Global well-posedness in L^2 for the periodic Benjamin-Ono equation

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Abstract. We prove that the Benjamin-Ono equation is globally well-posed in $H^s(\mathbb{T})$ for $s \geq 0$. Moreover we show that the associated flow-map is Lipschitz on every bounded set of $H^s_0(\mathbb{T})$, $s \geq 0$, and even real-analytic in this space for small times. This result is sharp in the sense that the flow-map (if it can be defined and coincides with the standard flow-map on $H^\infty_0(\mathbb{T})$) cannot be of class $C^{1+\alpha}$, $\alpha > 0$, from $H^s_0(\mathbb{T})$ into $H^s_0(\mathbb{T})$ as soon as $s < 0$.

1 Introduction, main results and notations

1.1 Introduction

In this paper we continue our study (see [18]) of the Cauchy problem for the Benjamin-Ono equation on the circle

$$\text{(BO)} \qquad \begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u - u\partial_x u = 0 \ , \ (t,x) \in I\!R \times \mathbb{T} \ , \\ u(0,x) = u_0(x) \ , \end{cases}$$

where $\mathbb{T} = I\!R/2\pi\mathbb{Z}$, u is real-valued and $\mathcal{H}$ is the Hilbert transform defined for 2π-periodic functions with mean value zero by

$$\widehat{\mathcal{H}(f)}(0) := 0 \quad \text{and} \quad \widehat{\mathcal{H}(f)}(\xi) := -i\,\mathrm{sgn}(\xi)\hat{f}(\xi), \quad \xi \in \mathbb{Z}^* \qquad .$$

The Benjamin-Ono equation arises as a model for long internal gravity waves in deep stratified fluids, see [3]. This equation possesses a Lax pair structure (cf. [2], [9]) and thus has got an infinite number of conservation laws. These conservation laws permit to control the $H^{n/2}$-norms, $n \in \mathbb{N}$, and thus to derive global well-posedness results in Sobolev spaces. The Cauchy problem on the real line has been extensively studied these last years (cf. [22], [1], [13], [21], [20], [16], [14]). Recently, T. Tao [23] has pushed the well-posedness

theory to $H^1(\mathbb{R})$ by using an appropriate gauge transform. This approach has been improved very recently in [6] and [12] where respectively $H^s(\mathbb{R})$, $s > 0$, and $L^2(\mathbb{R})$ are reached.
In the periodic setting, the local well-posedness of (BO) is known in $H^s(\mathbb{T})$ for $s > 3/2$ (cf. [1], [13]), by standard compactness methods which do not take advantage of the dispersive effects of the equation. Thanks to the conservation laws mentioned above and an interpolation argument, this leads to global well-posedness in $H^s(\mathbb{T})$ for $s > 3/2$ (cf. [1]). Very recently, F. Ribaud and the author [19] have improved the global well-posedness result to $H^1(\mathbb{T})$ by using the gauge transform introduced by T. Tao [23] combining with Strichartz estimates derived in [3] for the Schrödinger group on the one-dimensional torus. In [18] this approach combined with estimates in Bourgain type spaces leads to a global well-posedness result in the energy space $H^{1/2}(\mathbb{T})$. Recall that the Momentum and the Energy of the Benjamin-Ono equation are respectively given by

$$M(u) := \int_{\mathbb{T}} u^2 \quad \text{and} \quad E(u) := \frac{1}{2}\int_{\mathbb{T}} |D_x^{1/2} u|^2 + \frac{1}{6}\int_{\mathbb{T}} u^3 \quad . \tag{1}$$

The aim of this paper is to improve the local and global well-posedness to $L^2(\mathbb{T})$.

1.2 Notations

For $x, y \in \mathbb{R}$, $x \sim y$ means that there exists C_1, $C_2 > 0$ such that $C_1|x| \leq |y| \leq C_2|x|$ and $x \lesssim y$ means that there exists $C_2 > 0$ such that $|x| \leq C_2|y|$. For a Banach space X, we denote by $\|\cdot\|_X$ the norm in X.
We will use the same notations as in [7] and [8] to deal with Fourier transform of space periodic functions with a large period λ. $(d\xi)_\lambda$ will be the renormalized counting measure on $\lambda^{-1}\mathbb{Z}$:

$$\int a(\xi)\,(d\xi)_\lambda := \frac{1}{\lambda}\sum_{\xi\in\lambda^{-1}\mathbb{Z}} a(\xi) \quad .$$

As written in [8], $(d\xi)_\lambda$ is the counting measure on the integers when $\lambda = 1$ and converges weakly to the Lebesgue measure when $\lambda \to \infty$. In all the text, all the Lebesgue norms in ξ will be with respect to the measure $(d\xi)_\lambda$. For a $(2\pi\lambda)$-periodic function φ, we define its space Fourier transform on $\lambda^{-1}\mathbb{Z}$ by

$$\hat{\varphi}(\xi) := \int_{\mathbb{R}/(2\pi\lambda)\mathbb{Z}} e^{-i\xi x} f(x)\,dx, \quad \forall \xi \in \lambda^{-1}\mathbb{Z} \quad .$$

We denote by $V(\cdot)$ the free group associated with the linearized Benjamin-Ono equation,

$$\widehat{V(t)\varphi}(\xi) := e^{-i\xi|\xi|t}\,\hat{\varphi}(\xi), \quad \xi \in \lambda^{-1}\mathbb{Z} \quad .$$

We define the Sobolev spaces H^s_λ for $(2\pi\lambda)$-periodic functions by

$$\|\varphi\|_{H^s_\lambda} := \|\langle\xi\rangle^s \widehat{\varphi}(\xi)\|_{L^2_\xi} = \|J^s_x\varphi\|_{L^2_\lambda} \quad ,$$

where $\langle\cdot\rangle := (1+|\cdot|^2)^{1/2}$ and $\widehat{J^s_x\varphi}(\xi) := \langle\xi\rangle^s\widehat{\varphi}(\xi)$.
For $s \geq 0$, the closed subspace of zero mean value functions of H^s_λ will be denoted by $H^s_{0,\lambda}$ (it is equipped with the H^s_λ-norm).
The Lebesgue spaces L^q_λ, $1 \leq q \leq \infty$, will be defined as usually by

$$\|\varphi\|_{L^q_\lambda} := \Big(\int_{I\!R/(2\pi\lambda)\mathbb{Z}} |\varphi(x)|^q\,dx\Big)^{1/q}$$

with the obvious modification for $q = \infty$.
In the same way, for a function $u(t,x)$ on $I\!R \times I\!R/(2\pi\lambda)\mathbb{Z}$, we define its space-time Fourier transform by

$$\hat{u}(\tau,\xi) := \mathcal{F}_{t,x}(u)(\tau,\xi) := \int_{I\!R}\int_{I\!R/(2\pi\lambda)\mathbb{Z}} e^{-i(\tau t+\xi x)}\,u(t,x)\,dxdt, \quad \forall(\tau,\xi) \in I\!R\times\lambda^{-1}\mathbb{Z} \quad .$$

We define the Bourgain spaces $X^{b,s}_\lambda$, $Z^{b,s}_\lambda$, A_λ and Y^s_λ of $(2\pi\lambda)$-periodic (in x) functions respectively endowed with the norm

$$\|u\|_{X^{b,s}_\lambda} := \|\langle\tau+\xi|\xi|\rangle^b\langle\xi\rangle^s\hat{u}\|_{L^2_{\tau,\xi}} = \|\langle\tau\rangle^b\langle\xi\rangle^s\mathcal{F}_{t,x}(V(-t)u)\|_{L^2_{\tau,\xi}}\,, \tag{2}$$

$$\|u\|_{Z^{b,s}_\lambda} := \|\langle\tau+\xi|\xi|\rangle^b\langle\xi\rangle^s\hat{u}\|_{L^2_\xi L^1_\tau} = |\langle\tau\rangle^b\langle\xi\rangle^s\mathcal{F}_{t,x}(V(-t)u)\|_{L^2_\xi L^1_\tau}\,, \tag{3}$$

$$\|u\|_{A^b_\lambda} \;:=\; \|\langle\tau+\xi|\xi|\rangle^b\hat{u}\|_{L^1_{\tau,\xi}} = \|\langle\tau\rangle^b\mathcal{F}_{t,x}(V(-t)u)\|_{L^1_{\tau,\xi}} \tag{4}$$

and

$$\|u\|_{Y^s_\lambda} := \|u\|_{X^{1/2,s}_\lambda} + \|u\|_{Z^{0,s}_\lambda} \quad , \tag{5}$$

where we will denote A^0_λ simply by A_λ. Recall that $Y^s_\lambda \hookrightarrow Z^{0,s}_\lambda \hookrightarrow C(I\!R;H^s_\lambda)$.
We will also need the homogeneous semi-norm of $\dot{X}^{b,s}_\lambda$ defined by

$$\|u\|_{\dot{X}^{b,s}_\lambda} := \||\tau+\xi|\xi||^b|\xi|^s\hat{u}\|_{L^2_{\tau,\xi}}\,.$$

$L^p_t L^q_\lambda$ will denote the Lebesgue spaces

$$\|u\|_{L^p_t L^q_\lambda} := \Big(\int_{I\!R} \|u(t,\cdot)\|^p_{L^q_\lambda}\, dt\Big)^{1/p}$$

with the obvious modification for $p = \infty$.

Let $u = \sum_{j\geq 0} \Delta_j u$ be a classical smooth non homogeneous Littlewood-Paley decomposition in space of u, Supp $\mathcal{F}_x(\Delta_0 u) \subset I\!R \times [-2, 2]$ and

$$\text{Supp } \mathcal{F}_x(\Delta_j u) \subset I\!R \times [-2^{j+1}, -2^{j-1}] \cup I\!R \times [2^{j-1}, 2^{j+1}]), \quad j \geq 1 \quad .$$

We defined the Besov type space $\tilde{L}^4_{t,\lambda}$ by

$$\|u\|_{\tilde{L}^4_{t,\lambda}} := \Big(\sum_{k\geq 0} \|\Delta_k u\|^2_{L^4_{t,\lambda}}\Big)^{1/2} \tag{6}$$

Note that by the Littlewood-Paley square function theorem and Minkowski inequality,

$$\|u|_{L^4_{t,\lambda}} \sim \Big\|\Big(\sum_{k=0}^{\infty} (\Delta_k u)^2\Big)^{1/2}\Big\|_{L^4_{t,\lambda}} \lesssim \Big(\sum_{k=0}^{\infty} \|\Delta_k u\|^2_{L^4_{t,\lambda}}\Big)^{1/2} = \|u\|_{\tilde{L}^4_{t,\lambda}}$$

and thus $\tilde{L}^4_{t,\lambda} \hookrightarrow L^4_{t,\lambda}$.

We will work in the function spaces N_λ and M^s_λ respectively defined by

$$\|u\|_{N_\lambda} := \|u\|_{Z^{0,0}_\lambda} + \|Q_3 u\|_{X^{7/8,-1}_\lambda} + \|\chi_{[-4,4]}(t)\, u\|_{\tilde{L}^4_{t,\lambda}}$$

and

$$\|w\|_{M^s_\lambda} := \|w\|_{Y^s_\lambda} + \|Q_1 w\|_{X^{1,-1}_\lambda} \quad ,$$

where Q_a, $a \geq 0$, denotes the projection on the spatial Fourier modes of absolute value greater than a .

Finally, for any function space B_λ and any $T > 0$, we denote by $B_{T,\lambda}$ the corresponding restriction in time space endowed with the norm

$$\|u\|_{B_{T,\lambda}} := \inf_{v\in B_\lambda} \{\|v\|_{B_\lambda},\, v(\cdot) \equiv u(\cdot) \text{ on }]0, T[\,\}\ .$$

It is worth noticing that the map $u \mapsto \overline{u}$ is an isometry in all our function spaces.

We will denote by P_+ and P_- the projection on respectiveley the positive and the negative spatial Fourier modes. Moreover, for $a \geq 0$, we will denote by P_a, Q_a, $P_{>a}$ and $P_{<a}$ the projection on respectively the spatial Fourier modes of absolute value equal or less than a, the spatial Fourier modes of absolute value greater than a, the spatial Fourier modes larger than a and the spatial Fourier modes smaller than a.

1.3 Main result

Our well-posedness theorem reads :

Theorem 1.1 *For all $u_0 \in H^s(\mathbb{T})$ with $0 \leq s \leq 1/2$ and all $T > 0$, there exists a solution u of the Benjamin-Ono equation (BO) satisfying*

$$u \in C([0,T]; H^s(\mathbb{T})) \cap N_{T,1} \quad \text{and} \quad P_+(e^{-i\partial_x^{-1}\tilde{u}/2}\tilde{u}) \in X_{T,1}^{1/2,s} \tag{7}$$

where

$$\tilde{u} := u(t, x - t \fint u_0) - \fint u_0 \quad \text{and} \quad \widehat{\partial_x^{-1}} := \frac{1}{i\xi},\ \xi \in \mathbb{Z}^* \quad .$$

This solution is unique in the class (7).
Moreover $u \in C_b(\mathbb{R}, L^2(\mathbb{T}))$ and the map $u_0 \mapsto u$ is continuous from $H^s(\mathbb{T})$ into $C([0,T], H^s(\mathbb{T}))$ and Lipschitz on every bounded set from $H_0^s(\mathbb{T})$ into $C([0,T], H_0^s(\mathbb{T}))$.

Note that the result for $s \geq 1/2$ is established in [18]. Before stating our ill-posedness result let us make some comments on Theorem 1.1.

Remark 1.1 *We are not able to prove that for any solution u of (BO) belonging to $C([0,T]; H^s(\mathbb{T})) \cap N_{T,\lambda}$, the function $P_+(e^{-i\partial_x^{-1}\tilde{u}/2}\tilde{u})$ belongs to $X_{T,\lambda}^{1/2,s}$. This is why we have to add this condition in our uniqueness class. Note however that any solution that are limit in $C([0,T]; H^s(\mathbb{T}))$ of smooth solutions belongs to this class. Therefore, our solution satisfies also the following (weaker) uniqueness notion used in [12] : it is the unique solution that is a limit in $C([0,T]; H^s)$ of smooth solutions to (BO).*

Remark 1.2 *Actually, we prove that the flow-map is Lipschitz on every bounded subset of any hyperplan of $H^s(\mathbb{T})$ of functions with a fixed mean value.*

Remark 1.3 *The fact that u is real-valued is crucial to derive the equation (20) on w. So, it does not seem that our approach can be adapted to prove the local existence of complex-valued solutions. On the other hand, it seems that a slight modification of the proof in [18] can lead to the local-wellposedness in $H^{1/2}(\mathbb{T})$ for the complex-valued version of (BO).*

Let us now state our ill-posedness issue.

Theorem 1.2 *For $s \geq 0$ and $t \in [0,1]$ the flow-map constructed by Theorem 1.1 is real-analytic from $H^s_0(\mathbb{T})$ into $H^s_0(\mathbb{T})$. On the other hand, for any $t \in]0,1[$ and any $\alpha > 0$, the flow-map (if it can be defined and coincides with the standard flow-map on $H^\infty_0(\mathbb{T})$) cannot be of class $C^{1+\alpha}$ from $H^s_0(\mathbb{T})$ into $H^s_0(\mathbb{T})$ as soon as $s < 0$.*

The main tools to prove Theorem 1.1 are the gauge transformation of T. Tao and the Fourier restriction spaces introduced by Bourgain. Recall that in order to solve (BO), T. Tao [23] performed a kind of complex Cole-Hopf transformation[1] $W = P_+(e^{-iF/2})$, where F is a primitive of u. In the periodic setting, requiring that u has mean value zero, we can take $F = \partial_x^{-1} u$ the unique zero mean value primitive of u. By the mean value theorem, it is then easy to check that the above gauge transformation is Lipschitz from L^2_λ to L^∞_λ. This property, which is not true on the real line, is crucial to derive the smoothness of the flow-map. The equation satisfied by $w = \partial_x W$ takes the form

$$w_t - iw_{xx} = \partial_x P_+(WP_- u_x) + ...$$

which looks quite good since such nonlinear term enjoys a strong smoothing effect on u in Bourgain spaces. On the other hand, when one wants to inverse the gauge transformation, one gets something like

$$u = e^{iF} w + ...$$

which is not so good since multiplication by gauge function as e^{iF} behaves not so well in Bourgain spaces[2]. Actually, the "bad" regularity of u in the scale of Bourgain spaces is the main obstruction in going below $H^{1/2}(\mathbb{T})$ in [18]. In this work we substitute the above expression of u in the equation satisfied by w. u still appears but only under the form $e^{\mp iF/2}$ which possesses more regularities. On the other hand we have now to treat the multiplication by such functions in Bourgain spaces when estimating w. Note that in the case $s = 0$ there is an additional difficulty mainly since we would like to control $\mathcal{F}^{-1}_{t,x}(|\hat{u}|)$ in $L^4_{t,x}$ whereas we only have a control on u in this space. This difficulty is overcome by noticing that actually u belongs to a smaller space than $L^4_{t,x}$ which is $\tilde{L}^4_{t,x}$ (see (6)).

Concerning Theorem 1.2, the fact that the flow-map (if it can be defined) cannot be of class C^3 in $H^s_0(\mathbb{T})$, $s < 0$, can be obtained in the classical way for dispersive equations posed on $\mathbb{T}$ (cf. [5]). To prove that it cannot be of

[1] Note that projecting (BO) on the non negative frequencies, one gets the following equation : $\partial_t(P_+u) - i\partial_x^2 P_+ u = -P_+(uu_x)$

[2] Let us note that Bourgain spaces do not enjoy an algebra property

class $C^{1+\alpha}$, we somehow combine the bad behavior of the third iterate with the real-analyticity result in $L^2(\mathbb{T})$.

This paper is organized as follows: In the next section we recall some linear estimates in Bourgain type spaces. In Section 3 we introduce the gauge transform and state the key nonlinear estimates. In Section 4, we prove the estimates on the gauge function w whereas the estimates on u are proven in Section 5. In Section 4 we derive uniform bounds for small initial data solutions and show a Lipschitz bound on the solution-map $u_0 \mapsto u$. The proof of Theorem 1.1 and Theorem 1.2 are completed respectively in Section 6 and Section 7. Note that the proof of some technical lemmas needed in Sections 4-5 can be found in the appendix.

2 Linear Estimates

One of the main ingredient is the following linear estimate due to Bourgain [4].

$$\|v\|_{L^4(]-\pi,\pi[)L^4_1} \lesssim \|v\|_{X^{3/8,0}_{]-\pi,\pi[,1}} \quad . \tag{8}$$

This estimate is proved in [4] (see also [18] for a shorter proof) for Bourgain spaces of functions on $\mathbb{T}^2$ associated with the Schrödinger group. The result for Bourgain space of functions on $I\!R \times \mathbb{T}$ can be proven in exactly the same way (this can be easily seen in the short proof presented in [18]). The corresponding estimate for the Benjamin-Ono group follows by writting v as the sum of its positive and negative spatial modes parts. The estimate for any period $\lambda \geq 1$ follows directly from dilation arguments. Indeed for any $v \in X^{3/8,0}_1$, setting $v_\lambda := \lambda^{-1} v(\lambda^{-2}t, \lambda^{-1}x)$, it is easy to see that $v_\lambda \in X^{3/8,0}_\lambda$ satisfies

$$\|v_\lambda\|_{L^4_{t,\lambda}} = \lambda^{-1/4}\|v\|_{L^4_{t,1}}\,, \quad \|v_\lambda\|_{\dot{X}^{3/8,0}_\lambda} = \lambda^{-1/4}\|v\|_{\dot{X}^{3/8,0}_\lambda} \text{ and } \|v_\lambda\|_{L^2_{t,\lambda}} = \lambda^{1/2}\|v\|_{L^2_{t,1}}\,.$$

From (8) we infer that for any function belonging to $X^{3/8,0}_\lambda$ with $\lambda \geq 1$, it holds

$$\|v\|_{L^4_{t,\lambda}} \lesssim \|v\|_{X^{3/8,0}_\lambda} \quad . \tag{9}$$

Let us now state some estimates for the free group and the Duhamel operator. Let $\psi \in C^\infty_0([-2,2])$ be a time function such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ on $[-1,1]$. The following linear estimates are well-known (cf. [4], [10]).

Lemma 2.1 *For all $\varphi \in H^s_\lambda$ and all $R > 0$, it holds :*

$$\|\psi(t)V(t)\varphi\|_{Y^s_\lambda} \lesssim \|\varphi\|_{H^s_\lambda} \quad , \tag{10}$$

$$\|\psi(t/R)V(t)\varphi\|_{Z^{0,s}_\lambda} \lesssim \|\varphi\|_{H^s_\lambda} \quad , \tag{11}$$

$$\|\psi(t/R)V(t)\varphi\|_{A_\lambda} \lesssim \|\hat{\varphi}\|_{L^1_\xi} \quad , \tag{12}$$

where it is worth noticing that the implicit constants in (11) and (12) do not depend on R.

Proof. (10) and (11) are classical. (12) can be obtained in the same way. Since $V(t)$ commutes with any time function and

$$\mathcal{F}_{x,t}(V(t)w(t,\cdot)) = \hat{w}(\tau - \xi|\xi|, \xi) \quad ,$$

we infer that

$$\begin{aligned} \|\psi(t/R)V(t)\varphi\|_{A_\lambda} &= \|V(t)\psi(t/R)\varphi\|_{A_\lambda} = \|\mathcal{F}_{t,x}(\psi(\cdot/R)\varphi)\|_{L^1_{\tau,\xi}} \\ &= \|\hat{\psi}(\cdot)\|_{L^1_\tau}\|\hat{\varphi}\|_{L^1_\xi} \lesssim \|\hat{\varphi}\|_{L^1_\xi} \quad . \end{aligned}$$

Note that we will use (11)-(12) with $R = \lambda^2$ to estimate the low modes of u in (28).

