

Global well-posedness in L^2 for the periodic Benjamin-Ono equation

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Abstract. We prove that the Benjamin-Ono equation is globally well-posed in $H^s(\mathbb{T})$ for $s \geq 0$. Moreover we show that the associated flow-map is Lipschitz on every bounded set of $\dot{H}^s(\mathbb{T})$, $s \geq 0$, and even real-analytic in this space for small times. This result is sharp in the sense that the flow-map (if it can be defined and coincides with the standard flow-map on $H^\infty(\mathbb{T})$) cannot be of class $C^{1+\alpha}$, $\alpha > 0$, from $\dot{H}^s(\mathbb{T})$ into $\dot{H}^s(\mathbb{T})$ as soon as $s < 0$.

1 Introduction, main results and notations

1.1 Introduction

In this paper we continue our study (see [18]) of the Cauchy problem for the Benjamin-Ono equation on the circle

$$(BO) \quad \begin{cases} \partial_t u + \mathcal{H} \partial_x^2 u - u \partial_x u = 0, & (t, x) \in \mathbb{R} \times \mathbb{T}, \\ u(0, x) = u_0(x), \end{cases}$$

where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, u is real-valued and \mathcal{H} is the Hilbert transform defined for 2π -periodic functions with mean value zero by

$$\widehat{\mathcal{H}(f)}(0) = 0 \quad \text{and} \quad \widehat{\mathcal{H}(f)}(\xi) = -i \operatorname{sgn}(\xi) \hat{f}(\xi), \quad \xi \in \mathbb{Z}^* .$$

The Benjamin-Ono equation arises as a model for long internal gravity waves in deep stratified fluids, see [3]. This equation possesses a Lax pair structure (cf. [2], [9]) and thus has got an infinite number of conservation laws. These conservation laws permit to control the $H^{n/2}$ -norms, $n \in \mathbb{N}$, and thus to derive global well-posedness results in Sobolev spaces. The Cauchy problem on the real line has been extensively studied these last years (cf. [23], [1], [13], [22], [21], [16], [14]). Recently, T. Tao [24] has pushed the well-posedness

theory to $H^1(\mathbb{R})$ by using an appropriate gauge transform. This approach has been improved very recently in [6] and [12] where respectively $H^s(\mathbb{R})$, $s > 0$, and $L^2(\mathbb{R})$ are reached.

In the periodic setting, the local well-posedness of (BO) is known in $H^s(\mathbb{T})$ for $s > 3/2$ (cf. [1], [13]), by standard compactness methods which do not take advantage of the dispersive effects of the equation. Thanks to the conservation laws mentioned above and an interpolation argument, this leads to global well-posedness in $H^s(\mathbb{T})$ for $s > 3/2$ (cf. [1]). Very recently, F. Ribaud and the author [20] have improved the global well-posedness result to $H^1(\mathbb{T})$ by using the gauge transform introduced by T. Tao [24] combining with Strichartz estimates derived in [3] for the Schrödinger group on the one-dimensional torus. In [18] this approach combined with estimates in Bourgain type spaces leads to a global well-posedness result in the energy space $H^{1/2}(\mathbb{T})$. Recall that the Momentum and the Energy of the Benjamin-Ono equation are respectively given by

$$M(u) = \int_{\mathbb{T}} u^2 \quad \text{and} \quad E(u) = \frac{1}{2} \int_{\mathbb{T}} |D_x^{1/2} u|^2 + \frac{1}{6} \int_{\mathbb{T}} u^3 \quad . \quad (1)$$

The aim of this paper is to improve the local and global well-posedness to $L^2(\mathbb{T})$.

1.2 Notations

For $x, y \in \mathbb{R}$, $x \sim y$ means that there exists $C_1, C_2 > 0$ such that $C_1|x| \leq |y| \leq C_2|x|$ and $x \lesssim y$ means that there exists $C_2 > 0$ such that $|x| \leq C_2|y|$. For a Banach space X , we denote by $\|\cdot\|_X$ the norm in X .

We will use the same notations as in [7] and [8] to deal with Fourier transform of space periodic functions with a large period λ . $(d\xi)_\lambda$ will be the renormalized counting measure on $\lambda^{-1}\mathbb{Z}$:

$$\int a(\xi) (d\xi)_\lambda = \frac{1}{\lambda} \sum_{\xi \in \lambda^{-1}\mathbb{Z}} a(\xi) \quad .$$

As written in [8], $(d\xi)_\lambda$ is the counting measure on the integers when $\lambda = 1$ and converges weakly to the Lebesgue measure when $\lambda \rightarrow \infty$. In all the text, all the Lebesgue norms in ξ will be with respect to the measure $(d\xi)_\lambda$. For a $(2\pi\lambda)$ -periodic function φ , we define its space Fourier transform on $\lambda^{-1}\mathbb{Z}$ by

$$\hat{\varphi}(\xi) = \int_{\mathbb{R}/(2\pi\lambda)\mathbb{Z}} e^{-i\xi x} f(x) dx, \quad \forall \xi \in \lambda^{-1}\mathbb{Z} \quad .$$

We denote by $V(\cdot)$ the free group associated with the linearized Benjamin-Ono equation,

$$\widehat{V(t)\varphi}(\xi) = e^{-i\xi|\xi|t} \widehat{\varphi}(\xi), \quad \xi \in \lambda^{-1}\mathbb{Z} \quad .$$

We define the Sobolev spaces H_λ^s for $(2\pi\lambda)$ -periodic functions by

$$\|\varphi\|_{H_\lambda^s} = \|\langle \xi \rangle^s \varphi(\xi)\|_{L_\xi^2} = \|J_x^s \varphi\|_{L_\lambda^2} \quad ,$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$ and $\widehat{J_x^s \varphi}(\xi) = \langle \xi \rangle^s \widehat{\varphi}(\xi)$.

For $s \geq 0$, the closed subspace of zero mean value functions of H_λ^s will be denoted by \dot{H}_λ^s .

The Lebesgue spaces L_λ^q , $1 \leq q \leq \infty$, will be defined as usually by

$$\|\varphi\|_{L_\lambda^q} = \left(\int_{\mathbb{R}/(2\pi\lambda)\mathbb{Z}} |\varphi(x)|^q dx \right)^{1/q}$$

with the obvious modification for $q = \infty$.

In the same way, for a function $u(t, x)$ on $\mathbb{R} \times \mathbb{R}/(2\pi\lambda)\mathbb{Z}$, we define its space-time Fourier transform by

$$\widehat{u}(\tau, \xi) = \mathcal{F}_{t,x}(u)(\tau, \xi) = \int_{\mathbb{R}} \int_{\mathbb{R}/(2\pi\lambda)\mathbb{Z}} e^{-i(\tau t + \xi x)} u(t, x) dx dt, \quad \forall (\tau, \xi) \in \mathbb{R} \times \lambda^{-1}\mathbb{Z} \quad .$$

We define the Bourgain spaces $X_\lambda^{b,s}$, $Z_\lambda^{b,s}$, A_λ and Y_λ^s of $(2\pi\lambda)$ -periodic (in x) functions respectively endowed with the norm

$$\|u\|_{X_\lambda^{b,s}} = \|\langle \tau + \xi|\xi| \rangle^b \langle \xi \rangle^s \widehat{u}\|_{L_{\tau,\xi}^2} = \|\langle \tau \rangle^b \langle \xi \rangle^s \mathcal{F}_{t,x}(V(-t)u)\|_{L_{\tau,\xi}^2} \quad , \quad (2)$$

$$\|u\|_{Z_\lambda^{b,s}} = \|\langle \tau + \xi|\xi| \rangle^b \langle \xi \rangle^s \widehat{u}\|_{L_\xi^2 L_\tau^1} = \|\langle \tau \rangle^b \langle \xi \rangle^s \mathcal{F}_{t,x}(V(-t)u)\|_{L_\xi^2 L_\tau^1} \quad , \quad (3)$$

$$\|u\|_{A_\lambda^b} = \|\langle \tau + \xi|\xi| \rangle^b \widehat{u}\|_{L_{\tau,\xi}^1} = \|\langle \tau \rangle^b \mathcal{F}_{t,x}(V(-t)u)\|_{L_{\tau,\xi}^1} \quad (4)$$

and

$$\|u\|_{Y_\lambda^s} = \|u\|_{X_\lambda^{1/2,s}} + \|u\|_{Z_\lambda^{0,s}} \quad , \quad (5)$$

where we will denote A_λ^0 simply by A_λ . Recall that $Y_\lambda^s \hookrightarrow Z_\lambda^{0,s} \hookrightarrow C(\mathbb{R}; H_\lambda^s)$.

We will also need the homogeneous semi-norm of $X_\lambda^{b,s}$ defined by

$$\|u\|_{\dot{X}_\lambda^{b,s}} = \|\langle \tau + \xi|\xi| \rangle^b |\xi|^s \widehat{u}\|_{L_{\tau,\xi}^2} \quad .$$

$L_t^p L_\lambda^q$ will denote the Lebesgue spaces

$$\|u\|_{L_t^p L_\lambda^q} = \left(\int_{\mathbb{R}} \|u(t, \cdot)\|_{L_\lambda^q}^p dt \right)^{1/p}$$

with the obvious modification for $p = \infty$.

Let $u = \sum_{j \geq 0} \Delta_j u$ be a classical smooth non homogeneous Littlewood-Paley decomposition in space of u , $\text{Supp } \mathcal{F}_x(\Delta_0 u) \subset \mathbb{R} \times [-2, 2]$ and

$$\text{Supp } \mathcal{F}_x(\Delta_j u) \subset \mathbb{R} \times [-2^{j+1}, -2^{j-1}] \cup \mathbb{R} \times [2^{j-1}, 2^{j+1}], \quad j \geq 1 \quad .$$

We defined the Besov type space $\tilde{L}_{t,\lambda}^4$ by

$$\|u\|_{\tilde{L}_{t,\lambda}^4} = \left(\sum_{k \geq 0} \|\Delta_k u\|_{L_{t,\lambda}^4}^2 \right)^{1/2} \quad (6)$$

We will work in the function spaces N_λ and M_λ^s respectively defined by

$$\|u\|_{N_\lambda} = \|u\|_{Z_\lambda^{0,0}} + \|Q_3 u\|_{X_\lambda^{7/8,-1}} + \|\chi_{[-4,4]}(t) u\|_{L_{t,\lambda}^4} + \|\chi_{[-4,4]}(t) u\|_{\tilde{L}_{t,\lambda}^4}$$

and

$$\|w\|_{M_\lambda^s} = \|w\|_{Y_\lambda^s} + \|Q_1 w\|_{X_\lambda^{1,-1}} \quad .$$

Finally, for any function space B_λ and any $T > 0$, we denote by $B_{T,\lambda}$ the corresponding restriction in time space endowed with the norm

$$\|u\|_{B_{T,\lambda}} = \inf_{v \in B_\lambda} \{ \|v\|_{B_\lambda}, v(\cdot) \equiv u(\cdot) \text{ on }]0, T[\} .$$

It is worth noticing that the map $u \mapsto \bar{u}$ is an isometry in all our function spaces. For all function spaces of $(2\pi\lambda)$ -periodic functions, we will drop the index λ when $\lambda = 1$.

We will denote by P_+ and P_- the projection on respectively the positive and the negative spatial Fourier modes. Moreover, for $a \geq 0$, we will denote by P_a , Q_a , $P_{>a}$ and $P_{<a}$ the projection on respectively the spatial Fourier modes of absolute value equal or less than a , the spatial Fourier modes of absolute value greater than a , the spatial Fourier modes larger than a and the spatial Fourier modes smaller than a .

1.3 Main result

Our well-posedness theorem reads :

Theorem 1.1 For all $u_0 \in H^s(\mathbb{T})$ with $0 \leq s \leq 1/2$ and all $T > 0$, there exists a solution u of the Benjamin-Ono equation (BO) satisfying

$$u \in C([0, T]; H^s(\mathbb{T})) \cap N_{T, \lambda} \quad \text{and} \quad P_+(e^{-i\partial_x^{-1}\tilde{u}/2}\tilde{u}) \in X_{T, \lambda}^{1/2, s} \quad (7)$$

where

$$\tilde{u} = u(t, x - t \int u_0) - \int u_0 \quad \text{and} \quad \widehat{\partial_x^{-1}} = \frac{1}{i\xi}, \quad \xi \in \mathbb{Z}^* \quad .$$

This solution is unique in the class (7).

Moreover $u \in C_b(\mathbb{R}, L^2(\mathbb{T}))$ and the map $u_0 \mapsto u$ is continuous from $H^s(\mathbb{T})$ into $C([0, T], H^s(\mathbb{T}))$ and Lipschitz on every bounded set from $\dot{H}^s(\mathbb{T})$ into $C([0, T], \dot{H}^s(\mathbb{T}))$.

Note that the result for $s \geq 1/2$ is established in [18]. Before stating our ill-posedness result let us make some comments on Theorem 1.1.

Remark 1.1 We are not able to prove that for any solution u of (BO) belonging to $C([0, T]; H^s(\mathbb{T})) \cap N_{T, \lambda}$, the function $P_+(e^{-i\partial_x^{-1}\tilde{u}/2}\tilde{u})$ belongs to $X_{T, \lambda}^{1/2, s}$. This is why we have to add this condition in our uniqueness class. Note however that any solution that are limit in $C([0, T]; H^s(\mathbb{T}))$ of smooth solutions belongs to this class. Therefore, our solution satisfies also the following (weaker) uniqueness notion used in [12] : it is the unique solution that is a limit in $C([0, T]; H^s)$ of smooth solutions to (BO).

