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SETS OF K-RECURRENCE BUT NOT (K+1)-RECURRENCE

NIKOS FRANTZIKINAKIS, EMMANUEL LESIGNE, AND MÁTÉ WIERDL

We dedicate this paper to Y. Katznelson. Our work began at the conference organized for his 70th birthday, and we wish to honor him for his fundamental contribution to ergodic theory.

Abstract. For every $k \in \mathbb{N}$, we produce a set of integers which is k-recurrent but not $(k+1)$-recurrent. This extends a result of Furstenberg who produced a 1-recurrent set which is not 2-recurrent. We discuss a similar result for convergence of multiple ergodic averages. We also point out a combinatorial consequence related to Szemerédi's theorem.

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1. Introduction and main results

In his seminal paper [3], Furstenberg gave an ergodic theoretic proof of the famous theorem of Szemerédi claiming that every integer subset with positive density contains arbitrarily long arithmetic progressions. Furstenberg proved this by showing the following multiple recurrence property for measure preserving systems:

Theorem 1.1 (Furstenberg). *Let $(X, \mathcal{X}, \mu, T)$ be a finite measure preserving dynamical system and $A \in \mathcal{X}$ be a set with $\mu(A) > 0$. Then for every $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that*

$$\mu(A \cap T^{-n}A \cap \cdots \cap T^{-nk}A) > 0.$$

This motivated the following definition:

Definition 1.2. Let $(X, \mathcal{X}, \mu, T)$ be a probability preserving dynamical system. We say that $S \subset \mathbb{N}$ is a set of *k-recurrence for the system $(X, \mathcal{X}, \mu, T)$* if for every $A \in \mathcal{X}$ with $\mu(A) > 0$, there exists $n \in S$ such that

$$\mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A) > 0.$$

We say that $S \subset \mathbb{N}$ is a set of *k-recurrence* if it is a set of k-recurrence for every probability preserving dynamical system.

The corresponding definition in topological dynamics is:

Definition 1.3. We say that $S \subset \mathbb{N}$ is a set of *k-topological recurrence* if for every homeomorphism T of a compact metric space X such that the topological dynamical system (X, T) is minimal and for every nonempty open subset U of X, there exists $n \in S$ such that

$$U \cap T^{-n}U \cap \cdots \cap T^{-kn}U \neq \emptyset.$$

If S is infinite then the difference set $S - S = \{s_1 - s_2 \colon s_1, s_2 \in S\}$ is easily shown to be a set of 1-recurrence. By appropriately choosing S, Furstenberg constructed, in [4, pages 177-178], a set of 1-recurrence that is not a set of 2-recurrence. Constructing sets of 2-recurrence is much harder, in fact all the examples known turned out to be sets of k-recurrence for every k. This raised the question, first stated explicitly by Bergelson in [2]:

Question. *Let $k \geq 2$ be an integer. Does there exist a set of $(k-1)$-recurrence that is not a set of k-recurrence?*

The main objective of this article is to show that the answer is affirmative. The examples that we construct are very explicit:

Theorem A. *Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. We define*

$$S_k = \big\{n \in \mathbb{N} \colon \{n^k \alpha\} \in [1/4, 3/4]\big\},$$

where $\{a\}$ denotes the fractional part of a. Then S_k is a set of $(k-1)$-recurrence but not a set of k-recurrence.

It will appear in the proof that not only the set S_k is a set of $(k-1)$-recurrence for powers of a single transformation, but that it is a set of $(k-1)$-recurrence for families of commuting transformations (see definition in Section 4). Moreover, the set S_k is not a set of k-topological recurrence (but it is a set of $(k-1)$-topological recurrence since it is a set of $(k-1)$-recurrence).