Lemma 2.2 *For all $G \in X^{-1/2,s}_\lambda \cap Z^{-1,s}_\lambda$, it holds*

$$\|\psi(t)\int_0^t V(t-t')G(t')\,dt'\|_{Y^s_\lambda} \lesssim \|G\|_{X^{-1/2,s}_\lambda} + \|G\|_{Z^{-1,s}_\lambda} \quad . \tag{13}$$

$$\|\psi(t)\int_0^t V(t-t')G(t')\,dt'\|_{A_\lambda} \lesssim \|G\|_{A^{-1}_\lambda} \quad . \tag{14}$$

Let us recall that (13)-(14) are direct consequences of the following one dimensional (in time) inequalities (cf. [10] and [11]): for any function $f \in \mathcal{S}(I\!R)$, it holds

$$\|\psi(t)\int_0^t f(t')\,dt'\|_{H^{1/2}_t} \lesssim \|f\|_{H^{-1/2}_t} + \Big\|\frac{\mathcal{F}_t(f)}{\langle\tau\rangle}\Big\|_{L^1_\tau}$$

and

$$\Big\|\mathcal{F}_t\Big(\psi(t)\int_0^t f(t')\,dt'\Big)\Big\|_{L^1_\tau} \lesssim \Big\|\frac{\mathcal{F}_t(f)}{\langle\tau\rangle}\Big\|_{L^1_\tau} \quad .$$

3 Gauge transform and nonlinear estimates

3.1 Gauge transform

Let $\lambda \geq 1$ and u be a smooth $(2\pi\lambda)$-periodic solution of (BO) with initial data u_0. In the sequel, we assume that $u(t)$ has mean value zero for all time. Otherwise we do the change of unknown :

$$v(t,x) := u(t, x - t \fint u_0) - \fint u_0 \quad , \tag{15}$$

where $\fint u_0 := P_0(u_0) = \frac{1}{2\pi\lambda}\int_{I\!R/(2\pi\lambda)\mathbb{Z}} u_0$ is the mean value of u_0. It is easy to see that v satisfies (BO) with $u_0 - \fint u_0$ as initial data and since $\fint v$ is preserved by the flow of (BO), $v(t)$ has mean value zero for all time. We define $F = \partial_x^{-1} u$ which is the periodic, zero mean value, primitive of u,

$$\hat{F}(0) = 0 \quad \text{ and } \widehat{F}(\xi) = \frac{1}{i\xi}\hat{u}(\xi), \quad \xi \in \lambda^{-1}\mathbb{Z}^* \quad .$$

Following T. Tao [23], we introduce the gauge transform

$$W := P_+(e^{-iF/2}) \quad . \tag{16}$$

Since F satisfies

$$F_t + \mathcal{H}F_{xx} = \frac{F_x^2}{2} - \frac{1}{2}\fint F_x^2 = \frac{F_x^2}{2} - \frac{1}{2}P_0(F_x^2) \quad ,$$

we can check that $w := W_x = -\frac{i}{2}P_+(e^{-iF/2}F_x) = -\frac{i}{2}P_+(e^{-iF/2}u)$ satisfies

$$\begin{aligned} w_t - iw_{xx} &= -\partial_x P_+\Big[e^{-iF/2}\Big(P_-(F_{xx}) - \frac{i}{4}P_0(F_x^2)\Big)\Big] \\ &= -\partial_x P_+\Big(WP_-(u_x)\Big) + \frac{i}{4}P_0(F_x^2)w \quad . \end{aligned} \tag{17}$$

On the other hand, one can write u as

$$u = e^{iF/2}e^{-iF/2}F_x = 2i\, e^{iF/2}\partial_x(e^{-iF/2}) = 2ie^{iF/2}w + 2ie^{iF/2}\partial_x P_-(e^{-iF/2}) \quad . \tag{18}$$

Recalling that u is real-valued, we get

$$u = \overline{u} = -2ie^{-iF/2}\overline{w} - 2ie^{-iF/2}\partial_x\overline{P_-(e^{-iF/2})}$$

and thus

$$P_-(u) = -2iP_-\Big(e^{-iF/2}\overline{w}\Big) - 2iP_-\Big(e^{-iF/2}\partial_x P_+(e^{iF/2})\Big) \tag{19}$$

since $\overline{P_-(v)} = P_+(\overline{v})$ for any complex-valued function v. Substituing (19) in (17), we obtain the following equation satisfied by w :

$$\begin{aligned} w_t - iw_{xx} &= 2i\partial_x P_+\Big(W\partial_x P_-(e^{-iF/2}\overline{w})\Big) \\ &\quad +2i\partial_x P_+\Big[W\partial_x P_-\Big(e^{-iF/2}\partial_x P_+(e^{iF/2})\Big)\Big] + \frac{i}{4}P_0(F_x^2)W_x \quad (20) \end{aligned}$$

Note also that it follows from (18) that

$$\begin{aligned} P_{>1}u &= 2iP_{>1}\Big(e^{iF/2}w\Big) + 2iP_{>1}\Big(e^{iF/2}\partial_x P_-(e^{-iF/2})\Big) \\ &= 2iP_{>1}\Big(e^{iF/2}w\Big) + 2iP_{>1}\Big(P_{>1}(e^{iF/2})\partial_x P_-(e^{-iF/2})\Big) \quad . \quad (21) \end{aligned}$$

To end this section is we state the crucial nonlinear estimates on u and w that will be proven in the next two sections. It is worth noticing that in all the estimates, we will replace the exponential function (if it appears) by its entire serie and prove the absolute convergence of the resulting serie. Even if this approach can appear unecessary to prove the well-posedness result, it will be very useful in order to derive the analyticity of the flow-map. On the other hand it will in some estimates cause the appearance of a factor $e^{\|\widehat{\partial_x^{-1}u_0}\|_{L^1_\xi}}$ that could be avoid otherwise.

Proposition 3.1 *Let $u \in L^\infty_1 H^s_{0,\lambda} \cap N_{1,\lambda}$ be a solution of (BO) and $w \in X^{1/2,s}_{1,\lambda}$ satisfying (17)-(18). Then for $0 \le s \le 1/2$, it holds*

$$\begin{aligned} \|w\|_{M^s_{1,\lambda}} &\lesssim (1+\|u_0\|_{L^2_\lambda})e^{\|\widehat{\partial_x^{-1}u_0}\|_{L^1_\xi}}\|u_0\|_{H^s_\lambda} \\ &\quad +\|w\|_{X^{1/2,s}_{1,\lambda}}\Big(\|u\|_{N_{1,\lambda}} + \|w\|_{X^{1/2,0}_{1,\lambda}}\Big)e^{\tilde{K}} \quad , \quad (22) \end{aligned}$$

$$\|u\|_{N_{1,\lambda}} \lesssim \|u_0\|_{L^2_\lambda} + \Big(\|w\|_{M^0_{1,\lambda}} + \|u\|^2_{N_{1,\lambda}}\Big)e^{\tilde{K}} \quad (23)$$

and

$$\|u\|_{L^\infty_1 H^s_\lambda} \lesssim \|u_0\|_{L^2_\lambda} + \Big(\|w\|_{M^s_{1,\lambda}} + \|u\|^2_{N_{1,\lambda}}\Big)e^{\tilde{K}} \quad (24)$$

where

$$\tilde{K} = C\left(\|\widehat{\partial_x^{-1}u_0}\|_{L^1_\xi} + \|u\|_{N_{1,\lambda}} + \|u\|^2_{N_{1,\lambda}}\right) \quad . \quad (25)$$

for some universal constant $C > 1$.

From Proposition 3.1 we will deduce uniform bounds for smooth solutions of (BO) with small data (see Proposition 6.1). This will be the key point to derive the local well-posedness result.

4 Proof of the estimate on w

In this section, we will need the two following technical lemmas. The first one, which is proven in the appendix, enables to treat the multiplication with the gauge function $e^{-iF/2}$ in the Sobolev spaces whereas the second one (see the appendix of [18] for a proof), shows that, due to the frequency projections, we can share derivatives when taking the H^s-norm of the second term of the right-hand side to (20) or (21).

Lemma 4.1 *Let $2 \leq q \leq 4$. Let h be function of H^1_λ and let $g \in L^q_\lambda$ such that $J^\alpha_x g \in L^q_\lambda$ with $0 < \alpha \leq 1/2$. Then it holds*

$$\|J^\alpha(hg)\|_{L^q_\lambda} \lesssim \|J^\alpha_x g\|_{L^q_\lambda}(\|h\|_{L^\infty_\lambda} + \|h_x\|_{L^2_\lambda}) \quad . \tag{26}$$

Lemma 4.2 *Let $\alpha \geq 0$ and $1 < q < \infty$ then*

$$\Big\| D^\alpha_x P_+\Big(f P_- \partial_x g\Big)\Big\|_{L^q_\lambda} \lesssim \|D^{\gamma_1}_x f\|_{L^{q_1}_\lambda}\, \|D^{\gamma_2}_x g\|_{L^{q_2}_\lambda} \quad , \tag{27}$$

with $1 < q_i < \infty$, $1/q_1 + 1/q_2 = 1/q$ and $\left\{ \begin{array}{l} \gamma_1 \geq \alpha,\ \gamma_2 \geq 0 \\ \gamma_1 + \gamma_2 = \alpha + 1 \end{array} \right.$.

4.1 Choice of the extensions outside $]0,1[$

Let us introduce the following extensions outside the time interval $]0,1[$. Let $\tilde{\tilde{w}}$ be a zero-mean value extension of w satisfying $\|\tilde{\tilde{w}}\|_{X^{1/2,0}_\lambda} \leq 2\|w\|_{X^{1/2,0}_{1,\lambda}}$ with $\tilde{\tilde{w}} = P_+(\tilde{\tilde{w}})$, $\tilde{W}$ be an extension of W satisfying $\|\tilde{W}_x\|_{X^{1/2,s}_\lambda} \leq 2\|w\|_{X^{1/2,s}_{1,\lambda}}$ with $\tilde{W} = P_+(\tilde{W})$ and let $\tilde{w} := \tilde{W}_x$. We will also need a suitable extention $\tilde{F}$ of F. To construct $\tilde{F}$ we proceed as follows : we take $\tilde{\tilde{u}}$ a zero-mean value extension of u in N_λ such that $\|\tilde{\tilde{u}}\|_{N_\lambda} \leq 2\|u\|_{N_{1,\lambda}}$ and define $\tilde{u}$ by setting $Q_3\tilde{u} = \psi Q_3\tilde{\tilde{u}}$ and

$$P_3\tilde{u} = \psi(t/\lambda^2)\, P_3 V(t)u_0 + \frac{\psi(t)}{2} P_3\Big[\int_0^t V(t-t')\partial_x(\psi\tilde{\tilde{u}}(t'))^2\, dt'\Big] \quad . \tag{28}$$

The factor λ above will be very useful in (66) to compensate a factor λ coming from the L^2_λ-norm of $\partial_x^{-1}u_0$. It is clear that $\tilde{u} \equiv u$ on $[0,1]$ and $\mathcal{f}\,\tilde{u} = 0$ on $\mathbb{R}$ and thus we can set $\tilde{F} = \partial_x^{-1}\tilde{u}$.

By the Duhamel formulation of (20), for $0 \leq t \leq 1$, we have

$$w(t) \;=\; \psi(t)\Big[V(t)w(0) + 2i\int_0^t V(t-t')\partial_x P_+\Big((\psi\tilde{W})\partial_x P_-(e^{-i\tilde{F}/2}\psi\overline{\tilde{\tilde{w}}})\Big)$$

$$+2i\int_0^t V(t-t')\partial_x P_+\Big[(\psi\tilde{W})\partial_x P_-\Big(e^{-i\tilde{F}/2}\partial_x P_+(e^{i\tilde{F}/2})\Big)\Big]$$
$$+\frac{i}{4}\int_0^t V(t-t')\Big(P_0(\tilde{u}^2)\psi\tilde{W}_x\Big)(t')\,dt'\Big] \quad . \tag{29}$$

To obtain the desired estimates we will first apply Lemmas 2.1-2.2 to (29) and then apply Lemmas 4.3-4.5 below with $W := \psi\tilde{W}$, $F := \tilde{F}$ and $v := \psi\overline{\tilde{\tilde{w}}}$. Note that since $\tilde{\tilde{w}} = P_+\tilde{\tilde{w}}$, we have $\overline{\tilde{\tilde{w}}} = P_-\overline{\tilde{\tilde{w}}}$ and thus $v = P_-v$. Moreover, W and v being supported in time in $\{t \in I\!R, |t| \leq 2\}$, $W = \psi_2 W$ and $v = \psi_2 v$ where $\psi_2(\cdot) = \psi(\cdot/2)$ and ψ is the cut-off in time function defined in Section 2.

4.2 Some multilinear estimates

The main tool for proving (22) are three multilinear estimates. These estimates enlight the good behavior in Bourgain spaces of the terms of the right-hand side of (20). In the following lemmas W, $w := \partial_x W$ and v are assumed to be supported in time in $[-2, 2]$ and we set $\psi_2(\cdot) = \psi(\cdot/2)$ (see above).

Lemma 4.3 *For any $s \geq 0$ and $0 < \varepsilon \ll 1$,*

$$\Big\|\partial_x P_+\Big[W\partial_x P_-\Big(e^{-iF/2}\partial_x P_+(e^{iF/2})\Big)\Big]\Big\|_{X_\lambda^{-1/2+\varepsilon,s}} \lesssim \|w\|_{X_\lambda^{1/2,s}}\|\psi_2\, F_x\|^2_{L^4_{t,\lambda}}\, e^{C\,\|F\|_{L^\infty_{t,\lambda}}} \quad . \tag{30}$$

Proof. As written above, we will actually prove (30) with as left-hand side member (Note that the factor $e^{\|F\|_{L^\infty_{t,\lambda}}}$ in the right-hand side of (30) could be avoid otherwise):

$$\sum_{k\geq 1}\sum_{l\geq 1}\frac{1}{k!}\frac{1}{l!}\Big\|\partial_x P_+\Big[W\partial_x P_-\Big(F^k\partial_x P_+F^l\Big)\Big]\Big\|_{X_\lambda^{-1/2+\varepsilon,s}} \quad .$$

Note that, according to the support in time of W, the expression contained in norm remains unchanged by multiplication with the cut-off in time function ψ_2. Setting

$$g = \partial_x P_-\Big(\psi_2 F^k\partial_x P_+(\psi_2 F^l)\Big)\ ,$$

it follows from Lemma 4.2 that

$$\|g\|_{L^2_{t,\lambda}} \lesssim \|\psi_2\partial_x(F^k)\|_{L^4_{t,\lambda}}\|\psi_2\partial_x(F^l)\|_{L^4_{t,\lambda}} \lesssim k\,l\|\psi_2 F_x\|^2_{L^4_{t,\lambda}}\|F\|^{k+l-2}_{L^\infty_{t,\lambda}}\ .$$

It thus suffices to prove that

$$\|\partial_x P_+(WP_-g)\|_{X^{-1/2+\varepsilon,s}} \lesssim \|w\|_{X^{1/2,s}} \|g\|_{L^2_{t,\lambda}} \quad . \tag{31}$$

By duality it is equivalent to estimate

$$I \;=\; \Big| \int_A \xi\, \hat{h}(\tau,\xi) \xi_1^{-1} \hat{w}(\tau_1,\xi_1) \hat{g}(\tau_2,\xi_2) \Big|$$

where $(\tau_2,\xi_2) = (\tau-\tau_1, \xi-\xi_1)$, and due to the frequency projections

$$A = \{ (\tau,\tau_1,\xi,\xi_1) \in I\!R^2 \times (\lambda^{-1}\mathbb{Z})^2, \quad \xi \geq 1/\lambda,\, \xi_1 \geq 1/\lambda,\, \xi - \xi_1 \leq -1/\lambda \quad \} \quad .$$

Note that in the domain of integration above,

$$\xi_1 \geq |\xi - \xi_1| \quad \text{ and } \quad \xi_1 \geq \xi \quad . \tag{32}$$

It thus folllows that

$$I \lesssim \int_A \langle \xi \rangle^{-s} |\hat{h}(\tau,\xi)| \langle \xi_1 \rangle^s |\hat{w}(\tau_1,\xi_1)| |\hat{g}(\tau_2,\xi_2)|$$

and on account of (9),

$$\begin{aligned} I \;&\lesssim\; \|\mathcal{F}^{-1}(\langle \xi \rangle^{-s} |\hat{h}|)\|_{L^4_{t,\lambda}} \|\mathcal{F}^{-1}(\langle \xi \rangle^s |\hat{w}|)\|_{L^4_{t,\lambda}} \|\mathcal{F}^{-1}(|\hat{g}|)\|_{L^2_{t,\lambda}} \\ &\lesssim\; \|h\|_{X^{3/8,-s}_\lambda} \|w\|_{X^{1/2,s}_\lambda} \|g\|_{L^2_{t,\lambda}} \end{aligned}$$

which proves (30).

Lemma 4.4 *For any $s \geq 0$ it holds*

$$\begin{aligned} &\Big\| \partial_x P_+ \Big(W \partial_x P_- (e^{-iF/2} P_- v) \Big) \Big\|_{X^{-1/2,s}_\lambda} \;\lesssim\; \|w\|_{X^{1/2,s}_\lambda} \|v\|_{X^{1/2,0}_\lambda}\, e^{C\,\|F\|_{A_\lambda}} \\ &\Big(1 + \|P_3 F\|_{\dot{X}^{1,0}_\lambda} + \|P_{>3} F_x\|_{X^{7/8,-1}_\lambda} + \|F\|_{A_\lambda} + \|\psi_2 F_x\|_{\tilde{L^4}_{t,\lambda}} \Big) \, . \end{aligned} \tag{33}$$

Proof. Again we will in fact prove (33) with as left-hand side member :

$$\Big\| \partial_x P_+ \Big(W \partial_x P_- v) \Big) \Big\|_{X^{-1/2,s}_\lambda} + \sum_{k \geq 1} \frac{1}{k!} \Big\| \partial_x P_+ \Big(W \partial_x P_- (F^k P_- v) \Big) \Big\|_{X^{-1/2,s}_\lambda} \quad .$$

The first term of the above inequality is estimated in ([18], Lemma 3.3) by

$$\Big\| \partial_x P_+ \Big(W \partial_x P_- v) \Big) \Big\|_{X^{-1/2,s}_\lambda} \lesssim \|w\|_{X^{1/2,s}} \|v\|_{X^{1/2,0}_\lambda} \quad . \tag{34}$$

By duality it thus remains to estimate

$$D_k \;=\; \Big|\int_B \xi\,\hat h(\tau,\xi)\xi_1^{-1}\hat w(\tau_1,\xi_1)(\xi-\xi_1)\widehat{P_-v}(\tau_2,\xi_2)\prod_{i=3}^{k+2}\hat F(\tau_i,\xi_i)\Big| \quad (35)$$

where $(\tau_{k+2},\xi_{k+2})=(\tau,\xi)-\sum_{i=1}^{k+1}(\tau_i,\xi_i)$, and due to the frequency projections

$$B=\{(\tau,\tau_1,..,\tau_{k+1},\xi,\xi_1,..,\xi_{k+1})\in I\!R^{k+2}\times(\lambda^{-1}\mathbb{Z})^{k+2},\\ \xi_1\geq\xi\geq 1/\lambda,\,\xi-\xi_1\leq -1/\lambda\quad\}\quad .$$

First splitting D_k into the two following terms

$$I_k=\Big|\int_B \xi\,\hat h(\tau,\xi)\xi_1^{-1}\hat w(\tau_1,\xi_1)(\xi-\xi_1)\widehat{P_{\{2^{10}k\}}P_-v}(\tau_2,\xi_2)\prod_{i=3}^{k+2}\hat F(\tau_i,\xi_i)\Big|$$

and

$$J_k=\Big|\int_B \xi\,\hat h(\tau,\xi)\xi_1^{-1}\hat w(\tau_1,\xi_1)(\xi-\xi_1)\widehat{Q_{\{2^{10}k\}}P_-v}(\tau_2,\xi_2)\prod_{i=3}^{k+2}\hat F(\tau_i,\xi_i)\Big|$$

we observe that

$$\begin{aligned}I_k&\lesssim \|\mathcal{F}^{-1}(\langle\xi\rangle^{-s}|\hat h|)\|_{L^4_{t,\lambda}}\|\mathcal{F}^{-1}(\langle\xi\rangle^{s}|\hat w|)\|_{L^4_{t,\lambda}}\Big\|\partial_x\Big((P_{\{2^{10}k\}}P_-v)F^k\Big)\Big\|_{L^2_{t,\lambda}}\\ &\lesssim k\,\|h\|_{X^{3/8,-s}_\lambda}\|w\|_{X^{1/2,s}_\lambda}(\|v\|_{L^2_{t,\lambda}}\|F\|_{L^\infty_{t,\lambda}}+\|v\|_{L^4_{t,\lambda}}\|\psi_2F_x\|_{L^4_{t,\lambda}})\|F\|^{k-1}_{L^\infty_{t,\lambda}} \quad (36)\end{aligned}$$

since obviously,

$$\begin{aligned}\Big\|\partial_x\Big((P_{\{2^{10}k\}}P_-v)F^k\Big)\Big\|_{L^2_{t,\lambda}}&\lesssim \|P_{\{2^{10}k\}}P_-v_x\|_{L^2_{t,\lambda}}\|F\|^k_{L^\infty_{t,\lambda}}\\ &\quad+k\|P_{\{2^{10}k\}}P_-v\|_{L^4_{t,\lambda}}\|\psi_2F_x\|_{L^4_{t,\lambda}}\|F\|^{k-1}_{L^\infty_{t,\lambda}}\\ &\lesssim k(\|v\|_{L^2_{t,\lambda}}\|F\|_{L^\infty_{t,\lambda}}+\|v\|_{L^4_{t,\lambda}}\|\psi_2F_x\|_{L^4_{t,\lambda}})\|F\|^{k-1}_{L^\infty_{t,\lambda}}\quad .\end{aligned}$$

