



HAL
open science

A remark on 'Some numerical results in complex differential geometry'

Kefeng Liu, Xiaonan Ma

► **To cite this version:**

Kefeng Liu, Xiaonan Ma. A remark on 'Some numerical results in complex differential geometry'. 2006. hal-00016826v1

HAL Id: hal-00016826

<https://hal.science/hal-00016826v1>

Preprint submitted on 11 Jan 2006 (v1), last revised 18 Jan 2006 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A REMARK ON 'SOME NUMERICAL RESULTS IN COMPLEX DIFFERENTIAL GEOMETRY'

KEFENG LIU AND XIAONAN MA

ABSTRACT. In this note we verify certain statement about the operator Q_K constructed by Donaldson in [3] by using the full asymptotic expansion of Bergman kernel obtained in [2] and [4].

In order to find explicit numerical approximation of Kähler-Einstein metric of projective manifolds, Donaldson introduced in [3] various operators with good properties to approximate classical operators. See the discussions in Section 4.2 of [3] for more details related to our discussion. In this note we verify certain statement of Donaldson about the operator Q_K in Section 4.2 by using the full asymptotic expansion of Bergman kernel derived in [2, Theorem 4.18] and [4, §3.4]. Such statement is needed for the convergence of the approximation procedure.

Let (X, ω, J) be a compact Kähler manifold of $\dim_{\mathbb{C}} X = n$, and let (L, h^L) be a holomorphic Hermitian line bundle on X . Let ∇^L be the holomorphic Hermitian connection on (L, h^L) with curvature R^L . We assume that

$$(1) \quad \frac{\sqrt{-1}}{2\pi} R^L = \omega.$$

Let $g^{TX}(\cdot, \cdot) := \omega(\cdot, J\cdot)$ be the Riemannian metric on TX induced by ω, J . Let dv_X be the Riemannian volume form of (TX, g^{TX}) , then $dv_X = \omega^n/n!$. Let $d\nu$ be any volume form on X . Let η be the positive function on X defined by

$$(2) \quad dv_X = \eta d\nu.$$

The L^2 -scalar product $\langle \cdot \rangle_{\nu}$ on $\mathcal{C}^{\infty}(X, L^p)$, the space of smooth sections of L^p , is given by

$$(3) \quad \langle \sigma_1, \sigma_2 \rangle_{\nu} := \int_X \langle \sigma_1(x), \sigma_2(x) \rangle_{L^p} d\nu(x).$$

Let $P_{\nu,p}(x, x')$ ($x, x' \in X$) be the smooth kernel of the orthogonal projection from $(\mathcal{C}^{\infty}(X, L^p), \langle \cdot \rangle_{\nu})$ onto $H^0(X, L^p)$, the space of the holomorphic sections of L^p on X , with respect to $d\nu(x')$. Note that $P_{\nu,p}(x, x') \in L_x^p \otimes L_{x'}^{p*}$. Following [3, §4], set

$$(4) \quad K_p(x, x') := |P_{\nu,p}(x, x')|_{h_x^{L^p} \otimes h_{x'}^{L^{p*}}}, \quad R_p := (\dim H^0(X, L^p))/\text{Vol}(X, \nu),$$

here $\text{Vol}(X, \nu) := \int_X d\nu$. Set $\text{Vol}(X, dv_X) := \int_X dv_X$.

Let Q_{K_p} be the integral operator associated to K_p which is defined for $f \in \mathcal{C}^{\infty}(X)$,

$$(5) \quad Q_{K_p}(f)(x) := \frac{1}{R_p} \int_X K_p(x, y) f(y) d\nu(y).$$

Let Δ be the (positive) Laplace operator on (X, g^{TX}) acting on the functions on X . We denote by $|\cdot|_{L^2}$ the L^2 -norm on the function on X with respect to dv_X .

Theorem 1. *There exists a constant $C > 0$ such that for any $f \in \mathcal{C}^\infty(X)$, $p \in \mathbb{N}$,*

$$(6) \quad \begin{aligned} & \left| \left(Q_{K_p} - \frac{\text{Vol}(X, \nu)}{\text{Vol}(X, dv_X)} \exp\left(-\frac{\Delta}{4\pi p}\right) \eta \right) f \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}, \\ & \left| \left(\frac{\Delta}{p} Q_{K_p} - \frac{\text{Vol}(X, \nu)}{\text{Vol}(X, dv_X)} \frac{\Delta}{p} \exp\left(-\frac{\Delta}{4\pi p}\right) \eta \right) f \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}. \end{aligned}$$

Moreover, (6) is uniform in that there is an integer s such that if all data h^L , dv run over a set which are bounded in \mathcal{C}^s and that g^{TX} , dv_X are bounded from below, then the constant C is independent of h^L , dv .