Remark 1.2 Actually, we prove that the flow-map is Lipschitz on every bounded subset of any hyperplan of $H^s(\mathbb{T})$ of functions with a fixed mean value.

Remark 1.3 The fact that u is real-valued is crucial to derive the equation (20) on w . So, it does not seem that our approach can be adapted to prove the local existence of complex-valued solutions. On the other hand, it seems that a slight modification of the proof in [18] can lead to the local-wellposedness in $H^{1/2}(\mathbb{T})$ for the complex-valued version of (BO).

Let us now state our ill-posedness issue.

Theorem 1.2 For $s \geq 0$ and $t \in [0, 1]$ the flow-map constructed by Theorem 1.1 is real-analytic from $\dot{H}^s(\mathbb{T})$ into $\dot{H}^s(\mathbb{T})$. On the other hand, for any $t \in]0, 1[$ and any $\alpha > 0$, the flow-map (if it can be defined and coincides with the standard flow-map on $H^\infty(\mathbb{T})$) cannot be of class $C^{1+\alpha}$ from $\dot{H}^s(\mathbb{T})$ into $\dot{H}^s(\mathbb{T})$ as soon as $s < 0$.

The main tools to prove Theorem 1.1 are the gauge transformation of T. Tao and the Fourier restriction spaces introduced by Bourgain. Recall that in order to solve (BO), T. Tao [24] performed a kind of complex Cole-Hopf transformation¹ $W = P_+(e^{-iF/2})$, where F is a primitive of u . In the periodic setting, requiring that u has mean value zero, we can take $F = \partial_x^{-1}u$ the unique zero mean value primitive of u . By the mean value theorem, it is then easy to check that the above gauge transformation is Lipschitz from L_λ^2 to L_λ^∞ . This property, which is not true on the real line, is crucial to derive the smoothness of the flow-map. The equation satisfied by $w = \partial_x W$ takes the form

$$w_t - iw_{xx} = \partial_x P_+(WP_-u_x) + \dots$$

which looks quite good since such nonlinear term enjoys a strong smoothing effect on u in Bourgain spaces. On the other hand, when one wants to inverse the gauge transformation, one gets something like

$$u = e^{iF}w + \dots$$

which is not so good since multiplication by gauge function as e^{iF} behaves not so well in Bourgain spaces². Actually, the “bad” regularity of u in the scale of Bourgain spaces is the main obstruction in going below $H^{1/2}(\mathbb{T})$ in [18]. In this work we substitute the above expression of u in the equation satisfied by w . u still appears but only under the form $e^{\mp iF/2}$ which possesses more regularities. On the other hand we have now to treat the multiplication by such functions in Bourgain spaces when estimating w . Note that in the case $s = 0$ there is an additional difficulty mainly since we would like to control $\mathcal{F}_{t,x}^{-1}(|\hat{u}|)$ in $L_{t,x}^4$ whereas we only have a control on u in this space. This difficulty is overcome by noticing that actually u belongs to a smaller space than $L_{t,x}^4$ which is $L_{t,x}^4 \cap \tilde{L}_{t,x}^4$ (see (6)).

Concerning Theorem 1.2, the fact that the flow-map (if it can be defined) cannot be of class C^3 in $\dot{H}^s(\mathbb{T})$, $s < 0$, can be obtained in the classical way for periodic equations (cf. [5]). To prove that it cannot be of class $C^{1+\alpha}$ we somehow combine the bad behavior of the third iterate with the real-analyticity result in $L^2(\mathbb{T})$.

This paper is organized as follows: In the next section we recall some linear estimates in Bourgain type spaces. In Section 3 we introduce the gauge transform and state the key nonlinear estimates. In Section 4, we prove the

¹Note that projecting (BO) on the non negative frequencies, one gets the following equation : $\partial_t(P_+u) - i\partial_x^2 P_+u = -P_+(uu_x)$

²Let us note that Bourgain spaces do not enjoy an algebra property

estimates on the gauge function w whereas the estimates on u are proven in Section 5. In Section 4 we derive uniform bounds for small initial data solutions and show a Lipschitz bound on the solution-map $u_0 \mapsto u$. The proof of Theorem 1.1 and Theorem 1.2 are completed respectively in Section 6 and Section 7. Note that the proof of some technical lemmas needed in Sections 4-5 can be found in the appendix.

2 Linear Estimates

One of the main ingredient is the following linear estimate due to Bourgain [4].

$$\|v\|_{L^4(\mathbb{R})_{[-\frac{\lambda^2}{2}, \frac{\lambda^2}{2}]} L_\lambda^4} \lesssim \|v\|_{X_\lambda^{3/8,0}(\mathbb{R})_{[-\frac{\lambda^2}{2}, \frac{\lambda^2}{2}]} L_\lambda} . \quad (8)$$

This estimate is proved in [4] (see also [18] for a shorter proof) for Bourgain spaces associated with the Schrödinger group and for a period equal to 1. The corresponding estimate for the Benjamin-Ono group follows by writing v as the sum of its positive and negative spatial modes parts. The estimate for any period $\lambda \geq 1$ follows directly from dilation arguments. From (8) we infer that for any function belonging to $X_\lambda^{3/8,0}$ it holds

$$\|v\|_{L_{t,\lambda}^4} \lesssim \|v\|_{X_\lambda^{3/8,0}} . \quad (9)$$

Indeed, for any fixed $v \in L_{t,\lambda}^4 \cap X_\lambda^{3/8,0}$ there exists $k \in \mathbb{N}^*$ such that

$$\|v\|_{L^4(\mathbb{R})_{[-\frac{k^2\lambda^2}{2}, \frac{k^2\lambda^2}{2}]} L_\lambda^4} \lesssim \|v\|_{L^4(\mathbb{R})_{[-\frac{k^2\lambda^2}{2}, \frac{k^2\lambda^2}{2}]} L_{k\lambda}^4}$$

and thus according to (8) the following chain of inequalities holds

$$\begin{aligned} \|v\|_{L_{t,\lambda}^4} &\lesssim \|v\|_{L^4(\mathbb{R})_{[-\frac{k^2\lambda^2}{2}, \frac{k^2\lambda^2}{2}]} L_\lambda^4} = \frac{1}{k} \|v\|_{L^4(\mathbb{R})_{[-\frac{k^2\lambda^2}{2}, \frac{k^2\lambda^2}{2}]} L_{k\lambda}^4} \\ &\lesssim \frac{1}{k} \|v\|_{X_\lambda^{3/8,0}(\mathbb{R})_{[-\frac{k^2\lambda^2}{2}, \frac{k^2\lambda^2}{2}]} L_{k\lambda}} = \|v\|_{X_\lambda^{3/8,0}(\mathbb{R})_{[-\frac{k^2\lambda^2}{2}, \frac{k^2\lambda^2}{2}]} L_\lambda} \lesssim \|v\|_{X_\lambda^{3/8,0}} . \end{aligned}$$

(9) follows then by density.

Let us now state some estimates for the free group and the Duhamel operator. Let $\psi \in C_0^\infty([-2, 2])$ be a time function such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ on $[-1, 1]$. The following linear estimates are well-known (cf. [4], [10]).

Lemma 2.1 *For all $\varphi \in H_\lambda^s$ and all $R > 0$, it holds :*

$$\|\psi(t)V(t)\varphi\|_{Y_\lambda^s} \lesssim \|\varphi\|_{H_\lambda^s} , \quad (10)$$

$$\|\psi(t/R)V(t)\varphi\|_{Z_\lambda^{0,s}} \lesssim \|\varphi\|_{H_\lambda^s} \quad , \quad (11)$$

$$\|\psi(t/R)V(t)\varphi\|_{A_\lambda} \lesssim \|\hat{\varphi}\|_{L_\xi^1} \quad . \quad (12)$$

Proof. (10) and (11) are classical. (12) can be obtained in the same way. Since $V(t)$ commutes with any time function and

$$\mathcal{F}_{x,t}(V(t)w(t, \cdot)) = \hat{w}(\tau - \xi|\xi|, \xi) \quad ,$$

we infer that

$$\begin{aligned} \|\psi(t/R)V(t)\varphi\|_{A_\lambda} &= \|V(t)\psi(t/R)\varphi\|_{A_\lambda} = \|\mathcal{F}_{t,x}(\psi(\cdot/R)\varphi)\|_{L_{\tau,\xi}^1} \\ &\lesssim \|\hat{\psi}(\cdot)\|_{L_\tau^1} \|\hat{\varphi}\|_{L_\xi^1} \lesssim \|\hat{\varphi}\|_{L_\xi^1} \quad , \end{aligned}$$

Note that we will use (11)-(12) with $R = \lambda^2$ to estimate the low modes of u in (57).

Lemma 2.2 *For all $G \in X_\lambda^{-1/2,s} \cap Z_\lambda^{-1,s}$, it holds*

$$\|\psi(t) \int_0^t V(t-t')G(t') dt'\|_{Y_\lambda^s} \lesssim \|G\|_{X_\lambda^{-1/2,s}} + \|G\|_{Z_\lambda^{-1,s}} \quad . \quad (13)$$

$$\|\psi(t) \int_0^t V(t-t')G(t') dt'\|_{A_\lambda} \lesssim \|G\|_{A_\lambda^{-1}} \quad . \quad (14)$$

Let us recall that (13)-(14) are direct consequences of the following one dimensional (in time) inequalities (cf. [10]): for any function $f \in \mathcal{S}(\mathbb{R})$, it holds

$$\|\psi(t) \int_0^t f(t') dt'\|_{H_t^{1/2}} \lesssim \|f\|_{H_t^{-1/2}} + \left\| \frac{\mathcal{F}_t(f)}{\langle \tau \rangle} \right\|_{L_\tau^1}$$

and

$$\left\| \mathcal{F}_t \left(\psi(t) \int_0^t f(t') dt' \right) \right\|_{L_\tau^1} \lesssim \left\| \frac{\mathcal{F}_t(f)}{\langle \tau \rangle} \right\|_{L_\tau^1} \quad .$$

3 Gauge transform and nonlinear estimates

3.1 Gauge transform

Let $\lambda \geq 1$ and u be a smooth $(2\pi\lambda)$ -periodic solution of (BO) with initial data u_0 . In the sequel, we assume that $u(t)$ has mean value zero for all time. Otherwise we do the change of unknown :

$$v(t, x) = u(t, x - t \int u_0) - \int u_0 \quad , \quad (15)$$

where $\int u_0 = P_0(u_0) = \frac{1}{2\pi\lambda} \int_{\mathbb{R}/(2\pi\lambda)\mathbb{Z}} u_0$ is the mean value of u_0 . It is easy to see that v satisfies (BO) with $u_0 - \int u_0$ as initial data and since $\int v$ is preserved by the flow of (BO), $v(t)$ has mean value zero for all time. We define $F = \partial_x^{-1}u$ which is the periodic, zero mean value, primitive of u ,

$$\widehat{F}(0) = 0 \quad \text{and} \quad \widehat{F}(\xi) = \frac{1}{i\xi} \widehat{u}(\xi), \quad \xi \in \lambda^{-1}\mathbb{Z}^* \quad .$$

Following T. Tao [24], we introduce the gauge transform

$$W = P_+(e^{-iF/2}) \quad . \quad (16)$$

Since F satisfies

$$F_t + \mathcal{H}F_{xx} = \frac{F_x^2}{2} - \frac{1}{2} \int F_x^2 = \frac{F_x^2}{2} - \frac{1}{2} P_0(F_x^2) \quad ,$$

we can check that $w = W_x = -\frac{i}{2}P_+(e^{-iF/2}F_x) = -\frac{i}{2}P_+(e^{-iF/2}u)$ satisfies

$$\begin{aligned} w_t - iw_{xx} &= -\partial_x P_+ \left[e^{-iF/2} \left(P_-(F_{xx}) - \frac{i}{4} P_0(F_x^2) \right) \right] \\ &= -\partial_x P_+ \left(W P_-(u_x) \right) + \frac{i}{4} P_0(F_x^2) w \quad . \end{aligned} \quad (17)$$

On the other hand, one can write u as

$$u = e^{iF/2} e^{-iF/2} F_x = 2i e^{iF/2} \partial_x (e^{-iF/2}) = 2i e^{iF/2} w + 2i e^{iF/2} \partial_x P_-(e^{-iF/2}) \quad . \quad (18)$$

Recalling that u is real-valued, we get

$$u = \bar{u} = -2i e^{-iF/2} \bar{w} - 2i e^{-iF/2} \overline{\partial_x P_-(e^{-iF/2})}$$

and thus

$$P_-(u) = -2i P_-(e^{-iF/2} \bar{w}) - 2i P_-(e^{-iF/2} \partial_x P_+(e^{iF/2})) \quad (19)$$

since $\overline{P_-(v)} = P_+(\bar{v})$ for any complex-valued function v . Substituting (19) in (17), we obtain the following equation satisfied by w :

$$\begin{aligned} w_t - iw_{xx} &= 2i \partial_x P_+ \left(W \partial_x P_-(e^{-iF/2} \bar{w}) \right) \\ &\quad + 2i \partial_x P_+ \left[W \partial_x P_-(e^{-iF/2} \partial_x P_+(e^{iF/2})) \right] + \frac{i}{4} P_0(F_x^2) W_x \end{aligned} \quad (20)$$

Note also that it follows from (18) that

$$\begin{aligned} P_{>1}u &= 2iP_{>1}\left(e^{iF/2}w\right) + 2iP_{>1}\left(e^{iF/2}\partial_x P_-(e^{-iF/2})\right) \\ &= 2iP_{>1}\left(e^{iF/2}w\right) + 2iP_{>1}\left(P_{>1}(e^{iF/2})\partial_x P_-(e^{-iF/2})\right) \quad . \quad (21) \end{aligned}$$

To end this section is we state the crucial nonlinear estimates on u and w that will be proven in the next two sections. It is worth noticing that in all the estimates, we will replace the exponential function (if it appears) by its entire serie and prove the absolute convergence of the resulting serie. Even if this approach can appear unnecessary to prove the well-posedness result, it will be very useful in order to derive the analyticity of the flow-map.