It thus remains to estimate J_k. Note that since (32) holds on B, setting

$$B_1=\{(\tau,\tau_1,..,\tau_{k+1},\xi,\xi_1,..,\xi_{k+1})\in B,\,|\xi|\leq 2^{10}k \text{ or } |\xi-\xi_1|\leq 2^{10}k\,\}$$

we get thanks to (9),

$$\begin{aligned}J_{k_{/B_1}}&\lesssim k\,\|\mathcal{F}^{-1}(\langle\xi\rangle^{-s}|\hat h|)\|_{L^4_{t,\lambda}}\|\mathcal{F}^{-1}(\langle\xi\rangle^{s}|\hat w|)\|_{L^4_{t,\lambda}}\|(Q_{\{2^{10}k\}}P_-v)F^k\|_{L^2_{t,\lambda}}\\ &\lesssim k\,\|h\|_{X^{3/8,-s}_\lambda}\|w\|_{X^{1/2,s}_\lambda}\|v\|_{L^2_{t,\lambda}}\|F\|^k_{L^\infty_{t,\lambda}}\quad . \quad (37)\end{aligned}$$

It thus suffices to control

$$J_{k_{/B_2}} = \Big| \int_{B_2} \xi\, \hat{h}(\tau,\xi)\xi_1^{-1}\hat{w}(\tau_1,\xi_1)(\xi-\xi_1)\widehat{(Q_{\{2^{10}k\}}P_- v)}(\tau_2,\xi_2) \prod_{i=3}^{k+2} \hat{F}(\tau_i,\xi_i)\Big| \tag{38}$$

where

$$B_2 = \{(\tau,\tau_1,..,\tau_{k+1},\xi,\xi_1,..,\xi_{k+1}) \in B,\, \xi > 2^{10}k,\, \xi-\xi_1 < -2^{10}k\}\,.$$

One of the main difficulties will be that we do not have a control on $\|\mathcal{F}_{t,x}^{-1}(|\hat{F}_x|)\|_{L^4_{t,\lambda}}$ but only on $\|F_x\|_{L^4_{t,\lambda}}$. This can be overcame when $s>0$ but causes a kind of logarithmic divergence when $s=0$. To control $J_{k_{/B_2}}$ we will have to use the stronger norm $\tilde{L}^4_{t,\lambda}$ of F_x. To simplify the notation we denote $Q_{\{2^{10}k\}}P_- v$ by $\tilde{v}$. Since we cannot "force" the integrant to be non negative in (38), we have to act carefully. We notice that using Littlewood-Paley decomposition (see (6)) we can rewrite $Q_{2^{10}k}(\tilde{v}F^k)$ as

$$\begin{aligned} Q_{\{2^{10}k\}}(\tilde{v}F^k) &= Q_{\{2^{10}k\}}\Big(\sum_{i_2\geq 8+\alpha(k)} \Delta_{i_2}(\tilde{v}) \sum_{i_3\geq i_2-6-\alpha(k)} \Delta_{i_3}(F) \sum_{0\leq i_4,..,i_{k+2}\leq i_3} n(i_3,..,i_{k+2}) \prod_{j=4}^{k+2}\Delta_{i_j}(F)\Big) \\ &\quad + Q_{\{2^{10}k\}}\Big(\sum_{i_2\geq 8+\alpha(k)} \Delta_{i_2}(\tilde{v}) \sum_{0\leq i_3<i_2-6-\alpha(k)} \Delta_{i_3}(F) \sum_{0\leq i_4,..,i_{k+2}\leq i_3} n(i_3,..,i_{k+2}) \prod_{j=4}^{k+2}\Delta_{i_j}(F)\Big) \\ &= G_1+G_2 \quad , \end{aligned}$$

where $\alpha(k)$ denotes the entire part of $\ln(k)/\ln(2)$ and $n(i_3,..,i_{k+2})$ is an integer belonging to $\{1,..,k\}$ (Note for instance that $n(i_3,..,i_{k+2})=1$ for $i_3=i_4=\cdot\cdot=i_{k+2}$ and $n(i_3,..,i_{k+2})=k$ for $i_3\neq i_4\neq\cdot\cdot\neq i_{k+2}$). From (38) we thus infer that

$$\begin{aligned} J_{k_{/B_2}} &\lesssim \sum_{i=1}^{2}\int_{B_1} \xi\,|\hat{h}(\tau,\xi)|\xi_1^{-1}|\hat{w}(\tau_1,\xi_1||\xi-\xi_1||\widehat{G_i}(\tau-\tau_1,\xi-\xi_1)| \\ &\lesssim \Lambda_1+\Lambda_2\,. \end{aligned} \tag{39}$$

•*Estimate on* Λ_1. Thanks to the definition of B and (9), we easily obtain

$$\begin{aligned} \Lambda_1 &\lesssim \|\mathcal{F}^{-1}(\langle\xi\rangle^{-s}|\hat{h}|)\|_{L^4_{t,\lambda}}\|\mathcal{F}^{-1}(\langle\xi\rangle^{s}|\hat{w}|)\|_{L^4_{t,\lambda}}\|\partial_x G_1\|_{L^2_{t,\lambda}} \\ &\lesssim \|h\|_{X^{3/8,-s}_\lambda}\|w\|_{X^{1/2,s}_\lambda}\|\partial_x G_1\|_{L^2_{t,\lambda}} \quad . \end{aligned}$$

On the other hand, using the frequency support of the functions, we infer that for $q \geq 9+\alpha(k)$,

$$\Delta_q(G_1) = Q_{\{2^{10}k\}}\Delta_q\Big(\sum_{\substack{i_3\geq q-8-\alpha(k)\\ i_3\geq 2}}\Delta_{i_3}(F)\sum_{\substack{i_2\geq 8+\alpha(k)\\ i_2\leq i_3+6+\alpha(k)}}\Delta_{i_2}(\tilde{v})\sum_{0\leq i_4,..,i_{k+2}\leq i_3} n(i_3,..,i_{k+2})\prod_{j=4}^{k+2}\Delta_{i_j}(F)\Big)$$

and thus

$$\|\Delta_q G_1\|_{L^2_{t,\lambda}} \lesssim \sum_{\substack{i_3\geq q-8-\alpha(k)\\ i_3\geq 2}}\|\psi_2\,\Delta_{i_3}F\|_{L^4_{t,\lambda}}\|\tilde{v}\|_{L^4_{t,\lambda}}\Big\|\sum_{0\leq i_4,..,i_{k+2}\leq i_3} n(i_3,..,i_{k+2})\prod_{j=4}^{k+2}\Delta_{i_j}(F)\Big\|_{L^\infty_{t,\lambda}}\,.$$

But

$$\begin{aligned}\Big\|\sum_{0\leq i_4,..,i_{k+2}\leq i_3} n(i_3,..,i_{k+2})\prod_{j=4}^{k+2}\Delta_{i_j}(F)\Big\|_{L^\infty_{t,\lambda}} &\lesssim k\Big\|\sum_{0\leq i_4,..,i_{k+2}}|\widehat{\Delta_{i_4}(F)}|*..*|\widehat{\Delta_{i_{k+2}}(F)}|\Big\|_{L^1_{\tau,\xi}}\\ &\lesssim k\Big\|\Big(\sum_{i_4\geq 0}|\widehat{\Delta_{i_4}(F)}|\Big)*..*\Big(\sum_{i_{k+2}\geq 0}|\widehat{\Delta_{i_{k+2}}(F)}|\Big)\Big\|_{L^1_{\tau,\xi}}\\ &\lesssim k\,\|F\|^{k-1}_{A_\lambda}\quad .\end{aligned}\tag{40}$$

Therefore,

$$\begin{aligned}\|\partial_x G_1\|^2_{L^2_{t,\lambda}} &\sim \sum_{q\geq 9+\alpha(k)}2^{2q}\|\Delta_q G_1\|^2_{L^2_{t,\lambda}}\\ &\lesssim k^2\|\tilde{v}\|^2_{L^4_{t,\lambda}}\|F\|^{2(k-1)}_{A_\lambda}\sum_{q\geq 9+\alpha(k)}\Big(\sum_{\substack{j\geq q-8-\alpha(k)\\ j\geq 2}}2^{(q-j)}2^j\|\psi_2\Delta_j F\|_{L^4_{t,\lambda}}\Big)^2\end{aligned}$$

But, by the definition of the norm $\tilde{L}^4_{t,\lambda}$ [3] (see (6)), for $j\geq 2$, $2^j\|\Delta_j F\|_{L^4_{t,\lambda}}\lesssim \gamma_j\|F_x\|_{\tilde{L}^4_{t,\lambda}}$ with $\|(\gamma_j)\|_{l^2(\mathbb{N})}\lesssim 1$. Hence, by Young inequality,

$$\sum_{\substack{j\geq q-8-\alpha(k)\\ j\geq 2}}2^{(q-j)}2^j\|\psi_2\Delta_j F\|_{L^4_{t,\lambda}}\lesssim k\gamma_q\|\psi_2 F_x\|_{\tilde{L}^4_{t,\lambda}}$$

and thus

$$\|\partial_x G_1\|_{L^2_{t,\lambda}}\lesssim k^2\|\tilde{v}\|_{L^4_{t,\lambda}}\|\psi_2 F_x\|_{\tilde{L}^4_{t,\lambda}}\|F\|^{k-1}_{A_\lambda}\quad .$$

[3] Note that we could avoid the $\tilde{L}^4_{t,\lambda}$-norm here by invoking the Littlewood-Paley square function theorem in the estimate on G_1

Therefore, the following estimate holds

$$\Lambda_1 \lesssim k^2 \|h\|_{X_\lambda^{3/8,-s}} \|w\|_{X_\lambda^{1/2,s}} \|v\|_{L^4_{t,\lambda}} \|\psi_2 F_x\|_{\tilde{L}^4_{t,\lambda}} \|F\|^{k-1}_{A_\lambda} \tag{41}$$

• *Estimate on* Λ_2. We rewrite G_2 as

$$\begin{aligned}
G_2 &= Q_{\{2^{10}k\}}\Big(\sum_{i_2 \geq 8+\alpha(k)} \Delta_{i_2}(\tilde{v}) \sum_{1 \leq i_3 < i_2 - 6 - \alpha(k)} \Delta_{i_3}(F) \sum_{0 \leq i_4, .., i_{k+2} \leq i_3} n(i_3, .., i_{k+2}) \prod_{j=4}^{k+2} \Delta_{i_j}(F) \Big) \\
&\quad + Q_{\{2^{10}k\}}\Big(\sum_{i_2 \geq 8+\alpha(k)} \Delta_{i_2}(\tilde{v}) (\Delta_0(F))^k \Big) \\
&= \sum_{p \geq 1} \Big[Q_{\{2^{10}k\}}\Big(\sum_{\substack{i_2 \geq 8+\alpha(k) \\ i_2 > p+6+\alpha(k)}} \Delta_{i_2}(\tilde{v}) \Delta_p(F) \sum_{0 \leq i_4, .., i_{k+2} \leq p} n(i_3, .., i_{k+2}) \prod_{j=4}^{k+2} \Delta_{i_j}(F) \Big) \Big] \\
&\quad + Q_{\{2^{10}k\}}\Big(\sum_{i_2 \geq 8+\alpha(k)} \Delta_{i_2}(\tilde{v}) (\Delta_0(F))^k \Big) \\
&= \sum_{p \geq 1} H_p + L \quad .
\end{aligned}$$

it is thus clear that

$$\begin{aligned}
\Lambda_2 &\lesssim \sum_{p \geq 1} \Big| \int_{B_2} \xi \, |\hat{h}(\tau, \xi)| \xi_1^{-1} |\hat{w}(\tau_1, \xi_1)| |\xi - \xi_1| |\widehat{H_p}(\tau - \tau_1, \xi - \xi_1)| \\
&\qquad + \int_{B_2} \xi \, |\hat{h}(\tau, \xi)| \xi_1^{-1} |\hat{w}(\tau_1, \xi_1)| |\xi - \xi_1| |\hat{L}(\tau - \tau_1, \xi - \xi_1)| \\
&= \Lambda_{21} + \Lambda_{22} \quad .
\end{aligned}$$

We rewrite Λ_{21} as the sum of two terms :

$$\begin{aligned}
\Lambda_{21} &= \sum_{p \geq 1} \int_{B_2} \chi_{\{|\xi| \leq 2^{p+6+\alpha(k)}\}} \xi \, |\hat{h}(\tau, \xi)| \xi_1^{-1} |\hat{w}(\tau_1, \xi_1)| |\xi - \xi_1| |\widehat{H_p}(\tau - \tau_1, \xi - \xi_1)| \\
&\quad + \sum_{p \geq 1} \int_{B_2} \chi_{\{|\xi| > 2^{p+6+\alpha(k)}\}} \xi \, |\hat{h}(\tau, \xi)| \xi_1^{-1} |\hat{w}(\tau_1, \xi_1)| |\xi - \xi_1| |\widehat{H_p}(\tau - \tau_1, \xi - \xi_1)| \\
&= \Lambda_{21}^1 + \Lambda_{21}^2 \quad .
\end{aligned}$$

Let us explain the idea of this dichotomy. In the domain of integration of Λ_{21}^1, the frequency ξ of $\hat{h}$ is controlled by the maximum of the $|\xi_i|$, $i = 3, .., k+1$, and thus we can in some sens exchange the derivative on h with a derivative on F. On the other hand, in the domain of integration of Λ_{21}^2, $|\xi|$ and $|\xi_2|$

are very large with respect to $|\xi_3|,..,|\xi_{k+1}|$ and then we have a good non-resonant relation (similar to the non-resonant relation used in [18] to prove the bilinear estimate (34)) that enables to regain one derivative.

• *Estimate on* Λ^1_{21}. Using a Littlewood-Paley decomposition of h, we get thanks to (32) and Cauchy-Schwarz inequality in p

$$\begin{aligned}
\Lambda^1_{21} &\lesssim \sum_{p\geq 1}\sum_{q=-7-\alpha(k)}^{p-8-\alpha(k)}\int_{B_2} 2^{p-q}|\widehat{\Delta_{p-q}h}(\tau,\xi)||\hat{w}(\tau_1,\xi_1)||\widehat{H_p}(\tau-\tau_1,\xi-\xi_1)| \\
&\lesssim \sum_{q=-7-\alpha(k)}^{\infty} 2^{-q}\sum_{p\geq q+8+\alpha(k)}\int_{B_2}|\widehat{\Delta_{p-q}h}(\tau,\xi)||\hat{w}(\tau_1,\xi_1)|2^p|\widehat{H_p}(\tau-\tau_1,\xi-\xi_1)| \\
&\lesssim \sup_{q\geq -7-\alpha(k)} k\sum_{p\geq q+8+\alpha(k)}\int_{B_2}|\widehat{\Delta_{p-q}h}(\tau,\xi)||\hat{w}(\tau_1,\xi_1)|2^p|\widehat{H_p}(\tau-\tau_1,\xi-\xi_1)| \\
&\lesssim k\|\mathcal{F}^{-1}(\langle\xi\rangle^s|\hat{w}|)\|_{L^4_{t,\lambda}}\Big(\sum_{p\geq 1}\|\mathcal{F}^{-1}(\langle\xi\rangle^{-s}|\widehat{\Delta_p h}|)\|^2_{L^4_{t,\lambda}}\Big)^{1/2}\Big(\sum_{p\geq 1}2^{2p}\|H_p\|^2_{L^2_{t,\lambda}}\Big)^{1/2}
\end{aligned}$$

Note that $\tilde{L}^4_{t,\lambda}\hookrightarrow X^{3/8,0}$ since by (9), for any function $z\in X^{3/8,0}_\lambda$,

$$\Big(\sum_{p\geq 1}\|\mathcal{F}^{-1}(|\widehat{\Delta_p z}|)\|^2_{L^4_{t,\lambda}}\Big)^{1/2}\lesssim\Big(\sum_{p\geq 1}\|\Delta_p z\|^2_{X^{3/8,0}_\lambda}\Big)^{1/2}\lesssim\|z\|_{X^{3/8,0}_\lambda}\,. \tag{42}$$

Moreover since, according to the frequency localization of the functions,

$$\Delta_q H_p = Q_{2^{10}k}\Big(\sum_{\substack{i_2\geq 8+\alpha(k)\\ q-1\leq i_2\leq q+1}}\Delta_{i_2}(\tilde{v})\Delta_p(F)\sum_{0\leq i_4,..,i_{k+2}\leq p}n(p,i_4,..,i_{k+2})\prod_{j=4}^{k+2}\Delta_{i_j}(F)\Big)$$

we infer from (40) and (42) that

$$\begin{aligned}
\|H_p\|^2_{L^2_{t,\lambda}} &\sim \sum_{q\geq p+9+\alpha(k)}\|\Delta_q H_p\|^2_{L^2_{t,\lambda}} \\
&\lesssim k^2\|F\|^{2(k-1)}_{A_\lambda}\|\psi_2\Delta_p F\|^2_{L^4_{t,\lambda}}\sum_{q\geq 1}\|\Delta_q\tilde{v}\|^2_{L^4_{t,\lambda}} \\
&\lesssim k^2\|F\|^{2(k-1)}_{A_\lambda}\|v\|^2_{X^{1/2,0}}\|\psi_2\Delta_p F\|^2_{L^4_{t,\lambda}}\quad.
\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}
\Lambda^1_{21} &\lesssim k\|h\|_{X^{3/8,-s}_\lambda}\|w\|_{X^{1/2,s}_\lambda}\|F\|^{k-1}_{A_\lambda}\|v\|_{X^{1/2,0}_\lambda}\Big(\sum_{p\geq 1}2^{2p}\|\psi_2\Delta_p F\|^2_{\tilde{L}^4_{t,\lambda}}\Big)^{1/2} \\
&\lesssim k\,\|h\|_{X^{3/8,-s}_\lambda}\|w\|_{X^{1/2,s}_\lambda}\|F\|^{k-1}_{A_\lambda}\|v\|_{X^{1/2,0}_\lambda}\|\psi_2 F_x\|_{\tilde{L}^4_{t,\lambda}}\quad.
\end{aligned}\tag{43}$$

• *Estimate on* Λ_{21}^2 *and* Λ_{22}. Since clearly , $\sum_{p\geq 0}|\widehat{\Delta_p(f)}(\tau,\xi)|\leq 2|\hat{f}(\tau,\xi)|$ for any $f\in L^2_{t,\lambda}$, we infer that

$$\begin{aligned}\Lambda_{21}^2 &\lesssim k\sum_{p\geq 1}\sum_{i_2\geq p+7+\alpha(k)}\int_{B_3}\xi|\widehat{h}(\tau,\xi)|\xi_1^{-1}|\hat{w}(\tau_1,\xi_1)||\xi-\xi_1||\widehat{\Delta_{i_2}(\tilde{v})}(\tau_2,\xi_2)||\widehat{\Delta_p(F)}(\tau_3,\xi_3)|\\ &\qquad\qquad\sum_{i_4,..,i_{k+2}\geq 0}\prod_{j=4}^{k+2}|\widehat{\Delta_{i_j}(F)}(\tau_j,\xi_j)|\\ &\lesssim k\int_{B_3}\xi\,|\hat{h}(\tau,\xi)|\xi_1^{-1}|\hat{w}(\tau_1,\xi_1)||\xi-\xi_1||\hat{v}(\tau_2,\xi_2)|\prod_{i=3}^{k+2}|\hat{F}(\tau_i,\xi_i)|\\ &= \tilde{J}_{k/B_3}\end{aligned}$$

where

$$B_3=\{(\tau,\tau_1,..,\tau_{k+1},\xi,\xi_1,..,\xi_{k+1})\in B_1,\ \xi_2\leq -2^{10}k,\ \min(|\xi|,|\xi_2|)>10k\max_{i=3,..,k+2}|\xi_i|\}\quad .$$

In the same way, it is easy to check that $\Lambda_{22}\lesssim\tilde{J}_{k/B_3}$. We set $\sigma=\sigma(\tau,\xi)=\tau-\xi|\xi|$ and $\sigma_i=\sigma(\tau_i,\xi_i)$, $i=1,..,k+2$. Noticing that on B_3 the sign of ξ, ξ_1 and ξ_2 are known, we get the following algebraic relation :

$$\begin{aligned}\sigma-\sum_{i=1}^{k+2}\sigma_i &= (\sum_{i=1}^{k+2}\xi_i)^2-\xi_1^2+\xi_2^2-\sum_{i=3}^{k+2}\xi_i|\xi_i|\\ &= 2\xi_2\sum_{i=1}^{k+2}\xi_i+2\xi_1\sum_{i=3}^{k+2}\xi_i-\sum_{i=3}^{k+2}\xi_i|\xi_i|+\Big(\sum_{i=3}^{k+2}\xi_i\Big)^2\\ &= 2\xi_2\xi+2\xi_1\sum_{i=3}^{k+2}\xi_i-\sum_{i=3}^{k+2}\xi_i|\xi_i|+\Big(\sum_{i=3}^{k+2}\xi_i\Big)^2\quad .\end{aligned}\tag{44}$$

Note that on B_3, we have $(\sum_{i=3}^{k+2}\xi_i)^2\leq 10^{-2}|\xi_2\xi|$, $\sum_{i=3}^{k+2}\xi_i|\xi_i|\leq 10^{-2}|\xi_2\xi|$ and

$$\frac{1}{2}|\xi_2|\leq|\xi-\xi_1|\leq|\xi_2|+\sum_{i=3}^{k+2}|\xi_i|\leq 2|\xi_2|\quad .\tag{45}$$

