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A REMARK ON 'SOME NUMERICAL RESULTS IN COMPLEX DIFFERENTIAL GEOMETRY'

KEFENG LIU AND XIAONAN MA

ABSTRACT. In this note we verify certain statement about the operator Q_K constructed by Donaldson in [3] by using the full asymptotic expansion of Bergman kernel obtained in [2] and [4].

In order to find explicit numerical approximation of Kähler-Einstein metric of projective manifolds, Donaldson introduced in [3] various operators with good properties to approximate classical operators. See the discussions in Section 4.2 of [3] for more details related to our discussion. In this note we verify certain statement of Donaldson about the operator Q_K in Section 4.2 by using the full asymptotic expansion of Bergman kernel derived in [2, Theorem 4.18] and [4, §3.4]. Such statement is needed for the convergence of the approximation procedure.

Let (X, ω, J) be a compact Kähler manifold of $\dim_{\mathbb{C}} X = n$, and let (L, h^L) be a holomorphic Hermitian line bundle on X . Let ∇^L be the holomorphic Hermitian connection on (L, h^L) with curvature R^L . We assume that

$$(1) \quad \frac{\sqrt{-1}}{2\pi} R^L = \omega.$$

Let $g^{TX}(\cdot, \cdot) := \omega(\cdot, J\cdot)$ be the Riemannian metric on TX induced by ω, J . Let dv_X be the Riemannian volume form of (TX, g^{TX}) , then $dv_X = \omega^n/n!$. Let $d\nu$ be any volume form on X . Let η be the positive function on X defined by

$$(2) \quad dv_X = \eta d\nu.$$

The L^2 -scalar product $\langle \cdot \rangle_{\nu}$ on $\mathcal{C}^{\infty}(X, L^p)$, the space of smooth sections of L^p , is given by

$$(3) \quad \langle \sigma_1, \sigma_2 \rangle_{\nu} := \int_X \langle \sigma_1(x), \sigma_2(x) \rangle_{L^p} d\nu(x).$$

Let $P_{\nu,p}(x, x')$ ($x, x' \in X$) be the smooth kernel of the orthogonal projection from $(\mathcal{C}^{\infty}(X, L^p), \langle \cdot \rangle_{\nu})$ onto $H^0(X, L^p)$, the space of the holomorphic sections of L^p on X , with respect to $d\nu(x')$. Note that $P_{\nu,p}(x, x') \in L_x^p \otimes L_{x'}^{p*}$. Following [3, §4], set

$$(4) \quad K_p(x, x') := |P_{\nu,p}(x, x')|_{h_x^{L^p} \otimes h_{x'}^{L^{p*}}}, \quad R_p := (\dim H^0(X, L^p))/\text{Vol}(X, \nu),$$

here $\text{Vol}(X, \nu) := \int_X d\nu$. Set $\text{Vol}(X, dv_X) := \int_X dv_X$.

Let Q_{K_p} be the integral operator associated to K_p which is defined for $f \in \mathcal{C}^{\infty}(X)$,

$$(5) \quad Q_{K_p}(f)(x) := \frac{1}{R_p} \int_X K_p(x, y) f(y) d\nu(y).$$

Let Δ be the (positive) Laplace operator on (X, g^{TX}) acting on the functions on X . We denote by $|\cdot|_{L^2}$ the L^2 -norm on the function on X with respect to dv_X .

Theorem 1. *There exists a constant $C > 0$ such that for any $f \in \mathcal{C}^\infty(X)$, $p \in \mathbb{N}$,*

$$(6) \quad \begin{aligned} & \left| \left(Q_{K_p} - \frac{\text{Vol}(X, \nu)}{\text{Vol}(X, dv_X)} \exp\left(-\frac{\Delta}{4\pi p}\right) \eta \right) f \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}, \\ & \left| \left(\frac{\Delta}{p} Q_{K_p} - \frac{\text{Vol}(X, \nu)}{\text{Vol}(X, dv_X)} \frac{\Delta}{p} \exp\left(-\frac{\Delta}{4\pi p}\right) \eta \right) f \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}. \end{aligned}$$

Moreover, (6) is uniform in that there is an integer s such that if all data h^L , dv run over a set which are bounded in \mathcal{C}^s and that g^{TX} , dv_X are bounded from below, then the constant C is independent of h^L , dv .