Proof. We explain at first the full asymptotic expansion of $P_{\nu,p}(x, x')$ from [2, Theorem 4.18'] and [4, §3.4]. For more details on our approach we also refer the readers to the recent book [5].

Let $E = \mathbb{C}$ be the trivial holomorphic line bundle on X . Let h^E the metric on E defined by $|1|_{h^E}^2 = 1$, here 1 is the canonical unity element of E . We identify canonically L^p to $L^p \otimes E$ by Section 1.

As in [4, §3.4], let h_ω^E be the metric on E defined by $|1|_{h_\omega^E}^2 = \eta^{-1}$, here 1 is the canonical unity element of E . Let $\langle \cdot, \cdot \rangle_\omega$ be the Hermitian product on $\mathcal{C}^\infty(X, L^p \otimes E) = \mathcal{C}^\infty(X, L^p)$ induced by h^L , h_ω^E , dv_X as in (3). Then by (2),

$$(7) \quad (\mathcal{C}^\infty(X, L^p \otimes E), \langle \cdot, \cdot \rangle_\omega) = (\mathcal{C}^\infty(X, L^p), \langle \cdot, \cdot \rangle_\nu).$$

Observe that $H^0(X, L^p \otimes E)$ does not depend on g^{TX} , h^L or h^E . If $P_{\omega,p}(x, x')$, $(x, x' \in X)$ denotes the smooth kernel of the orthogonal projection $P_{\omega,p}$ from $(\mathcal{C}^\infty(X, L^p \otimes E), \langle \cdot, \cdot \rangle_\omega)$ onto $H^0(X, L^p \otimes E) = H^0(X, L^p)$ with respect to $dv_X(x)$, from (2), as in [4, (3.38)], we have

$$(8) \quad P_{\nu,p}(x, x') = \eta(x') P_{\omega,p}(x, x').$$

For $f \in \mathcal{C}^\infty(X)$, set

$$(9) \quad \begin{aligned} K_{\omega,p}(x, x') &= |P_{\omega,p}(x, x')|_{(h^{L^p} \otimes h_\omega^E)_x \otimes (h^{L^p*} \otimes h_\omega^{E*})_{x'}}^2, \\ (K_{\omega,p}f)(x) &= \int_X K_{\omega,p}(x, y) f(y) dv_X(y). \end{aligned}$$

Then from (4), (8),

$$(10) \quad K_p(x, x') = \eta^2(x') K_{\omega,p}(x, x'),$$

and from (2) and (5),

$$(11) \quad Q_{K_p}(f)(x) = \frac{1}{R_p} \int_X K_{\omega,p}(x, y) \eta(y) f(y) dv_X(y).$$

Now for the kernel $P_{\omega,p}(x, x')$, we can apply the full asymptotic expansion [2, Theorem 4.18']. In fact let $\bar{\partial}^{L^p \otimes E, * \omega}$ be the formal adjoint of the Dolbeault operator $\bar{\partial}^{L^p \otimes E}$ on the Dolbeault complex $\Omega^{0, \bullet}(X, L^p \otimes E)$ with the scalar product induced by g^{TX} , h^L , h_ω^E , dv_X as in (3), and set

$$(12) \quad D_p = \sqrt{2}(\bar{\partial}^{L^p \otimes E} + \bar{\partial}^{L^p \otimes E, * \omega}).$$

Then $H^0(X, L^p \otimes E) = \text{Ker } D_p$ for p large enough, and D_p is a Dirac operator, as $g^{TX}(\cdot, \cdot) = \omega(\cdot, J\cdot)$ is a Kähler metric on TX .

Let ∇^E be the holomorphic Hermitian connection on (E, h_ω^E) . Let ∇^{TX} be the Levi-Civita connection on (TX, g^{TX}) . Let R^E, R^{TX} be the corresponding curvatures.

Let a^X be the injectivity radius of (X, g^{TX}) . We fix $\varepsilon \in]0, a^X/4[$. We denote by $B^X(x, \varepsilon)$ and $B^{T_x X}(0, \varepsilon)$ the open balls in X and $T_x X$ with center x and radius ε . We identify $B^{T_x X}(0, \varepsilon)$ with $B^X(x, \varepsilon)$ by using the exponential map of (X, g^{TX}) .