Hence, $\xi_1\leq 2\max(|\xi|,|\xi-\xi_1|)\leq 4\max(|\xi|,|\xi_2|)$ and $|\xi_1\sum_{i=3}^{k+2}\xi_i|\leq 2|\xi_2\xi|/5$. We thus deduce from (44) that the following non-resonant relation holds

$$\max_{i=1,..,k+2}(|\sigma|,|\sigma_i|)\gtrsim|\xi\xi_2|/k\quad .\tag{46}$$

It remains to divide B_3 in subregions according to the indice where the maximum is reached in (46). Thanks to (32),

$$\tilde{J}_{k_{/B_3}} \lesssim \int_{B_2} |\xi\xi_2|^{1/2} |\hat{h}(\tau,\xi)||\hat{w}(\tau_1,\xi_1)||\hat{v}(\tau_2,\xi_2)| \prod_{i=3}^{k+2} |\hat{F}(\tau_i,\xi_i)| \quad .$$

• $|\sigma|$ dominant. By (32) and (46), Plancherel, Holder inequality and (9), we infer that

$$\begin{aligned}\tilde{J}_{k_{/B_3}} &\lesssim k\|\mathcal{F}^{-1}(\langle\sigma\rangle^{1/2}\langle\xi\rangle^{-s}|\hat{h}|)\|_{L^2_{t,\lambda}} \|\mathcal{F}^{-1}(\langle\xi\rangle^s|\hat{w}|)\|_{L^4_{t,\lambda}} \|\mathcal{F}^{-1}(|\hat{v}|)\|_{L^4_{t,\lambda}} \|\mathcal{F}^{-1}(|\hat{F}|)\|^k_{L^\infty_{t,\lambda}} \\ &\lesssim k\|h\|_{X^{1/2,-s}_\lambda} \|w\|_{X^{1/2,s}_\lambda} \|v\|_{X^{1/2,0}_\lambda} \|F\|^k_{A_\lambda} \quad . \end{aligned} \tag{47}$$

• $|\sigma_1|$ or $|\sigma_2|$ dominant. It is easy to see that in the same way

$$\tilde{J}_{k_{/B_3}} \lesssim k\|h\|_{X^{3/8,-s}_\lambda} \|w\|_{X^{1/2,s}_\lambda} \|v\|_{X^{1/2,0}_\lambda} \|F\|^k_{A_\lambda} \quad . \tag{48}$$

• $|\sigma_i|$, $i \geq 3$, dominant. By Plancherel, Holder inequality and (9), we infer that

$$\begin{aligned}\tilde{J}_{k_{/B_3}} \lesssim & k^{1/2} \|\mathcal{F}^{-1}(\langle\xi\rangle^{-s}|\hat{h}|)\|_{L^4_{t,\lambda}} \|\mathcal{F}^{-1}(\langle\xi\rangle^s|\hat{w}|)\|_{L^4_{t,\lambda}} \\ & \|\mathcal{F}^{-1}(|\hat{v}|)\|_{L^4_{t,\lambda}} \|\mathcal{F}^{-1}(|\sigma|^{1/2}\chi_{\{|\sigma|\gtrsim 1\}}|\hat{F}|)\|_{L^4_{t,\lambda}} \|\mathcal{F}^{-1}(|\hat{F}|)\|^{k-1}_{L^\infty_{t,\lambda}} \\ \lesssim & k\|h\|_{X^{3/8,-s}_\lambda} \|w\|_{X^{1/2,s}_\lambda} \|v\|_{X^{1/2,0}_\lambda} (\|P_3F\|_{\dot{X}^{1,0}_\lambda} + \|P_{>3}F\|_{X^{7/8,0}_\lambda}) \|F\|^{k-1}_{A_\lambda} \end{aligned} \tag{49}$$

where we use that $|\sigma_i| \gtrsim |\xi\xi_2|/k \gtrsim 1$ on B_3 to get an homogeneous Bourgain type norm on P_3F. It has some importance when using dilations argument since the L^2-norm of P_3F is surcritical and thus behaves badly for such arguments. Since clearly, $\|P_{>3}F\|_{X^{7/8,0}_\lambda} \lesssim \|P_{>3}F_x\|_{X^{7/8,-1}_\lambda}$, gathering (36), (37), (41), (43), (47), (48) and (49), (33) follows.

Lemma 4.5 *For any $s \geq 0$ it holds*

$$\begin{aligned}&\Big\|\partial_x P_+\Big(W\partial_x P_-(e^{-iF/2}P_-v)\Big)\Big\|_{Z^{-1,s}_\lambda} \lesssim \|w\|_{X^{1/2,s}_\lambda} \|v\|_{X^{1/2,0}_\lambda} e^{C\|F\|_{A_\lambda}} \\ &\Big(1 + \|P_3F\|_{\dot{X}^{1,0}_\lambda} + \|P_{>3}F_x\|_{X^{7/8,-1}_\lambda} + \|F\|_{A_\lambda} + \|\psi_2 F_x\|_{\tilde{L}^4_{t,\lambda}}\Big) \, . \end{aligned} \tag{50}$$

Proof. The proof of this lemma essentially follows the one of Lemma 4.4 and thus will be only sketched. We estimate

$$\Big\|\partial_x P_+\Big(W\partial_x P_-v\Big)\Big\|_{Z^{-1,s}_\lambda} + \sum_{k\geq 1} \frac{1}{k!}\Big\|\partial_x P_+\Big(W\partial_x P_-(F^k P_-v)\Big)\Big\|_{Z^{-1,s}_\lambda} \, .$$

Again, the first term of the above inequality is estimated in ([18], Lemma 3.4) by

$$\Big\|\partial_x P_+\Big(W\partial_x P_-v)\Big)\Big\|_{Z_\lambda^{-1,s}} \lesssim \|w\|_{X_\lambda^{1/2,s}}\|v\|_{X_\lambda^{1/2,0}} \tag{51}$$

To estimate the second term we first note that by Cauchy-Schwarz in τ,

$$\Big\|\frac{\langle\xi\rangle^s}{\langle\sigma\rangle}\mathcal{F}\Big[\partial_x P_+\Big(W\partial_x P_-(F^k P_-v)\Big)\Big]\Big\|_{L^2_\xi L^1_\tau} \lesssim \Big\|\partial_x P_+\Big(W\partial_x P_-(F^k P_-v)\Big)\Big\|_{X_\lambda^{-1/2+\varepsilon,s}},\ \varepsilon>0.$$

On account of (36), (37), (41), (43), (48) and (49), this last term is controlled by the right-hand side of (50) except in the region B_3 with $|\sigma|$ dominant. Moreover, in the region $\{\xi_1 \le 1\}$, using (32) and then (9) we infer that

$$\begin{aligned}\Big\|\partial_x P_+\Big(W\partial_x P_-(F^k P_-v)\Big)\Big\|_{X_\lambda^{-1/2+\varepsilon,s}} &\lesssim \|\mathcal{F}^{-1}_{t,x}(|w|)\|_{L^4_{t,\lambda}}\|\mathcal{F}^{-1}_{t,x}(|v|)\|_{L^4_{t,\lambda}}\|\mathcal{F}^{-1}_{t,x}(|F|)\|^k_{L^\infty_{t,\lambda}}\\ &\lesssim \|w\|_{X_\lambda^{1/2,0}}\|v\|_{X_\lambda^{1/2,0}}\|F\|^k_{A_\lambda}\quad .\end{aligned}$$

It thus remains to treat the region B_3 with $\xi_1 \ge 1$ and $|\sigma|$ dominant. To handle with this region we proceed as in [8]. The proof is very similar to the one of Lemma 3.4 in [18].
By (44) in this region we have

$$\langle\sigma\rangle \gtrsim \langle\xi\xi_2\rangle/k \gtrsim 1\quad . \tag{52}$$

Therefore (50) will be proven if we show the following inequality:

$$J_k \lesssim k\|\tilde{w}\|_{L^2_{t,\lambda}}\|\tilde{v}\|_{L^2_{t,\lambda}}\|\hat{F}\|^k_{L^1_{\tau,\xi}} \tag{53}$$

with

$$J_k = \Big\|\int_{C(\tau,\xi)} \frac{\langle\xi\rangle^s\xi\langle\xi_1\rangle^{-s}\xi_1^{-1}|\widehat{\tilde{w}}(\tau_1,\xi_1)||\xi-\xi_1||\widehat{\tilde{v}}(\tau_2,\xi_2)|\prod_{i=3}^{k+2}|\hat{F}(\tau_i,\xi_i)|}{\langle\sigma\rangle\langle\sigma_1\rangle^{1/2}\langle\sigma\rangle^{1/2}}\Big\|_{L^2_\xi L^1_\tau} \tag{54}$$

and

$$\begin{aligned}C(\tau,\xi) \;=\; &\Big\{(\tau_1,..,\tau_{k+1},\xi_1,..,\xi_{k+1})\in I\!R^{k+1}\times(\lambda^{-1}\mathbb{Z})^{k+1},\\ &(\tau,\tau_1,..,\tau_{k+1},\xi,\xi_1,..,\xi_{k+1})\in B_3,\ |\xi_1|>1,\ \max_{i=1,..,k+2}(\langle\sigma_i\rangle)\le\langle\sigma\rangle\Big\}\end{aligned} \tag{55}$$

• The subregion $\max(|\sigma_1|,|\sigma_2|) \ge (\xi|\xi_2|)^{\frac{1}{16}}$. We will assume that $\max(|\sigma_1|,|\sigma_2|) = |\sigma_1|$ since the other case can be treated in exactly the same way. By (52)

and (45), recalling that on the domain of integration $\xi_1 \geq \max(\xi, |\xi - \xi_1|)$, we infer that

$$J_k \lesssim k \Big\| \int_{C_1(\tau,\xi)} \frac{|\widehat{\widetilde{w}}(\tau_1,\xi_1)||\widehat{\widetilde{v}}(\tau_2,\xi_2)| \prod_{i=3}^{k+2} |\hat{F}(\tau_i,\xi_i)|}{\langle\sigma\rangle^{1/2+\frac{1}{128}} \langle\sigma_1\rangle^{3/8} \langle\sigma\rangle^{1/2}} \Big\|_{L^2_\xi L^1_\tau}$$

where

$$C_1(\tau,\xi) = \{(\tau_1,\xi_1) \in C(\tau,\xi),\ |\sigma_1| \geq (\xi|\xi_2|)^{\frac{1}{16}}\} \quad .$$

Applying Cauchy-Schwarz in τ we obtain thanks to (9),

$$\begin{aligned} J_k &\lesssim k \Big\| \int_{C_1(\tau,\xi)} \frac{|\widehat{\widetilde{w}}(\tau_1,\xi_1)||\widehat{\widetilde{v}}(\tau_2,\xi_2)| \prod_{i=3}^{k+2} |\hat{F}(\tau_i,\xi_i)|}{\langle\sigma_1\rangle^{3/8} \langle\sigma\rangle^{1/2}} \Big\|_{L^2_{\xi,\tau}} \\ &\lesssim k \Big\| \mathcal{F}^{-1}\Big(\frac{|\widehat{\widetilde{w}}|}{\langle\sigma\rangle^{3/8}}\Big) \Big\|_{L^4_{t,\lambda}} \Big\| \mathcal{F}^{-1}\Big(\frac{|\widehat{\widetilde{v}}|}{\langle\sigma\rangle^{1/2}}\Big) \Big\|_{L^4_{t,\lambda}} \|\mathcal{F}^{-1}(|\hat{F}|)\|^k_{L^\infty_{t,\lambda}} \\ &\lesssim k \|\tilde{w}\|_{L^2_{t,\lambda}} \|\tilde{v}\|_{L^2_{t,\lambda}} \|\hat{F}\|^k_{L^1_{\tau,\xi}} \quad . \end{aligned}$$

• The subregion $\max(|\sigma_1|, |\sigma_2|) \leq (\xi|\xi_2|)^{\frac{1}{16}}$. Changing the $\tau, \tau_1, .., \tau_{k+1}$ integrals in $\tau_1, .., \tau_{k+2}$ integrals in (54) and using (45) and (52), we infer that

$$\begin{aligned} J_k \lesssim k \Big\| \chi_{\{\xi\geq 1\}} \int_{D(\xi)} \xi_1^{-1} \int_{\tau_1 = -\xi_1^2 + O(|\xi\,\xi_2|^{1/16})} \frac{|\widehat{\widetilde{w}}(\tau_1,\xi_1)|}{\langle \tau_1 + |\xi_1|\xi_1\rangle^{1/2}} \\ \int_{\tau_2 = \xi_2^2 + O(|\xi\,\xi_2|^{1/16})} \frac{|\widehat{\widetilde{v}}(\tau_2,\xi_2)|}{\langle \tau_2 + |\xi_2|\xi_2\rangle^{1/2}} \int_{\tau_3,..,\tau_{k+2}} \prod_{i=3}^{k+2} |\hat{F}(\tau_i,\xi_i)| \Big\|_{L^2_\xi} \end{aligned}$$

with

$$D(\xi) = \{(\xi_1, .., \xi_{k+1}) \in (\lambda^{-1}\mathbb{Z})^{k+1}, \xi_1 \geq 1,\ \xi - \xi_1 \leq -1/\lambda\} \, .$$

Applying Cauchy-Schwarz inequality in τ_1 and τ_2 and recalling that $\xi_1 \geq 1$ we get

$$J_k \lesssim k \Big\| \chi_{\{\xi\geq 1\}} \int_{D(\xi)} \langle\xi_1\rangle^{-1} (\xi|\xi_2|)^{\frac{1}{8}} K_1(\xi_1) K_2(\xi_2) \prod_{i=3}^{k+2} K(\xi_i) \Big\|_{L^2_\xi}$$

where

$$K_1(\xi) = \Big(\int_\tau \frac{|\widehat{\widetilde{w}}(\tau,\xi)|^2}{\langle \tau + |\xi|\xi\rangle} \Big)^{1/2}, \quad K_2(\xi) = \Big(\int_\tau \frac{|\widehat{\widetilde{v}}(\tau,\xi)|^2}{\langle \tau + |\xi|\xi\rangle} \Big)^{1/2} \text{ and } K(\xi) = \int_\tau |\hat{F}(\tau,\xi)| \, .$$

Therefore, by using (45) and (32), Hölder and then Cauchy-Schwarz inequalities,

$$\begin{aligned}
J_k &\lesssim k\Big\| \langle\xi\rangle^{-\frac{3}{4}} \int_{(\xi_3,..,\xi_{k+2})\in(\lambda^{-1}\mathbb{Z})^k} \prod_{i=3}^{k+2} K(\xi_i) \int_{\xi_1\in\lambda^{-1}\mathbb{Z}} K_1(\xi_1)K_2(\xi_2)\Big\|_{L^2_\xi} \\
&\lesssim k\Big\| \int_{(\xi_3,..,\xi_{k+2})\in(\lambda^{-1}\mathbb{Z})^k} \prod_{i=3}^{k+2} K(\xi_i) \int_{\xi_1\in\lambda^{-1}\mathbb{Z}} K_1(\xi_1)K_2(\xi_2)\Big\|_{L^\infty_\xi} \\
&\lesssim k\int_{(\xi_3,..,\xi_{k+2})\in(\lambda^{-1}\mathbb{Z})^k} \prod_{i=3}^{k+2} K(\xi_i) \Big(\int_{\xi\in\lambda^{-1}\mathbb{Z}} K_1(\xi)^2\Big)^{1/2} \Big(\int_{\xi\in\lambda^{-1}\mathbb{Z}} K_2(\xi)^2\Big)^{1/2} \\
&\lesssim k\|\tilde{w}\|_{X^{-1/2,0}_\lambda} \|\tilde{v}\|_{X^{-1/2,0}_\lambda} \|\hat{F}\|^k_{L^1_{\tau,\xi}} \\
&\lesssim k\|\tilde{w}\|_{L^2_{t,\lambda}} \|\tilde{v}\|_{L^2_{t,\lambda}} \|\hat{F}\|^k_{L^1_{\tau,\xi}} \quad , \qquad (56)
\end{aligned}$$

4.3 End of the proof of (22).

It remains to treat the third term of the right-hand side of (20). Observe that by Cauchy-Schwarz inequality in τ, Sobolev inequalities in time and Minkowski inequality,

$$\|P_0(u^2)w\|_{Z^{-1,s}_\lambda} + \|P_0(u^2)w\|_{X^{-1/2,s}_\lambda} \lesssim \|P_0(u^2)w\|_{X^{-1/2+\varepsilon',s}_\lambda} \lesssim \|P_0(u^2)w\|_{L^{1+\varepsilon}_t H^s_\lambda} \quad ,$$

for some $0 < \varepsilon, \varepsilon' \ll 1$. Assuming that w is supported in time in $[-2,2]$, by Hölder inequality in time and (9) we get

$$\|P_0(u^2)w\|_{L^{1+\varepsilon}_t H^s_\lambda} \lesssim \|J^s_x w\|_{L^4_{t,\lambda}} \|\psi_2^2 P_0(u^2)\|_{L^2_t L^4_\lambda} \lesssim \|w\|_{X^{1/2,s}_\lambda} \|\psi_2^2 P_0(u^2)\|_{L^2_{t,\lambda}} \quad ,$$

where we used that $\|1\|_{L^4_\lambda} \leq \|1\|_{L^2_\lambda}$ since $\lambda \geq 1$. Hence, the following estimate holds:

$$\|P_0(u^2)w\|_{Z^{-1,s}_\lambda} + \|P_0(u^2)w\|_{X^{-1/2,s}_\lambda} \lesssim \|w\|_{X^{1/2,s}_\lambda} \|\psi_2 u\|^2_{L^4_{t,\lambda}} \quad . \qquad (57)$$

Therefore, combining Lemmas 2.1-2.2, 4.3-4.4-4.5 and (57), we infer that for $s \geq 0$, the extension w^* of w defined by (29) satisfies

$$\begin{aligned}
\|w^*\|_{Y^s_\lambda} &\lesssim \|w(0)\|_{H^s_\lambda} + \|\tilde{W}_x\|_{X^{1/2,s}_\lambda} e^{C\|\tilde{F}\|_{A_\lambda}} \Big[\|\tilde{u}\|^2_{L^4_{t,\lambda}} + \|\tilde{\tilde{w}}\|_{X^{1/2,0}_\lambda} \\
&\qquad \Big(\|P_3\tilde{F}\|_{\dot{X}^{1,0}_\lambda} + \|P_{>3}\tilde{F}_x\|_{X^{7/8,-1}_\lambda} + \|\tilde{F}\|_{A_\lambda} + \|\psi_2\tilde{F}_x\|_{\tilde{L}^4_{t,\lambda}} \Big) \Big] \\
&\lesssim \|w(0)\|_{H^s_\lambda} + \|\ w\|_{X^{1/2,s}_{1,\lambda}} \Big(\|u\|^2_{N_{1,\lambda}} + \|\ w\|_{X^{1/2,0}_{1,\lambda}} \Big) e^{C\,(\|\widehat{\partial_x^{-1}u_0}\|_{L^1_\xi} + \|u\|_{N_{1,\lambda}} + \|u\|^2_{N_{1,\lambda}})}
\end{aligned}$$

where in the last step we used Lemma 4.6 below to estimate $\|P_3\tilde{F}\|_{\dot{X}^{1,0}_\lambda} + \|P_3\tilde{F}\|_{A_\lambda}$ and that, by Cauchy-Schwarz in ξ,

$$\|P_{>3}\tilde{F}\|_{A_\lambda} \lesssim \|\widehat{\tilde{F}_x}\|_{L^2_\xi L^1_\tau} \quad . \tag{58}$$

Lemma 4.6 *Let $\tilde{\tilde{u}} \in N_{1,\lambda}$ and let $P_3\tilde{u}$ be defined as in (28). Then $P_3\tilde{F} = P_3\partial_x^{-1}\tilde{u}$ satisfies :*

$$\|P_3\tilde{F}\|_{A_\lambda} \lesssim \|\widehat{\partial_x^{-1}u_0}\|_{L^1_\xi} + \|\tilde{\tilde{u}}\|^2_{N_\lambda} \quad , \tag{59}$$

$$\|P_3\tilde{F}\|_{\dot{X}^{1,0}_\lambda} \lesssim \|u_0\|_{L^2_\lambda} + \|\tilde{\tilde{u}}\|^2_{N_\lambda} \tag{60}$$

and

$$\|P_3\tilde{F}_x\|_{N_\lambda} \lesssim \|u_0\|_{L^2_\lambda} + \|\tilde{\tilde{u}}\|^2_{N_\lambda} \quad . \tag{61}$$

Moreover, $\forall\ 0 < \alpha < 3$,

$$\|\psi_2 P_3 Q_\alpha \tilde{F}\|_{X^{7/8,0}_\lambda} \lesssim \frac{1}{\alpha}\|u_0\|_{L^2_\lambda} + \|\tilde{\tilde{u}}\|^2_{N_\lambda} \quad . \tag{62}$$

We postponed the proof of this lemma to the end of this section.