We also answer the corresponding question for sets of k-convergence:

Definition 1.4. A set $S = \{a_1 < a_2 < \ldots\} \subset \mathbb{N}$ is called a set of *k-convergence* if for every probability preserving dynamical system $(X, \mathcal{X}, \mu, T)$ and functions $f_1, \ldots, f_k \in L^\infty(\mu)$, the averages

$$\frac{1}{N}\sum_{n=1}^{N} T^{a_n} f_1 \cdot \ldots \cdot T^{ka_n} f_k$$

converge in $L^2(\mu)$ as $N \to \infty$.

Host and Kra in [7] (see also Ziegler's work in [9] for an alternate proof) showed that $S = \mathbb{N}$ is a set of k-convergence for every k. We show:

Theorem B. *Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let*

$$I_j = \begin{cases} n \in [2^j, 2^{j+1}] \colon \{n^k a\} \in [1/10, 2/10] \text{ if } j \text{ is even} \\ n \in [2^j, 2^{j+1}] \colon \{n^k a\} \in [5/10, 6/10] \text{ if } j \text{ is odd} \end{cases}$$

and define

$$S'_k = \bigcup_{j=1}^{\infty} I_j.$$

Then S'_k is a set of $(k-1)$-convergence but not a set of k-convergence.

It will be clear from the proof that S'_k is also a set of $(k-1)$-recurrence but not a set of k-recurrence.

The strategy of the proof of the theorems is as follows. In Section 3 we use some elementary considerations in order to show that S_k is not a set of k-recurrence and S'_k is not a set of k-convergence. The basic observation is that if S is a set of k-recurrence/convergence then the set consisting of the k-th powers of elements of S has good 1-recurrence/convergence properties. In Section 4 we prove a multiple ergodic theorem (Proposition 4.2) that enables us to show that S_k is a set of $(k-1)$-recurrence and S'_k is a set of $(k-1)$-convergence.

Note that, as we already did in this introduction, we denote by $\mathbb{N}$ the set of *positive* integers.

2. A combinatorial consequence

A set $S \subset \mathbb{N}$ is called intersective if for every integer subset Λ with positive density we have $\Lambda \cap (\Lambda + n) \neq \emptyset$ for some $n \in S$. More generally we define:

Definition 2.1. A set $S \subset \mathbb{N}$ is *k-intersective* if every integer subset with positive density contains at least one arithmetic progression of length $k+1$ and common difference in S.

In [6, pages 528-529] it is shown that:

Proposition 2.2. *A set $S \subset \mathbb{N}$ is k-intersective if and only if it is a set of k-recurrence.*

We conclude from Theorem A that:

Corollary 2.3. *Let $k \geq 2$. There exists a set that is $(k-1)$-intersective but not k-intersective.*

3. Bad sets for k-recurrence and k-convergence

We will use the following elementary fact:

Lemma 3.1. *For every $k, n \in \mathbb{N}$ we have*

$$\sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}\binom{in}{l} = \begin{cases} 1 & \text{if} \quad l=0, \\ 0 & \text{if} \quad 1 \leq l < k, \\ (-1)^{k+1}n^k & \text{if} \quad l=k. \end{cases}$$

Proof. Consider the function

$$f(x) = (1-x^n)^k = \sum_{i=0}^{k}(-1)^i\binom{k}{i}x^{in}$$

Using the first representation for $f(x)$ and differentiating using the chain and product rule it is easy to check that

$$f^{(l)}(x) = \begin{cases} (1-x^n)P_l(x) & \text{if} \quad 0 \leq l < k, \\ (1-x^n)P_l(x) + (-1)^l n^l\, l!\, x^{l(n-1)} & \text{if} \quad l=k, \end{cases}$$

for some polynomials $P_l(x)$, $l = 0, \ldots, k-1$. Setting $x = 1$ gives

$$f^{(l)}(1) = \begin{cases} 0 & \text{if} \quad 0 \leq l < k, \\ (-1)^l n^l\, l! & \text{if} \quad l=k. \end{cases} \tag{3.1}$$

Using the second representation for $f(x)$ and differentiating we find

$$f^{(l)}(x) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{in}{l}\, l!\, x^{in-l}.$$

Setting $x = 1$ gives

$$f^{(l)}(1) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{in}{l}\, l!, \tag{3.2}$$

for every nonnegative integer l. Comparing equations (3.1) and (3.2) we get the advertised result. □

The following result will enable us to show that the set S_k is bad for k-recurrence and S'_k is bad for k-convergence. To better illustrate the idea, after proving the proposition, we explain how the argument works for $k = 2$.