Proof. We explain at first the full asymptotic expansion of $P_{\nu,p}(x, x')$ from [2, Theorem 4.18'] and [4, §3.4]. For more details on our approach we also refer the readers to the recent book [5].

Let $E = \mathbb{C}$ be the trivial holomorphic line bundle on X . Let h^E the metric on E defined by $|1|_{h^E}^2 = 1$, here 1 is the canonical unity element of E . We identify canonically L^p to $L^p \otimes E$ by Section 1.

As in [4, §3.4], let h_ω^E be the metric on E defined by $|1|_{h_\omega^E}^2 = \eta^{-1}$, here 1 is the canonical unity element of E . Let $\langle \cdot, \cdot \rangle_\omega$ be the Hermitian product on $\mathcal{C}^\infty(X, L^p \otimes E) = \mathcal{C}^\infty(X, L^p)$ induced by h^L , h_ω^E , dv_X as in (3). Then by (2),

$$(7) \quad (\mathcal{C}^\infty(X, L^p \otimes E), \langle \cdot, \cdot \rangle_\omega) = (\mathcal{C}^\infty(X, L^p), \langle \cdot, \cdot \rangle_\nu).$$

Observe that $H^0(X, L^p \otimes E)$ does not depend on g^{TX} , h^L or h^E . If $P_{\omega,p}(x, x')$, $(x, x' \in X)$ denotes the smooth kernel of the orthogonal projection $P_{\omega,p}$ from $(\mathcal{C}^\infty(X, L^p \otimes E), \langle \cdot, \cdot \rangle_\omega)$ onto $H^0(X, L^p \otimes E) = H^0(X, L^p)$ with respect to $dv_X(x)$, from (2), as in [4, (3.38)], we have

$$(8) \quad P_{\nu,p}(x, x') = \eta(x') P_{\omega,p}(x, x').$$

For $f \in \mathcal{C}^\infty(X)$, set

$$(9) \quad \begin{aligned} K_{\omega,p}(x, x') &= |P_{\omega,p}(x, x')|_{(h^{L^p} \otimes h_\omega^E)_x \otimes (h^{L^p*} \otimes h_\omega^{E*})_{x'}}^2, \\ (K_{\omega,p}f)(x) &= \int_X K_{\omega,p}(x, y) f(y) dv_X(y). \end{aligned}$$

Then from (4), (8),

$$(10) \quad K_p(x, x') = \eta^2(x') K_{\omega,p}(x, x'),$$

and from (2) and (5),

$$(11) \quad Q_{K_p}(f)(x) = \frac{1}{R_p} \int_X K_{\omega,p}(x, y) \eta(y) f(y) dv_X(y).$$

Now for the kernel $P_{\omega,p}(x, x')$, we can apply the full asymptotic expansion [2, Theorem 4.18']. In fact let $\bar{\partial}^{L^p \otimes E, * \omega}$ be the formal adjoint of the Dolbeault operator $\bar{\partial}^{L^p \otimes E}$ on the Dolbeault complex $\Omega^{0, \bullet}(X, L^p \otimes E)$ with the scalar product induced by g^{TX} , h^L , h_ω^E , dv_X as in (3), and set

$$(12) \quad D_p = \sqrt{2}(\bar{\partial}^{L^p \otimes E} + \bar{\partial}^{L^p \otimes E, * \omega}).$$

Then $H^0(X, L^p \otimes E) = \text{Ker } D_p$ for p large enough, and D_p is a Dirac operator, as $g^{TX}(\cdot, \cdot) = \omega(\cdot, J\cdot)$ is a Kähler metric on TX .

Let ∇^E be the holomorphic Hermitian connection on (E, h_ω^E) . Let ∇^{TX} be the Levi-Civita connection on (TX, g^{TX}) . Let R^E, R^{TX} be the corresponding curvatures.

Let a^X be the injectivity radius of (X, g^{TX}) . We fix $\varepsilon \in]0, a^X/4[$. We denote by $B^X(x, \varepsilon)$ and $B^{T_x X}(0, \varepsilon)$ the open balls in X and $T_x X$ with center x and radius ε . We identify $B^{T_x X}(0, \varepsilon)$ with $B^X(x, \varepsilon)$ by using the exponential map of (X, g^{TX}) .