Proposition 3.1 *Let $L_1^\infty \dot{H}_\lambda^s \cap N_{1,\lambda}$ be a solution of (BO) and $w \in X_{1,\lambda}^{1/2,s}$ satisfying (17)-(18). Then for $0 \leq s \leq 1/2$, it holds*

$$\begin{aligned} \|w\|_{M_{1,\lambda}^s} &\lesssim (1 + \|u_0\|_{L_\lambda^2})e^{\widehat{\|\partial_x^{-1}u_0\|_{L_\xi^1}}} \|u_0\|_{H_\lambda^s} \\ &\quad + \|w\|_{X_{1,\lambda}^{1/2,s}} \left(\|u\|_{N_{1,\lambda}} + \|w\|_{X_{1,\lambda}^{1/2,0}} \right) e^{\tilde{K}} \quad , \quad (22) \end{aligned}$$

$$\|u\|_{N_{1,\lambda}} \lesssim \|u_0\|_{L_\lambda^2} + \left(\|w\|_{M_{1,\lambda}^0} + \|u\|_{N_{1,\lambda}}^2 \right) e^{\tilde{K}} \quad , \quad (23)$$

and

$$\|u\|_{L_1^\infty H_\lambda^s} \lesssim \|u_0\|_{L_\lambda^2} + \left(\|w\|_{M_{1,\lambda}^s} + \|u\|_{N_{1,\lambda}}^2 \right) e^{\tilde{K}} \quad . \quad (24)$$

where

$$\tilde{K} = C \left(\widehat{\|\partial_x^{-1}u_0\|_{L_\xi^1}} + \|u\|_{N_{1,\lambda}}^2 \right) \quad . \quad (25)$$

for some universal constant $C > 0$.

4 Proof of the estimate on w

In this section, we will need the two following technical lemmas. The first one, which is proven in the appendix, enables to treat the multiplication with the gauge function $e^{-iF/2}$ in the Sobolev spaces whereas the second one (see the appendix of [18] for a proof), shows that, due to the frequency projections, we can share derivatives when taking the H^s -norm of the second term of the right-hand side to (20) or (21).

Lemma 4.1 *Let $2 \leq q \leq 4$. Let h be function of H_λ^1 and let $g \in L_\lambda^q$ such that $J_x^\alpha g \in L_\lambda^q$ with $0 < \alpha \leq 1/2$. Then it holds*

$$\|J^\alpha(hg)\|_{L_\lambda^q} \lesssim \|J_x^\alpha g\|_{L_\lambda^q} (\|h\|_{L_\lambda^\infty} + \|h_x\|_{L_\lambda^2}) \quad . \quad (26)$$

Lemma 4.2 *Let $\alpha \geq 0$ and $1 < q < \infty$ then*

$$\left\| D_x^\alpha P_+ \left(f P_- \partial_x g \right) \right\|_{L_\lambda^q} \lesssim \|D_x^{\gamma_1} f\|_{L_\lambda^{q_1}} \|D_x^{\gamma_2} g\|_{L_\lambda^{q_2}} \quad , \quad (27)$$

with $1 < q_i < \infty$, $1/q_1 + 1/q_2 = 1/q$ and $\begin{cases} \gamma_1 \geq \alpha, \gamma_2 \geq 0 \\ \gamma_1 + \gamma_2 = \alpha + 1 \end{cases}$.

4.1 Some multilinear estimates

The main tool for proving (22) are three multilinear estimates. These estimates enlight the good behavior in Bourgain spaces of the terms of the right-hand side of (20). In the following, we assume that W and v are supported in time in $[-2, 2]$ and we set $\psi_2(\cdot) = \psi(\cdot/2)$.

Lemma 4.3 *For any $s \geq 0$ and $0 < \varepsilon \ll 1$,*

$$\left\| \partial_x P_+ \left[W \partial_x P_- \left(e^{-iF/2} \partial_x P_+ \left(e^{iF/2} \right) \right) \right] \right\|_{X_\lambda^{-1/2+\varepsilon,s}} \lesssim \|w\|_{X_\lambda^{1/2,s}} \|\psi_2 F_x\|_{L_{t,\lambda}^4}^2 e^{\|F\|_{L_{t,\lambda}^\infty}} \quad . \quad (28)$$

Proof. As written above, we will actually prove (28) with as left-hand side member :

$$\sum_{k \geq 1} \sum_{l \geq 1} \frac{1}{k!} \frac{1}{l!} \left\| \partial_x P_+ \left[W \partial_x P_- \left(F^k \partial_x P_+ F^l \right) \right] \right\|_{X_\lambda^{-1/2+\varepsilon,s}} \quad .$$

Setting

$$g = \partial_x P_- \left(\psi_2 F^k \partial_x P_+ (\psi_2 F^l) \right) ,$$

it follows from Lemma 4.2 that

$$\|g\|_{L_{t,\lambda}^2} \lesssim \|\psi_2 \partial_x (F^k)\|_{L_{t,\lambda}^4} \|\psi_2 \partial_x (F^l)\|_{L_{t,\lambda}^4} \lesssim k l \|\psi_2 F_x\|_{L_{t,\lambda}^4}^2 \|F\|_{L_{t,\lambda}^\infty}^{k+l-2} .$$

It thus suffices to prove that

$$\|\partial_x P_+ (W P_- g)\|_{X_\lambda^{-1/2+\varepsilon,s}} \lesssim \|w\|_{X_\lambda^{1/2,s}} \|g\|_{L_{t,\lambda}^2} \quad . \quad (29)$$

By duality it is equivalent to estimate

$$I = \left| \int_A \xi \hat{h}(\tau, \xi) \xi_1^{-1} \hat{w}(\tau_1, \xi_1) \hat{g}(\tau_2, \xi_2) \right|$$

where $(\tau_2, \xi_2) = (\tau - \tau_1, \xi - \xi_1)$, and due to the frequency projections

$$A = \{(\tau, \tau_1, \xi, \xi_1) \in \mathbb{R}^2 \times (\lambda^{-1}\mathbb{Z})^2, \quad \xi \geq 1/\lambda, \xi_1 \geq 1/\lambda, \xi - \xi_1 \leq -1/\lambda \} \quad .$$

Note that in the domain of integration above,

$$\xi_1 \geq |\xi - \xi_1| \quad \text{and} \quad \xi_1 \geq \xi \quad . \quad (30)$$

It thus follows that

$$I \lesssim \int_A \langle \xi \rangle^{-s} |\hat{h}(\tau, \xi)| \langle \xi_1 \rangle^s |\hat{w}(\tau_1, \xi_1)| |\hat{g}(\tau_2, \xi_2)|$$

and on account of (9),

$$\begin{aligned} I &\lesssim \|\mathcal{F}^{-1}(\langle \xi \rangle^{-s} |\hat{h}|)\|_{L_{t,\lambda}^4} \|\mathcal{F}^{-1}(\langle \xi \rangle^s |\hat{w}|)\|_{L_{t,\lambda}^4} \|\mathcal{F}^{-1}(|\hat{g}|)\|_{L_{t,\lambda}^2} \\ &\lesssim \|h\|_{X_\lambda^{3/8, -s}} \|w\|_{X_\lambda^{1/2, s}} \|g\|_{L_{t,\lambda}^2} \end{aligned}$$

which proves (28).

Lemma 4.4 *For any $s \geq 0$ it holds*

$$\begin{aligned} \left\| \partial_x P_+ \left(W \partial_x P_- (e^{-iF/2} P_- v) \right) \right\|_{X^{-1/2, s}} &\lesssim \|w\|_{X^{1/2, s}} \|v\|_{X^{1/2, 0}} e^{C\|F\|_{A_\lambda}} \\ &\left(1 + \|P_3 F\|_{\dot{X}_\lambda^{1, 0}} + \|P_{>3} F_x\|_{\dot{X}_\lambda^{7/8, -1}} + \|F\|_{A_\lambda} + \|\psi_2 F_x\|_{L_{t,\lambda}^4} + \|\psi_2 F_x\|_{\tilde{L}_{t,\lambda}^4} \right) . \quad (31) \end{aligned}$$

Proof. Again we will in fact prove (31) with as left-hand side member :

$$\left\| \partial_x P_+ \left(W \partial_x P_- v \right) \right\|_{X_\lambda^{-1/2, s}} + \sum_{k \geq 1} \frac{1}{k!} \left\| \partial_x P_+ \left(W \partial_x P_- (F^k P_- v) \right) \right\|_{X_\lambda^{-1/2, s}} \quad .$$

The first term of the above inequality is estimated in ([18], Lemma 3.3) by

$$\left\| \partial_x P_+ \left(W \partial_x P_- v \right) \right\|_{X_\lambda^{-1/2, s}} \lesssim \|w\|_{X^{1/2, s}} \|v\|_{X_\lambda^{1/2, 0}} \quad . \quad (32)$$

By duality it thus remains to estimate

$$I_k = \left| \int_B \xi \hat{h}(\tau, \xi) \xi_1^{-1} \hat{w}(\tau_1, \xi_1) (\xi - \xi_1) \widehat{P_- v}(\tau_2, \xi_2) \prod_{i=3}^{k+2} \hat{F}(\tau_i, \xi_i) \right| \quad (33)$$

where $(\tau_{k+2}, \xi_{k+2}) = (\tau, \xi) - \sum_{i=1}^{k+1} (\tau_i, \xi_i)$, and due to the frequency projections

$$B = \{(\tau, \tau_1, \dots, \tau_{k+1}, \xi, \xi_1, \dots, \xi_{k+1}) \in \mathbb{R}^{k+2} \times (\lambda^{-1}\mathbb{Z})^{k+2}, \\ \xi_1 \geq \xi \geq 1/\lambda, \xi - \xi_1 \leq -1/\lambda \} \quad .$$

We observe that

$$\begin{aligned} & \left| \int_B \xi \hat{h}(\tau, \xi) \xi_1^{-1} \hat{w}(\tau_1, \xi_1) (\xi - \xi_1) \widehat{(P_{\{2^{10}k\}} P_- v)}(\tau_2, \xi_2) \prod_{i=3}^{k+2} \hat{F}(\tau_i, \xi_i) \right| \\ & \lesssim \|\mathcal{F}^{-1}(\langle \xi \rangle^{-s} |\hat{h}|)\|_{L_{t,\lambda}^4} \|\mathcal{F}^{-1}(\langle \xi \rangle^s |\hat{w}|)\|_{L_{t,\lambda}^4} \left\| \partial_x \left((P_{\{2^{10}k\}} P_- v) F^k \right) \right\|_{L_{t,\lambda}^2} \\ & \lesssim k \|h\|_{X_\lambda^{3/8, -s}} \|w\|_{X_\lambda^{1/2, s}} (\|v\|_{L_{t,\lambda}^2} \|F\|_{L_{t,\lambda}^\infty} + \|v\|_{L_{t,\lambda}^4} \|\psi_2 F_x\|_{L_{t,\lambda}^4}) \|F\|_{L_{t,\lambda}^\infty}^{k-1}, \quad (34) \end{aligned}$$

since obviously,

$$\begin{aligned} \left\| \partial_x \left((P_{\{2^{10}k\}} P_- v) F^k \right) \right\|_{L_{t,\lambda}^2} & \lesssim \|P_{\{2^{10}k\}} P_- v_x\|_{L_{t,\lambda}^2} \|F\|_{L_{t,\lambda}^\infty}^k + k \|P_{\{2^{10}k\}} P_- v\|_{L_{t,\lambda}^4} \|\psi_2 F_x\|_{L_{t,\lambda}^4} \|F\|_{L_{t,\lambda}^\infty}^{k-1} \\ & \lesssim k (\|v\|_{L_{t,\lambda}^2} \|F\|_{L_{t,\lambda}^\infty} + \|v\|_{L_{t,\lambda}^4} \|\psi_2 F_x\|_{L_{t,\lambda}^4}) \|F\|_{L_{t,\lambda}^\infty}^{k-1} \quad . \end{aligned}$$

It thus remains to prove (33) with $P_- v$ replaced by $Q_{\{2^{10}k\}} P_- v$. Note also that since (30) holds on B , for $|\xi| \leq 2^{10}k$ or $|\xi - \xi_1| \leq 2^{10}k$ we get thanks to (9),

$$\begin{aligned} I_{k/B} & \lesssim k \|\mathcal{F}^{-1}(\langle \xi \rangle^{-s} |\hat{h}|)\|_{L_{t,\lambda}^4} \|\mathcal{F}^{-1}(\langle \xi \rangle^s |\hat{w}|)\|_{L_{t,\lambda}^4} \|(Q_{\{2^{10}k\}} P_- v) F^k\|_{L_{t,\lambda}^2} \\ & \lesssim k \|h\|_{X_\lambda^{3/8, -s}} \|w\|_{X_\lambda^{1/2, s}} \|v\|_{L_{t,\lambda}^2} \|F\|_{L_{t,\lambda}^\infty}^k \quad . \quad (35) \end{aligned}$$