Proposition 3.2. *If $S = \{a_1 < a_2 < \ldots\} \subset \mathbb{N}$ is a set of k-recurrence then $S^k = \{a_1^k < a_2^k < \cdots\}$ is a set of 1-recurrence for all circle rotations, and if S is a set of k-convergence then S^k is a set of 1-convergence.*

Remarks. Note that in the above proposition we only claim that S^k is a set of 1-recurrence for rotations of the circle. It is clear that the argument we give in the proof below can be extended to show that if S is a set of k-recurrence, then S^k is a set of 1-recurrence for all translations of multidimensional tori. In fact, it is an *unsolved problem* whether a set S being of k-recurrence implies that S^k is a set of 1-recurrence.

Here is a related unsolved problem of Katznelson from [8]: is it true that a set of 1-recurrence for all translations of multidimensional tori is, in fact, a set of 1-topological recurrence ?

Proof of Proposition 3.2. (i) Let S be a set of k-recurrence. It suffices to show that for every $\alpha \in [0, 1)$ and $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\left\|a_n^k \alpha\right\| \leq \varepsilon$, where $\|a\|$ is the distance of a to the closest integer. To do this we use the k-recurrence property for some appropriately chosen system. Given $\alpha \in [0, 1)$, we define the measure preserving transformation R acting on $\mathbb{T}^k$ with the Haar measure λ as follows:

$$R(t_1, \ldots, t_k) = (t_1 + \alpha, t_2 + t_1, \ldots, t_k + t_{k-1}).$$

Then for $j = 1, \ldots, k$ the k-th coordinate of $R^{jn}(t_1, \ldots, t_k)$ is

$$c_{j,n,k} = t_k + \binom{jn}{1} t_{k-1} + \binom{jn}{2} t_{k-2} + \cdots + \binom{jn}{k-1} t_1 + \binom{jn}{k} \alpha. \tag{3.3}$$

By Lemma 3.1 we have

$$\sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} c_{j,n,k} = t_k + (-1)^{k+1} n^k \alpha \tag{3.4}$$

for all $n \in \mathbb{N}$. Let $U_\varepsilon = B(0, \varepsilon/2^k)$. If S is a set of k-recurrence then there exists an $n_0 \in S$ such that

$$U_\varepsilon \cap R^{-n_0} U_\varepsilon \cap \cdots \cap R^{-kn_0} U_\varepsilon \neq \emptyset.$$

Let $(t_1, \ldots, t_k)$ be an element of the intersection and $c_{j,n}$, $j = 1, \ldots, k$, be given by (3.3). We have $\|c_{j,n_0,k}\| \leq \varepsilon/2^k$, $j = 1, \ldots, k$, and $\|t_k\| \leq \varepsilon/2^k$. Using (3.4) we get that $\left\|n_0^k \alpha\right\| \leq \varepsilon$, completing the proof.