We fix $x_0 \in X$. For $Z \in B^{T_{x_0} X}(0, \varepsilon)$ we identify L_Z, E_Z and $(L^p \otimes E)_Z$ to L_{x_0}, E_{x_0} and $(L^p \otimes E)_{x_0}$ by parallel transport with respect to the connections ∇^L, ∇^E and $\nabla^{L^p \otimes E}$ along the curve $\gamma_Z : [0, 1] \ni u \rightarrow \exp_{x_0}^X(uZ)$. Then under our identification, $P_{\omega, p}(Z, Z')$ is a function on $Z, Z' \in T_{x_0} X, |Z|, |Z'| \leq \varepsilon$, we denote it by $P_{\omega, p, x_0}(Z, Z')$. Let $\pi : TX \times_X TX \rightarrow X$ be the natural projection from the fiberwise product of TX on X . Then we can view $P_{\omega, p, x_0}(Z, Z')$ as a smooth function on $TX \times_X TX$ (which is defined for $|Z|, |Z'| \leq \varepsilon$) by identifying a section $S \in \mathcal{C}^\infty(TX \times_X TX, \pi^* \text{End}(E))$ with the family $(S_x)_{x \in X}$, where $S_x = S|_{\pi^{-1}(x)}, \text{End}(E) = \mathbb{C}$.

We choose $\{w_i\}_{i=1}^n$ an orthonormal basis of $T_{x_0}^{(1,0)} X$, then $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \bar{w}_j)$ and $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \bar{w}_j), j = 1, \dots, n$ forms an orthonormal basis of $T_{x_0} X$. We use the coordinates on $T_{x_0} X \simeq \mathbb{R}^{2n}$ where the identification is given by

$$(13) \quad (Z_1, \dots, Z_{2n}) \in \mathbb{R}^{2n} \longrightarrow \sum_i Z_i e_i \in T_{x_0} X.$$

In what follows we also introduce the complex coordinates $z = (z_1, \dots, z_n)$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. By [2, (4.114)] (cf. [4, (1.91)]), set

$$(14) \quad P^N(Z, Z') = \exp\left(-\frac{\pi}{2} \sum_i (|z_i|^2 + |z'_i|^2 - 2z_i \bar{z}'_i)\right).$$

Then P^N is the classical Bergman kernel on \mathbb{C}^n (cf. [4, Remark 1.14]) and

$$(15) \quad |P^N(Z, Z')|^2 = e^{-\pi|Z-Z'|^2}.$$

By [2, Proposition 4.1], for any $l, m \in \mathbb{N}, \varepsilon > 0$, there exists $C_{l, m, \varepsilon} > 0$ such that for $p \geq 1, x, x' \in X$,

$$(16) \quad |P_{\omega, p}(x, x')|_{\mathcal{C}^m(X \times X)} \leq C_{l, m, \varepsilon} p^{-l} \quad \text{if } d(x, x') \geq \varepsilon.$$

Here the \mathcal{C}^m -norm is induced by $\nabla^L, \nabla^E, \nabla^{TX}$ and h^L, h^E, g^{TX} .

By [2, Theorem 4.18'], there exist $J_r(Z, Z')$ polynomials in Z, Z' , such that for any $k, m, m' \in \mathbb{N}$, there exist $N \in \mathbb{N}, C > 0, C_0 > 0$ such that for $\alpha, \alpha' \in \mathbb{N}^n, |\alpha| + |\alpha'| \leq m, Z, Z' \in T_{x_0} X, |Z|, |Z'| \leq \varepsilon, x_0 \in X, p \geq 1$,

$$(17) \quad \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left(\frac{1}{p^n} P_{\omega, p, x_0}(Z, Z') - \sum_{r=0}^k (J_r P^N)(\sqrt{p}Z, \sqrt{p}Z') p^{-r/2} \right) \right|_{\mathcal{C}^{m'}(X)} \leq C p^{-(k+1-m)/2} (1 + |\sqrt{p}Z| + |\sqrt{p}Z'|)^N \exp(-C_0 \sqrt{p}|Z - Z'|) + \mathcal{O}(p^{-\infty}).$$

Here $\mathcal{C}^{m'}(X)$ is the $\mathcal{C}^{m'}$ norm for the parameter $x_0 \in X$. The term $\mathcal{O}(p^{-\infty})$ means that for any $l, l_1 \in \mathbb{N}$, there exists $C_{l, l_1} > 0$ such that its \mathcal{C}^{l_1} -norm is dominated by $C_{l, l_1} p^{-l}$. (In fact, by [2, Theorems 4.6 and 4.17, (4.117)] (cf. [4, Theorem 1.18, (1.31)]),

the polynomials $J_r(Z, Z')$ have the same parity as r and $\deg J_r(Z, Z') \leq 3r$, whose coefficients are polynomials in R^{TX} , R^E and their derivatives of order $\leq r-1$.