We fix $x_0 \in X$. For $Z \in B^{T_{x_0} X}(0, \varepsilon)$ we identify L_Z, E_Z and $(L^p \otimes E)_Z$ to L_{x_0}, E_{x_0} and $(L^p \otimes E)_{x_0}$ by parallel transport with respect to the connections ∇^L, ∇^E and $\nabla^{L^p \otimes E}$ along the curve $\gamma_Z : [0, 1] \ni u \rightarrow \exp_{x_0}^X(uZ)$. Then under our identification, $P_{\omega, p}(Z, Z')$ is a function on $Z, Z' \in T_{x_0} X, |Z|, |Z'| \leq \varepsilon$, we denote it by $P_{\omega, p, x_0}(Z, Z')$. Let $\pi : TX \times_X TX \rightarrow X$ be the natural projection from the fiberwise product of TX on X . Then we can view $P_{\omega, p, x_0}(Z, Z')$ as a smooth function on $TX \times_X TX$ (which is defined for $|Z|, |Z'| \leq \varepsilon$) by identifying a section $S \in \mathcal{C}^\infty(TX \times_X TX, \pi^* \text{End}(E))$ with the family $(S_x)_{x \in X}$, where $S_x = S|_{\pi^{-1}(x)}, \text{End}(E) = \mathbb{C}$.

We choose $\{w_i\}_{i=1}^n$ an orthonormal basis of $T_{x_0}^{(1,0)} X$, then $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \bar{w}_j)$ and $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \bar{w}_j), j = 1, \dots, n$ forms an orthonormal basis of $T_{x_0} X$. We use the coordinates on $T_{x_0} X \simeq \mathbb{R}^{2n}$ where the identification is given by

$$(13) \quad (Z_1, \dots, Z_{2n}) \in \mathbb{R}^{2n} \longrightarrow \sum_i Z_i e_i \in T_{x_0} X.$$

In what follows we also introduce the complex coordinates $z = (z_1, \dots, z_n)$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. By [2, (4.114)] (cf. [4, (1.91)]), set

$$(14) \quad P^N(Z, Z') = \exp\left(-\frac{\pi}{2} \sum_i (|z_i|^2 + |z'_i|^2 - 2z_i \bar{z}'_i)\right).$$

Then P^N is the classical Bergman kernel on \mathbb{C}^n (cf. [4, Remark 1.14]) and

$$(15) \quad |P^N(Z, Z')|^2 = e^{-\pi|Z-Z'|^2}.$$

By [2, Proposition 4.1], for any $l, m \in \mathbb{N}, \varepsilon > 0$, there exists $C_{l, m, \varepsilon} > 0$ such that for $p \geq 1, x, x' \in X$,

$$(16) \quad |P_{\omega, p}(x, x')|_{\mathcal{C}^m(X \times X)} \leq C_{l, m, \varepsilon} p^{-l} \quad \text{if } d(x, x') \geq \varepsilon.$$

Here the \mathcal{C}^m -norm is induced by $\nabla^L, \nabla^E, \nabla^{TX}$ and h^L, h^E, g^{TX} .

By [2, Theorem 4.18'], there exist $J_r(Z, Z')$ polynomials in Z, Z' , such that for any $k, m, m' \in \mathbb{N}$, there exist $N \in \mathbb{N}, C > 0, C_0 > 0$ such that for $\alpha, \alpha' \in \mathbb{N}^n, |\alpha| + |\alpha'| \leq m, Z, Z' \in T_{x_0} X, |Z|, |Z'| \leq \varepsilon, x_0 \in X, p \geq 1$,

$$(17) \quad \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left(\frac{1}{p^n} P_{\omega, p, x_0}(Z, Z') - \sum_{r=0}^k (J_r P^N)(\sqrt{p}Z, \sqrt{p}Z') p^{-r/2} \right) \right|_{\mathcal{C}^{m'}(X)} \leq C p^{-(k+1-m)/2} (1 + |\sqrt{p}Z| + |\sqrt{p}Z'|)^N \exp(-C_0 \sqrt{p}|Z - Z'|) + \mathcal{O}(p^{-\infty}).$$

Here $\mathcal{C}^{m'}(X)$ is the $\mathcal{C}^{m'}$ norm for the parameter $x_0 \in X$. The term $\mathcal{O}(p^{-\infty})$ means that for any $l, l_1 \in \mathbb{N}$, there exists $C_{l, l_1} > 0$ such that its \mathcal{C}^{l_1} -norm is dominated by $C_{l, l_1} p^{-l}$. (In fact, by [2, Theorems 4.6 and 4.17, (4.117)] (cf. [4, Theorem 1.18, (1.31)]),

the polynomials $J_r(Z, Z')$ have the same parity as r and $\deg J_r(Z, Z') \leq 3r$, whose coefficients are polynomials in R^{TX} , R^E and their derivatives of order $\leq r-1$.