It thus suffices to control

$$\tilde{I}_{k/B_1} = \left| \int_{B_1} \xi \hat{h}(\tau, \xi) \xi_1^{-1} \hat{w}(\tau_1, \xi_1) (\xi - \xi_1) \widehat{(Q_{\{2^{10}k\}} P_- v)}(\tau_2, \xi_2) \prod_{i=3}^{k+2} \hat{F}(\tau_i, \xi_i) \right| \quad (36)$$

where

$$B_1 = \{(\tau, \tau_1, \dots, \tau_{k+1}, \xi, \xi_1, \dots, \xi_{k+1}) \in B, \xi > 2^{10}k, \xi - \xi_1 < -2^{10}k \}$$

One of the main difficulties will be that we do not have a control on $\|\mathcal{F}_{t,x}^{-1}(|\hat{F}_x|)\|_{L_{t,\lambda}^4}$ but only on $\|F_x\|_{L_{t,\lambda}^4}$. This can be overcome when $s > 0$ but causes a kind of logarithmic divergence when $s = 0$. To control \tilde{I}_{k/B_1} we will have to use

the complementary norm $\tilde{L}_{t,\lambda}^4$ of F_x . To simplify the notation we denote $Q_{\{2^{10}k\}}P_-v$ by \tilde{v} . Since we cannot “force” the integrant to be non negative in (36), we have to act carefully. We notice that using Littlewood-Paley decomposition (see (6)) we can rewrite $Q_{2^{10}k}(\tilde{v}F^k)$ as

$$\begin{aligned} Q_{\{2^{10}k\}}(\tilde{v}F^k) &= Q_{\{2^{10}k\}}\left(\sum_{i_2 \geq 8+\alpha(k)} \Delta_{i_2}(\tilde{v}) \sum_{i_3 \geq i_2-6-\alpha(k)} \Delta_{i_3}(F) \sum_{0 \leq i_4, \dots, i_{k+2} \leq i_3} n(i_3, \dots, i_{k+2}) \prod_{j=4}^{k+2} \Delta_{i_j}(F)\right) \\ &\quad + Q_{\{2^{10}k\}}\left(\sum_{i_2 \geq 8+\alpha(k)} \Delta_{i_2}(\tilde{v}) \sum_{0 \leq i_3 < i_2-6-\alpha(k)} \Delta_{i_3}(F) \sum_{0 \leq i_4, \dots, i_{k+2} \leq i_3} n(i_3, \dots, i_{k+2}) \prod_{j=4}^{k+2} \Delta_{i_j}(F)\right) \\ &= G_1 + G_2 \quad , \end{aligned}$$

where $\alpha(k)$ denotes the entire part of $\ln(k)/\ln(2)$ and $n(i_3, \dots, i_{k+2})$ is an integer belonging to $\{1, \dots, k\}$ (Note for instance that $n(i_3, \dots, i_{k+2}) = 1$ for $i_3 = i_4 = \dots = i_{k+2}$ and $n(i_3, \dots, i_{k+2}) = k$ for $i_3 \neq i_4 \neq \dots \neq i_{k+2}$). From (36) we thus infer that

$$\begin{aligned} \tilde{I}_{k/B_1} &\lesssim \sum_{i=1}^2 \int_{B_1} \xi |\hat{h}(\tau, \xi)| \xi_1^{-1} |\hat{w}(\tau_1, \xi_1)| |\xi - \xi_1| |\widehat{G_i}(\tau - \tau_1, \xi - \xi_1)| \\ &\lesssim \Lambda_1 + \Lambda_2 \quad . \end{aligned} \quad (37)$$

• *Estimate on Λ_1 .* Thanks to the definition of B and (9), we easily obtain

$$\begin{aligned} \Lambda_1 &\lesssim \|\mathcal{F}^{-1}(\langle \xi \rangle^{-s} |\hat{h}|)\|_{L_{t,\lambda}^4} \|\mathcal{F}^{-1}(\langle \xi \rangle^s |\hat{w}|)\|_{L_{t,\lambda}^4} \|\partial_x G_1\|_{L_{t,\lambda}^2} \\ &\lesssim \|h\|_{X_\lambda^{3/8, -s}} \|w\|_{X_\lambda^{1/2, s}} \|\partial_x G_1\|_{L_{t,\lambda}^2} \quad . \end{aligned}$$

On the other hand, using the frequency support of the functions, we infer that for $q \geq 9 + \alpha(k)$,

$$\Delta_q(G_1) = Q_{\{2^{10}k\}} \Delta_q \left(\sum_{\substack{i_3 \geq q-8-\alpha(k) \\ i_3 \geq 2}} \Delta_{i_3}(F) \sum_{\substack{i_2 \geq 8+\alpha(k) \\ i_2 \leq i_3+6+\alpha(k)}} \Delta_{i_2}(\tilde{v}) \sum_{0 \leq i_4, \dots, i_{k+2} \leq i_3} n(i_3, \dots, i_{k+2}) \prod_{j=4}^{k+2} \Delta_{i_j}(F) \right)$$

and thus

$$\|\Delta_q G_1\|_{L_{t,\lambda}^2} \lesssim \sum_{i_3 \geq q-8-\alpha(k)} \|\psi_2 \Delta_{i_3} F\|_{L_{t,\lambda}^4} \|\tilde{v}\|_{L_{t,\lambda}^4} \left\| \sum_{0 \leq i_4, \dots, i_{k+2} \leq i_3} n(i_3, \dots, i_{k+2}) \prod_{j=4}^{k+2} \Delta_{i_j}(F) \right\|_{L_{t,\lambda}^\infty} \quad .$$

But

$$\left\| \sum_{0 \leq i_4, \dots, i_{k+2} \leq i_3} n(i_3, \dots, i_{k+2}) \prod_{j=4}^{k+2} \Delta_{i_j}(F) \right\|_{L_{t,\lambda}^\infty} \lesssim k \left\| \sum_{0 \leq i_4, \dots, i_{k+2}} |\widehat{\Delta_{i_4}(F)}| * \dots * |\widehat{\Delta_{i_{k+2}}(F)}| \right\|_{L_{\tau,\xi}^1}$$

$$\begin{aligned}
&\lesssim k \left\| \left(\sum_{i_4 \geq 0} |\widehat{\Delta_{i_4}(F)}| \right) * \dots * \left(\sum_{i_{k+2} \geq 0} |\widehat{\Delta_{i_{k+2}}(F)}| \right) \right\|_{L^1_{\tau, \xi}} \\
&\lesssim k \|F\|_{A_\lambda}^{k-1} . \tag{38}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\partial_x G_1\|_{L^2_{t, \lambda}}^2 &\sim \sum_{q \geq 9 + \alpha(k)} 2^{2q} \|\Delta_q G_1\|_{L^2_{t, \lambda}}^2 \\
&\lesssim k^2 \|\tilde{v}\|_{L^4_{t, \lambda}}^2 \|F\|_{A_\lambda}^{2(k-1)} \sum_{q \geq 9 + \alpha(k)} \left(\sum_{\substack{j \geq q - 8 - \alpha(k) \\ j \geq 2}} 2^{(q-j)2^j} \|\psi_2 \Delta_j F\|_{L^4_{t, \lambda}} \right)^2
\end{aligned}$$

But, by the definition of the norm $\tilde{L}^4_{t, \lambda}$ ³ (see (6)), for $j \geq 2$, $2^j \|\Delta_j F\|_{L^4_{t, \lambda}} \lesssim \gamma_j \|F_x\|_{\tilde{L}^4_{t, \lambda}}$ with $\|(\gamma_j)\|_{l^2(\mathbb{N})} \lesssim 1$. Hence, by Young inequality,

$$\sum_{\substack{j \geq q - 8 - \alpha(k) \\ j \geq 2}} 2^{(q-j)2^j} \|\psi_2 \Delta_j F\|_{L^4_{t, \lambda}} \lesssim k \gamma_j \|\psi_2 F_x\|_{\tilde{L}^4_{t, \lambda}}$$

and thus

$$\|\partial_x G_1\|_{L^2_{t, \lambda}} \lesssim k^2 \|\tilde{v}\|_{L^4_{t, \lambda}} \|\psi_2 F_x\|_{\tilde{L}^4_{t, \lambda}} \|F\|_{A_\lambda}^{k-1} .$$

Therefore, the following estimate holds

$$\Lambda_1 \lesssim k^2 \|h\|_{X_\lambda^{3/8, -s}} \|w\|_{X_\lambda^{1/2, s}} \|v\|_{L^4_{t, \lambda}} \|\psi_2 F_x\|_{\tilde{L}^4_{t, \lambda}} \|F\|_{A_\lambda}^{k-1} \tag{39}$$

• *Estimate on Λ_2 .* We rewrite G_2 as

$$\begin{aligned}
G_2 &= Q_{\{2^{10}k\}} \left(\sum_{i_2 \geq 8 + \alpha(k)} \Delta_{i_2}(\tilde{v}) \sum_{1 \leq i_3 < i_2 - 6 - \alpha(k)} \Delta_{i_3}(F) \sum_{0 \leq i_4, \dots, i_{k+2} \leq i_3} n(i_3, \dots, i_{k+2}) \prod_{j=4}^{k+2} \Delta_{i_j}(F) \right) \\
&\quad + Q_{\{2^{10}k\}} \left(\sum_{i_2 \geq 8 + \alpha(k)} \Delta_{i_2}(\tilde{v}) (\Delta_0(F))^k \right) \\
&= \sum_{p \geq 1} \left[Q_{\{2^{10}k\}} \left(\sum_{\substack{i_2 \geq 8 + \alpha(k) \\ i_2 > p + 6 + \alpha(k)}} \Delta_{i_2}(\tilde{v}) \Delta_p(F) \sum_{0 \leq i_4, \dots, i_{k+2} \leq p} n(i_3, \dots, i_{k+2}) \prod_{j=4}^{k+2} \Delta_{i_j}(F) \right) \right] \\
&\quad + Q_{\{2^{10}k\}} \left(\sum_{i_2 \geq 8 + \alpha(k)} \Delta_{i_2}(\tilde{v}) (\Delta_0(F))^k \right) \\
&= \sum_{p \geq 1} H_p + L .
\end{aligned}$$

³Note that we could avoid the $\tilde{L}^4_{t, \lambda}$ -norm here by invoking the Littlewood-Paly square function theorem in the estimate on G_1

it is thus clear that

$$\begin{aligned}\Lambda_2 &\lesssim \sum_{p \geq 1} \left| \int_{B_1} \xi |\hat{h}(\tau, \xi)| \xi_1^{-1} |\hat{w}(\tau_1, \xi_1)| |\xi - \xi_1| |\widehat{H}_p(\tau - \tau_1, \xi - \xi_1)| \right. \\ &\quad \left. + \int_{B_1} \xi |\hat{h}(\tau, \xi)| \xi_1^{-1} |\hat{w}(\tau_1, \xi_1)| |\xi - \xi_1| |\widehat{L}(\tau - \tau_1, \xi - \xi_1)| \right| \\ &= \Lambda_{21} + \Lambda_{22} \quad .\end{aligned}$$

Let us rewrite Λ_{21} as the sum of two terms :

$$\begin{aligned}\Lambda_{21} &= \sum_{p \geq 1} \int_{B_1} \chi_{\{|\xi| \leq 2^{p+6+\alpha(k)}\}} \xi |\hat{h}(\tau, \xi)| \xi_1^{-1} |\hat{w}(\tau_1, \xi_1)| |\xi - \xi_1| |\widehat{H}_p(\tau - \tau_1, \xi - \xi_1)| \\ &\quad + \sum_{p \geq 1} \int_{B_1} \chi_{\{|\xi| > 2^{p+6+\alpha(k)}\}} \xi |\hat{h}(\tau, \xi)| \xi_1^{-1} |\hat{w}(\tau_1, \xi_1)| |\xi - \xi_1| |\widehat{H}_p(\tau - \tau_1, \xi - \xi_1)| \\ &= \Lambda_{21}^1 + \Lambda_{21}^2 \quad .\end{aligned}$$

• *Estimate on Λ_{21}^1 .* Using a Littlewood-Paley decomposition of h , we get thanks to (30) and Cauchy-Schwarz inequality in p

$$\begin{aligned}\Lambda_{21}^1 &\lesssim \sum_{p \geq 1} \sum_{s=-7-\alpha(k)}^{p-8-\alpha(k)} \int_{B_1} 2^{p-s} |\widehat{\Delta_{p-s} h}(\tau, \xi)| |\hat{w}(\tau_1, \xi_1)| |\widehat{H}_p(\tau - \tau_1, \xi - \xi_1)| \\ &\lesssim \sum_{s=-7-\alpha(k)}^{\infty} 2^{-s} \sum_{p \geq s+8+\alpha(k)} \int_{B_1} |\widehat{\Delta_{p-s} h}(\tau, \xi)| |\hat{w}(\tau_1, \xi_1)| |\xi - \xi_1| 2^p |\widehat{H}_p(\tau - \tau_1, \xi - \xi_1)| \\ &\lesssim \sup_{s \geq -7-\alpha(k)} k \sum_{p \geq s+8+\alpha(k)} \int_{B_1} |\widehat{\Delta_{p-s} h}(\tau, \xi)| |\hat{w}(\tau_1, \xi_1)| |\xi - \xi_1| 2^p |\widehat{H}_p(\tau - \tau_1, \xi - \xi_1)| \\ &\lesssim k \|\mathcal{F}^{-1}(\langle \xi \rangle^s |\hat{w}|)\|_{L_{t,\lambda}^4} \left(\sum_{p \geq 1} \|\mathcal{F}^{-1}(\langle \xi \rangle^{-s} |\widehat{\Delta_p h}|)\|_{L_{t,\lambda}^4}^2 \right)^{1/2} \left(\sum_{p \geq 1} 2^{2p} \|H_p\|_{L_{t,\lambda}^2}^2 \right)^{1/2}\end{aligned}$$