On the other hand, obviously,

$$\|P_{>1}w^*\|_{X^{1,-1}_\lambda} \lesssim \|P_{>1}\partial_x^{-1}w^*\|_{X^{1,0}_\lambda}$$

and from (29) we deduce that $w^* = \psi w^{**}$ where w^{**} satisfies (20) with W, w and F respectively replaced by $\psi\tilde{W}$, $\psi\tilde{\tilde{w}}$ and $\tilde{F}$ in the right-hand side member. Therefore using Lemma 4.2 and expanding the exponential function we infer that

$$\begin{aligned}
\|P_{>1}w^*\|_{X^{1,-1}_\lambda} &\lesssim \|w^*\|_{L^\infty_t L^2_\lambda} + \|\partial_t(P_{>1}\partial_x^{-1}w^{**}) + \mathcal{H}\partial_x^2(P_{>1}\partial_x^{-1}w^{**})\|_{L^2_{t,\lambda}} \\
&\lesssim \|w^*\|_{Y^s_\lambda} + \|\tilde{w}\|_{L^4_{t,\lambda}}\Big(\|\tilde{\tilde{w}}\|_{L^4_{t,\lambda}} + \|\psi_2\tilde{F}_x\|_{L^4_{t,\lambda}} + \|\psi_2\tilde{F}_x\|^2_{L^4_{t,\lambda}}\Big)e^{C\|\tilde{F}\|_{A_\lambda}} \\
\lesssim \|w(0)\|_{H^s_\lambda} + \|\ w\|_{X^{1/2,s}_{1,\lambda}}&\Big(\|u\|_{N_{1,\lambda}} + \|\ w\|_{X^{1/2,0}_{1,\lambda}}\Big)e^{C\,(\|\widehat{\partial_x^{-1}u_0}\|_{L^1} + \|u\|_{N_{1,\lambda}} + \|u\|^2_{N_{1,\lambda}})}
\end{aligned}$$

Finally, using Lemma 4.1 we infer that for $0 \le s \le 1/2$,

$$\begin{aligned}
\|w(0)\|_{H^s_\lambda} &= \|\partial_x P_+ e^{-i\partial_x^{-1}u_0}\|_{H^s_\lambda} = \frac{1}{2}\|P_+(u_0 e^{-i\partial_x^{-1}u_0})\|_{H^s_\lambda} \\
&\lesssim \sum_{k\ge 0}\frac{1}{k!}\|u_0(\partial_x^{-1}u_0)^k\|_{H^s_\lambda} \\
&\lesssim \|u_0\|_{H^s_\lambda}(1 + \|u_0\|_{L^2_\lambda})e^{\|\partial_x^{-1}u_0\|_{L^\infty_\lambda}}
\end{aligned} \tag{63}$$

which ends the proof of (22).

Proof of Lemma 4.6. From (28), $P_3\tilde{F} = P_3\tilde{F}^1 + P_3\tilde{F}^2$ where $P_3\tilde{F}^1 = \psi(t/\lambda^2)V(t)P_3\partial_x^{-1}u_0$ and $P_3\tilde{F}^2 = \psi(t)P_3\tilde{\tilde{F}}^2$ with

$$P_3\tilde{\tilde{F}}^2(0) = 0 \text{ and } P_3\tilde{\tilde{F}}^2{}_t + \mathcal{H}\partial_x^2 P_3\tilde{\tilde{F}}^2 = P_3\Big[(\psi\tilde{\tilde{u}})^2/2 - P_0((\psi\tilde{\tilde{u}})^2)/2\Big] \quad . \tag{64}$$

Therefore

$$\|P_3\tilde{F}^2\|_{\dot{X}^{1,0}_\lambda} \lesssim \|\psi_t\|_{L^2_t}\|V(t)P_3\tilde{\tilde{F}}^2\|_{L^\infty_t L^2_\lambda} + \|\psi\|_{L^\infty_t}\|P_3\tilde{\tilde{F}}^2{}_t + \mathcal{H}\partial_x^2 P_3\tilde{\tilde{F}}^2\|_{L^2_{t,\lambda}}$$

and (64) leads to

$$\|P_3\tilde{F}^2\|_{\dot{X}^{1,0}_1} \lesssim \|\psi\tilde{\tilde{u}}\|^2_{L^4_{t,\lambda}} \lesssim \|\tilde{\tilde{u}}\|^2_{N_\lambda} \tag{65}$$

On the other hand, by the definition of $P_3\tilde{F}^1$,

$$\begin{aligned}
\|P_3\tilde{F}^1\|_{\dot{X}^{1,0}_1} &= \Big\|\partial_t\Big(\psi(\cdot/\lambda^2)P_3\partial_x^{-1}u_0\Big)\Big\|_{L^2_{t,\lambda}} \\
&\lesssim \lambda^{-2}\|\psi_t(\cdot/\lambda^2)\|_{L^2_t}\|\partial_x^{-1}u_0\|_{L^2_\lambda} \\
&\lesssim \lambda^{-1}\,\lambda\|u_0\|_{L^2_\lambda} \lesssim \|u_0\|_{L^2_\lambda} \quad .
\end{aligned} \tag{66}$$

Moreover, from (64) and Lemma 2.1-2.2 we deduce that

$$\|\widehat{P_3\tilde{F}}\|_{L^1_{\tau,\xi}} \lesssim \|\widehat{P_3F_0}\|_{L^1_\xi} + \Big\|\chi_{\{|\xi|\le 3\}}\frac{\widehat{\psi\tilde{\tilde{u}}} * \widehat{\psi\tilde{\tilde{u}}}}{\langle\sigma\rangle}\Big\|_{L^1_{\tau,\xi}} + \Big\|\frac{\mathcal{F}_t\Big(P_0((\psi\tilde{\tilde{u}})^2)\Big)}{\langle\sigma\rangle}\Big\|_{L^1_\tau}$$

Applying Cauchy-Schwarz inequality in τ and ξ , it follows that

$$\begin{aligned}
\|\widehat{P_3\tilde{F}}\|_{L^1_{\tau,\xi}} &\lesssim \|\widehat{P_3F_0}\|_{L^1_\xi} + \Big\|\widehat{\psi\tilde{\tilde{u}}} * \widehat{\psi\tilde{\tilde{u}}}\Big\|_{L^2_{\tau,\xi}} + \|P_0((\psi\tilde{\tilde{u}})^2)\|_{L^2_t} \\
&\lesssim \|\widehat{P_3F_0}\|_{L^1_\xi} + \|\psi\tilde{\tilde{u}}\|^2_{L^4_{t,\lambda}} \\
&\lesssim \|\widehat{P_3F_0}\|_{L^1_\xi} + \|\tilde{\tilde{u}}\|^2_{N_\lambda} \quad .
\end{aligned}$$

To get (62) we notice that by classical linear estimates in Bourgain spaces (cf. [10]) and (28) we have

$$\|\psi_2 P_3 Q_\alpha\tilde{F}\|_{X^{7/8,0}_\lambda} \lesssim \|Q_\alpha\partial_x^{-1}u_0\|_{L^2_\lambda} + \|(\psi\tilde{\tilde{u}})^2\|_{X^{-1/8,0}_\lambda} \lesssim \frac{1}{\alpha}\|u_0\|_{L^2_\lambda} + \|\psi\tilde{\tilde{u}}\|^2_{L^4_{t,\lambda}} \ .$$

It remains to get the estimate (61) on $\|P_3\tilde{F}_x\|_{N_\lambda}$. But this is straightforward by combining (28), Lemmas 2.1-2.2 and Sobolev inequality in time for evaluating the $Z^{0,0}_\lambda$-norm and by writing $\|\chi_{[-4,4]}(t)P_3\tilde{F}_x\|_{\tilde{L}^4_{t,\lambda}} \lesssim \|\chi_{[-4,4]}(t)P_3\tilde{F}_x\|_{L^\infty_t L^2_\lambda}$ and then using the unitarity of $V(t)$ in L^2_λ (see (70) below).

5 Proof of the estimates on u

In this section we prove estimates (23) and (24) of Proposition 3.1. We will need the following lemma, proven in the appendix, which enables to treat the multiplication with the gauge function $e^{-iF/2}$ in $\tilde{L}^4_{t,\lambda}$.

Lemma 5.1 *Let $z \in L^\infty_t H^1_\lambda$ and let $v \in \tilde{L}^4_{t,\lambda}$ then*

$$\|zv\|_{\tilde{L^4}_{t,\lambda}} \lesssim (\|z\|_{L^\infty_{t,\lambda}} + \|z_x\|_{L^\infty_t L^2_\lambda})\|v\|_{\tilde{L}^4_{t,\lambda}} \quad . \tag{67}$$

5.1 Proof of (24)

Since u is real-valued, it holds

$$\|J^s_x u\|_{L^p_1 L^q_\lambda} \lesssim \|P_1 u\|_{L^p_1 L^q_\lambda} + \|D^s_x P_{>1} u\|_{L^p_1 L^q_\lambda} \quad .$$

To estimate the high modes part, we use (21) where we expand the exponential function. Hence, we write

$$\begin{aligned} \|D^s_x P_{>1} u\|_{L^\infty_1 L^2_\lambda} \quad &\lesssim \quad \sum_{k\geq 0} \frac{1}{k!} \|D^s_x (F^k w)\|_{L^\infty_1 L^2_\lambda} \\ &\quad + \sum_{k\geq 1}\sum_{l\geq 1} \frac{1}{k!\,l!} \Big\| D^s_x P_{>1}\Big(P_{>1}(F^k)\partial_x P_-(F^l)\Big)\Big\|_{L^\infty_1 L^2_\lambda} . \end{aligned} \tag{68}$$

From (68), Lemmas 4.1 and 4.2, Sobolev inequalities and (9), we infer that for $0 \leq s \leq 1/2$,

$$\begin{aligned} \|D^s_x P_{>1} u\|_{L^\infty_1 L^2_\lambda} \quad &\lesssim \quad \sum_{k\geq 0} \frac{1}{k!} (\|F^k\|_{L^\infty_{1,\lambda}} + \|\partial_x (F^k)\|_{L^\infty_1 L^2_\lambda}) \|J^s_x w\|_{L^\infty_1 L^2_\lambda} \\ &\quad + \sum_{k\geq 1}\sum_{l\geq 1} \frac{1}{k!\,l!} \|D^{5s/4}_x P_{>1}(F^k)\|_{L^\infty_1 L^{4/s}_\lambda} \|D^{1-s/4}_x P_-(F^l)\|_{L^\infty_1 L^{\frac{4}{2-s}}_\lambda} \\ &\lesssim \quad \sum_{k\geq 0} \frac{1}{k!} (\|F\|^k_{L^\infty_{1,\lambda}} + k\|F\|^{k-1}_{L^\infty_{1,\lambda}} \|F_x\|_{L^\infty_1 L^2_\lambda}) \|w\|_{Y^s_{1,\lambda}} \\ &\quad + \sum_{k\geq 1}\sum_{l\geq 1} \frac{1}{k!\,l!} \|D^{s+1/2}_x \ P_{>1}(F^k)\|_{L^\infty_1 L^2_\lambda} \|\partial_x P_-(F^l)\|_{L^\infty_1 L^2_\lambda} \end{aligned} \tag{69}$$

with

$$\Big\| D^{s+1/2}_x \ P_{>1}(F^k)\|_{L^\infty_1 L^2_\lambda} \lesssim \|\partial_x (F^k)\|_{L^\infty_1 L^2_\lambda} \lesssim k\|F\|^{k-1}_{L^\infty_{1,\lambda}} \|F_x\|_{L^\infty_1 L^2_\lambda}$$

and

$$\|\partial_x(F^l)\|_{L_1^\infty L_\lambda^2} \lesssim l\,\|F\|_{L_{1,\lambda}^\infty}^{l-1}\|F_x\|_{L_1^\infty L_\lambda^2}\ .$$

On the other hand, by the Duhamel formulation of the equation, the unitarity of $V(t)$ in L_λ^2, the continuity of $\partial_x P_1$ in L_λ^2 and Sobolev inequalities, we get

$$\|P_1 u\|_{L_1^\infty L_\lambda^2} \lesssim \|u_0\|_{L_\lambda^2} + \|u^2\|_{L_1^1 L_\lambda^2} \lesssim \|u_0\|_{L_\lambda^2} + \|u\|_{L_{1,\lambda}^4}^2\ . \tag{70}$$

This completes the proof of (24).

5.2 Proof of (23)

Remark 5.1 *It would considerably simplify the estimates on u if we were able to prove that there exists $C>0$ such that for any $v\in N_{1,\lambda}$ there exists an extension $\tilde v$ of v satisfying :*

$$\|\tilde v\|_{X_\lambda^{7/8,-1}} \le C\,\|v\|_{X_{1,\lambda}^{7/8,-1}},\ \|\tilde v\|_{Z_\lambda^{0,0}} \le C\,\|v\|_{Z_{1,\lambda}^{0,0}} \text{ and } \|\tilde v\|_{\tilde L_{t,\lambda}^4} \le C\,\|v\|_{\tilde L_{1,\lambda}^4}\ .$$

Indeed, we could then take different extensions of u according to the part of the N_λ-norm we want to estimate. Note, in particular, that taking the extension $P_{>3}\tilde u$ of $P_{>3}u$ defined by

$$P_{>3}\tilde u = \psi(t)\Big[V(t)P_{>3}u_0 + \frac{1}{2}\int_0^t V(t-t')P_{>3}\partial_x(\psi u)^2(t')\,dt'\Big]$$

we directly get

$$\|P_{>3}u\|_{X_{1,\lambda}^{7/8,-1}} \lesssim \|u\|_{L_{1,\lambda}^4}^2 + \|u\|_{L_1^\infty L_\lambda^2} \lesssim \|u\|_{N_{1,\lambda}}^2 + \|u_0\|_{L_\lambda^2}\ .$$

We start by constructing our extension F^* of F. To construct the high modes part, we first need some how to inverse the map $F\mapsto W$. From (16) we infer that

$$P_{>1}W = P_{>1}(e^{-iF/2}) = e^{-iF/2} - P_{\le 1}(e^{-iF/2})$$

By decomposing F in $Q_1F + P_1F$, we obtain

$$e^{-iQ_1F/2} = e^{iP_1F/2}\Big(P_{>1}W + P_{\le 1}(e^{-iF/2})\Big)$$

and thus

$$\begin{aligned} P_{>3}F &= 2iP_{>3}\Big[e^{iP_1F/2}\Big(P_{>1}W - P_{\leq 1}(e^{-iF/2})\Big)\Big] \\ &\quad -2iP_{>3}\Big(e^{-iQ_1F/2} + iQ_1F/2\Big) \quad . \end{aligned} \tag{71}$$

Now, let $\tilde{W}$ be an extension of W such that $\|\tilde{W}_x\|_{M^0_\lambda} \leq 2\|W_x\|_{M^0_{1,\lambda}}$ and $\tilde{F}$ be the extension of F defined in the last section. We set

$$\begin{aligned} P_{>3}F^* &= 2i\psi\, P_{>3}\Big[e^{iP_1\tilde{F}/2}\Big(P_{>1}(\psi\tilde{W}) - P_{\leq 1}(e^{-i\tilde{F}/2})\Big)\Big] \\ &\quad -2i\psi\, P_{>3}\Big(e^{-iQ_1\tilde{F}/2} + iQ_1\tilde{F}/2\Big) \quad , \end{aligned} \tag{72}$$

$P_{<-3}F^* = \overline{P_{>3}F^*}$ and $P_3F^* = P_3\tilde{F}$. It is clear that by construction $F^* \equiv F$ on $[0,1]$. Note that by (61), in Lemma 4.6, we already have an estimate on the low-modes part P_3F^*. Moreover, combining estimate (59) with (58), we infer that

$$\|\tilde{F}\|_{A_\lambda} \lesssim \|\widehat{\partial_x^{-1}u_0}\|_{L^1_\xi} + \|u\|^2_{N_{1,\lambda}} + \|u\|_{N_{1,\lambda}} \quad . \tag{73}$$

To estimate the high-modes part, for convenience, we drop the $\tilde{}$ in the right-hand side of (72). In the remaining of this section we assume that W is supported in time in $[-2,2]$.

5.2.1 Estimate on the $\tilde{L}^4_{t,\lambda}$ -norm

Differentiating (72) with respect to x and expanding the exponential function, we get

$$\begin{aligned} \|P_{>3}F^*_x\|_{\tilde{L}^4_{t,\lambda}} &\lesssim \sum_{k\geq 0}\frac{1}{k!}\Big(k\Big\|(P_1F_x)(P_1F)^{k-1}P_{>1}W\Big\|_{\tilde{L}^4_{t,\lambda}} + \Big\|(P_1F)^kP_{>1}W_x\Big\|_{\tilde{L}^4_{t,\lambda}}\Big) \\ &+\sum_{k\geq 0}\sum_{l\geq 0}\frac{1}{k!}\frac{1}{l!}\Big[k\Big\|\psi\, P_{>3}\Big((P_1F_x)(P_1F)^{k-1}P_{\leq 1}(F^l)\Big)\Big\|_{\tilde{L}^4_{t,\lambda}} + l\Big\|\psi\, P_{>3}\Big((P_1F)^kP_{\leq 1}(F^{l-1}F_x)\Big)\Big\|_{\tilde{L}^4_{t,\lambda}}\Big] \\ &\quad +\sum_{k\geq 2}\frac{1}{(k-1)!}\|\psi\,(Q_1F)^{k-1}Q_1F_x\|_{\tilde{L}^4_{t,\lambda}} \end{aligned}$$

We notice that by the frequency projections,

$$P_{>3}\Big((P_1F_x)(P_1F)^{k-1}P_{\leq 1}(F^l)\Big) \text{ and } P_{>3}\Big((P_1F)^kP_{\leq 1}(F^{l-1}F_x)\Big) \quad . \tag{74}$$

vanish for $k \leq 2$. Moreover, decomposing P_1F as $P_1Q_{\frac{1}{k-1}}F + P_{\frac{1}{k-1}}F$ we infer that for $k \geq 3$ the two terms appearing in (74) are respectively equal

to

$$P_{>3}\Big[(P_1F_x)(P_1Q_{\frac{1}{k-1}}F)P_{\leq 1}(F^l)G\Big] \text{ and } P_{>3}\Big[(P_1F)(P_1Q_{\frac{1}{k-1}}F)P_{\leq 1}(F^{l-1}F_x)G\Big]$$

with

$$G=\sum_{q=1}^{k-1} C_{k-1}^q (P_1Q_{\frac{1}{k-1}}F)^{q-1}(P_{\frac{1}{k-1}}F)^{k-1-q} \quad .$$

Note that G can be also written as

$$G=\sum_{j=0}^{k-2} \frac{C_{k-1}^{j+1}}{C_{k-2}^j} C_{k-2}^j \,(P_1Q_{\frac{1}{k-1}}F)^j(P_{\frac{1}{k-1}}F)^{k-2-j}$$

and thus it is not too hard to see that

$$\|G\|_{L^\infty_{t,\lambda}} \lesssim \|G\|_{A_\lambda} \lesssim (k-1)\|F\|_{A_\lambda}^{k-2} \quad . \tag{75}$$

Therefore, using that, by Sobolev inequalities,

$$\|Q_{\frac{1}{k-1}}P_1F\|_{L^\infty_{t,\lambda}} \lesssim (k-1)\|F_x\|_{L^\infty_t L^2_\lambda} \quad ,$$

using Lemma 5.1 and the embedding $X_\lambda^{1/2,0} \hookrightarrow \tilde{L}^4_{2,\lambda}$ (see (42)), we infer that

$$\begin{aligned}
\|P_{>3}F^*\|_{\tilde{L}^4_{t,\lambda}} &\lesssim \Big[(\|F_x\|_{L^\infty_t L^2_\lambda}+1)\|W_x\|_{\tilde{L}^4_{t,\lambda}} \\
&\quad +\|F_x\|_{L^\infty_t L^2_\lambda}\|\psi_2 F_x\|_{\tilde{L}^4_{t,\lambda}}\Big] e^{2\|F\|_{L^\infty_{t,\lambda}}} \\
&\lesssim \Big(\|w\|_{M^0_{1,\lambda}}(1+\|u\|_{N_{1,\lambda}})+\|u\|^2_{N_{1,\lambda}}\Big)\, e^{\tilde{K}} \quad . \qquad (76)
\end{aligned}$$

where $\tilde{K}$ is defined as in (25).