Now we claim that in (17),

$$(18) \quad J_0 = 1, \quad J_1(Z, Z') = 0.$$

In fact, let $dv_{T_{x_0}X}$ be the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$, and κ be the function defined by

$$(19) \quad dv_X(Z) = \kappa(x_0, Z)dv_{T_{x_0}X}(Z).$$

Then (also cf. [4, (1.31)])

$$(20) \quad \kappa(x_0, Z) = 1 + \frac{1}{6} \langle R_{x_0}^{TX}(Z, e_i)Z, e_i \rangle_{x_0} + \mathcal{O}(|Z|^3).$$

As we only work on $\mathcal{C}^\infty(X, L^p \otimes E)$, by [2, (4.115)], we get the first equation in (18).

Recall that in the normal coordinate, after the rescaling $Z \rightarrow Z/t$ with $t = \frac{1}{\sqrt{p}}$, we get an operator \mathcal{L}_t from the restriction of D_p^2 on $\mathcal{C}^\infty(X, L^p \otimes E)$ which has the following formal expansion (cf. [2, (1.104)], [4, Theorem 1.4]),

$$(21) \quad \mathcal{L}_t = \mathcal{L} + \sum_{r=1}^{\infty} \mathcal{Q}_r t^r.$$

Now, from [2, Theorem 5.1] (or [4, (1.87), (1.97)]),

$$(22) \quad \mathcal{L} = \sum_{j=1}^n (-2\frac{\partial}{\partial z_i} + \pi \bar{z}_i)(2\frac{\partial}{\partial \bar{z}_i} + \pi z_i), \quad \mathcal{Q}_1 = 0.$$

(In fact, $P^N(Z, Z')$ is the smooth kernel of the orthogonal projection from $L^2(\mathbb{R})$ onto $\text{Ker}(\mathcal{L})$). Thus from [2, (4.107)] (cf. [4, (1.111)]), (20) and (22) we get the second equation of (18).

Note that $|P_{\omega, p, x_0}(Z, Z')|^2 = P_{\omega, p, x_0}(Z, Z') \overline{P_{\omega, p, x_0}(Z, Z')}$, thus from (9), (17) and (18), there exist $J'_r(Z, Z')$ polynomials in Z, Z' such that

$$(23) \quad \left| \frac{1}{p^{2n+1}} \Delta_Z \left(K_{\omega, p, x_0}(Z, Z') - \left(1 + \sum_{r=2}^k p^{-r/2} J'_r(\sqrt{p}Z, \sqrt{p}Z') \right) e^{-\pi p|Z-Z'|^2} \right) \right| \leq Cp^{-(k+1)/2} (1 + |\sqrt{p}Z| + |\sqrt{p}Z'|)^N \exp(-C_0\sqrt{p}|Z-Z'|) + \mathcal{O}(p^{-\infty}).$$

For a function $f \in \mathcal{C}^\infty(X)$, we denote it as $f(x_0, Z)$ a family (with parameter x_0) of function on Z in the normal coordinate near x_0 . Now, for any polynomial $Q_{x_0}(Z')$, we define the operator

$$(24) \quad (\mathcal{Q}_p f)(x_0) = p^n \int_{|Z'| \leq \varepsilon} Q_{x_0}(\sqrt{p}Z') e^{-\pi p|Z'|^2} f(x_0, Z') dv_X(x_0, Z').$$

Then we observe that there exists $C_1 > 0$ such that for any $p \in \mathbb{N}$, $f \in \mathcal{C}^\infty(X)$, we have

$$(25) \quad |\mathcal{Q}_p f|_{L^2} \leq C_1 |f|_{L^2}.$$

In fact,

$$\begin{aligned}
 (26) \quad |\mathcal{Q}_p f|_{L^2}^2 &\leq \int_X dv_X(x_0) \left\{ p^n \left(\int_{|Z'| \leq \varepsilon} |Q_{x_0}(\sqrt{p}Z')| e^{-\pi p|Z'|^2} dv_X(x_0, Z') \right) \right. \\
 &\quad \left. \times p^n \left(\int_{|Z'| \leq \varepsilon} |Q_{x_0}(\sqrt{p}Z')| e^{-\pi p|Z'|^2} |f(x_0, Z')|^2 dv_X(x_0, Z') \right) \right\} \\
 &\leq C' \int_X dv_X(x_0) p^n \int_{|Z'| \leq \varepsilon} |Q_{x_0}(\sqrt{p}Z')| e^{-\pi p|Z'|^2} |f(x_0, Z')|^2 dv_X(x_0, Z') \\
 &\leq C_1 |f|_{L^2}^2.
 \end{aligned}$$