(ii) Suppose that $S = \{a_1 < a_2 < \cdots\}$ is a set of k-convergence. Let R be the transformation defined in the proof of part (i). For $j = 1, \ldots, k$ define the functions

$$f_j(t_1, \ldots, t_k) = e\Big((-1)^{j+1}\binom{k}{j}t_k\Big).$$

Since S is a set of k-convergence the averages

$$\frac{1}{N}\sum_{n=1}^{N} R^{a_n} f_1 \cdot \ldots \cdot R^{ka_n} f_k = e\big(t_k\big) \cdot \frac{1}{N}\sum_{n=1}^{N} e((-1)^{k+1}a_n^k\alpha)$$

converge in $L^2(\lambda)$ as $N \to \infty$. Hence, for every $\alpha \in [0,1)$ the averages

$$\frac{1}{N}\sum_{n=1}^{N} e(a_n^k\alpha)$$

converge as $N \to \infty$. The spectral theorem gives that for every measure preserving system $(X, \mathcal{X}, \mu, T)$ and $f \in L^2(\mu)$ the averages

$$\frac{1}{N}\sum_{n=1}^{N} T^{a_n^k} f$$

converge in $L^2(\mu)$ as $N \to \infty$, completing the proof. □

We now give an example illustrating how the argument of Proposition 3.2 works for $k = 2$:

Example. (i) Suppose that S is a set of 2-recurrence. Let $\alpha \in [0,1)$ and $\varepsilon > 0$. The transformation $R\colon \mathbb{T}^2 \to \mathbb{T}^2$ is defined by

$$R(t_1, t_2) = (t_1 + \alpha, t_2 + t_1).$$

Then

$$R^n(t_1, t_2) = \Big(t_1 + n\alpha, t_2 + nt_1 + \binom{n}{2}\alpha\Big).$$

Since S is a set of 2-recurrence there exists an $n_0 \in S$ such that

$$U_\varepsilon \cap R^{-n_0}U_\varepsilon \cap R^{-2n_0}U_\varepsilon \neq \emptyset,$$

where $U_\varepsilon = B(0, \varepsilon/4)$. If (t_1, t_2) is an element of the intersection then

$$\|t_2\| \leq \varepsilon/4, \ \|c_1\| \leq \varepsilon/4, \quad \|c_2\| \leq \varepsilon/4, \tag{3.5}$$

where

$$c_1 = t_2 + n_0t_1 + \binom{n_0}{2}\alpha, \ c_2 = t_2 + 2n_0t_1 + \binom{2n_0}{2}\alpha.$$

Since

$$2c_1 - c_2 = t_2 - n_0^2\alpha$$

we get from (3.5) that $\|n_0^2\alpha\| \leq \varepsilon$. This implies that S^2 is a set of recurrence for circle rotations.

(ii) Suppose that $S = \{a_1 < a_2 < \cdots\}$ is a set of 2-convergence. We define the transformation $R\colon \mathbb{T}^2 \to \mathbb{T}^2$ as in part (i) and the functions

$$f_1(t_1, t_2) = e(2t_2),\ f_2(t_1, t_2) = e(-t_2).$$

Then

$$\frac{1}{N}\sum_{n=1}^{N} R^{a_n} f_1 \cdot R^{2a_n} f_2 = e(t_2) \cdot \frac{1}{N}\sum_{n=1}^{N} e(-a_n^2\alpha).$$

Since S is a set of 2-convergence it follows that the averages

$$\frac{1}{N}\sum_{n=1}^{N} e(a_n^2\alpha)$$

converge as $N \to \infty$ for every $\alpha \in [0, 1)$. The spectral theorem gives that S^2 is a set of 2-convergence.

Corollary 3.3. *The set S_k of Theorem A is* **not** *a set of k-recurrence and the set S'_k of Theorem B is* **not** *a set of k-convergence.*

Proof. By definition, S_k^k is not a set of recurrence for the rotation by α, so by Proposition 3.2 we have that S_k is not a set of k-recurrence.

