mathcal{A} is the σ -algebra of sets fixed by

all elements $R_h \times R'_h$, $h \in G$ and therefore the first part of Proposition 6 has been shown.

In order to prove the second part, simply notice that if $C(T)$ does not contain any nontrivial compact subgroup, then the subgroup $\{R_h\}_{h \in G}$ is reduced to the identity and therefore so is $\{R'_h\}_{h \in G}$. It follows that $\mathcal{A} = \mathcal{B} \otimes \mathcal{A}_2$. \square

Assume now that T is an ergodic automorphism of a probability standard Borel space $(\Omega, \mathcal{B}, \mu)$. Assume that \mathcal{H} is a compact subgroup of $C(T)$ and let $\mathcal{A} = \mathcal{A}(\mathcal{H})$ be the factor determined by \mathcal{H} , that is \mathcal{A} is the σ -algebra of sets fixed by all elements of \mathcal{H} . It is well-known (see for example [10], Chapter 6, Section 3) that in this case we can represent T in the form of a skew product R_ϕ acting on the space $(Z \times \mathcal{H}, \mathcal{A} \otimes \mathcal{B}(\mathcal{H}), \rho \otimes m_{\mathcal{H}})$, where $Z = \Omega/\mathcal{A}$, $m_{\mathcal{H}}$ stands for Haar measure on \mathcal{H} , $\phi : Z \rightarrow \mathcal{H}$ is measurable and $R_\phi(z, h) = (Rz, \phi(z)h)$. Notice that the map $\tau_g(z, h) = (z, hg)$ is an element of the centralizer of R_ϕ .

Lemma 7 *The σ -algebra $(\mathcal{A} \otimes \mathcal{B}(\mathcal{H})) \otimes \{\emptyset, Z \times \mathcal{H}\}$ is not contained in the factor of the \mathbb{Z}^2 -product action of the form (5) for $T = S = R_\phi$ determined by*

$$F : (Z \times \mathcal{H})^2 \rightarrow Z \times \mathcal{H}, \quad F((z_1, h_1), (z_2, h_2)) = (z_1, h_1 h_2^{-1}).$$

Moreover, the map F satisfies the R_ϕ -univalence property.

Proof.

Note that for each $g \in \mathcal{H}$, $F \circ (\tau_g \times \tau_g) = F$. It follows that the \underline{T} -factor determined by F is contained in the σ -algebra of sets fixed by all the elements $\tau_g \times \tau_g$ ($g \in \mathcal{H}$) of $C(\underline{T})$ and clearly the only sets from $(\mathcal{A} \otimes \mathcal{B}(\mathcal{H})) \otimes \{\emptyset, Z \times \mathcal{H}\}$ which are $\tau_g \times \tau_g$ for all $g \in \mathcal{H}$ are sets of the form $Z_0 \times \mathcal{H} \times Z \times \mathcal{H}$ for some $Z_0 \in \mathcal{A}$.

Finally, notice that if

$$F((z_1, h_1), (z_2, h_2)) = F((z'_1, h'_1), (z_2, h_2))$$

then $(z_1, h_1 h_2^{-1}) = (z'_1, h'_1 h_2^{-1})$, so $z_1 = z'_1$ and $h_1 = h'_1$. \square

Proof of Theorem 1. The necessity follows directly from Proposition 6. For the sufficiency, given T with a nontrivial subgroup in the centralizer we only need to indicate S and a real-valued measurable function F with the T -univalence property so that T cannot be filtered out from the factor defined by F of the \mathbb{Z}^2 -product action defined by T and S . It follows from Lemma 7 that such S and F exist because the function F in Lemma 7 takes its values in a standard Borel space $Z \times \mathcal{H}$ which (as a Borel space) is isomorphic to \mathbb{R} . The proof of Theorem 1 is now complete. \square

Application. Assume that T is the automorphism canonically associated to an ergodic stationary process $(X_n)_{n \in \mathbb{Z}}$ and T has no non-trivial compact subgroup in the centralizer. Assume moreover that $f((x_n), (y_n)) = f(x_0, y_0)$ satisfies $f(x, \cdot) \neq f(x', \cdot)$ for $x \neq x'$. Then we will filter the process $(X_n)_{n \in \mathbb{Z}}$ from any \mathbb{Z}^2 -random field $Z_{n,m} = f(X_n, Y_m)$, where $(Y_m)_{m \in \mathbb{Z}}$ is an arbitrary ergodic stationary process and $(X_n, Y_m)_{m,n \in \mathbb{Z}}$ is also stationary.

