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# A composition of different $q$ nonextensive systems with the normalized expectation based on escort probability

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## Abstract

This is a study of composition rule and temperature definition for nonextensive systems containing different  $q$  subsystems. The physical meaning of the multiplier  $\beta$  associated with the energy expectation in the optimization of Tsallis entropy is investigated for the formalism with normalized expectation given by escort probability. This study is carried out for two possible cases: the case of the approximation of additive energy; and the case of nonadditive energy prescribed by an entropy composition rule for different  $q$  systems.

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## 1 Introduction

The extension of the nonextensive statistics (NS) theory developed initially for systems having same  $q$  value in different formalisms[1, 2, 3, 4] to the cases of different  $q$  subsystems has been a major concern of the scientists in or interested by this domain[5, 6]. The key point in this problematic is the composition of different  $q$  systems into a total system and the interpretation of the multiplier  $\beta$  (measurable temperature) associated with the expectation of energy in the optimization of Tsallis entropy  $S_q$ [1]. Although some mathematical details are still under investigation[8, 9], the physical definition of  $\beta$  has been satisfactorily clarified for the same  $q$  subsystems (see [7]

for a general review). Essential of  $\beta$  interpretation in NS is to find, for the nonextensive subsystems in equilibrium or stationary states (with local equilibrium) optimizing the  $S_q$  of composite system, the right quantities that are equal in every subsystems. For example, for thermal equilibrium or local equilibrium, the temperature  $T$  must be equal in every subsystems in order to be measurable, i.e.,  $T(A) = T(B) = \dots$  (zeroth law) where  $A$  and  $B$  represent subsystems. For dynamic and mechanical equilibrium, the pressure  $P$  (or equivalent quantities) must be intensive and equal everywhere, i.e.,  $P(A) = P(B) = \dots$ [7].

In this work, we limit ourself in the discussion of thermal equilibrium and local thermal equilibrium (at the point of measure) concerning temperature. As is well known, for same  $q$  subsystems, the starting point is the composition rule of entropy given by

$$S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B) \quad (1)$$

where the total system  $A + B$  is composed of two subsystems  $A$  and  $B$ . However, for the description of composite systems containing different  $q$  subsystems, this rule should be extended in order to include different  $q$ 's.

The fundamental reasons why this rule must be extended have been previously discussed[7, 10, 11]. For the two formalisms of NS with unnormalized energy expectation  $U_q = \sum_i p_i^q E_i$  ( $\sum_i p_i = 1$ )[7, 10] and  $U_q = \sum_i p_i E_i$  ( $\sum_i p_i^q = 1$ )[11], respectively, a possible extension of Eq.(1) has been proposed, i.e.,

$$\begin{aligned} (1 - q_{A+B})S_q(A + B) &= (1 - q_A)S_{q_A}(A) + (1 - q_B)S_{q_B}(B) \\ &+ (1 - q_A)(1 - q_B)S_{q_A}(A)S_{q_B}(B), \end{aligned} \quad (2)$$

where  $q$  is the parameter for  $A + B$ ,  $q_A$  for  $A$  and  $q_B$  for  $B$ .

In view of the fundamental importance and the wide application[12, 13, 14, 15, 16] of the formalism of NS based on the normalized energy expectation defined with the escort probability ( $U_q = \sum_i p_i^q E_i / \sum_i p_i^q$  with  $\sum_i p_i = 1$ )[3], it would be useful to see the possibility of extending Eq.(1) to different  $q$  subsystems.

This work is a trial of the above approach within this formalism which has an energy probability distribution given by[3]

$$p_i = \frac{[1 - (1 - q) \frac{\beta}{\sum_i p_i^q} (E_i - U_q)]^{\frac{1}{1-q}}}{Z_q} \quad (3)$$

where  $\beta$  is the Lagrange multiplier associated with  $U_q = \frac{\sum_i p_i^q E_i}{\sum_i p_i^q}$ ,  $E_i$  is the energy of the state  $i$ ,  $w$  the total number of states, the partition function is given by  $Z_q = \sum_i^w [1 - (1 - q) \frac{\beta}{\sum_i^w p_i^q} (E_i - U_q)]^{\frac{1}{1-q}}$  or  $Z_q = \sum_i^w [1 - (1 - q) \frac{\beta}{\sum_i^w p_i^q} (E_i - U_q)]^{\frac{q}{1-q}}$ , and we have  $\sum_i p_i^q = Z_q^{1-q}$ . For same  $q$  nonextensive systems having Eq.(1) and (approximately) additive energy  $U_q(A + B) = U_q(A) + U_q(B)$ , the measurable temperature  $T$  has been defined by  $\frac{1}{T} = [1 + (1 - q)S_q]^{-1} \frac{\partial S_q}{\partial U_q} = Z_q^{q-1} \frac{\partial S_q}{\partial U_q} = Z_q^{q-1} \beta$  [12, 13, 14, 15, 16] (Boltzmann constant  $k_B = 1$ ).