Note that $\tilde{L}_{t,\lambda}^4 \leftrightarrow X^{3/8,0}$ since by (9), for any function $z \in X_\lambda^{3/8,0}$,

$$\left(\sum_{p \geq 1} \|\mathcal{F}^{-1}(|\widehat{\Delta_p z}|)\|_{L_{t,\lambda}^4}^2 \right)^{1/2} \lesssim \left(\sum_{p \geq 1} \|\Delta_p z\|_{X^{3/8,0}}^2 \right)^{1/2} \lesssim \|z\|_{X^{3/8,0}} \quad . \quad (40)$$

Moreover since, according to the frequency localization of the functions,

$$\Delta_q H_p = Q_{2^{10}k} \left(\sum_{\substack{i_2 \geq 8+\alpha(k) \\ q-1 \leq i_2 \leq q+1}} \Delta_{i_2}(\tilde{v}) \Delta_p(F) \sum_{0 \leq i_4, \dots, i_{k+2} \leq p} n(p, i_4, \dots, i_{k+2}) \prod_{j=4}^{k+2} \Delta_{i_j}(F) \right)$$

we infer from (38) and (40) that

$$\begin{aligned}
\|H_p\|_{L_{t,\lambda}^2}^2 &\sim \sum_{q \geq p+9+\alpha(k)} \|\Delta_q H_p\|_{L_{t,\lambda}^2}^2 \\
&\lesssim k^2 \|F\|_{A_\lambda}^{2(k-1)} \|\psi_2 \Delta_p F\|_{L_{t,\lambda}^4}^2 \sum_{q \geq 1} \|\Delta_q \tilde{v}\|_{L_{t,\lambda}^4}^2 \\
&\lesssim k^2 \|F\|_{A_\lambda}^{2(k-1)} \|v\|_{X^{1/2,0}}^2 \|\psi_2 \Delta_p F\|_{L_{t,\lambda}^4}^2 .
\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}
\Lambda_{21}^1 &\lesssim k \|h\|_{X^{3/8,-s}} \|w\|_{X^{1/2,s}} \|F\|_{A_\lambda}^{k-1} \|v\|_{X^{1/2,0}} \left(\sum_{p \geq 1} 2^{2p} \|\psi_2 \Delta_p F\|_{L_{t,\lambda}^4}^2 \right)^{1/2} \\
&\lesssim k \|h\|_{X^{3/8,-s}} \|w\|_{X^{1/2,s}} \|F\|_{A_\lambda}^{k-1} \|v\|_{X^{1/2,0}} \|\psi_2 F_x\|_{\tilde{L}_{t,\lambda}^4} . \quad (41)
\end{aligned}$$

• *Estimate on Λ_{21}^2 and Λ_{22} .* Since clearly , $\sum_{p \geq 0} |\widehat{\Delta_p(f)}(\tau, \xi)| \leq 2|\hat{f}(\tau, \xi)|$ for any $f \in L_{t,\lambda}^2$, we infer that

$$\begin{aligned}
\Lambda_{21}^2 &\lesssim k \sum_{p \geq 1} \sum_{i_2 \geq p+7+\alpha(k)} \int_{B_2} \xi |\hat{h}(\tau, \xi)| \xi_1^{-1} |\hat{w}(\tau_1, \xi_1)| |\xi - \xi_1| |\widehat{\Delta_{i_2}(\tilde{v})}(\tau_2, \xi_2)| |\widehat{\Delta_p(F)}(\tau_3, \xi_3)| \\
&\quad \sum_{i_4, \dots, i_{k+2} \geq 0} \prod_{j=4}^{k+2} |\widehat{\Delta_{i_j}(F)}(\tau_j, \xi_j)| \\
&\lesssim k \int_{B_2} \xi |\hat{h}(\tau, \xi)| \xi_1^{-1} |\hat{w}(\tau_1, \xi_1)| |\xi - \xi_1| |\hat{v}(\tau_2, \xi_2)| \prod_{i=3}^{k+2} |\hat{F}(\tau_i, \xi_i)| \\
&= \tilde{I}_{k/B_2}
\end{aligned}$$

where

$$B_2 = \{(\tau, \tau_1, \dots, \tau_{k+1}, \xi, \xi_1, \dots, \xi_{k+1}) \in B_1, \xi_2 \leq -2^{10}k, \min(|\xi|, |\xi_2|) > 10k \max_{i=3, \dots, k+2} |\xi_i|\} .$$

In the same way, it is easy to check that $\Lambda_{22} \lesssim \tilde{I}_{k/B_2}$. We set $\sigma = \sigma(\tau, \xi) = \tau - \xi|\xi|$ and $\sigma_i = \sigma(\tau_i, \xi_i)$, $i = 1, \dots, k+2$. Noticing that on B_2 the sign of ξ , ξ_1 and ξ_2 are known, we get the following algebraic relation :

$$\sigma - \sum_{i=1}^{k+2} \sigma_i = \left(\sum_{i=1}^{k+2} \xi_i \right)^2 - \xi_1^2 + \xi_2^2 - \sum_{i=3}^{k+2} \xi_i |\xi_i|$$

$$\begin{aligned}
&= 2\xi_2 \sum_{i=1}^{k+2} \xi_i + 2\xi_1 \sum_{i=3}^{k+2} \xi_i - \sum_{i=3}^{k+2} \xi_i |\xi_i| + \left(\sum_{i=3}^{k+2} \xi_i \right)^2 \\
&= 2\xi_2 \xi + 2\xi_1 \sum_{i=3}^{k+2} \xi_i - \sum_{i=3}^{k+2} \xi_i |\xi_i| + \left(\sum_{i=3}^{k+2} \xi_i \right)^2 . \quad (42)
\end{aligned}$$

Note that on B_2 , we have $(\sum_{i=3}^{k+2} \xi_i)^2 \leq 10^{-2} |\xi_2 \xi|$, $\sum_{i=3}^{k+2} \xi_i |\xi_i| \leq 10^{-2} |\xi_2 \xi|$ and

$$\frac{1}{2} |\xi_2| \leq |\xi - \xi_1| \leq |\xi_2| + \sum_{i=3}^{k+2} |\xi_i| \leq 2|\xi_2| . \quad (43)$$

Hence, $\xi_1 \leq \max(|\xi|, |\xi - \xi_1|) \leq 2 \max(|\xi|, |\xi_2|)$ and $|\xi_1 \sum_{i=3}^{k+2} \xi_i| \leq |\xi_2 \xi|/5$. It thus follows from (42) that

$$\max_{i=1, \dots, k+2} (|\sigma|, |\sigma_i|) \gtrsim |\xi \xi_2|/k . \quad (44)$$

It remains to divide B_2 in subregions according to the indice where the maximum is reached in (44). Thanks to (30),

$$\tilde{I}_{k/B_2} \lesssim \int_{B_2} |\xi \xi_2|^{1/2} |\hat{h}(\tau, \xi)| |\hat{w}(\tau_1, \xi_1)| |\hat{v}(\tau_2, \xi_2)| \prod_{i=3}^{k+2} |\hat{F}(\tau_i, \xi_i)| .$$

• σ dominant. By (30) and (44), Plancherel, Holder inequality and (9), we infer that

$$\begin{aligned}
\tilde{I}_{k/B_2} &\lesssim k \|\mathcal{F}^{-1}(\langle \sigma \rangle^{1/2} \langle \xi \rangle^{-s} |\hat{h}|)\|_{L_{t,\lambda}^2} \|\mathcal{F}^{-1}(\langle \xi \rangle^s |\hat{w}|)\|_{L_{t,\lambda}^4} \|\mathcal{F}^{-1}(|\hat{v}|)\|_{L_{t,\lambda}^4} \|\mathcal{F}^{-1}(|\hat{F}|)\|_{L_{t,\lambda}^\infty}^k \\
&\lesssim k \|h\|_{X_\lambda^{1/2, -s}} \|w\|_{X_\lambda^{1/2, s}} \|v\|_{X_\lambda^{1/2, 0}} \|F\|_{A_\lambda}^k . \quad (45)
\end{aligned}$$

• σ_1 or σ_2 dominant. It is easy to see that in the same way

$$\tilde{I}_{k/B_2} \lesssim k \|h\|_{X_\lambda^{3/8, -s}} \|w\|_{X_\lambda^{1/2, s}} \|v\|_{X_\lambda^{1/2, 0}} \|F\|_{A_\lambda}^k . \quad (46)$$

• σ_i , $i \geq 3$, dominant. By Plancherel, Holder inequality and (9), we infer that

$$\begin{aligned}
\tilde{I}_{k/B_2} &\lesssim k^{1/2} \|\mathcal{F}^{-1}(\langle \xi \rangle^{-s} |\hat{h}|)\|_{L_{t,\lambda}^4} \|\mathcal{F}^{-1}(\langle \xi \rangle^s |\hat{w}|)\|_{L_{t,\lambda}^4} \\
&\quad \|\mathcal{F}^{-1}(|\hat{v}|)\|_{L_{t,\lambda}^4} \|\mathcal{F}^{-1}(|\sigma|^{1/2} \chi_{\{|\sigma| \gtrsim 1\}} |\hat{F}|)\|_{L_{t,\lambda}^4} \|\mathcal{F}^{-1}(|\hat{F}|)\|_{L_{t,\lambda}^\infty}^{k-1} \\
&\lesssim k \|h\|_{X_\lambda^{3/8, -s}} \|w\|_{X_\lambda^{1/2, s}} \|v\|_{X_\lambda^{1/2, 0}} (\|P_3 F\|_{\dot{X}_\lambda^{1,0}} + \|P_{>3} F\|_{X_\lambda^{7/8,0}}) \|F\|_{A_\lambda}^{k-1} \quad (47)
\end{aligned}$$

where we use that $|\sigma_i| \gtrsim |\xi\xi_2|/k \gtrsim 1$ on B_2 to get an homogeneous Bourgain type norm on P_3F . It has some importance when using dilations argument since the L^2 -norm of P_3F is surcritical and thus behaves badly for such arguments. Since clearly, $\|P_{>3}F\|_{X_\lambda^{7/8,0}} \lesssim \|P_{>3}F_x\|_{X_\lambda^{7/8,-1}}$, gathering (34), (35), (39), (41), (45), (46) and (47), (31) follows.

Lemma 4.5 *For any $s \geq 0$ it holds*

$$\left\| \partial_x P_+ \left(W \partial_x P_- (e^{-iF/2} P_- v) \right) \right\|_{Z^{-1,s}} \lesssim \|w\|_{X_\lambda^{1/2,s}} \|v\|_{X_\lambda^{1/2,0}} e^{C\|F\|_{A_\lambda}} \\ \left(1 + \|P_3F\|_{\dot{X}_\lambda^{1,0}} + \|P_{>3}F_x\|_{X_\lambda^{7/8,-1}} + \|F\|_{A_\lambda} + \|\psi_2 F_x\|_{L_{t,\lambda}^4} + \|\psi_2 F_x\|_{\tilde{L}_{t,\lambda}^4} \right). \quad (48)$$

Proof. We estimate

$$\left\| \partial_x P_+ \left(W \partial_x P_- v \right) \right\|_{Z^{-1,s}} + \sum_{k \geq 1} \frac{1}{k!} \left\| \partial_x P_+ \left(W \partial_x P_- (F^k P_- v) \right) \right\|_{Z^{-1,s}}.$$

Again, the first term of the above inequality is estimated in ([18], Lemma 3.4) by

$$\left\| \partial_x P_+ \left(W \partial_x P_- v \right) \right\|_{Z^{-1,s}} \lesssim \|w\|_{X^{1/2,s}} \|v\|_{X^{1/2,0}} \quad (49)$$

To estimate the second term we first note that by Cauchy-Schwarz in τ ,

$$\left\| \frac{\langle \xi \rangle^s}{\langle \sigma \rangle} \mathcal{F} \left[\partial_x P_+ \left(W \partial_x P_- (F^k P_- v) \right) \right] \right\|_{L_\xi^2 L_\tau^2} \lesssim \left\| \partial_x P_+ \left(W \partial_x P_- (F^k P_- v) \right) \right\|_{X_\lambda^{-1/2+\varepsilon,s}}, \quad \varepsilon > 0.$$

On account of (34), (35), (39), (41), (46) and (47), this last term is controlled by the right-hand side of (48) except in the region B_2 with σ dominant. Moreover, in the region $\{\xi_1 \leq 1\}$, using (30) and then (9) we infer that

$$\left\| \partial_x P_+ \left(W \partial_x P_- (F^k P_- v) \right) \right\|_{X_\lambda^{-1/2+\varepsilon,s}} \lesssim \|\mathcal{F}_{t,x}^{-1}(|w|)\|_{L_{t,\lambda}^4} \|\mathcal{F}_{t,x}^{-1}(|v|)\|_{L_{t,\lambda}^4} \|\mathcal{F}_{t,x}^{-1}(|F|)\|_{L_{t,\lambda}^\infty}^k \\ \lesssim \|w\|_{X_\lambda^{1/2,0}} \|v\|_{X_\lambda^{1/2,0}} \|F\|_{A_\lambda}^k.$$

It thus remains to treat the region B_2 with $\xi_1 \geq 1$ and σ dominant. To handle with this region we proceed as in [8]. The proof is very similar to the one of Lemma 3.4 in [18].