Now we claim that in (17),

$$(18) \quad J_0 = 1, \quad J_1(Z, Z') = 0.$$

In fact, let $dv_{T_{x_0}X}$ be the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$, and κ be the function defined by

$$(19) \quad dv_X(Z) = \kappa(x_0, Z)dv_{T_{x_0}X}(Z).$$

Then (also cf. [4, (1.31)])

$$(20) \quad \kappa(x_0, Z) = 1 + \frac{1}{6} \langle R_{x_0}^{TX}(Z, e_i)Z, e_i \rangle_{x_0} + \mathcal{O}(|Z|^3).$$

As we only work on $\mathcal{C}^\infty(X, L^p \otimes E)$, by [2, (4.115)], we get the first equation in (18).

Recall that in the normal coordinate, after the rescaling $Z \rightarrow Z/t$ with $t = \frac{1}{\sqrt{p}}$, we get an operator \mathcal{L}_t from the restriction of D_p^2 on $\mathcal{C}^\infty(X, L^p \otimes E)$ which has the following formal expansion (cf. [2, (1.104)], [4, Theorem 1.4]),

$$(21) \quad \mathcal{L}_t = \mathcal{L} + \sum_{r=1}^{\infty} \mathcal{Q}_r t^r.$$

Now, from [2, Theorem 5.1] (or [4, (1.87), (1.97)]),

$$(22) \quad \mathcal{L} = \sum_{j=1}^n (-2\frac{\partial}{\partial z_i} + \pi \bar{z}_i)(2\frac{\partial}{\partial \bar{z}_i} + \pi z_i), \quad \mathcal{Q}_1 = 0.$$

(In fact, $P^N(Z, Z')$ is the smooth kernel of the orthogonal projection from $L^2(\mathbb{R})$ onto $\text{Ker}(\mathcal{L})$). Thus from [2, (4.107)] (cf. [4, (1.111)]), (20) and (22) we get the second equation of (18).

Note that $|P_{\omega, p, x_0}(Z, Z')|^2 = P_{\omega, p, x_0}(Z, Z') \overline{P_{\omega, p, x_0}(Z, Z')}$, thus from (9), (17) and (18), there exist $J'_r(Z, Z')$ polynomials in Z, Z' such that

$$(23) \quad \left| \frac{1}{p^{2n+1}} \Delta_Z \left(K_{\omega, p, x_0}(Z, Z') - \left(1 + \sum_{r=2}^k p^{-r/2} J'_r(\sqrt{p}Z, \sqrt{p}Z') \right) e^{-\pi p|Z-Z'|^2} \right) \right| \leq Cp^{-(k+1)/2} (1 + |\sqrt{p}Z| + |\sqrt{p}Z'|)^N \exp(-C_0\sqrt{p}|Z-Z'|) + \mathcal{O}(p^{-\infty}).$$

For a function $f \in \mathcal{C}^\infty(X)$, we denote it as $f(x_0, Z)$ a family (with parameter x_0) of function on Z in the normal coordinate near x_0 . Now, for any polynomial $Q_{x_0}(Z')$, we define the operator

$$(24) \quad (\mathcal{Q}_p f)(x_0) = p^n \int_{|Z'| \leq \varepsilon} Q_{x_0}(\sqrt{p}Z') e^{-\pi p|Z'|^2} f(x_0, Z') dv_X(x_0, Z').$$

Then we observe that there exists $C_1 > 0$ such that for any $p \in \mathbb{N}$, $f \in \mathcal{C}^\infty(X)$, we have

$$(25) \quad |\mathcal{Q}_p f|_{L^2} \leq C_1 |f|_{L^2}.$$

In fact,

$$\begin{aligned}
 (26) \quad |\mathcal{Q}_p f|_{L^2}^2 &\leq \int_X dv_X(x_0) \left\{ p^n \left(\int_{|Z'| \leq \varepsilon} |Q_{x_0}(\sqrt{p}Z')| e^{-\pi p|Z'|^2} dv_X(x_0, Z') \right) \right. \\
 &\quad \left. \times p^n \left(\int_{|Z'| \leq \varepsilon} |Q_{x_0}(\sqrt{p}Z')| e^{-\pi p|Z'|^2} |f(x_0, Z')|^2 dv_X(x_0, Z') \right) \right\} \\
 &\leq C' \int_X dv_X(x_0) p^n \int_{|Z'| \leq \varepsilon} |Q_{x_0}(\sqrt{p}Z')| e^{-\pi p|Z'|^2} |f(x_0, Z')|^2 dv_X(x_0, Z') \\
 &\leq C_1 |f|_{L^2}^2.
 \end{aligned}$$