5.2.2 Estimate on the $Z^{0,0}_\lambda$ -norm

Now, using again the frequency projections and that A_λ is clearly an algebra, we deduce from (72) and (75) that

$$\begin{aligned}
\|P_{>3}F^*_x\|_{Z^{0,0}_\lambda} &\lesssim \sum_{k\geq 0}\frac{1}{k!}\|P_1F\|^{k-1}_{A_\lambda}\Big(k\|P_1F_x\|_{Z^{0,0}_\lambda}\|P_{>1}W\|_{A_\lambda}+\|P_1F\|_{A_\lambda}\|W_x\|_{Z^{0,0}_\lambda}\Big) \\
&\quad +\sum_{k\geq 3}\sum_{l\geq 0}\frac{k}{k!\,l!}(k+l)\|Q_{\frac{1}{k-1}}P_1F\|_{A_\lambda}\|F_x\|_{Z^{0,0}_\lambda}\|F\|^{k+l-2}_{A_\lambda} \\
&\quad +\sum_{k\geq 2}\frac{k}{k!}\|F_x\|_{Z^{0,0}_\lambda}\|Q_1F\|^{k-1}_{A_\lambda} \quad .
\end{aligned}$$

Using that, by Cauchy-Schwarz in ξ,

$$\|P_{>1}W\|_{A_\lambda} \lesssim \|W_x\|_{Z^{0,0}_\lambda} \text{ and } \|Q\widehat{_{\frac{1}{k-1}}P_1}F\|_{A_\lambda} \lesssim (k-1)\,\|F_x\|_{Z^{0,0}_\lambda}$$

we infer that

$$\begin{aligned}\|P_{>3}F^*_x\|_{Z^{0,0}_\lambda} &\lesssim \Big(\|F_x\|_{N_\lambda}\|W_x\|_{Z^{0,0}_\lambda} + \|W_x\|_{Z^{0,0}_\lambda} + \|F_x\|^2_{N_\lambda}\Big)e^{2\|F\|_{A_\lambda}} \\ &\lesssim \Big(\|w\|_{M^0_{1,\lambda}}(1+\|u\|_{N_{1,\lambda}}) + \|u\|^2_{N_{1,\lambda}}\Big)\,e^{\tilde{K}} \quad . \end{aligned} \tag{77}$$

5.2.3 Estimate on the $X^{7/8,-1}_\lambda$-norm

It remains to estimate the $X^{7/8,-1}_\lambda$-norm of $P_{>3}F^*_x$. Note that obviously

$$\|P_{>3}F^*_x\|_{X^{7/8,-1}_\lambda} \sim \|P_{>3}F^*\|_{X^{7/8,0}_\lambda}$$

From (72) we infer that

$$\begin{aligned}\|P_{>3}F^*\|_{X^{7/8,0}_\lambda} &\lesssim \sum_{k\geq 0}\frac{1}{k!}\Big\|P_{>3}\Big((P_1F)^kP_{>1}W\Big)\Big\|_{X^{7/8,0}_\lambda} \\ &\quad + \sum_{k\geq 3}\sum_{l\geq 0}\frac{1}{k!\,l!}\Big\|\psi\,P_{>3}\Big((P_1F)^kP_{\leq 1}(F^l)\Big)\Big\|_{X^{7/8,0}_\lambda} \\ &\quad + \sum_{k\geq 2}\frac{1}{k!}\Big\|\psi\,P_3\Big(Q_1F)^k\Big)\Big\|_{X^{7/8,0}_\lambda} \\ &= \sum_{k\geq 0}\frac{1}{k!}\,I_k + \sum_{k\geq 3}\sum_{l\geq 0}\frac{1}{k!\,l!}\,J_{k,l} + \sum_{k\geq 2}\frac{1}{k!}\,L_k \quad . \end{aligned}$$

Let us estimate I_k, $J_{k,l}$ and L_k, one by one.

i) Estimate on I_k. First note that for $k=0$, we have directly

$$I_0 \lesssim \|P_{>1}W\|_{X^{7/8,0}_\lambda} \lesssim \|w\|_{X^{7/8,-1}_\lambda} \lesssim \|w\|_{M^0_{1,\lambda}} \quad . \tag{78}$$

Now, for $k\geq 1$,

$$I_k = \Big\|\chi_{\{\xi\geq 3\}}\langle\sigma\rangle^{7/8}\int_{I\!R^k\times(\lambda^{-1}\mathbb{Z})^k}\widehat{P_1F}(\tau_1,\xi_1)..\widehat{P_1F}(\tau_k,\xi_k)\,\widehat{P_{>1}W}(\tau_{k+1},\xi_{k+1})\Big\|_{L^2_{\tau,\xi}}$$

where $\sigma = \tau + \xi|\xi|$ and $\Big(\sum_{i=1}^{k+1}\tau_i, \sum_{i=1}^{k+1}\xi_i\Big) = (\tau,\xi)$.
We divide $I\!R^{k+1}\times(\lambda^{-1}\mathbb{Z})^{k+1}$ in different regions.

• The region $|\sigma| \le 2^{10}k$. In this region, clearly,

$$I_k \lesssim k\|P_{>1}W\|_{L_t^\infty L_\lambda^2}\|F\|_{A_\lambda}^k \lesssim k\|w\|_{M^0_{1,\lambda}}\|F\|_{A_\lambda}^k \quad . \tag{79}$$

• The region $\{2^4k\,|\tau_{k+1}+\xi_{k+1}|\xi_{k+1}|| \ge |\sigma| \text{ and } |\sigma| > 2^{10}k\}$. In this region it is easy to see that

$$I_k \lesssim k\|P_{>1}W\|_{X_\lambda^{7/8,0}}\|F\|_{A_\lambda}^k \lesssim k\|Q_1w\|_{X_\lambda^{7/8,-1}}\|F\|_{A_\lambda}^k \lesssim k\|w\|_{M^0_{1,\lambda}}\|F\|_{A_\lambda}^k \quad .(80)$$

• The region $\{\exists i \in \{1,..,k\}, \quad 2^4k\,|\tau_i+\xi_i|\xi_i|| \ge \langle\sigma\rangle \text{ and } |\sigma| > 2^{10}k\}$. Then we have

$$\begin{aligned} I_k &\lesssim k\,\|P_1F\|_{\dot X_\lambda^{7/8,0}}\|P_1F\|_{A_\lambda}^{k-1}\|P_{>1}W\|_{A_\lambda} \\ &\lesssim k\,\|P_1F\|_{\dot X_\lambda^{1,0}}\|P_1F\|_{A_\lambda}^{k-1}\|P_{>1}W_x\|_{Z_\lambda^{0,0}} \\ &\lesssim k\|w\|_{M^0_{1,\lambda}}\|P_1F\|_{\dot X_\lambda^{1,0}}\|P_1F\|_{A_\lambda}^{k-1} \\ &\lesssim k\,\|w\|_{M^0_{1,\lambda}}(\|u\|_{N_{1,\lambda}}+\|u\|_{N_{1,\lambda}}^2)\|F\|_{A_\lambda}^{k-1} \end{aligned} \tag{81}$$

where we used (60) in the last step.
• The region $\{|\sigma| \ge 2^4k \max_{i=1,..,k+1}|\tau_i+\xi_i|\xi_i|| \text{ and } |\sigma| > 2^{10}k\}$. In this region, since $\xi \ge 0$, we have

$$\langle\sigma\rangle \le 2|\sigma| \le 4\Big|\sigma - \sum_{i=1}^{k+1}(\tau_i - \xi_i|\xi_i|)\Big| = \Big|\Big(\sum_{i=1}^{k+1}\xi_i\Big)^2 - \sum_{i=1}^{k+1}\xi_i|\xi_i|\Big| \quad . \tag{82}$$

Let us denote by $|\xi_{i_1}| = \max|\xi_i|$ and $|\xi_{i_2}| = \max_{i\neq i_1}|\xi_i|$. We claim that (82) implies

$$\langle\sigma\rangle \le 2^5k^2\,|\xi_{i_1}|\,|\xi_{i_2}| \quad . \tag{83}$$

Indeed, either $2k|\xi_{i_2}| \ge |\xi_{i_1}|$ and

$$\langle\sigma\rangle \le 4\Big(\Big(\sum_{i=1}^{k+1}|\xi_i|\Big)^2 + \sum_{i=1}^{k+1}|\xi_i|^2\Big) \le 2^5k^2\,|\xi_{i_1}|\,|\xi_{i_2}|$$

or $|\xi_{i_1}| \ge 2k|\xi_{i_2}|$ and then ξ and ξ_{i_1} have the same sign so that

$$\langle\sigma\rangle \le 4\Big(\sum_{i\neq i_1}|\xi_1|^2 + \Big(\sum_{i\neq i_1}|\xi_i|\Big)^2 + 2|\xi_{i_1}|\sum_{i\neq i_1}|\xi_i|\Big) \le 2^4k^2|\xi_{i_1}|\,|\xi_{i_2}| \quad .$$

From (83), we infer that in this region,

$$\begin{aligned} I_k &\lesssim k\|P_1F_x\|_{A_\lambda}\|P_{>1}W_x\|_{L^2_{t,\lambda}}\|F\|^{k-1}_{A_\lambda} + k(k-1)\|P_1F_x\|^2_{A_\lambda}\|P_{>1}W\|_{L^2_{t,\lambda}}\|F\|^{k-2}_{A_\lambda} \\ &\lesssim k^2\|P_{>1}W_x\|_{L^2_{t,\lambda}}\|F\|^{k-1}_{A_\lambda} \\ &\lesssim k^2\|w\|_{M^0_{1,\lambda}}\|F\|^{k-1}_{A_\lambda} \quad . \end{aligned} \tag{84}$$

ii) Estimate on $J_{k,l}$. Proceeding as in the treatment of the terms in (74), we can write $J_{k,l}$ as

$$J_{k,l} = \Big\| P_{>3}\Big((\psi_2 P_1 Q_{\frac{2}{k-1}}F)^2 P_{\leq 1}(F^l)\, G\Big\|_{X^{7/8,0}_\lambda}$$

where

$$G = \sum_{q=2}^{k} C^q_k (P_1 Q_{\frac{2}{k-1}}F)^{q-2}(P_{\frac{2}{k-1}}F)^{k-q} = \sum_{j=0}^{k-2} \frac{C^{j+2}_k}{C^j_{k-2}} C^j_{k-2}(P_1 Q_{\frac{2}{k-1}}F)^j (P_{\frac{2}{k-1}}F)^{k-2-j} \quad .$$

Clearly

$$|\hat{G}| \lesssim k(k-1)|\widehat{P_1F}| * \cdots |\widehat{P_1F}| \tag{85}$$

and thus

$$\|G\|_{A_\lambda} \lesssim k^2\|P_1F\|^{k-2}_{A_\lambda}\,. \tag{86}$$

Note first that we can assume that $|\sigma| \geq 2^{10}(k+l)$ since otherwise obviously,

$$\begin{aligned} J_{k,l} &\lesssim k^2\|\psi_2 Q_{\frac{2}{k-1}}P_1F\|^2_{L^4_{t,\lambda}}\|F\|^{k+l-2}_{A_\lambda} \lesssim k^4\|\psi_2 F_x\|^2_{L^4_{t,\lambda}}\|F\|^{k+l-2}_{A_\lambda} \\ &\lesssim k^4\|u\|^2_{N_{1,\lambda}}\|F\|^{k+l-2}_{A_\lambda} \quad . \end{aligned} \tag{87}$$

We have thus to estimate

$$\tilde{J}_{k,l} = \Big\| \chi_{\{\xi\geq 3\}}\chi_{\{|\sigma|\geq 2^{10}k\}}\langle\sigma\rangle^{7/8}\mathcal{F}_{t,x}\Big((\psi_2 Q_{\frac{2}{k-1}}P_1F)^2 P_{\leq 1}(F^l)\, G\Big)(\tau,\xi)\Big\|_{L^2_{\tau,\xi}}$$

where $\sigma = \tau + \xi|\xi|$.
As in Lemma 4.4, one of the difficulties is that we do not know if $\mathcal{F}^{-1}_{t,x}(|\hat{F}|)$ belongs to $L^4_{t,\lambda}$. Using again the Littlewood-Paley decomposition it can be seen that for $l \geq 2$,

$$F^l = \sum_{i_1\geq i_2\geq 0} \Delta_{i_1}(F)\Delta_{i_2}(F) \sum_{0\leq i_3,..,i_l\leq i_2} n(i_1,..,i_l)\prod_{j=3}^{l}\Delta_{i_j}(F)\,, \tag{88}$$

where $n(i_1,..,i_l)$ is an integer belonging to $\{1,..,l(l-1)\}$ (Note for instance that $n(i_1,..,i_l)=1$ for $i_1=\cdot\cdot=i_l$ and $n(i_1,..,i_l)=l(l-1)$ for $i_1\neq\cdot\cdot\neq i_l$). We set

$$H_{j,q,l}=\Delta_j(F)\Delta_q(F)\sum_{0\leq i_3,..,i_l\leq q} n(j,q,i_3,..,i_l)\prod_{m=3}^{l}\Delta_{i_m}(F) \quad .$$

It is clear that for $l\geq 2$,

$$\begin{aligned}\tilde{J}_{k,l} &\lesssim \sum_{j\geq q\geq 1}\Big\|\chi_{\{\ \xi\geq 3\ /\ |\sigma|\geq 2^{10}k\ \}}\langle\sigma\rangle^{7/8}\mathcal{F}_{t,x}\Big((\psi_2 Q_{\frac{2}{k-1}}P_1F)^2\,G\,P_{\leq 1}(H_{j,q,l})\Big)\Big\|_{L^2_{\tau,\xi}}\\ &\quad +l\Big\|\chi_{\{\ \xi\geq 3\ /\ |\sigma|\geq 2^{10}k\ \}}\langle\sigma\rangle^{7/8}\sum_{j\geq 0}\mathcal{F}_{t,x}\Big((\psi_2 Q_{\frac{2}{k-1}}P_1F)^2\,G\,P_{\leq 1}\Big(\Delta_j(F)\Delta_0(F)^{l-1}\Big)\Big\|_{L^2_{\tau,\xi}}\\ &= \Lambda_{k,l}+\Gamma_{k,l} \quad .\end{aligned}$$

Let us write $\Lambda_{k,l}$ as the sum of two terms :

$$\begin{aligned}\Lambda_{k,l} &= \sum_{j\geq q\geq 1}\Big\|\chi_{\{\xi\geq 3\}}\chi_{\{|\sigma|\in D_k^1\}}\langle\sigma\rangle^{7/8}\mathcal{F}_{t,x}\Big((\psi_2 Q_{\frac{2}{k-1}}P_1F)^2 G P_{\leq 1}(H_{j,q,l})\Big)\Big\|_{L^2_{\tau,\xi}}\\ &\quad +\sum_{j\geq q\geq 1}\Big\|\chi_{\{\xi\geq 3,\,|\sigma|\in D_k^2\}}\langle\sigma\rangle^{7/8}\mathcal{F}_{t,x}\Big((\psi_2 Q_{\frac{2}{k-1}}P_1F)^2 G P_{\leq 1}((H_{j,q,l})\Big)\Big\|_{L^2_{\tau,\xi}}\\ &= \Lambda^1_{k,l}+\Lambda^2_{k,l} \quad ,\end{aligned}$$

with

$$D_k^1=[2^{10}(k+l),(k+l)^2 2^{8+j+q}[\ \text{and}\ D_k^2=[\max\Big((2^{10}(k+l),(k+l)^2 2^{8+j+q}\Big),+\infty[\quad .$$

From the definition of $H_{j,q,l}$, (86) and (40) we infer that for $l\geq 2$,

$$\begin{aligned}\Lambda^1_{k,l} &\lesssim k^2\sum_{j=1}^{\infty}\sum_{q=1}^{j}2^{7j/8}2^{7q/8}\,\|F\|^k_{A_\lambda}\|\psi_2^2\,H_{j,q,l}\|_{L^2_{\tau,\xi}}\\ &\lesssim (kl)^2\|F\|^{k+l-2}_{A_\lambda}\Big(\sum_{j=1}^{\infty}2^{7j/8}\,\|\psi_2\Delta_j F\|_{L^4_{t,\lambda}}\Big)^2\\ &\lesssim (kl)^2\|F\|^{k+l-2}_{A_\lambda}\|\psi_2 F_x\|^2_{L^4_{t,\lambda}}\lesssim (kl)^2\|F\|^{k+l-2}_{A_\lambda}\|F_x\|^2_{N_\lambda} \quad . \quad (89)\end{aligned}$$

On the other hand , using (85), it is easy to check that for $l\geq 2$,

$$\begin{aligned}\Lambda^2_{k,l} &\lesssim (kl)^2\Big\|\chi_{\{\xi\geq 3\}}\langle\sigma\rangle^{7/8}\int_{\mathbb{R}^{k+l-1}\times(\lambda^{-1}\mathbb{Z})^{k+l-1}}\chi_{\{|\sigma|>2^5(k+l)^2|\xi_{i_1}\xi_{i_2}|\}}|\widehat{Q_{\frac{2}{k-1}}P_1}F(\tau_1,\xi_1)|\\ &\quad |\mathcal{F}_{t,x}\Big(\psi_2 Q_{\frac{2}{k-1}}P_1F\Big)(\tau_2,\xi_2)|\,|\widehat{P_1F}(\tau_3,\xi_3)||\widehat{P_1F}(\tau_k,\xi_k)||\widehat{F}(\tau_{k+1},\xi_{k+1})|..|\widehat{F}(\tau_{k+l},\xi_{k+l})|\Big\|_{L^2_{\tau,\xi}}\end{aligned}$$

where $|\xi_{i_1}| = \max |\xi_i|$ and $|\xi_{i_2}| = \max_{i\neq i_1} |\xi_i|$. But the same considerations as in (82)-(83) ensure that $2^{10}(k+l) \leq |\sigma| \leq 10(k+l)\max_{i=1,..,k+l} |\tau_i - \xi_i|\xi_i||$ in the region of integration above. Therefore, according to Lemma 4.6,

$$\begin{aligned}
\Lambda_{k,l}^2 &\lesssim (kl)^2(k+l)\|\psi_2 Q_{\frac{2}{k-1}} P_1 F\|_{A_\lambda}^2 \|F\|_{A_\lambda}^{k+l-3} \|F\|_{\dot{X}_\lambda^{7/8,0}} \\
&\quad + (kl)^2(k+l)\|\psi_2 Q_{\frac{2}{k-1}} P_1 F\|_{A_\lambda} \|F\|_{A_\lambda}^{k+l-2} \|\psi_2 Q_{\frac{2}{k-1}} P_1 F\|_{X_\lambda^{7/8,0}} \\
&\lesssim (kl)^2(k+l)k\|F_x\|_{Z_\lambda^{0,0}} \|F\|_{A_\lambda}^{k+l-3} \\
&\quad \Big(k\|F_x\|_{Z_\lambda^{0,0}} (\|P_3 F\|_{\dot{X}_\lambda^{1,0}} + \|Q_3 F\|_{X_\lambda^{7/8,0}}) + \|F\|_{A_\lambda} \|\psi_2 Q_{\frac{2}{k-1}} P_1 F\|_{X_\lambda^{7/8,0}} \Big) \\
&\lesssim (k+l)^5 \|u\|_{N_{1,\lambda}}^2 (\|u\|_{N_{1,\lambda}} + \|F\|_{A_\lambda})(1 + \|u\|_{N_{1,\lambda}}) \|F\|_{A_\lambda}^{k+l-3} . \qquad (90)
\end{aligned}$$

It remains to estimate $\Gamma_{k,l}$ for $l \geq 2$. We notice that

$$\begin{aligned}
\Gamma_{k,l} &\lesssim k^2 l \Big\| \int_{\mathbb{R}^{k+l-1} \times (\lambda^{-1}\mathbb{Z})^{k+l-1}} |\mathcal{F}_{t,x}\Big(\psi_2 Q_{\frac{2}{k-1}} P_1 F\Big)(\tau_1, \xi_1)| |\mathcal{F}_{t,x}\Big(\psi_2 Q_{\frac{2}{k-1}} P_1 F\Big)(\tau_2, \xi_2)| \\
&\qquad \prod_{i=3}^{k} |\widehat{P_1 F}(\tau_i, \xi_i)| \; |\hat{F}(\tau_{k+1}, \xi_{l+1})| \prod_{i=k+2}^{k+l} |\widehat{P_3 F}(\tau_i, \xi_i)| \Big\|_{L^2_{\tau,\xi}}
\end{aligned}$$

which can be estimated in the same way we did for I_k. More precisely, in the region, $2^4(k+l) \max_{i=1,..,k+l} |\tau_i - \xi_i|\xi_i|| \geq |\sigma|$ we easily get as above

$$\Gamma_{k,l} \lesssim (k+l)^5 \|u\|_{N_{1,\lambda}}^2 (\|u\|_{N_{1,\lambda}} + \|F\|_{A_\lambda})(1 + \|u\|_{N_{1,\lambda}}) \Big) \|F\|_{A_\lambda}^{k+l-3}, \qquad (91)$$

and in the region $|\sigma| \geq 2^4(k+l) \max_{i=1,..,k+l} |\tau_i - \xi_i|\xi_i||$ we infer from (83) that

$$\tilde{J}_{k,l} \lesssim (k+l)^5 \|u\|_{N_{1,\lambda}}^2 \|F\|_{A_\lambda}^{k+l-2} . \qquad (92)$$

Finally, we notice that $\tilde{J}_{k,0}$ and $\tilde{J}_{k,1}$ with $k \geq 3$ can be estimated exactly in the same way.

iii) Estimate on L_k This term can be treated in the same way as the preceding one and is even much simpler. Since $k \geq 2$ we can decompose $Q_1(F)^k$ as we did for F^l in (88) and then proceed exactly in the same way as for $J_{k,l}$. We get

$$L_k \lesssim k^5 \|u\|_{N_{1,\lambda}}^k (1 + \|u\|_{N_{1,\lambda}}^k) \qquad (93)$$

Gathering (78)-(81), (84)-(87) and (89)-(93), we finally deduce that

$$\|P_{>3} F_x^*\|_{X_\lambda^{7/8,-1}} \lesssim \Big(\|w\|_{M_{1,\lambda}^0} (1 + \|u\|_{N_{1,\lambda}}) + \|u\|_{N_{1,\lambda}}^2 \Big) e^{\tilde{K}} \qquad (94)$$

which ends the proof of (23).

6 Uniform estimates and Lipschitz bound for small initial data

6.1 Uniform estimate for small initial data

We are now ready to state the following crucial proposition on the uniform boundedness of small smooth solutions to (BO).