5 Final remarks

The problem we have considered in this paper is under which assumptions on f we can filter the process $(X_n)_{n \in \mathbb{Z}}$ from the random \mathbb{Z}^2 -field $(Z_{n,m})_{n,m \in \mathbb{Z}}$ defined by

$$(Z_{n,m}) = (f((X_{n+k})_{k \in \mathbb{Z}}, (Y_{m+\ell})_{\ell \in \mathbb{Z}})) .$$

We have already noticed that both our filtering results work in case $f((x_n), (y_n)) = x_0 + y_0$ that is in case of the \mathbb{Z}^2 -version of the original problem of Furstenberg. In this particular case yet another argument can be used. Indeed, we have $Z_{n,m} = X_n + Y_m$, hence the process $(X_n - X_{n-1})_{n \in \mathbb{Z}}$ is a function of $(Z_{n,m})_{n,m \in \mathbb{Z}}$ and the only thing we have to prove is that the original process $(X_n)_{n \in \mathbb{Z}}$ is measurable with respect to the difference process $(X_n - X_{n-1})_{n \in \mathbb{Z}}$. By passing now to the pure ergodic theory language we face the following problem. Given an ergodic automorphism T of a standard probability Borel space $(\Omega, \mathcal{B}, \mu)$ assume that a function $f : \Omega \rightarrow \mathbb{R}$ is measurable with respect to a T -invariant sub- σ -algebra $\mathcal{A} \subset \mathcal{B}$ and let

$$(16) \quad f = g - g \circ T,$$

where g is \mathcal{B} -measurable. Is it now true that g is in fact also \mathcal{A} -measurable? The answer is positive, however the proof is non-trivial and in the general case it is highly “non-algorithmic”. Indeed, the proof follows immediately from a characterization of so called additive coboundaries (these are functions f satisfying (16) for some measurable g) due to Moore and Schmidt (see [15]). Namely, a measurable function $f : \Omega \rightarrow \mathbb{R}$ is an additive coboundary if and only if the sequence $((f^{(n)})_*(\mu))_{n \in \mathbb{N}}$ of distributions of the functions

$$f^{(n)}(\omega) = f(\omega) + f(T\omega) + \dots + f(T^{n-1}\omega),$$

$n \geq 0$, is uniformly tight. The proof of the result by Moore and Schmidt is quite complicated: first it turns out that the problem of being an additive coboundary is equivalent to the problem e^{icf} of being a multiplicative coboundary for “sufficiently many” $c \in \mathbb{R}$ (see [15] but also e.g. [13] for probably the simplest proof of this fact) and then still some equicontinuity argument together with a fixed point theorem is needed. For other proofs of the above characterizations of additive coboundaries see [2] and [3] (the proofs in these papers are neither of “algorithmic” type). Note in passing that if we assume that f and g in (16)

are L^1 -functions then we “filter” g from the process $(f \circ T^n)_{n \in \mathbb{Z}}$ quite easily. Indeed, first of all notice that in (16), g can be given only up to an additive constant, so we can assume that $\int_X g d\mu = 0$. Then

$$\frac{1}{N} \sum_{n=0}^{N-1} f^{(n)} = g - \frac{1}{N} \sum_{n=0}^{N-1} g \circ T^n,$$

so by the pointwise ergodic theorem, $g = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f^{(n)}$ a.s. (and also in L^1).

6 Appendix

6.1 Spectral properties of a contraction

Let \mathcal{H} be a separable Hilbert space and $Q : \mathcal{H} \rightarrow \mathcal{H}$ a contraction, *i.e.* $\|Q\| \leq 1$. By a unitary *dilation* of Q we mean a unitary operator U acting on a certain Hilbert space \mathcal{H}_0 containing \mathcal{H} and such that for each $n \in \mathbb{N}$,

$$Q^n = \text{Pr}_{\mathcal{H}} \circ U^n|_{\mathcal{H}},$$

where by $\text{Pr}_{\mathcal{H}}$ we denoted the orthogonal projection on \mathcal{H} . If, additionally $\mathcal{H}_0 = \bigvee_{n=0}^{\infty} U^n \mathcal{H}$ then we call U a *minimal* unitary dilation of Q . It is known (see [6], Theorem 4.2 Chapter I. and Proposition 6.1 Chapter II.) that for an arbitrary contraction there exists a minimal unitary dilation, moreover

$$(17) \quad \left\{ \begin{array}{l} \text{if } U \text{ is a minimal unitary dilation of a contraction } Q \text{ then} \\ U \text{ and } Q \text{ have the same eigenvalues of modulus 1} \\ \text{with the same eigenvectors.} \end{array} \right.$$

The following lemma gives a description of invariant vectors for tensor products of contractions. This result is classical for unitary operators (see e.g. Furstenberg’s book [8], Section 4.4).