By (42) in this region we have

$$\langle \sigma \rangle \gtrsim \langle \xi\xi_2 \rangle / k \gtrsim 1. \quad (50)$$

Therefore (48) will be proven if we show the following inequality:

$$J_k \lesssim k \|\tilde{w}\|_{L^2_{t,\lambda}} \|\tilde{v}\|_{L^2_{t,\lambda}} \|\hat{F}\|_{L^1_{\tau,\xi}}^k \quad (51)$$

with

$$J_k = \left\| \int_{C(\tau,\xi)} \frac{\langle \xi \rangle^s \xi \langle \xi_1 \rangle^{-s} \xi_1^{-1} |\widehat{w}(\tau_1, \xi_1)| |\xi - \xi_1| |\widehat{v}(\tau_2, \xi_2)| \prod_{i=3}^{k+2} |\hat{F}(\tau_i, \xi_i)|}{\langle \sigma \rangle \langle \sigma_1 \rangle^{1/2} \langle \sigma \rangle^{1/2}} \right\|_{L^2_{\xi} L^1_{\tau}} \quad (52)$$

and

$$C(\tau, \xi) = \left\{ (\tau_1, \dots, \tau_{k+1}, \xi_1, \dots, \xi_{k+1}) \in \mathbb{R}^{k+1} \times (\lambda^{-1}\mathbb{Z})^{k+1}, \right. \\ \left. (\tau, \tau_1, \dots, \tau_{k+1}, \xi, \xi_1, \dots, \xi_{k+1}) \in B_2, |\xi_1| > 1, \max_{i=1, \dots, k+2} (\langle \sigma_i \rangle) \leq \langle \sigma \rangle \right\} \quad (53)$$

• The subregion $\max(|\sigma_1|, |\sigma_2|) \geq (\xi|\xi_2|)^{\frac{1}{16}}$. We will assume that $\max(|\sigma_1|, |\sigma_2|) = |\sigma_1|$ since the other case can be treated in exactly the same way. By (50) and (43), recalling that on the domain of integration $\xi_1 \geq \max(\xi, |\xi - \xi_1|)$, we infer that

$$J_k \lesssim k \left\| \int_{C_1(\tau,\xi)} \frac{|\widehat{w}(\tau_1, \xi_1)| |\widehat{v}(\tau_2, \xi_2)| \prod_{i=3}^{k+2} |\hat{F}(\tau_i, \xi_i)|}{\langle \sigma \rangle^{1/2 + \frac{1}{128}} \langle \sigma_1 \rangle^{3/8} \langle \sigma \rangle^{1/2}} \right\|_{L^2_{\xi} L^1_{\tau}}$$

where

$$C_1(\tau, \xi) = \{(\tau_1, \xi_1) \in C(\tau, \xi), |\sigma_1| \geq (\xi|\xi_2|)^{\frac{1}{16}}\} .$$

Applying Cauchy-Schwarz in τ we obtain thanks to (9),

$$\begin{aligned} J_k &\lesssim k \left\| \int_{C_1(\tau,\xi)} \frac{|\widehat{w}(\tau_1, \xi_1)| |\widehat{v}(\tau_2, \xi_2)| \prod_{i=3}^{k+2} |\hat{F}(\tau_i, \xi_i)|}{\langle \sigma_1 \rangle^{3/8} \langle \sigma \rangle^{1/2}} \right\|_{L^2_{\xi, \tau}} \\ &\lesssim k \left\| \mathcal{F}^{-1} \left(\frac{|\widehat{w}|}{\langle \sigma \rangle^{3/8}} \right) \right\|_{L^4_{t,\lambda}} \left\| \mathcal{F}^{-1} \left(\frac{|\widehat{v}|}{\langle \sigma \rangle^{1/2}} \right) \right\|_{L^4_{t,\lambda}} \|\mathcal{F}^{-1}(|\hat{F}|)\|_{L^{\infty}_{t,\lambda}}^k \\ &\lesssim k \|\tilde{w}\|_{L^2_{t,\lambda}} \|\tilde{v}\|_{L^2_{t,\lambda}} \|\hat{F}\|_{L^1_{\tau,\xi}}^k . \end{aligned}$$

• The subregion $\max(|\sigma_1|, |\sigma_2|) \leq (\xi|\xi_2|)^{\frac{1}{16}}$. Changing the $\tau, \tau_1, \dots, \tau_{k+1}$ integrals in $\tau_1, \dots, \tau_{k+2}$ integrals in (52) and using (43) and (50), we infer that

$$J_k \lesssim k \left\| \chi_{\{\xi \geq 1\}} \int_{D(\xi)} \xi_1^{-1} \int_{\tau_1 = -\xi_1^2 + O(|\xi \xi_2|^{1/16})} \frac{|\widehat{w}(\tau_1, \xi_1)|}{\langle \tau_1 + |\xi_1| \xi_1 \rangle^{1/2}} \right\|$$

$$\int_{\tau_2 = \xi_2^2 + O(|\xi \xi_2|^{1/16})} \frac{|\widehat{v}(\tau_2, \xi_2)|}{\langle \tau_2 + |\xi_2| \xi_2 \rangle^{1/2}} \int_{\tau_3, \dots, \tau_{k+2}} \prod_{i=3}^{k+2} |\widehat{F}(\tau_i, \xi_i)| \Big\|_{L_\xi^2}$$

with

$$D(\xi) = \{(\xi_1, \dots, \xi_{k+1}) \in (\lambda^{-1}\mathbb{Z})^{k+1}, \xi_1 \geq 1, \xi - \xi_1 \leq -1/\lambda\}.$$

Applying Cauchy-Schwarz inequality in τ_1 and τ_2 and recalling that $\xi_1 \geq 1$ we get

$$J_k \lesssim k \left\| \chi_{\{\xi \geq 1\}} \int_{D(\xi)} \langle \xi_1 \rangle^{-1} (\xi |\xi_2|)^{\frac{1}{8}} K_1(\xi_1) K_2(\xi_2) \prod_{i=3}^{k+2} K(\xi_i) \right\|_{L_\xi^2}$$

where

$$K_1(\xi) = \left(\int_\tau \frac{|\widehat{w}(\tau, \xi)|^2}{\langle \tau + |\xi| \xi \rangle} \right)^{1/2}, \quad K_2(\xi) = \left(\int_\tau \frac{|\widehat{v}(\tau, \xi)|^2}{\langle \tau + |\xi| \xi \rangle} \right)^{1/2} \text{ and } K(\xi) = \int_\tau |\widehat{F}(\tau, \xi)|.$$

Therefore, by using (43) and (30), Hölder and then Cauchy-Schwarz inequalities,

$$\begin{aligned} J_k &\lesssim k \left\| \langle \xi \rangle^{-\frac{3}{4}} \int_{(\xi_3, \dots, \xi_{k+2}) \in (\lambda^{-1}\mathbb{Z})^k} \prod_{i=3}^{k+2} K(\xi_i) \int_{\xi_1 \in \lambda^{-1}\mathbb{Z}} K_1(\xi_1) K_2(\xi_2) \right\|_{L_\xi^2} \\ &\lesssim k \left\| \int_{(\xi_3, \dots, \xi_{k+2}) \in (\lambda^{-1}\mathbb{Z})^k} \prod_{i=3}^{k+2} K(\xi_i) \int_{\xi_1 \in \lambda^{-1}\mathbb{Z}} K_1(\xi_1) K_2(\xi_2) \right\|_{L_\xi^\infty} \\ &\lesssim k \int_{(\xi_3, \dots, \xi_{k+2}) \in (\lambda^{-1}\mathbb{Z})^k} \prod_{i=3}^{k+2} K(\xi_i) \left(\int_{\xi \in \lambda^{-1}\mathbb{Z}} K_1(\xi)^2 \right)^{1/2} \left(\int_{\xi \in \lambda^{-1}\mathbb{Z}} K_2(\xi)^2 \right)^{1/2} \\ &\lesssim k \| \tilde{w} \|_{X_\lambda^{-1/2,0}} \| \tilde{v} \|_{X_\lambda^{-1/2,0}} \| \widehat{F} \|_{L_{\tau,\xi}^1}^k \\ &\lesssim k \| \tilde{w} \|_{L_{t,\lambda}^2} \| \tilde{v} \|_{L_{t,\lambda}^2} \| \widehat{F} \|_{L_{\tau,\xi}^1}^k, \end{aligned} \tag{54}$$

4.2 End of the proof of (22).

We have not treat yet the third term of the right-hand side of (20). Observe that by Cauchy-Schwarz inequality in τ , Sobolev inequalities in time and Minkowski inequality,

$$\|P_0(u^2)w\|_{Z_\lambda^{-1,s}} + \|P_0(u^2)w\|_{X_\lambda^{-1/2,s}} \lesssim \|P_0(u^2)w\|_{X_\lambda^{-1/2+\varepsilon',s}} \lesssim \|P_0(u^2)w\|_{L_t^{1+\varepsilon} H_\lambda^s},$$

for some $0 < \varepsilon, \varepsilon' \ll 1$. Assuming that w is supported in time in $[-2, 2]$, by Hölder inequality in time and (9) we get

$$\|P_0(u^2)w\|_{L_t^{1+\varepsilon}H_\lambda^s} \lesssim \|J_x^s w\|_{L_{t,\lambda}^4} \|\psi_2^2 P_0(u^2)\|_{L_t^2 L_\lambda^4} \lesssim T^{1/8} \|w\|_{X_\lambda^{1/2,s}} \|\psi_2^2 P_0(u^2)\|_{L_{t,\lambda}^2},$$

where we used that $\|1\|_{L_\lambda^4} \leq \|1\|_{L_\lambda^2}$ since $\lambda \geq 1$. Hence, the following estimate holds:

$$\|P_0(u^2)w\|_{Z_\lambda^{-1,s}} + \|P_0(u^2)w\|_{X_\lambda^{-1/2,s}} \lesssim \|w\|_{X_\lambda^{1/2,s}} \|\psi_2 u\|_{L_{t,\lambda}^4}. \quad (55)$$

Now, by the Duhamel formulation of (17), for $0 \leq t \leq 1$, we have

$$\begin{aligned} w(t) &= \psi(t) \left[V(t)w(0) + 2i \int_0^t V(t-t') \partial_x P_+ \left((\psi \tilde{W}) \partial_x P_- (e^{-i\tilde{F}/2} \psi \tilde{w}) \right) \right. \\ &\quad \left. + 2i \int_0^t V(t-t') \partial_x P_+ \left[(\psi \tilde{W}) \partial_x P_- \left(e^{-i\tilde{F}/2} \partial_x P_+ (e^{i\tilde{F}/2}) \right) \right] \right. \\ &\quad \left. + \frac{i}{4} \int_0^t V(t-t') \left(P_0(\tilde{u}^2) \psi \tilde{W}_x \right) (t') dt' \right], \end{aligned} \quad (56)$$

where $\tilde{w} = P_+(\tilde{w})$ is a zero-mean value extension of w satisfying $\|\tilde{w}\|_{X_\lambda^{1/2,0}} \leq 2\|w\|_{X_{1,\lambda}^{1/2,0}}$, $\tilde{W} = P_+(\tilde{W})$ is an extension of W satisfying $\|\tilde{W}_x\|_{X_\lambda^{1/2,s}} \leq 2\|w\|_{X_{1,\lambda}^{1/2,s}}$, $\tilde{w} = \tilde{W}_x$, and \tilde{F} is a particular extension of F . To construct \tilde{F} we proceed as follows : we take \tilde{u} a zero-mean value extension of u in N_λ such that $\|\tilde{u}\|_{N_\lambda} \leq 2\|u\|_{N_{1,\lambda}}$ and define \tilde{u} by setting $Q_3 \tilde{u} = \psi Q_3 \tilde{u}$ and

$$P_3 \tilde{u} = \psi(\cdot/\lambda^2) P_3 V(t)u_0 + \frac{1}{2} P_3 \left[\int_0^t V(t-t') \partial_x (\psi \tilde{u}(t'))^2 dt' \right]. \quad (57)$$