Let $S'_k = \{a_1 < a_2 < \cdots\}$ and

$$A_N = \frac{1}{N}\sum_{n=1}^{N} e(a_n^k\alpha), \quad B_N = \frac{1}{N}\sum_{n=N+1}^{2N} e(a_n^k\alpha).$$

By the definition of S'_k we have that for j even the real part of B_{2^j} is positive and bounded away from zero, and for j odd the real part of B_{2^j} is negative. Hence, the sequence B_N does not converge as $N \to \infty$. Since $B_N = 2A_{2N} - A_N$ it follows that the sequence A_N does not converge as $N \to \infty$. By Proposition 3.2, S'_k is not a set of k-convergence. □

4. Good sets for k-recurrence and k-convergence

We will use the following elementary lemma ([1]):

Lemma 4.1 (Van der Corput). *Let $\{a_n\}_{n\in\mathbb{N}}$ be a bounded sequence of vectors in a Hilbert space. For each m we set*

$$b_m = \limsup_{N-M\to\infty} \Big| \frac{1}{N-M} \sum_{n=M+1}^{N} < a_{n+m}, a_n > \Big|.$$

If

$$\limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{m=M+1}^{N} b_m = 0$$

then

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M+1}^{N} a_n = 0$$

in norm.

Proposition 4.2. *Let $k \in \mathbb{N}$, $T_1, \dots, T_{k-1}$ be commuting measure preserving transformations acting on the probability space $(X, \mathcal{X}, \mu)$, and $p \in \mathbb{R}[t]$ be a polynomial of degree $\geq k$ with irrational leading coefficient. If $g\colon \mathbb{T} \to \mathbb{C}$ is Riemann integrable and $f_1, \dots, f_{k-1} \in L^\infty(\mu)$, then the difference*

$$\frac{1}{N-M} \sum_{n=M+1}^{N} T_1^n f_1 \cdot \ldots \cdot T_{k-1}^n f_{k-1} \cdot g(p(n)) - \frac{1}{N-M} \sum_{n=M+1}^{N} T_1^n f_1 \cdot \ldots \cdot T_{k-1}^n f_{k-1} \cdot \int_{\mathbb{T}} g \, d\lambda$$

converges to 0 in $L^2(\mu)$ as $N - M \to \infty$.

Proof. Using the standard estimation by continuous functions from above and below, it suffices to check the result when g is a continuous function. By Weierstrass approximation theorem of continuous functions by trigonometric polynomials, and using linearity, it suffices to check the result when $g(t) = e^{2\pi i l t}$ for some $l \in \mathbb{Z}$. The case $l = 0$ is trivial. If $l \neq 0$ the polynomial $q(n) = lp(n)$ satisfies the assumptions of our theorem, so it suffices to verify the result when $g(t) = e(t)$, where $e(t) = e^{2\pi i t}$.

We proceed by induction on the number of functions k. If $k = 1$ (empty product is 1) then

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M+1}^{N} e(p(n)) = 0$$

follows from Weyl's uniform distribution theorem. Assume that the result is true for $k-1$ functions. We will verify the result for k functions. We can assume that $\|f_k\|_\infty \leq 1$. We apply Van der Corput's Lemma on the Hilbert space $L^2(\mu)$ for the sequence of functions

$$a_n(x) = f_1(T_1^n x) \cdot \ldots \cdot f_k(T_k^n x) \cdot e(p(n)).$$

It suffices to verify that for every $m \in \mathbb{N}$ the averages

$$\frac{1}{N-M}\sum_{n=M+1}^{N}\int a_{n+m}(x)\cdot\overline{a_n(x)}\,d\mu =$$

$$\frac{1}{N-M}\sum_{n=M+1}^{N}\int g_1(T_1^n x)\cdot\ldots\cdot g_k(T_k^n x)\cdot e\big(p(n+m)-p(n)\big)\,d\mu,$$

where $g_i(x) = f_i(T_i^m x)\cdot\overline{f_i(x)}$, $i = 1,\ldots,k$, converge to 0 as $N-M\to\infty$. Introducing the notation $S_i = T_iT_k^{-1}$, $i = 1,\ldots,k-1$, and using Cauchy-Schwarz inequality, it suffices to prove that

$$\frac{1}{N-M}\sum_{n=M+1}^{N} g_1(S_1^n x)\cdot\cdot\ldots\cdot g_{k-1}(S_{k-1}^n x)\cdot e\big(p(n+m)-p(n)\big)$$

converge to zero in $L^2(\mu)$ as $N-M\to\infty$. But this follows from the induction hypothesis since the transformations S_i commute, and for $m\in\mathbb{N}$ the polynomial $q(n) = p(n+m)-p(n)$ has degree $\geq k-1$ and irrational leading coefficient. □

We remark that the non-uniform version ($M = 0$) of the previous result suffices for the proof of the next corollary[1]. The uniform version is only used to simplify the proof.