Lemma 8 *Assume that Q_i is a contraction of a separable Hilbert space \mathcal{H}^i , $i = 1, 2$. Set $Q = Q_1 \otimes Q_2$ and let $f \in \mathcal{H}^1 \otimes \mathcal{H}^2$ be an invariant vector for Q . Then*

$$f = \sum_{i,j} g_i \otimes h_j,$$

where g_i, h_j are eigenvectors for Q_1 and Q_2 respectively. Moreover, if in the above representation of f , g_i corresponds to an eigenvalue c then h_j corresponds to c^{-1} and in particular $|c| = 1$.

Proof.

Denote by U_1, U_2 minimal unitary dilations of Q_1 and Q_2 respectively. It can be easily verified that $U_1 \otimes U_2$ is a unitary dilation of $Q_1 \otimes Q_2$ and we have

$$f = (Q_1 \otimes Q_2)(f) = \Pr_{\mathcal{H}^1 \otimes \mathcal{H}^2} \circ (U_1 \otimes U_2)(f).$$

But $U_1 \otimes U_2$ is a unitary operator, so $\|f\| = \|(U_1 \otimes U_2)(f)\|$. Therefore,

$$\|\Pr_{\mathcal{H}^1 \otimes \mathcal{H}^2} \circ (U_1 \otimes U_2)(f)\| = \|(U_1 \otimes U_2)(f)\|.$$

It follows that $(U_1 \otimes U_2)(f) \in \mathcal{H}^1 \otimes \mathcal{H}^2$ and moreover

$$(U_1 \otimes U_2)(f) = f.$$

For unitary operators, the lemma we want to prove is already known. Hence we have

$$f = \sum_{i,j} g_i \otimes h_j,$$

where g_i and h_j are eigenfunctions for U_1 and U_2 respectively and they correspond to mutually inverse eigenvalues. However U_1 and U_2 are minimal dilations of Q_1 and Q_2 respectively. We just have to use (17) to complete the proof of the lemma. \square

6.2 Some properties of standard Borel spaces

Assume that (Ω, d) is a complete, separable, metric space and let \mathcal{B} stand for the σ -algebra of Borel subsets of Ω . The measurable space (Ω, \mathcal{B}) is called a *standard Borel space*. We have (see e.g. [1])

Theorem 2 (Kuratowski's Theorem) *Two standard Borel spaces (Ω, \mathcal{B}) and (Ω', \mathcal{B}') are isomorphic if and only if Ω and Ω' have the same cardinality.*

Theorem 3 (Measurable Image Theorem) *Suppose (Ω, \mathcal{B}) and (Ω', \mathcal{B}') are standard Borel spaces and that $f : \Omega \rightarrow \Omega'$ is measurable and 1-1, then $f(A) \in \mathcal{B}'$ for every $A \in \mathcal{B}$.*

We can now give proofs of Lemmas 4-6.

Proof of Lemma 4.

From the definition of a standard Borel space, it is clear that the σ -algebra of such a space is always countably generated. This property comes back via a factor map. Hence any sub- σ -algebra determined by a factor is countably generated.

If a sub- σ -algebra of (Ω, \mathcal{B}) is generated by a countable sub-family $\{F_n\}_{n \in \mathbb{N}}$ then it is determined by the factor map

$$(18) \quad \pi : \Omega \rightarrow \{0, 1\}^{\mathbb{N}}, \quad \omega \mapsto (\mathbf{1}_{F_n}(\omega))_{n \in \mathbb{N}}.$$

□

Proof of Lemma 5.

Let \mathcal{B}' be a sub- σ -algebra determined by a factor. It is the preimage of the Borel algebra of $\{0, 1\}^{\mathbb{N}}$ by a measurable map π of the type (18). The fact that \mathcal{B}' separates the points of Ω means exactly that the map π is one to one. By the Measurable Image Theorem, the image $\pi(\Omega)$ is a Borel set. The Borel σ -algebra of this Borel set is generated by the trace of cylinder subsets of $\{0, 1\}^{\mathbb{N}}$. Coming back via π , we conclude that the Borel σ -algebra \mathcal{B} of Ω is generated by the preimages of the cylinder subsets. But these preimages are in \mathcal{B}' . We conclude that $\mathcal{B}' = \mathcal{B}$. □

Let μ be a probability on the measurable space (Ω, \mathcal{B}) .

Proof of Lemma 6.

Consider the σ -algebra \mathcal{B} equipped with the pseudo-metric inherited from the $L^1(\mu)$ norm : $d(A, B) = \mu(A \Delta B)$. Because the σ -algebra \mathcal{B} is countably generated, the pseudo-metric space (\mathcal{B}, d) contains a countable dense subset. Hence the subspace (\mathcal{B}', d) contains also a countable dense subset. Let us call \mathcal{D} such a subset. It is not difficult to verify that the σ -algebra generated by \mathcal{D} coincides modulo μ with \mathcal{B}' . □

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