The factor λ above will be very useful in (68) to compensate a factor λ coming from the L_λ^2 -norm of $\partial_x^{-1}u_0$. It is clear that $\tilde{u} \equiv u$ on $[0, 1]$ and $\tilde{u} = 0$ on \mathbb{R} and thus we can set $\tilde{F} = \partial_x^{-1}\tilde{u}$. We will use Lemmas 4.4-4.5 with $v = \psi \tilde{w}$. Note that since $\tilde{w} = P_+ \tilde{w}$, we have $\tilde{w} = P_- \tilde{w}$. Therefore, combining Lemmas 2.1-2.2, 4.3-4.4-4.5 and (55), we infer that for $s \geq 0$, the extension w^* of w defined by (56) satisfies

$$\begin{aligned} \|w^*\|_{Y_\lambda^s} &\lesssim \|w(0)\|_{H_\lambda^s} + \|\tilde{W}_x\|_{X_\lambda^{1/2,s}} e^{C\|\tilde{F}\|_{A_\lambda}} \left[\|\tilde{u}\|_{L_{t,\lambda}^4}^2 + \|\tilde{w}\|_{X_\lambda^{1/2,0}} \right. \\ &\quad \left. \left(\|P_3 \tilde{F}\|_{\dot{X}_\lambda^{1,0}} + \|P_{>3} \tilde{F}_x\|_{X_\lambda^{7/8,-1}} + \|\tilde{F}\|_{A_\lambda} + \|\psi_2 \tilde{F}_x\|_{L_{t,\lambda}^4} + \|\psi_2 \tilde{F}_x\|_{\tilde{L}_{t,\lambda}^4} \right) \right] \\ &\lesssim \|w(0)\|_{H_\lambda^s} + \|w\|_{X_{1,\lambda}^{1/2,s}} \left(\|u\|_{N_{1,\lambda}}^2 + \|w\|_{X_{1,\lambda}^{1/2,0}} \right) e^{C(\|\partial_x^{-1}u_0\|_{L_\lambda^1} + \|u\|_{N_{1,\lambda}}^2)}. \end{aligned} \quad (58)$$

where in the last step we used Lemma 4.6 below to estimate $\|P_3\tilde{F}\|_{\dot{X}_\lambda^{1,0}} + \|P_3\tilde{F}\|_{A_\lambda}$ and that, by Cauchy-Schwarz in ξ ,

$$\|P_{>3}\tilde{F}\|_{A_\lambda} \lesssim \|\widehat{\tilde{F}_x}\|_{L_\xi^2 L_\tau^1} \quad . \quad (59)$$

Lemma 4.6 *Let $\tilde{u} \in N_{1,\lambda}$ and let $P_3\tilde{u}$ be defined as in (57). Then $P_3\tilde{F} = P_3\partial_x^{-1}\tilde{u}$ satisfies :*

$$\|P_3\tilde{F}\|_{A_\lambda} \lesssim \|\widehat{\partial_x^{-1}u_0}\|_{L_\xi^1} + \|\tilde{u}\|_{N_\lambda}^2 \quad , \quad (60)$$

$$\|P_3\tilde{F}\|_{\dot{X}_\lambda^{1,0}} \lesssim \|u_0\|_{L_\lambda^2} + \|\tilde{u}\|_{N_\lambda}^2 \quad (61)$$

and

$$\|P_3\tilde{F}_x\|_{N_\lambda} \lesssim \|u_0\|_{L_\lambda^2} + \|\tilde{u}\|_{N_\lambda}^2 \quad . \quad (62)$$

Moreover, $\forall 0 < \alpha < 3$,

$$\|\psi_2 P_3 Q_\alpha \tilde{F}\|_{X_\lambda^{7/8,0}} \lesssim \frac{1}{\alpha} \|u_0\|_{L_\lambda^2} + \|\tilde{u}\|_{N_\lambda}^2 \quad . \quad (63)$$

We postponed the proof of this lemma to the end of this section.

On the other hand, obviously,

$$\|P_{>1}w^*\|_{X_\lambda^{1,-1}} \lesssim \|P_{>1}\partial_x^{-1}w^*\|_{X_\lambda^{1,0}}$$

and from (56) we deduce that $w^* = \psi w^{**}$ where w^{**} satisfies (20) with W , w and F respectively replaced by $\psi\tilde{W}$, $\psi\tilde{w}$ and \tilde{F} . Therefore using Lemma 4.2 and expanding the exponential function we infer that

$$\begin{aligned} \|P_{>1}w^*\|_{X_\lambda^{1,-1}} &\lesssim \|w^*\|_{L_t^\infty L_\lambda^2} + \|\partial_t(P_{>1}\partial_x^{-1}w^{**}) + \mathcal{H}\partial_x^2(P_{>1}\partial_x^{-1}w^{**})\|_{L_{t,\lambda}^2} \\ &\lesssim \|w(0)\|_{L_\lambda^2} + \|\tilde{w}\|_{L_{t,\lambda}^4} \left(\|\tilde{w}\|_{L_{t,\lambda}^4} + \|\psi_2\tilde{F}_x\|_{L_{t,\lambda}^4} + \|\psi_2\tilde{F}_x\|_{L_{t,\lambda}^4}^2 \right) e^{C\|\tilde{F}\|_{A_\lambda}} \\ &\lesssim \|w(0)\|_{L_\lambda^2} + \|w\|_{X_{1,\lambda}^{1/2,s}} \left(\|u\|_{N_{1,\lambda}} + \|w\|_{X_{1,\lambda}^{1/2,0}} \right) e^{C(\|\widehat{\partial_x^{-1}u_0}\|_{L^1} + \|u\|_{N_{1,\lambda}}^2)} \end{aligned} \quad (64)$$

Finally, using Lemma 4.1 we infer that for $0 \leq s \leq 1/2$,

$$\begin{aligned} \|w(0)\|_{H_\lambda^s} &= \|\partial_x P_+ e^{-i\partial_x u_0}\|_{H_\lambda^s} = \frac{1}{2} \|P_+(u_0 e^{-i\partial_x u_0})\|_{H_\lambda^s} \\ &\lesssim \sum_{k \geq 0} \frac{1}{k!} \|u_0 (\partial_x^{-1} u_0)^k\|_{H_\lambda^s} \\ &\lesssim \|u_0\|_{H_\lambda^s} (1 + \|u_0\|_{L_\lambda^2}) e^{\|\partial_x^{-1} u_0\|_{L_\lambda^\infty}} \end{aligned} \quad (65)$$

which ends the proof of (22).

Proof of Lemma 4.6. From (57), $P_3\tilde{F} = P_3\tilde{F}^1 + P_3\tilde{F}^2$ where $P_3\tilde{F}^1 = \psi(t/\lambda^2)V(t)P_3\partial_x^{-1}u_0$ and

$$P_3\tilde{F}_t^2 + \mathcal{H}\partial_x^2 P_3\tilde{F}^2 = P_3\left[(\psi\tilde{u})^2/2 - P_0((\psi\tilde{u})^2)/2\right] . \quad (66)$$

But (66) leads to

$$\|P_3\tilde{F}^2\|_{\dot{X}_1^{1,0}} \lesssim \|\psi\tilde{u}\|_{L_\lambda^4}^2 \lesssim \|\tilde{u}\|_{N_\lambda}^2 \quad (67)$$

and by the definition of $P_3\tilde{F}^1$,

$$\begin{aligned} \|P_3\tilde{F}^1\|_{\dot{X}_1^{1,0}} &= \left\| \partial_t \left(\psi(\cdot/\lambda^2) P_3 \partial_x^{-1} u_0 \right) \right\|_{L_{t,\lambda}^2} \\ &\lesssim \lambda^{-2} \|\psi_t(\cdot/\lambda^2)\|_{L_t^2} \|\partial_x^{-1} u_0\|_{L_\lambda^2} \\ &\lesssim \lambda^{-1} \lambda \|u_0\|_{L_\lambda^2} \lesssim \|u_0\|_{L_\lambda^2} . \end{aligned} \quad (68)$$

Moreover, from (66) and Lemma 2.1-2.2 we deduce that

$$\|\widehat{P_3\tilde{F}}\|_{L_{\tau,\xi}^1} \lesssim \|\widehat{P_3F_0}\|_{L_\xi^1} + \left\| \chi_{\{|\xi| \leq 3\}} \frac{\widehat{\psi\tilde{u}} * \widehat{\psi\tilde{u}}}{\langle \sigma \rangle} \right\|_{L_{\tau,\xi}^1} + \left\| \frac{\mathcal{F}_t(P_0((\psi\tilde{u})^2))}{\langle \sigma \rangle} \right\|_{L_\tau^1}$$

Applying Cauchy-Schwarz inequality in τ and ξ , it follows that

$$\begin{aligned} \|\widehat{P_3\tilde{F}}\|_{L_{\tau,\xi}^1} &\lesssim \|\widehat{P_3F_0}\|_{L_\xi^1} + \left\| \widehat{\psi\tilde{u}} * \widehat{\psi\tilde{u}} \right\|_{L_{\tau,\xi}^2} + \|P_0((\psi\tilde{u})^2)\|_{L_t^2} \\ &\lesssim \|\widehat{P_3F_0}\|_{L_\xi^1} + \|\psi\tilde{u}\|_{L_{t,\lambda}^4}^2 \\ &\lesssim \|\widehat{P_3F_0}\|_{L_\xi^1} + \|\tilde{u}\|_{N_\lambda}^2 . \end{aligned}$$

To get (63) we notice that by classical linear estimates in Bourgain spaces (cf. [10]) and (57) we have

$$\|\psi_2 P_3 Q_\alpha \tilde{F}\|_{X_\lambda^{7/8,0}} \lesssim \|Q_\alpha \partial_x^{-1} u_0\|_{L_\lambda^2} + \|(\psi\tilde{u})^2\|_{X_\lambda^{-1/8,0}} \lesssim \frac{1}{\alpha} \|u_0\|_{L_\lambda^2} + \|\psi\tilde{u}\|_{L_{t,\lambda}^4}^2 .$$

It remains to get the estimate (62) on $\|P_3\tilde{F}_x\|_{N_\lambda}$. But this is straightforward by combining (57), Lemmas 2.1-2.2 for the $Z_\lambda^{0,0}$ -norm and by writing $\|\chi_{[-4,4]}(t)P_3\tilde{F}_x\|_{L_{t,\lambda}^4 \cap \tilde{L}_{t,\lambda}^4} \lesssim \|\chi_{[-4,4]}(t)P_3\tilde{F}_x\|_{L_t^\infty L_\lambda^2}$ and then using the unitarity of $V(t)$ in L_λ^2 (see (72) below) for the $L_{t,\lambda}^4$ and the $\tilde{L}_{t,\lambda}^4$ -norms.

5 Proof of the estimates on u

In this section we prove estimates (23) and (24) of Proposition 3.1. We will need the following lemma, proven in the appendix, which enables to treat the multiplication with the gauge function $e^{-iF/2}$ in $\tilde{L}_{t,\lambda}^4$.

Lemma 5.1 *Let $z \in L_t^\infty H_\lambda^1$ and let $v \in L_{t,\lambda}^4 \cap \tilde{L}_{t,\lambda}^4$ then*

$$\|zv\|_{\tilde{L}_{t,\lambda}^4} \lesssim (\|z\|_{L_{t,\lambda}^\infty} + \|z_x\|_{L_t^\infty L_\lambda^2})(\|v\|_{L_{t,\lambda}^4} + \|v\|_{\tilde{L}_{t,\lambda}^4}) \quad . \quad (69)$$

5.1 Proof of (24)

Since u is real-valued, it holds

$$\|J_x^s u\|_{L_1^p L_\lambda^q} \lesssim \|P_1 u\|_{L_1^p L_\lambda^q} + \|D_x^s P_{>1} u\|_{L_1^p L_\lambda^q} \quad .$$

To estimate the high modes part, we use (21) where we expand the exponential function. Hence, we write

$$\begin{aligned} \|D_x^s P_{>1} u\|_{L_1^\infty L_\lambda^2} &\lesssim \sum_{k \geq 0} \frac{1}{k!} \|D_x^s (F^k w)\|_{L_1^\infty L_\lambda^2} \\ &+ \sum_{k \geq 1} \sum_{l \geq 1} \frac{1}{k! l!} \left\| D_x^s P_{>1} \left(P_{>1} (F^k) \partial_x P_- (F^l) \right) \right\|_{L_1^\infty L_\lambda^2} \quad . \quad (70) \end{aligned}$$

From (70), Lemmas 4.1 and 4.2, Sobolev inequalities and (9), we infer that for $0 \leq s \leq 1/2$,

$$\begin{aligned} \|D_x^s P_{>1} u\|_{L_1^\infty L_\lambda^2} &\lesssim \sum_{k \geq 0} \frac{1}{k!} (\|F^k\|_{L_{1,\lambda}^\infty} + \|\partial_x (F^k)\|_{L_1^\infty L_\lambda^2}) \|J_x^s w\|_{L_1^\infty L_\lambda^2} \\ &+ \sum_{k \geq 1} \sum_{l \geq 1} \frac{1}{k! l!} \|D_x^{5s/4} P_{>1} (F^k)\|_{L_1^\infty L_\lambda^{4/s}} \|D_x^{1-s/4} P_- (F^l)\|_{L_1^\infty L_\lambda^{\frac{4}{2-s}}} \\ &\lesssim \sum_{k \geq 0} \frac{1}{k!} (\|F\|_{L_{1,\lambda}^\infty}^k + k \|F\|_{L_{1,\lambda}^\infty}^{k-1} \|F_x\|_{L_1^\infty L_\lambda^2}) \|w\|_{Y_{1,\lambda}^s} \\ &+ \sum_{k \geq 1} \sum_{l \geq 1} \frac{1}{k! l!} \|D_x^{s+1/2} P_{>1} (F^k)\|_{L_1^\infty L_\lambda^2} \|\partial_x P_- (F^l)\|_{L_1^\infty L_\lambda^2} \quad (71) \end{aligned}$$

with

$$\left\| D_x^{s+1/2} P_{>1} (F^k) \right\|_{L_1^\infty L_\lambda^2} \lesssim \|\partial_x (F^k)\|_{L_1^\infty L_\lambda^2} \lesssim k \|F\|_{L_{1,\lambda}^\infty}^{k-1} \|F_x\|_{L_1^\infty L_\lambda^2}$$

and

$$\|\partial_x(F^l)\|_{L_1^\infty L_\lambda^2} \lesssim l \|F\|_{L_{1,\lambda}^{\infty}}^{l-1} \|F_x\|_{L_1^\infty L_\lambda^2}.$$

On the other hand, by the Duhamel formulation of the equation, the unitarity of $V(t)$ in L_λ^2 , the continuity of $\partial_x P_1$ in L_λ^2 and Sobolev inequalities, we get

$$\|P_1 u\|_{L_1^\infty L_\lambda^2} \lesssim \|u_0\|_{L_\lambda^2} + \|u^2\|_{L_1^1 L_\lambda^2} \lesssim \|u_0\|_{L_\lambda^2} + \|u\|_{L_{1,\lambda}^4}^2. \quad (72)$$

This completes the proof of (24).