Proposition 6.1 *Let $0 \leq s \leq 1/2$ and $K \geq 1$ be given. There exists $0 < \varepsilon := \varepsilon(K) \sim e^{-8CK} < 1$ such that for any $u_0 \in H^\infty_{0,\lambda}$ with*

$$\|\widehat{\partial_x^{-1} u_0}\|_{L^1_\xi} \lesssim K \quad \text{and} \quad \|u_0\|_{L^2_\lambda} \lesssim \varepsilon^2 \quad ,$$

the emanating solution $u \in C(\mathbb{R}; H^\infty_{0,\lambda})$ to (BO) satisfies

$$\|u\|_{L^\infty_1 H^s_\lambda} \lesssim e^{2CK} \|u_0\|_{H^s_\lambda} \quad \text{and} \quad \|w\|_{M^s_{1,\lambda}} \lesssim e^K \|u_0\|_{H^s_\lambda} \quad . \tag{95}$$

Proof. For $K \geq 1$ given, let $B_{K,\lambda}$ be the small closed ball of L^2_λ defined by

$$B_{K,\lambda} := \left\{ \varphi \in L^2_\lambda, \ \|\widehat{\partial_x^{-1} \varphi}\|_{L^1_\xi} \lesssim K \text{ and } \|\varphi\|_{L^2_\lambda} \lesssim \varepsilon(K)^2 \right\} \tag{96}$$

where $0 < \varepsilon(K) \sim e^{-8CK} \ll 1$ ($C > 1$ is the universal constant appearing in (25)) only depends on K and the implicit constants contained in the estimates of the preceding sections. At this stage, it worth recalling that these implicit constants do not depend on the period λ.
We set $\varepsilon := \varepsilon(K)$. For u_0 belonging to $H^\infty_{0,\lambda} \cap B_{K,\lambda}$, we want to show that the emanating solution $u \in C(\mathbb{R}; H^\infty_{0,\lambda})$, given by the classical well-posedness results (cf. [1], [13]), satisfies

$$\|u\|_{N_{1,\lambda}} \lesssim e^{2CK} \varepsilon^2 \quad \text{and} \quad \|w\|_{M^0_{1,\lambda}} \lesssim e^K \varepsilon^2 \quad . \tag{97}$$

(95) then obviously follows from (97) together with (22) and (24).
Clearly, since u satisfies the equation, u belongs in fact to $C^\infty(\mathbb{R}; H^\infty_\lambda)$ and thus u and w belong to $M^\infty_{1,\lambda} \cap N_{1,\lambda}$. We are going to implement a bootstrap argument. Since we have chosen to take $T = 1$ we can not use any continuity argument in time but as in [6] we will apply a continuity argument on the space period. Recall that if $u(t,x)$ is a $2\lambda\pi$-periodic solution of (BO) on $[0,T]$ with initial data u_0 then $u_\beta(t,x) = \beta^{-1} u(\beta^{-2}t, \beta^{-1}x)$ is a $(2\pi\lambda\beta)$-periodic solution of (BO) on $[0, \beta^2 T]$ emanating from $u_{0,\beta} = \beta^{-1} u_0(\beta^{-1}x)$. Moreover, denoting by w_β the gauge transform of u_β, it is worth noticing that

$$w_\beta(t,x) = \beta^{-1} w(\beta^{-2}t, \beta^{-1}x) \quad . \tag{98}$$

Straightforward computations give

$$\|u_{0,\beta}\|_{L^2_{\lambda\beta}} = \beta^{-1/2}\|u_0\|_{L^2_\lambda} \text{ and } \|\widehat{\partial_x^{-1}u_{0,\beta}}\|_{L^1_\xi} = \|\widehat{\partial_x^{-1}u_0}\|_{L^1_\xi} \,. \tag{99}$$

Note that $\|\widehat{\partial_x^{-1}u_0}\|_{L^1_\xi}$ is invariant by the symmetry dilation of (BO). In the same way one can easily check that the $N_{1,\lambda\beta}$-norm of u_β and the $M^0_{1,\lambda\beta}$ norm of w_β tend to 0 as β tends to infinity. Hence, for β large enough, u_β and w_β satisfy

$$\|u_\beta\|_{N_{1,\lambda\beta}} + \|w_\beta\|_{M^0_{1,\lambda\beta}} \lesssim \varepsilon \,. \tag{100}$$

(22) then clearly ensures that $\|w_\beta\|_{M^0_{1,\lambda\beta}} \lesssim (1+\|u_{0,\beta}\|_{L^2_{\lambda\beta}})e^K\|u_{0,\beta}\|_{L^2_{\lambda\beta}}$ and (23)-(25) ensure that

$$\|u_\beta\|_{N_{1,\lambda\beta}} \lesssim (1+\|u_{0,\beta}\|_{L^2_{\lambda\beta}})e^{2CK}\|u_{0,\beta}\|_{L^2_{\lambda\beta}} \quad .$$

Therefore, by the assumptions on u_0 and (99), we finally get

$$\|u_\beta\|_{N_{1,\lambda\beta}} \lesssim e^{2CK}\beta^{-1/2}\varepsilon^2 \text{ and } \|w_\beta\|_{M^0_{1,\lambda\beta}} \lesssim e^K\beta^{-1/2}\varepsilon^2 \tag{101}$$

which, by the definition of ε, proves that

$$\|u_\beta\|_{N_{1,\lambda\beta}} + \|w_\beta\|_{M^0_{1,\lambda\beta}} \lesssim \beta^{-1/2}\varepsilon^{3/2} \quad .$$

$\beta \mapsto \|u_\beta\|_{N_{1,\lambda\beta}} + \|w_\beta\|_{M^0_{1,\lambda\beta}}$ being clearly continuous, a classical continuity argument in β ensures that we can take $\beta = 1$ in (101). This completes the proof of (97) and thus of (95).

6.2 Lipschitz bound

To prove the continuity of the solution as well as the continuity the flow-map we will derive a Lipschitz bound on the solution-map $u_0 \mapsto u$ for small solutions of (BO) (Note that up to now this map in only defined on H^∞_λ).

Let u_1 and u_2 be two solutions of (BO) in $N_{1,\lambda} \cap C([0,T];H^s_\lambda)$ associated with initial data φ_1 and φ_2 in $B_{K,\lambda} \cap H^s_\lambda$ such that their gauge transforms w_1 and w_2 belong to $M^s_{1,\lambda}$. We assume that they satisfy

$$\|u_i\|_{N_{1,\lambda}} + \|w_i\|_{M^0_{1,\lambda}} \lesssim \varepsilon^2, \quad i = 1,2 \quad , \tag{102}$$

where $0 < \varepsilon = \varepsilon(K) \ll 1$.
We set $W_i = P_+(e^{-iF_i/2})$ with $F_i = \partial_x^{-1}u_i$, $w_i = \partial_x W_i$, $v = u_1 - u_2$,

$Z = W_1 - W_2$ and $z = Z_x$.
It is easy to check that

$$\begin{aligned} v &= 2ie^{iF_1/2}\Big[z + \partial_x P_-\Big(e^{-iF_1/2} - e^{-iF_2/2}\Big)\Big] \\ &\quad +2i(e^{iF_1/2} - e^{iF_2/2})\Big(w_2 + \partial_x P_-(e^{-iF_2/2})\Big) \end{aligned} \tag{103}$$

and that z satifies

$$\begin{aligned} z_t - iz_{xx} &= -\partial_x P_+\Big[W_1 \partial_x P_-(v)\Big] - \partial_x P_+\Big[Z\, P_-(\partial_x u_2)\Big] \\ &\quad +\frac{i}{4}\Big(P_0(u_1^2)z + P_0(u_1^2 - u_2^2)w_2\Big)\,. \end{aligned} \tag{104}$$

As in the obtention of (20), we substitute (103) in (104) to get

$$\begin{aligned} z_t - iz_{xx} = {} & 2i\partial_x P_+\Big[W_1 \partial_x P_-(e^{-iF_1/2}\overline{z} + (e^{-iF_1/2} - e^{-iF_1/2})\overline{w_2})\Big] \\ & +2i\partial_x P_+\Big[W_1 \partial_x P_-\Big(e^{-iF_1/2}\partial_x P_+(e^{iF_1/2} - e^{iF_2/2})\Big)\Big] \\ & +2i\partial_x P_+\Big[W_1 \partial_x P_-\Big((e^{iF_1/2} - e^{iF_2/2})\partial_x P_+(e^{iF_2/2}\Big)\Big] + 2i\partial_x P_+\Big(Z\partial_x P_-(e^{-iF_2/2}\overline{w_2})\Big) \\ & +2i\partial_x P_+\Big[Z\partial_x P_-\Big(e^{-iF_2/2}\partial_x P_+(e^{iF_2/2})\Big)\Big] + \frac{i}{4}\Big(P_0(u_1^2)z + P_0(u_1^2 - u_2^2)w_2\Big)\,. \end{aligned}$$

This expression seems somewhat complicated but actually each term can be treated as in Section 4. We extend the functions w_i and F_i in the same way as in Section 4.3. To deal with the difference $e^{i\tilde{F}_1/2} - e^{i\tilde{F}_2/2}$ we use that formally

$$e^{i\tilde{F}_1/2} - e^{i\tilde{F}_2/2} = \sum_{k\in\mathbb{N}} \frac{(i/2)^k}{k!}(\tilde{F}_1^k - \tilde{F}_2^k) = \sum_{k\geq 1} \frac{(i/2)^k}{k!}(\tilde{F}_2 - \tilde{F}_2)\Big(\sum_{j=0}^{k-1} \tilde{F}_1^j \tilde{F}_2^{k-1-j}\Big)$$

Moreover, as in (59) we have

$$\|P_3(\tilde{F}_1 - \tilde{F}_2)\|_{A_\lambda} \lesssim \Big\|\mathcal{F}_x^{-1}\Big(\partial_x^{-1}(u_1(0) - u_2(0))\Big)\Big\|_{L^1_\xi} + \|\tilde{\tilde{u}}_1 - \tilde{\tilde{u}}_2\|_{N_\lambda}(\|\tilde{\tilde{u}}_1\|_{N_\lambda} + \|\tilde{\tilde{u}}_2\|_{N_\lambda})$$

and thus

$$\|\tilde{F}_1 - \tilde{F}_2\|_{A_\lambda} \lesssim \Big\|\mathcal{F}_x^{-1}\Big(\partial_x^{-1}(u_1(0) - u_2(0))\Big)\Big\|_{L^1_\xi} + \|\tilde{\tilde{u}}_1 - \tilde{\tilde{u}}_2\|_{N_\lambda}(1 + \|\tilde{\tilde{u}}_1\|_{N_\lambda} + \|\tilde{\tilde{u}}_2\|_{N_\lambda})\,.$$

Therefore, on account of Lemmas 2.1-2.2, 4.3-4.4 and (57), we infer that, for $0 \le s \le 1/2$,

$$\begin{aligned}\|z\|_{M^s_{1,\lambda}} \lesssim\ & \|z(0)\|_{H^s_\lambda} + e^{\tilde K_1+\tilde K_2}\Big[\|w_1\|_{X^{1/2,s}_{1,\lambda}}\Big(\|z\|_{X^{1/2,0}_{1,\lambda}}\|u_1\|_{N_{1,\lambda}} \\ & +(\|\widehat{\partial_x^{-1}v_0}\|_{L^1_\xi} + \|v\|_{N_{1,\lambda}} + \|v\|^2_{N_1,\lambda})(\|w_2\|_{X^{1/2,0}_{1,\lambda}} + \|u_1\|_{N_1,\lambda} + \|u_2\|_{N_1,\lambda})\Big) \\ & +\|z\|_{X^{1/2,s}_{1,\lambda}}(\|w_2\|_{X^{1/2,0}_{1,\lambda}} + \|u_2\|_{N_{1,\lambda}} + \|u_2\|^2_{N_{1,\lambda}}) + \|z\|_{X^{1/2,0}_{1,\lambda}}\|u_1\|^2_{N_{1,\lambda}} \\ & +\|v\|_{N_{1,\lambda}}\|u_2\|_{N_{1,\lambda}}\|w_2\|_{X^{1/2,0}_{1,\lambda}}\Big] \quad .\end{aligned}$$

where

$$\tilde K_1 + \tilde K_2 = C\Big(\|\widehat{\partial_x^{-1}u_1(0)}\|_{L^1_\xi} + \|\widehat{\partial_x^{-1}u_2(0)}\|_{L^1_\xi} + \|u_1\|^2_{N_{1,\lambda}} + \|u_2\|^2_{N_{1,\lambda}}\Big) \quad .$$

Thanks to (102) we thus obtain that

$$\begin{aligned}\|z\|_{M^s_{1,\lambda}} \lesssim\ & \Big(1 + \|\varphi_2\|_{L^2_\lambda} + \|\varphi_1\|_{L^2_\lambda}(1+\lambda^{1/2})\Big)\|\varphi_1-\varphi_2\|_{H^s_\lambda} \\ & +\varepsilon^2\, e^{2CK}\Big[\|w_1\|_{X^{1/2,s}_{1,\lambda}}(\|z\|_{X^{1/2,0}_{1,\lambda}} + \|\widehat{\partial_x^{-1}v(0)}\|_{L^1_\xi} + \|v\|_{N_{1,\lambda}}) \\ & +\|z\|_{X^{1/2,s}_{1,\lambda}} + \|v\|_{N_{1,\lambda}}\Big] \quad , \qquad (105)\end{aligned}$$

since, by Lemma 4.1, it can be easily seen that

$$\begin{aligned}\|z(0)\|_{H^s_\lambda} \lesssim\ & \|\varphi_1-\varphi_2\|_{H^s_\lambda}\Big(1 + \|\varphi_1\|_{L^2_\lambda} + \|\varphi_2\|_{L^2_\lambda}\Big) \\ & +\|e^{-iF_1(0)} - e^{-iF_2(0)}\|_{L^\infty_\lambda}\|\varphi_1\|_{H^s_\lambda}(1 + \|\varphi_1\|_{L^2_\lambda})\end{aligned}$$

with

$$\|e^{-iF_1(0)} - e^{-iF_2(0)}\|_{L^\infty_\lambda} \lesssim \|\partial_x^{-1}(\varphi_1-\varphi_2)\|_{L^\infty_\lambda} \lesssim \lambda^{1/2}\|\varphi_1-\varphi_2\|_{L^2_\lambda} \quad .$$

On the other hand, proceeding as in Section 5 and using (102), one can check that

$$\|v\|_{N_{1,\lambda}} \lesssim \|v(0)\|_{L^2_\lambda} + \Big[\|z\|_{M^0_{1,\lambda}} + \varepsilon^2\Big(\|\widehat{\partial_x^{-1}v(0)}\|_{L^1_\xi} + \|v\|_{N_{1,\lambda}}\Big)\Big]e^{2CK} \ . \quad (106)$$

Noticing that by Cauchy-Schwarz in ξ,

$$\|\widehat{\partial_x^{-1}v(0)}\|_{L^1_\xi} \lesssim \lambda^{1/2}\|\widehat{v(0)}\|_{L^2_\xi} \sim \lambda^{1/2}\|v(0)\|_{L^2_\lambda}$$

and gathering (105) and (106) we obtain

$$\|v\|_{N_{1,\lambda}} + \|z\|_{M^0_{1,\lambda}} \lesssim e^{2CK}(1+\varepsilon^2\lambda^{1/2})\|\varphi_1-\varphi_2\|_{L^2_\lambda} \quad . \tag{107}$$

Coming back to (105) this leads to

$$\|z\|_{M^s_{1,\lambda}} \lesssim e^{2CK}\Big(1+\varepsilon^2\lambda^{1/2}\Big)\|\varphi_1-\varphi_2\|_{H^s_\lambda} \quad . \tag{108}$$

Now, proceeding as in (18), we infer that

$$\begin{aligned} v &= \partial_x F_1 - \partial_x F_2 \\ &= 2ie^{iF_1/2}\Big[z+\partial_x P_-\Big(e^{-iF_1/2}-e^{-iF_2/2}\Big)\Big] + 2i(e^{iF_1/2}-e^{iF_2/2})\Big(w_2+\partial_x P_-(e^{-iF_2/2}\Big) \end{aligned}$$

and thus

$$\begin{aligned} P_{>1}v &= 2iP_{>1}(e^{iF_1/2}z) + 2iP_{>1}\Big[P_{>1}(e^{iF_1/2})\partial_x P_-\Big(e^{-iF_1/2}-e^{-iF_2/2}\Big)\Big] \\ &+2iP_{>1}\Big[(e^{iF_1/2}-e^{iF_2/2})w_2\Big] + 2iP_{>1}\Big[P_{>1}(e^{iF_1/2}-e^{iF_2/2})\partial_x P_-(e^{-iF_2/2}\Big] \end{aligned} \tag{109}$$

Therefore, by Lemmas 4.1-4.2, (102) and (25)

$$\begin{aligned} \|J^s_x Q_1 v\|_{L^\infty_1 L^2_\lambda} &\lesssim \Big(\|z\|_{Y^s_{1,\lambda}} + \varepsilon^2(\|v\|_{L^\infty_1 L^2_\lambda} + \varepsilon^2\|\partial_x^{-1}v\|_{L^\infty_1 L^\infty_\lambda})\Big)e^{\tilde K} \\ &\lesssim \Big(\|z\|_{Y^s_{1,\lambda}} + \varepsilon^2(1+\varepsilon^2\lambda^{1/2})\|v\|_{L^\infty_1 L^2_\lambda}\Big)e^{\tilde K} \end{aligned}$$

Since on the other hand (see (70)),

$$\|P_1\, v\|_{L^\infty_1 L^2_\lambda} \lesssim \|\varphi_1-\varphi_2\|_{L^2_\lambda} + \|v\|_{L^4_{1,\lambda}}\Big(\|u_1\|_{L^4_{1,\lambda}} + \|u_2\|_{L^4_{1,\lambda}}\Big), \tag{110}$$

we finally deduce from (106)-(107) that

$$\|J^s_x v\|_{L^\infty_1 L^2_\lambda} \lesssim e^{4CK}(1+\varepsilon^2\lambda^{1/2})^2\|\varphi_1-\varphi_2\|_{H^s_\lambda} \quad . \tag{111}$$

7 Proof of Theorem 1.1

We will first prove the local well-posedness result for small data, the result for arbitrary large data will then follow from scaling arguments.

7.1 Well-posedness for small initial data

For any $K \geq 1$ and $\lambda \geq 1$ given, let $u_0 \in B_{K,\lambda} \cap H^s_\lambda$ with $0 \leq s \leq 1/2$ and let $\{u_0^n\} \subset H^\infty_0(\mathbb{T}) \cap B_{K,\lambda}$ converging to u_0 in $H^s(\mathbb{T})$. We denote by u_n the solution of (BO) emanating from u_0^n. From standard existence theorems (see for instance [1], [13]), $u_n \in C(\mathbb{R}; H^\infty_{0,\lambda})$. According to (97) and (95), for all $n \in \mathbb{N}^*$,

$$\|u_n\|_{N_{1,\lambda}} + \|w_n\|_{M^0_{1,\lambda}} \lesssim e^{2CK}\, \varepsilon(K)^2 \tag{112}$$

and

$$\|u_n\|_{L^\infty_1 H^s_\lambda} + \|w_n\|_{M^s_{1,\lambda}} \lesssim e^{2C\|\widehat{\partial_x^{-1}u_0}\|_{L^1_\xi}}\, \|u_0\|_{H^s_\lambda}\ , \tag{113}$$

where $w_n = \partial_x P_+(e^{-iF_n/2})$ is the gauge transform of w_n. Note that this uniform bound would enable to prove the local existence for $s > 0$ by using weak convergences. On the other hand, for $s = 0$, weak convergences would not be sufficient to pass to the limit on the nonlinear term u^2. Actually, with (107) and (111) in hand, we observe that the approximative sequence u^n constructed for the local existence result is a Cauchy sequence in $C([0,1]; H^s_{0,\lambda}) \cap N_{1,\lambda}$ since the u_n satisty (97)-(95) and $u_{0,n}$ converges to u_0 in $H^s_{0,\lambda}$. Hence, u_n converges strongly to some u in $C([0,1]; H^s_{0,\lambda}) \cap N_{1,\lambda}$. This strong convergence permits to pass easily to the limit on the nonlinear term and thus u is a solution of (BO). Moreover, from (95) and (108) it follows that the sequence of gauge transforms w_n of u_n is a Cauchy sequence in $M^s_{1,\lambda}$. Hence $w_n = \partial_x P_+(e^{-iF_n/2})$ converges toward some function w in $Y^s_{1,\lambda}$ and from the strong convergence of u it is easy to check that $w = P_+(e^{-iF/2})$ with $F = \partial_x^{-1}u$.
Now let u^1 and u^2 be two solutions emanating from u_0 belonging to $N_{1,\lambda}$ such that their associated gauge functions belong to $X^{1/2,0}_\lambda$. According to (22), the gauge functions belong in fact to $M^0_{1,\lambda}$ and using the same dilation argument we used to prove the uniform boundness of the solution, we can show that for β large enough and $i = 1, 2$,

$$\|u^i_\beta\|_{N^0_{1,\lambda\beta}} + \|w^i_\beta\|_{M^0_{1,\lambda\beta}} \lesssim e^{2CK}\, \|u_{0,\beta}\|_{L^2_{\lambda\beta}} \lesssim e^{2CK}\beta^{-1/2}\varepsilon(K)^2$$

with $K = \|\widehat{\partial_x^{-1}u_{0,\beta}}\|_{L^1_\xi} + 1 = \|\widehat{\partial_x^{-1}u_0}\|_{L^1_\xi} + 1$. Therefore, for β large enough, (u^i_β, w^i_β) satisfies the smallness condition (102) with $\varepsilon = \varepsilon(K)$ and $u_{0\beta} \in B_{K,\lambda\beta}$. It then follows from (107) that $u^1_\beta \equiv u^2_\beta$ on $[0,1]$ and thus $u^1 \equiv u^2$ on $[0, 1/\beta^2]$. This proves the uniqueness result for initial data belonging to $B_{K,\lambda}$. Moreover, (111) clearly ensures that the flow-map is Lipschitz from $B_{K,\lambda} \cap H^s_\lambda$ into $C([0,1]; H^s_{0,\lambda})$.