Definition 4.3. We say that $S\subset\mathbb{N}$ is a set of *k-recurrence for commuting transformations* if whenever $T_1,\ldots,T_k$ are commuting measure preserving transformations acting on the probability space $(X,\mathcal{X},\mu)$ and $A\in\mathcal{X}$ with $\mu(A)>0$, there exists $n\in S$ such that

$$\mu(A\cap T_1^{-n}A\cap\cdots\cap T_k^{-n}A)>0.$$

Corollary 4.4. *The set S_k of Theorem A is a set of $(k-1)$-recurrence for commuting transformations and the set S'_k of Theorem B is a set of $(k-1)$-convergence.*

Proof. To show that S_k is a set of $(k-1)$-recurrence for commuting transformations we apply Proposition 4.2 for $g(t) = \mathbf{1}_{[1/4,3/4]}(t)$ and $p(n) = n^k\alpha$. We get that if $T_1,\ldots,T_{k-1}$ are commuting measure preserving transformations acting on the probability space $(X,\mathcal{X},\mu)$, and

[1]See our note at `http://www.csi.hu/mw/general_dyadic_construction_short.pdf`.

$A \in \mathcal{B}$ with $\mu(A) > 0$, then

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{S_k}(n) \cdot \mu(A \cap T_1^{-n} A \cap \cdots \cap T_{k-1}^{-n} A) =$$

$$\frac{1}{2} \cdot \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T_1^{-n} A \cap \cdots \cap T_{k-1}^{-n} A) > 0, \tag{4.1}$$

where positiveness of the limsup in equation (4.1) follows from the multiple recurrence theorem of Furstenberg and Katznelson [5]. Hence, S_k is a set of $(k-1)$-recurrence.

To show that S'_k is a set of $(k-1)$-convergence we apply Proposition 4.2 for $T_i = T^i$, $i = 1, \ldots, k-1$, $p(n) = n^k \alpha$, and $g = \mathbf{1}_{[1/10,2/10]}$ on intervals of the form $[2^j, 2^j + N)$, for large $N < 2^j$,when j is even, and $g = \mathbf{1}_{[5/10,6/10]}$ on intervals of the form $[2^j, 2^j + N)$, for large $N < 2^j$, when j is odd. We get that for every $f_1, \ldots, f_{k-1} \in L^\infty(\mu)$ the difference

$$\frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{S'_k}(n) \cdot T^n f_1 \cdot \ldots \cdot T^{(k-1)n} f_{k-1} - \frac{1}{10} \cdot \frac{1}{N} \sum_{n=1}^{N} T^n f_1 \cdot \ldots \cdot T^{(k-1)n} f_{k-1}$$

converges to zero in $L^2(\mu)$ as $N \to \infty$. We know from [7] that the averages

$$\frac{1}{N} \sum_{n=1}^{N} T^n f_1 \cdot \ldots \cdot T^{(k-1)n} f_{k-1}$$

converge in $L^2(\mu)$ as $N \to \infty$, so the set S'_k is a set of $(k-1)$-convergence. This completes the proof. □

The reason we cannot prove that S'_k is a set of $(k-1)$-convergence for commuting transformations is that we do not yet know the analogous convergence result for the averages

$$\frac{1}{N} \sum_{n=1}^{N} T_1^n f_1 \cdot \ldots \cdot T_k^n f_k,$$

for commuting measure preserving transformations (T_j).

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