Observe that in the normal coordinate, at $Z = 0$, $\Delta_Z = -\sum_{j=1}^{2n} \frac{\partial^2}{\partial Z_j^2}$. Thus

$$(27) \quad (\Delta_Z e^{-\pi p|Z-Z'|^2})|_{Z=0} = 4\pi p(n - \pi p|Z'|^2) e^{-\pi p|Z'|^2}.$$

Thus from (15), (17), (18) and (23), we get

$$\begin{aligned}
 (28) \quad &\left| p^{-n} K_{\omega,p} f - \int_{|Z'| \leq \varepsilon} e^{-\pi p|Z'|^2} f(x_0, Z') dv_X(x_0, Z') \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}, \\
 &\left| p^{-n-1} \Delta K_{\omega,p} f - 4\pi \int_{|Z'| \leq \varepsilon} (n - \pi p|Z'|^2) e^{-\pi p|Z'|^2} f(x_0, Z') dv_X(x_0, Z') \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}.
 \end{aligned}$$

Let $e^{-u\Delta}(x, x')$ be the smooth kernel of the heat operator $e^{-u\Delta}$ with respect to $dv_X(x')$. We still use the normal coordinate, by the heat kernel expansion in [1, Theorems 2.23, 2.26], there exist $\phi_{i,x_0}(Z')$ such that uniformly for $x_0 \in X$, $Z' \in T_{x_0}X$, $|Z'| \leq \varepsilon$, we have the following asymptotic expansion when $u \rightarrow 0$,

$$(29) \quad \left| \frac{\partial^l}{\partial u^l} \left(e^{-u\Delta}(0, Z') - (4\pi u)^{-n} \sum_{i=0}^k u^i \phi_{i,x_0}(Z') e^{-\frac{1}{4u}|Z'|^2} \right) \right| = \mathcal{O}(u^{k-n-l+1}),$$

with

$$(30) \quad \phi_{0,x_0}(Z') = 1.$$

Observe that

$$(31) \quad \frac{1}{p} \Delta \exp \left(-\frac{\Delta}{4\pi p} \right) = -\frac{1}{p} \left(\frac{\partial}{\partial u} e^{-u\Delta} \right) \Big|_{u=\frac{1}{4\pi p}}.$$

Now from (25), (28)–(31), we get

$$\begin{aligned}
 (32) \quad &\left| \left(p^{-n} K_{\omega,p} - \exp \left(-\frac{\Delta}{4\pi p} \right) \right) f \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}, \\
 &\left| \frac{1}{p} \left(p^{-n} \Delta K_{\omega,p} - \Delta \exp \left(-\frac{\Delta}{4\pi p} \right) \right) f \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}.
 \end{aligned}$$

Note that $R_p = \frac{\text{Vol}(X, dv_X)}{\text{Vol}(X, \nu)} p^n + \mathcal{O}(p^{n-1})$. From (11), (32), we get (6).

To get the last part of Theorem 1, as we noticed in [2, §4.5], the constants in (17) will be uniformly bounded under our condition, thus we can take C in (6) and (32) independent of h^L , dv . \square

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REFERENCES

- [1] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Grundle Math. Wiss. Band 298, Springer-Verlag, Berlin, 1992.
- [2] X. Dai, K. Liu, and X. Ma, *On the asymptotic expansion of Bergman kernel*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 3, 193–198. The full version: J. Differential Geom. to appear, math.DG/0404494.
- [3] S. K. Donaldson, *Some numerical results in complex differential geometry*, math.DG/0512625.
- [4] X. Ma and G. Marinescu, *Generalized Bergman kernels on symplectic manifolds*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 7, 493–498. The full version: math.DG/0411559.
- [5] ———, *Holomorphic Morse Inequalities and Bergman Kernels*, book in preparation, (2006).

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