7.2 The case of arbitrary large initial data

We use again the dilation invariance of (BO) to extend the result for arbitrary large data. Recall that that if $u(t,x)$ is a 2π-periodic solution of (BO) on $[0,T]$ with initial data u_0 then $u_\lambda(t,x) = \lambda^{-1}u(\lambda^{-2}t, \lambda^{-1}x)$ is a $(2\pi\lambda)$-periodic solution of (BO) on $[0,\lambda^2 T]$ emanating from $u_{0,\lambda} = \lambda^{-1}u_0(\lambda^{-1}x)$. Recall also that the associated gauge functions satisfy $w_\lambda(t,x) = \lambda^{-1}w(\lambda^{-2}t, \lambda^{-1}x)$. Let $u_0 \in H^s_0(\mathbb{T})$ with $0 \le s \le 1/2$. Note that

$$\|\widehat{\partial_x^{-1}u_0}\|_{L^1_\xi} \lesssim \|u_0\|_{L^2_1} \quad .$$

We thus set $K = \|u_0\|_{L^2_1} + 1$ and take

$$\lambda = \max\Big(1, \varepsilon(K)^{-4}\|u_0\|^2_{L^2_1}\Big) \ge 1$$

so that

$$\|u_{0,\lambda}\|_{L^2_\lambda} \le \lambda^{-1/2}\|u_0\|_{L^2_1} \le \varepsilon(K)^2 \, .$$

Recalling that $\|\widehat{\partial_x^{-1}u_{0,\lambda}}\|_{L^1_\xi} = \|\widehat{\partial_x^{-1}u_0}\|_{L^1_\xi}$, it follows that $u_{0,\lambda}$ belongs to $B_{K,\lambda}$ and so we are reduced to the case of small initial data. Therefore, there exists a unique solution $u_\lambda \in C([0,1]; H^s_{0,\lambda}) \cap N_{1,\lambda}$ of (BO) with $w_\lambda \in M^s_{1,\lambda}$. This proves the existence and uniqueness of the solution u of (BO) in the class

$$u \in C([0,T]; H^s_0(\mathbb{T})) \cap N_{T,1}, \quad w \in M^s_{T,1}$$

emanating from u_0 where $T = T(\|u_0\|_{L^2})$ and $\alpha \mapsto T(\alpha)$ is a non increasing function on $\mathbb{R}^*_+$. The fact that the flow-map is Lipschitz on every bounded set of $H^s_0(\mathbb{T})$ follows as well since λ only depends on $\|u_0\|_{L^2}$.

Note that the change of unknown (15) preserves the continuity of the solution and the continuity of the flow-map in $H^s(\mathbb{T})$. Moreover, the Lipschitz property (on bounded sets) of the flow-map is also preserved on the hyperplans of $H^s(\mathbb{T})$ of functions with fixed mean value. Finally, the global well-posedness result follows directly by combining the conservation of the L^2-norm and the local well-posedness result.

8 Proof of Theorem 1.2

8.1 Analycity of the flow-map

Let us prove the analyticity of the solution-map $\Psi : u_0 \mapsto u$ from $H^s_0(\mathbb{T})$ to $C([0,1]; H^s(\mathbb{T}))$ at the origin. Note that the other points of $H^s_0(\mathbb{T})$ could be

handle in the same way. Also we restrict ourself to the case $0 \leq s \leq 1/2$ but the case $s \geq 1/2$ can be treated in a similar way (in fact easier) by using the results of [18].
The analycity of the flow-map will be a direct consequence of the three following ingredients :
• The Lipschitz property of Ψ proven in Section 6.
• The fact that it appears only polynomial or analytic functions of u in the equations we deal with.
• We have an absolute convergence, in the norms we are interested in, of the serie obtained by replacing the analytic functions of u by their associated entire series.

So, let $\varphi \in H^s_0(\mathbb{T})$ with $\|\varphi\|_{H^s_1} = 1$ and let $\varepsilon > 0$ be a small real number to be fixed later. Taking $u_0 = \varepsilon\varphi$ we know from (107), (108) and (111) that, for ε small enough, there exists $c_1 > 0$ such that the corresponding solution u and its gauge transform w verify

$$\|u\|_{N_{1,1}} + \|u\|_{L^\infty_1 H^s_1} + \|w\|_{M^s_{1,1}} \leq c_1\,\varepsilon \quad , \tag{114}$$

Now let $C > 0$ be a universal constant we take very large (We can take for example $C > 0$ to be the exponential of the sum of all the implicit constants interfering in our estimates in Sections 4-5). According to (64) and (65), we get

$$\|P_3 u - \varepsilon V(t) P_3 \varphi\|_{N_{1,1}} \leq C(c_1\,\varepsilon)^2 \quad .$$

On the other hand, since $\partial_x^{-1}\varphi$ belongs to H^{s+1} which is an algebra, it holds in H^{s+1}_1

$$W(0) = P_+(e^{-i\varepsilon\partial_x^{-1}\varphi/2}) = 1 - \frac{i}{2}\varepsilon P_+(\partial_x^{-1}\varphi) + \sum_{k\geq 2}(\frac{-i\varepsilon}{2})^k \frac{1}{k!} P_+\Big((\partial_x^{-1}\varphi)^k\Big) \quad .$$

and thus

$$w(0) = -\frac{i}{2}\varepsilon P_+(\varphi) + \Lambda_\varepsilon \text{ with } \|\Lambda_\varepsilon\|_{H^s_1} \leq 4\varepsilon \quad .$$

Consequently,

$$V(t)w(0) = -\frac{i}{2}\varepsilon V(t) P_+\varphi + V(t)\Lambda_\varepsilon \text{ with } \|V(t)\Lambda_\varepsilon\|_{M^s_{1,1}} \leq C(4\varepsilon)^2 \quad .$$

Now according to (30), (33), (50) and (57), we infer that $\|w - V(t)w(0)\|_{M^s_{1,1}} \leq C(c_1\varepsilon)^2$ and thus

$$\|w + \frac{i}{2}\varepsilon V(t) P_+\varphi\|_{M^s_{1,1}} \leq 2C(c_1\varepsilon)^2 \quad . \tag{115}$$

It then follows from (21)-(69), (71), (77) and (94) that

$$P_{>3}(u) = 2iP_{>3}w + \tilde{\Lambda}_\varepsilon = \varepsilon V(t)P_{>3}(\varphi) + \tilde{\tilde{\Lambda}}_\varepsilon$$

for some function $\tilde{\tilde{\Lambda}}_\varepsilon$ satisfying $\|\tilde{\tilde{\Lambda}}_\varepsilon\|_{N_{1,1}} + \|\tilde{\tilde{\Lambda}}_\varepsilon\|_{L^\infty_{\bar{1}}H^s} \leq 3C(c_1\varepsilon)^2$.
We thus finally get,

$$\|u-\varepsilon V(t)\varphi\|_{N_{1,1}} + \|u-\varepsilon V(t)\varphi\|_{L^\infty_{\bar{1}}H^s_{\bar{1}}} + \|w+\frac{i}{2}\varepsilon V(t)P_{>3}\varphi\|_{M^s_{\bar{1},1}} \leq 6C(c_1\varepsilon)^2 \ . \tag{116}$$

In the same way, according to (20), expanding $e^{-iF/2}$ and $e^{iF/2}$ as in Section 4, with (114)-(116) in hand, we get

$$\begin{aligned} w &= -\frac{i}{2}\varepsilon V(t)P_+(\varphi) - \varepsilon^2\Big[\frac{1}{4}V(t)P_+(\varphi\partial_x^{-1}\varphi) + 2i\int_0^t V(t-t')\partial_x P_+\Big(W_1\partial_x P_-(\overline{w_1})\Big) \\ &\quad +\Lambda_\varepsilon \ . \end{aligned}$$

where

$$u_1 = V(t)\varphi,\ W_1 = -\frac{i}{2}V(t)P_+(\partial_x^{-1}\varphi),\ w_1 = \partial_x W_1$$

and $\|\Lambda_\varepsilon\|_{M^s_{\bar{1}}} \lesssim 6C^2\,(c_1\varepsilon)^3$ and so on ...
Iterating this process we obtain that there exists $\varepsilon_0 > 0$ such that the following asymptotic expansion of u in term of φ holds absolutely in $C([0,1];H^s(\mathbb{T}))$ for $0 < \varepsilon \leq \varepsilon_0$,

$$u = \sum_{k\geq 1}\varepsilon^k A_k(\varphi) \quad . \tag{117}$$

Here, $A_1(\varphi) = t \mapsto V(t)\varphi$ and more generaly A_k is a continuous k-linear operator from $H^s_0(\mathbb{T})$ to $C([0,1];H^s_0(\mathbb{T}))$. Therefore u is real-analytic and in particular C^∞ at the origin of $H^s_0(\mathbb{T})$. Moreover, since

$$u(t,\cdot) = \varepsilon U(t)\varphi + \frac{1}{2}\int_0^t V(t-t')\partial_x u^2(t')\,dt' \ ,$$

by identification we infer that

$$A_k(\varphi) = t \mapsto \frac{1}{2}\sum_{\substack{k_1,k_2\geq 1\\ k_1+k_2=k}}\int_0^t V(t-t')\partial_x\Big(A_{k_1}(\varphi)A_{k_2}(\varphi)\Big)(t')\,dt' \tag{118}$$

8.2 Non smoothness of the flow-map in $H^s(\mathbb{T})$, $s<0$.

Let us start by computing $A_k(t,\lambda\cos(Nx))$ for k=1,2,3. Of course,

$$A_1(t,cos(Nx))=\cos(Nx-N^2t) \quad .$$

Since $\partial_x\Big(A_1(t,cos(Nx))\Big)^2=-N\sin(2Nx-2N^2t)$ we infer that

$$\begin{aligned}
A_2(t,\cos(Nx)) &= \frac{1}{2}\int_0^t V(t-t')\partial_x\Big(A_1(t,cos(Nx))\Big)^2(t')\,dt\\
&= -\frac{N}{2}\int_0^t \sin\Big(2Nx-2N^2t'-4N^2(t-t')\Big)\,dt'\\
&= \frac{1}{4N}\Big[\cos(2Nx-2N^2t)-\cos(2Nx-4N^2t)\Big]
\end{aligned}$$

In the same way,

$$\begin{aligned}
\partial_x\Big(A_1(1,\cos(Nx))A_2(t,\cos(Nx))\Big) &= -\frac{1}{8}\Big[\sin(Nx-N^2t)-\sin(Nx-3N^2t)\Big]\\
&\quad -\frac{3}{8}\Big[\sin(3Nx-3N^2t)-\sin(3Nx-5N^2t)\Big]
\end{aligned}$$

and thus

$$\begin{aligned}
A_3(t,\cos(Nx)) &= \int_0^t V(t-t')\partial_x\Big(A_1(t,cos(Nx))A_2(t,\cos(Nx))\Big)(t')\,dt\\
&= -\frac{1}{8}\int_0^t\Big[\sin(Nx-N^2t)-\sin(Nx-3N^2t'-N^2(t-t'))\Big]\,dt'\\
&\quad -\frac{3}{8}\int_0^t\Big[\sin(3Nx-3N^2t'-9N^2(t-t'))-\sin(Nx-5N^2t'-N^2(t-t'))\Big]\,dt'\\
&= -\frac{t}{8}\sin(Nx-N^2t)\\
&\quad +\frac{1}{16N^2}\Big[\cos(Nx-3N^2t)-\cos(Nx-N^2t)\Big]\\
&\quad +\frac{1}{16N^2}\Big[\cos(3Nx-3N^2t)-\cos(3Nx-9N^2t)\Big]\\
&\quad -\frac{3}{32N^2}\Big[\cos(3Nx-5N^2t)-\cos(3Nx-9N^2t)\Big]
\end{aligned}$$

Therefore, setting $\Psi_N=N^{-s}\cos(Nx)$ it follows that

$$\|A_3(t,\Psi_N)\|_{H^s}\gtrsim t\,N^{-2s}\|\Psi_N\|_{H^s}^3$$

and from standard considerations (cf. [5]) the flow-map cannot be of class C^3 at the origin from $H^s_0(\mathbb{T})$ into $H^s_0(T)$ as soon as $s<0$. Moreover, by a direct induction argument it is not too hard to check that for any $k\geq 4$,

$$\|A_k(t,\cos(Nx))\|_{H^s}\leq \tilde{C}_k\, N^s \quad .$$

Therefore, for any fixed integer $K\geq 4$,

$$\Big\|\sum_{k=4}^{K+2} A_k(t,\varepsilon\cos(Nx))\Big\|_{H^s}\leq C_K\varepsilon^4 N^s \quad .$$

Now, taking as initial data $\varphi_N=\varepsilon_N\cos(Nx)$ with $0<\varepsilon_N\leq\varepsilon_0/2$, we know from (117) that the associated solution u_N can be written in $L^2(\mathbb{T})$ as

$$u_N(t,\cdot)=\sum_{k\geq 1}\varepsilon_N^k A_k(t,\cos(Nx)) \quad .$$

For N large enough and $s\leq 0$, we thus deduce from the computation of $A_2(t,\cos(Nx))$ and $A_3(t,\cos(Nx))$ above that

$$\begin{aligned}
\|u_N(t,\cdot)-V(t)\varphi_N\|_{H^s} &\gtrsim t\varepsilon_N^3\|\sin(Nx-N^2t)\|_{H^s}-2\varepsilon_N^2N^{s-1}-C_K\varepsilon_N^4N^s\\
&\quad -\tilde{C}\sum_{k=K+3}^{\infty}(\frac{\varepsilon_N}{\varepsilon_0})^k\|A_k(t,\varepsilon_0\cos(Nx))\|_{L^2}\\
&\gtrsim t\varepsilon_N^3\|\sin(Nx-N^2t)\|_{H^s}-2\varepsilon_N^2N^{s-1}-C_K\varepsilon_N^4N^s-C\varepsilon_N^{K+3}\\
&\gtrsim \varepsilon_N^3N^s\Big(t-\frac{2}{N\varepsilon_N}-C_K\varepsilon_N-C\varepsilon_N^KN^{-s}\Big) \quad .
\end{aligned}$$

For any $0<\alpha<1$ and $s<0$ fixed, we take $K>0$ such that

$$\frac{|s|}{K}<1 \quad \text{and} \quad \frac{4}{K}<\alpha \quad .$$

Setting

$$\varepsilon_N=\min\Big(\frac{\varepsilon_0}{2},\frac{t}{4C_K},(\frac{t\,N^s}{4C})^{\frac{1}{K}}\Big)$$

we infer that for N large enough,

$$\begin{aligned}
\|u_N(t,\cdot)-V(t)\varphi_N\|_{H^s} &\gtrsim t\varepsilon_N^3N^s\\
&\gtrsim t\varepsilon_N^{2-\alpha}N^{-\alpha s}\|\varphi_N\|_{H^s}^{1+\alpha}\\
&\gtrsim tN^{-\frac{\alpha s}{2}}\|\varphi_N\|_{H^s}^{1+\alpha} \quad .
\end{aligned}$$

It follows that the flow-map (if it coincides with the standard flow-map on $H^\infty_0(\mathbb{T})$) cannot be of class $C^{1+\alpha}$ at the origin from $H^s_0(\mathbb{T})$ into $H^s_0(\mathbb{T})$.

9 Appendix

9.1 Proof of Lemma 4.1

We separate the low and the high modes of h. To treat the high modes part, we observe that by Leibniz rule for fractional derivatives (cf. [15]) and Sobolev inequality,

$$\begin{aligned}\Big\|J_x^\alpha\Big(Q_1(h)\,g\Big)\Big\|_{L^q_\lambda} &\lesssim \Big\|J_x^\alpha Q_1(h)\Big\|_{L^{4/\alpha}_\lambda}\|g\|_{L^{\frac{4q}{4-\alpha q}}_\lambda} + \|h\|_{L^\infty_\lambda}\|J_x^\alpha g\|_{L^q_\lambda}\\ &\lesssim \Big\|J_x^{\alpha+1/2-\alpha/4}Q_1(h)\Big\|_{L^2_\lambda}\|J_x^{\alpha/4}g\|_{L^q_\lambda} + \|h\|_{L^\infty_\lambda}\|J_x^\alpha g\|_{L^q_\lambda}\\ &\lesssim (\|\partial_x h\|_{L^2_\lambda} + \|h\|_{L^\infty_\lambda})\|J_x^\alpha g\|_{L^q_\lambda}\ .\end{aligned}$$

On the other hand, one can easily check that

$$\Big\|J_x^1\Big(P_1(h)\,g\Big)\Big\|_{L^q_\lambda} \lesssim (\|h\|_{L^\infty_\lambda}+\|\partial_x h\|_{L^2_\lambda})\|J_x^1 g\|_{L^q_\lambda} \quad \text{and} \quad \|P_1(h)g\|_{L^q_\lambda} \lesssim \|h\|_{L^\infty_\lambda}\|g\|_{L^q_\lambda}\ .$$

Interpolating between this two estimates we obtain the desired estimate on the low modes part.

9.2 Proof of Lemma 5.1

Clearly the low modes part of zv can be estimated directly by an Holder inequality. Now, using the nonhomogeneous Littlewood-Paley decomposition, we get for $q \geq 8$,

$$\begin{aligned}\Delta_q(zv) &= \sum_{|i|\leq 2}\Delta_q\Big(\Delta_{q-i}(v)\sum_{j=0}^{q-i-2}\Delta_j(z)\Big)\\ &+\sum_{|i|\leq 2}\Delta_q\Big(\Delta_{q-i}(z)\sum_{j=0}^{q-i-2}\Delta_j(v)\Big) + \Delta_q\Big(\sum_{i\geq q-2}\sum_{|j|\leq 1}\Delta_{i-j}(v)\Delta_i(z)\Big)\quad .\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{q\geq 8}\|\Delta_q(zv)\|^2_{L^4_{t,\lambda}} &\lesssim \|z\|^2_{L^\infty_{t,\lambda}}\sum_{q\geq 4}\|\Delta_q(v)\|^2_{L^4_{t,\lambda}}\\ &+\|v\|_{L^4_{t,\lambda}}\Big(\sum_{q\geq 4}\|\Delta_q(z)\|^2_{L^\infty_{t,\lambda}} + \sum_{q\geq 4}\sum_{k\geq q-2}\|\Delta_k(z)\|^2_{L^\infty_{t,\lambda}}\Big)\ . \quad (119)\end{aligned}$$

The desired result follows since for $k \geq 2$,

$$\|\Delta_k(z)\|_{L^\infty_{t,\lambda}} \lesssim 2^{-k/4}\|z_x\|_{L^\infty_t L^2_\lambda}\ .$$

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References

[1] L. Abdelouhab, J. Bona, M. Felland, and J.C. Saut, Nonlocal models for nonlinear, dispersive waves, *Phys. D* **40** *(1989), 360–392.*

[2] M.J. Ablowitz and A.S. Fokas, The inverse scattering transform for the Benjamin- Ono equation, a pivot for multidimensional problems, *Stud. Appl. Math.* **68** *(1983), 1–10.*

[3] T.B. Benjamin, Internal waves of permanent form in fluids of great depth, *J. Fluid Mech.* **29** *(1967), 559–592.*

[4] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and application to nonlinear evolution equations I. The Schrödinger equation, *GAFA, 3 (1993), 157-178.*

[5] J. Bourgain, *Periodic Korteveg de Vries equation with measures as initial data* , Sel. Math. New. Ser. 3 (1993), pp. 115–159.

[6] N. Burq and F. Planchon, On well-posedness for the Benjamin-Ono equation, preprint 2005.

[7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Sharp global well-posedness results for periodic and non-periodic KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$, *J. Amer. Math. Soc.* **16** *(2003), 705-749.*

[8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Multilinear estimates for periodic KdV equations, and applications, *J. Funct. Analysis* **211** *(2004), 173-218.*

[9] R. Coifman and M. Wickerhauser, The scattering transform for the Benjamin-Ono equation, *Inverse Probl.* **6** *(1990), 825-860.*

[10] J. Ginibre, Le problème de Cauchy pour des EDP semi-linéaires périodiques en variables d'espace (d'après Bourgain), *in Séminaire Bourbaki 796, Astérique 237, 1995, 163–187.*

[11] J. Ginibre, Y. Tsutsumi and G. Velo, On the Cauchy problem for the Zakharov system, *J. Funct. Analysis* **133** *(1995), 50–68.*

[12] A.D. Ionescu and C.E. Kenig, Global well-posedness of the Benjamin-Ono equation in low-regularity spaces. *J. Amer. Math. Soc.* **20** *(2007), 753-798*

[13] J.R. Iorio, On the Cauchy problem for the Benjamin-Ono equation, *Comm. Partial Differential Equations* **11** *(10) (1986), 1031-1081.*

[14] C.E. Kenig and K. Koenig, On the local well-posedness of the Benjamin-Ono and modified Benjamin-Ono equations, *Math. Res. Lett.* **10** *(2003), 879-895.*

[15] C.E. Kenig, G. Ponce and , L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via contraction principle. Comm. and Pure Pure and Appl. Math. **46** (1993), 527-620.

[16] H. Koch and N. Tzvetkov, On the local well-posedness of the Benjamin-Ono equation in $H^s(\mathbb{R})$, *IMRN* **26** *(2003), 1449-1464.*

[17] H. Koch and N. Tzvetkov, Nonlinear wave interactions for the Benjamin-Ono equation, *IMRN* **30** *(2005), 1833-1847.*

[18] L. Molinet, Global well-posedness in the energy space for the Benjamin-Ono equation on the circle, *Math. Ann.* **337** *(2007), 353–383.*

[19] L. Molinet and F. Ribaud, Well-posedness in H^1 for the (generalized) Benjamin-Ono equation on the circle, *preprint.*

[20] L. Molinet, J.C. Saut and N. Tzvetkov, Ill-posedness issues for the Benjamin-Ono and related equations, *SIAM J. Math. Anal.* **33** *(4) (2001), 982-988.*

[21] G. Ponce, On the global well-posedness of the Benjamin-Ono equation, *Differential Integral Equations* **4** *(3) (1991), 527-542.*

[22] J.-C. Saut, Sur quelques généralisation de l'équation de Korteweg-de Vries, *J. Math. Pures Appl.* **58** *(1979), 21-61.*

[23] T. Tao, Global well-posedness of the Benjamin-Ono equation in $H^1(\mathbb{R})$, *J. Hyperbolic Differ. Equ.* **1** *(2004), 27-49.*