## 2 Different $q$ systems with additive energy

The formalism of NS with additive energy or other extensive variables is widely investigated for systems having Tsallis energy [12, 13, 14, 15, 16, 17]. Now let us suppose  $A$  and  $B$  have respectively  $q_A$  and  $q_B$  and Eq.(2) holds. When the composite systems  $(A + B)$  optimizing Tsallis entropy, we have

$$\frac{(1 - q_A) dS_{q_A}(A)}{Z_{q_A}^{1-q_A}(A)} + \frac{(1 - q_B) dS_{q_B}(B)}{Z_{q_B}^{1-q_B}(B)} = 0. \quad (4)$$

Now using the additive energy rule  $U_q(A + B) = U_q(A) + U_q(B)$  and the condition that the total energy  $U_q(A + B)$  is conserved, we get

$$\frac{(1 - q_A) \frac{\partial S_{q_A}(A)}{\partial U_{q_A}(A)}}{Z_{q_A}^{1-q_A}(A)} = \frac{(1 - q_B) \frac{\partial S_{q_B}(B)}{\partial U_{q_B}(B)}}{Z_{q_B}^{1-q_B}(B)}, \quad (5)$$

which means that at thermal equilibrium or local thermal equilibrium (at the point of measure) optimizing Tsallis entropy, the intensive quantity that is equal in both  $A$  and  $B$  is  $\beta' = \frac{(1-q) \frac{\partial S_q}{\partial U_q}}{Z_q^{1-q}} = \frac{(1-q)\beta}{Z_q^{1-q}}$ . So the physical (measurable) temperature  $T$  should be defined by

$$\frac{1}{T} = \frac{(1 - q) \frac{\partial S_q}{\partial U_q}}{Z_q^{1-q}}. \quad (6)$$

It is apparent that this definition only hold for  $q_A$  and  $q_B$  which are both different from unity. But for the case with one of  $q_A$  and  $q_B$  equal to one, the temperature may diverge if  $\frac{\partial S_q}{\partial U_q}$  is finite. For the general validity of NS and the concomitant thermodynamics for different  $q$  systems, this result should

be avoided. A possible way for avoiding this is to use nonadditive energy prescribed by the nonadditivity of entropy Eq.(2) as has been done for the cases of unnormalized expectation[7, 10].

### 3 Different $q$ systems with nonadditive energy

For the purpose of deducing the energy nonadditivity, we take into account the probability composition rule,

$$p_{ij}^q(A+B) = p_i^{q_A}(A)p_j^{q_B}(B) \quad (7)$$

produced by Eq.(2). Considering  $\sum_i p_i^q = Z_q^{1-q}$ , we can write

$$Z_q^{1-q}(A+B) = Z_{q_A}^{1-q_A}(A)Z_{q_B}^{1-q_B}(B). \quad (8)$$

which can be recast into

$$(1-q) \ln Z_q(A+B) = (1-q_A) \ln Z_{q_A}(A) + (1-q_B) \ln Z_{q_B}(B) \quad (9)$$

or in differential form:

$$\begin{aligned} & (1-q) \frac{1}{Z_q(A+B)} \frac{\partial Z_q(A+B)}{\partial U_q(A+B)} dU_q(A+B) \\ = & (1-q_A) \frac{1}{Z_{q_A}(A)} \frac{\partial Z_{q_A}(A)}{\partial U_{q_A}(A)} dU_{q_A}(A) + (1-q_B) \frac{1}{Z_{q_B}(B)} \frac{\partial Z_{q_B}(B)}{\partial U_{q_B}(B)} dU_{q_B}(B). \end{aligned} \quad (10)$$

Now from the definition of  $Z_q$  given in the introduction, we can find

$$\frac{\partial Z_q}{\partial U_q} = Z_q^q \beta. \quad (11)$$

Put this into Eq.(10) and considering  $dU_q(A+B) = 0$ , we see the following energy nonadditivity:

$$\frac{(1-q_A)dU_{q_A}(A)}{Z_{q_A}^{1-q_A}} + \frac{(1-q_B)dU_{q_B}(B)}{Z_{q_B}^{1-q_B}} = 0. \quad (12)$$

Comparing Eq.(12) to Eq.(4), we obtain

$$\frac{\partial S_{q_A}(A)}{\partial U_{q_A}(A)} = \frac{\partial S_{q_B}(B)}{\partial U_{q_B}(B)}. \quad (13)$$

Considering the relationships  $S_q = \frac{Z_q^{1-q}-1}{1-q}$  [3] and Eq.(11), we obtain  $\beta = \frac{\partial S_q}{\partial U_q}$ . So the Lagrange multiplier  $\beta$  in the distribution Eq.(3) should be taken as the measurable inverse temperature. Hence the definition of physical temperature with normalized energy expectation given by escort probability is the same as with the two unnormalized expectations mentioned above [10, 11] if we take into account the nonadditivity of energy prescribed by the entropy nonadditivity.

## 4 Conclusion

We have studied the composition of different  $q$  nonextensive systems and the definition of temperature for this kind of systems with normalized expectation of energy given by the escort probability. Essential of our approach is to find the quantity that is equal in all the subsystems at thermal equilibrium or stationary states optimizing Tsallis entropy. The starting point of this work is a nonadditivity rule of entropy Eq.(2) which generalized the rule of Eq.(1) for same  $q$  systems. The conclusion of this work is that, if we consider the nonadditivity of energy prescribed by the nonadditivity of entropy, the measurable (physical) temperature is give by  $\frac{1}{T} = \frac{\partial S_q}{\partial U_q}$ . This result is the same as obtained with the two unnormalized expectations of energy in our previous work [7, 10, 11].

Summarizing the works on these three possible formalisms of NS, the temperature definition is the same as in the conventional statistical mechanics. As a consequence, the form of the first and second laws of thermodynamics does not change in the nonextensive thermodynamics associated with these formalisms. However, it is not the case with other normalized expectations used in the case of same  $q$  subsystems where the physical temperature is given by  $\frac{\partial S_q}{\partial U_q}$  multiplied by a function of the partition function  $Z_q$  [7]. To our opinion, these formalisms with normalized expectations are not suitable for different  $q$  systems described by Eq.(2).

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