Observe that in the normal coordinate, at $Z = 0$, $\Delta_Z = -\sum_{j=1}^{2n} \frac{\partial^2}{\partial Z_j^2}$. Thus

$$(27) \quad (\Delta_Z e^{-\pi p|Z-Z'|^2})|_{Z=0} = 4\pi p(n - \pi p|Z'|^2) e^{-\pi p|Z'|^2}.$$

Thus from (15), (17), (18) and (23), we get

$$\begin{aligned}
 (28) \quad &\left| p^{-n} K_{\omega,p} f - \int_{|Z'| \leq \varepsilon} e^{-\pi p|Z'|^2} f(x_0, Z') dv_X(x_0, Z') \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}, \\
 &\left| p^{-n-1} \Delta K_{\omega,p} f - 4\pi \int_{|Z'| \leq \varepsilon} (n - \pi p|Z'|^2) e^{-\pi p|Z'|^2} f(x_0, Z') dv_X(x_0, Z') \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}.
 \end{aligned}$$

Let $e^{-u\Delta}(x, x')$ be the smooth kernel of the heat operator $e^{-u\Delta}$ with respect to $dv_X(x')$. We still use the normal coordinate, by the heat kernel expansion in [1, Theorems 2.23, 2.26], there exist $\phi_{i,x_0}(Z')$ such that uniformly for $x_0 \in X$, $Z' \in T_{x_0}X$, $|Z'| \leq \varepsilon$, we have the following asymptotic expansion when $u \rightarrow 0$,

$$(29) \quad \left| \frac{\partial^l}{\partial u^l} \left(e^{-u\Delta}(0, Z') - (4\pi u)^{-n} \sum_{i=0}^k u^i \phi_{i,x_0}(Z') e^{-\frac{1}{4u}|Z'|^2} \right) \right| = \mathcal{O}(u^{k-n-l+1}),$$

with

$$(30) \quad \phi_{0,x_0}(Z') = 1.$$

Observe that

$$(31) \quad \frac{1}{p} \Delta \exp\left(-\frac{\Delta}{4\pi p}\right) = -\frac{1}{p} \left(\frac{\partial}{\partial u} e^{-u\Delta} \right) \Big|_{u=\frac{1}{4\pi p}}.$$

Now from (25), (28)–(31), we get

$$\begin{aligned}
 (32) \quad &\left| \left(p^{-n} K_{\omega,p} - \exp\left(-\frac{\Delta}{4\pi p}\right) \right) f \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}, \\
 &\left| \frac{1}{p} \left(p^{-n} \Delta K_{\omega,p} - \Delta \exp\left(-\frac{\Delta}{4\pi p}\right) \right) f \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}.
 \end{aligned}$$

Note that $R_p = \frac{\text{Vol}(X, dv_X)}{\text{Vol}(X, \nu)} p^n + \mathcal{O}(p^{n-1})$. From (11), (32), we get (6).

To get the last part of Theorem 1, as we noticed in [2, §4.5], the constants in (17) will be uniformly bounded under our condition, thus we can take C in (6) and (32) independent of h^L , dv . \square

Acknowledgments. We thank Professor Simon Donaldson for useful communications.

REFERENCES

- [1] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Grundle Math. Wiss. Band 298, Springer-Verlag, Berlin, 1992.
- [2] X. Dai, K. Liu, and X. Ma, *On the asymptotic expansion of Bergman kernel*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 3, 193–198. The full version: J. Differential Geom. to appear, math.DG/0404494.
- [3] S. K. Donaldson, *Some numerical results in complex differential geometry*, math.DG/0512625.
- [4] X. Ma and G. Marinescu, *Generalized Bergman kernels on symplectic manifolds*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 7, 493–498. The full version: math.DG/0411559.
- [5] ———, *Holomorphic Morse Inequalities and Bergman Kernels*, book in preparation, (2006).

CENTER OF MATHEMATICAL SCIENCE, ZHEJIANG UNIVERSITY AND DEPARTMENT OF MATHEMATICS, UCLA, CA 90095-1555, USA (LIU@MATH.UCLA.EDU)

CENTRE DE MATHÉMATIQUES LAURENT SCHWARTZ, UMR 7640 DU CNRS, ECOLE POLYTECHNIQUE, 91128 PALAISEAU CEDEX, FRANCE (MA@MATH.POLYTECHNIQUE.FR)