5.2 Proof of (23)

Remark 5.1 *It would considerably simplify the estimates on u if we were able to prove that there exists $C > 0$ such that for any $v \in N_{1,\lambda}$ there exists an extension \tilde{v} of v satisfying :*

$$\|\tilde{v}\|_{X_\lambda^{7/8,-1}} \leq C \|v\|_{X_{1,\lambda}^{7/8,-1}}, \quad \|\tilde{v}\|_{Z_\lambda^{0,0}} \leq C \|v\|_{Z_{1,\lambda}^{0,0}} \quad \text{and} \quad \|\tilde{v}\|_{L_{t,\lambda}^4 \cap \tilde{L}_{t,\lambda}^4} \leq C \|v\|_{L_{1,\lambda}^4 \cap \tilde{L}_{1,\lambda}^4}.$$

Indeed, we could then take different extensions of u according to the part of the N_λ -norm we want to estimate. Note, in particular, that taking the extension $P_{>3}\tilde{u}$ of $P_{>3}u$ defined by

$$P_{>3}\tilde{u} = \psi \left[V(t)P_{>3}u_0 + \frac{1}{2} \int_0^t V(t-t')P_{>3}\partial_x(\psi u)^2(t') dt' \right]$$

we directly get

$$\|P_{>3}u\|_{X_{1,\lambda}^{7/8,-1}} \lesssim \|u\|_{L_{1,\lambda}^4}^2 + \|u\|_{L_1^\infty L_\lambda^2} \lesssim \|u\|_{N_{1,\lambda}}^2 + \|u_0\|_{L_\lambda^2}.$$

We start by constructing our extension F^* of F . To construct the high modes part, we first need some how to inverse the map $F \rightarrow W$. From (16) we infer that

$$P_{>1}W = P_{>1}(e^{-iF/2}) = e^{-iF/2} - P_{\leq 1}(e^{-iF/2})$$

By decomposing F in $Q_1F + P_1F$, we obtain

$$e^{-iQ_1F/2} = e^{iP_1F/2} \left(P_{>1}W - P_{\leq 1}(e^{-iF/2}) \right)$$

and thus

$$\begin{aligned} P_{>3}F &= 2iP_{>3}\left[e^{iP_1F/2}\left(P_{>1}W - P_{\leq 1}(e^{-iF/2})\right)\right] \\ &\quad - 2iP_{>3}\left(e^{-iQ_1F/2} + iQ_1F/2\right) . \end{aligned} \quad (73)$$

Now, let \tilde{W} be an extension of W such that $\|\tilde{W}_x\|_{M_\lambda^0} \leq 2\|W_x\|_{M_{1,\lambda}^0}$ and \tilde{F} be the extension of F defined in the last section. We set

$$\begin{aligned} P_{>3}F^* &= 2i\psi P_{>3}\left[e^{iP_1F/2}\left(P_{>1}(\psi\tilde{W}) - P_{\leq 1}(e^{-iF/2})\right)\right] \\ &\quad - 2i\psi P_{>3}\left(e^{-iQ_1F/2} + iQ_1F/2\right) , \end{aligned} \quad (74)$$

$P_{<-3}F^* = \overline{P_{>3}F^*}$ and $P_3F^* = P_3\tilde{F}$. It is clear that by construction $F^* \equiv F$ on $[0, 1]$. Note that by (62), in Lemma 4.6, we already have an estimate on the low-modes part P_3F^* . Moreover, combining estimate (60) with (59), we infer that

$$\|F^*\|_{A_\lambda} \lesssim \widehat{\|\partial_x^{-1}u_0\|_{L_\xi^1}} + \|u\|_{N_{1,\lambda}}^2 + \|u\|_{N_{1,\lambda}} . \quad (75)$$

To estimate the high-modes part, for convenience, we drop the \sim in the right-hand side of (74). In the remaining of this section we assume that W is supported in time in $[-2, 2]$.

5.2.1 Estimate on the $L_{t,\lambda}^4$ and $\tilde{L}_{t,\lambda}^4$ -norms

Differentiating (74) with respect to x and expanding the exponential function, we get

$$\begin{aligned} \|P_{>3}F^*\|_{L_{t,\lambda}^4} &\lesssim \sum_{k \geq 0} \frac{1}{k!} \left(k \left\| (P_1F_x)(P_1F)^{k-1}P_{>1}W \right\|_{L_{t,\lambda}^4} + \left\| (P_1F)^k P_{>1}W_x \right\|_{L_{t,\lambda}^4} \right) \\ &+ \sum_{k \geq 0} \sum_{l \geq 0} \frac{1}{k!} \frac{1}{l!} \left[k \left\| \psi P_{>3} \left((P_1F_x)(P_1F)^{k-1}P_{\leq 1}(F^l) \right) \right\|_{L_{t,\lambda}^4} + l \left\| \psi P_{>3} \left((P_1F)^k P_{\leq 1}(F^{l-1}F_x) \right) \right\|_{L_{t,\lambda}^4} \right] \\ &\quad + \sum_{k \geq 2} \frac{1}{(k-1)!} \left\| \psi (Q_1F)^{k-1} Q_1F_x \right\|_{L_{t,\lambda}^4} \end{aligned}$$

We notice that by the frequency projections,

$$P_{>3} \left((P_1F_x)(P_1F)^{k-1}P_{\leq 1}(F^l) \right) \text{ and } P_{>3} \left((P_1F)^k P_{\leq 1}(F^{l-1}F_x) \right) . \quad (76)$$

vanish for $k \leq 2$. Moreover, decomposing P_1F as $P_1Q_{\frac{1}{k-1}}F + P_{\frac{1}{k-1}}F$ we infer that for $k \geq 3$ the two terms appearing in (76) are respectively equal

to

$$P_{>3} \left[(P_1 F_x) (P_1 Q_{\frac{1}{k-1}} F) P_{\leq 1} (F^l) G \right] \text{ and } P_{>3} \left[(P_1 F) (P_1 Q_{\frac{1}{k-1}} F) P_{\leq 1} (F^{l-1} F_x) G \right]$$

with

$$G = \sum_{q=1}^{k-1} C_{k-1}^q (P_1 Q_{\frac{1}{k-1}} F)^{q-1} (P_{\frac{1}{k-1}} F)^{k-1-q} .$$

Note that G can be also written as

$$G = \sum_{j=0}^{k-2} \frac{C_{k-1}^{j+1}}{C_{k-2}^j} C_{k-2}^j (P_1 Q_{\frac{1}{k-1}} F)^j (P_{\frac{1}{k-1}} F)^{k-2-j}$$

and thus it is not too hard to see that

$$\|G\|_{L_{t,\lambda}^\infty} \lesssim \|G\|_{A_\lambda} \lesssim (k-1) \|F\|_{A_\lambda}^{k-2} . \quad (77)$$

Therefore, using that, by Sobolev inequalities,

$$\|Q_{\frac{1}{k-1}} P_1 F\|_{L_{t,\lambda}^\infty} \lesssim (k-1) \|F_x\|_{L_t^\infty L_\lambda^2} ,$$

we infer that

$$\begin{aligned} \|P_{>3} F^*\|_{L_{t,\lambda}^4} &\lesssim \left(\|F_x\|_{L_t^\infty L_\lambda^2} \|W_x\|_{L_{t,\lambda}^4} + \|W_x\|_{L_{t,\lambda}^4} + \|F_x\|_{L_t^\infty L_\lambda^2} \|\psi_2 F_x\|_{L_{t,\lambda}^4} \right) e^{2\|F\|_{A_\lambda}} \\ &\lesssim \left(\|w\|_{M_{1,\lambda}^0} (1 + \|u\|_{N_{1,\lambda}}) + \|u\|_{N_{1,\lambda}}^2 \right) e^{\tilde{K}} , \end{aligned} \quad (78)$$

where \tilde{K} is defined as in (25). In the same way, using Lemma 5.1 and the embedding $X_\lambda^{1/2,0} \hookrightarrow \tilde{L}_{2,\lambda}^4$ (see (40)), we infer that

$$\begin{aligned} \|P_{>3} F^*\|_{\tilde{L}_{t,\lambda}^4} &\lesssim \left[(\|F_x\|_{L_t^\infty L_\lambda^2} + 1) (\|W_x\|_{L_{t,\lambda}^4} + \|W_x\|_{\tilde{L}_{t,\lambda}^4}) \right. \\ &\quad \left. + \|F_x\|_{L_t^\infty L_\lambda^2} (\|\psi_2 F_x\|_{\tilde{L}_{t,\lambda}^4} + \|\psi_2 F_x\|_{L_{t,\lambda}^4}) \right] e^{2\|F\|_{L_{t,\lambda}^\infty}} \\ &\lesssim \left(\|w\|_{M_{1,\lambda}^0} (1 + \|u\|_{N_{1,\lambda}}) + \|u\|_{N_{1,\lambda}}^2 \right) e^{\tilde{K}} . \end{aligned} \quad (79)$$

5.2.2 Estimate on the $Z_\lambda^{0,0}$ -norm

Now, using again the frequency projections and that A_λ is clearly an algebra, we deduce from (74) and (77) that

$$\|P_{>3} F_x^*\|_{Z_\lambda^{0,0}} \lesssim \sum_{k \geq 0} \frac{1}{k!} \|P_1 F\|_{A_\lambda}^{k-1} \left(k \|P_1 F_x\|_{Z_\lambda^{0,0}} \|P_{>1} W\|_{A_\lambda} + \|P_1 F\|_{A_\lambda} \|W_x\|_{Z_\lambda^{0,0}} \right)$$

$$\begin{aligned}
& + \sum_{k \geq 3} \sum_{l \geq 0} \frac{k}{k! l!} (k+l) \|Q_{\frac{1}{k-1}} P_1 F\|_{A_\lambda} \|F_x\|_{Z_\lambda^{0,0}} \|F\|_{A_\lambda}^{k+l-2} \\
& + \sum_{k \geq 2} \frac{k}{k!} \|F_x\|_{Z_\lambda^{0,0}} \|Q_1 F\|_{A_\lambda}^{k-1} \quad .
\end{aligned}$$

Using that, by Cauchy-Schwarz in ξ ,

$$\|P_{>1} W\|_{A_\lambda} \lesssim \|W_x\|_{Z_\lambda^{0,0}} \quad \text{and} \quad \|Q_{\frac{1}{k-1}} \widehat{P_1 F}\|_{A_\lambda} \lesssim (k-1) \|F_x\|_{Z_\lambda^{0,0}}$$

we infer that

$$\begin{aligned}
\|P_{>3} F_x^*\|_{Z_\lambda^{0,0}} & \lesssim \left(\|F_x\|_{N_\lambda} \|W_x\|_{Z_\lambda^{0,0}} + \|W_x\|_{Z_\lambda^{0,0}} + \|F_x\|_{N_\lambda}^2 \right) e^{2\|F\|_{L_{t,\lambda}^\infty}} \\
& \lesssim \left(\|w\|_{M_{1,\lambda}^0} (1 + \|u\|_{N_{1,\lambda}}) + \|u\|_{N_{1,\lambda}}^2 \right) e^{\tilde{K}} \quad . \quad (80)
\end{aligned}$$

5.2.3 Estimate on the $X_\lambda^{7/8,-1}$ -norm

It remains to estimate the $X_\lambda^{7/8,-1}$ -norm of $P_{>3} F_x^*$. Note that obviously

$$\|P_{>3} F_x^*\|_{X_\lambda^{7/8,-1}} \sim \|P_{>3} F_x^*\|_{X_\lambda^{7/8,0}}$$

From (74) we infer that

$$\begin{aligned}
\|P_{>3} F_x^*\|_{X_\lambda^{7/8,0}} & \lesssim \sum_{k \geq 0} \frac{1}{k!} \left\| P_{>3} \left((P_1 F)^k P_{>1} W \right) \right\|_{X_\lambda^{7/8,0}} \\
& + \sum_{k \geq 3} \sum_{l \geq 0} \frac{1}{k! l!} \left\| \psi P_{>3} \left((P_1 F)^k P_{\leq 1} (F^l) \right) \right\|_{X_\lambda^{7/8,0}} \\
& + \sum_{k \geq 2} \frac{1}{k!} \left\| \psi P_3 \left(Q_1 F \right)^k \right\|_{X_\lambda^{7/8,0}} \\
& = \sum_{k \geq 0} \frac{1}{k!} I_k + \sum_{k \geq 3} \sum_{l \geq 0} \frac{1}{k! l!} J_{k,l} + \sum_{k \geq 2} \frac{1}{k!} L_k \quad .
\end{aligned}$$

Let us estimate I_k , $J_{k,l}$ and L_k , one by one.

i) Estimate on I_k . First note that for $k = 0$, we have obviously

$$I_0 \lesssim \|P_{>1} W\|_{X_\lambda^{7/8,0}} \lesssim \|w\|_{X_\lambda^{7/8,-1}} \lesssim \|w\|_{M_{1,\lambda}^0} \quad . \quad (81)$$

Now, for $k \geq 1$,

$$I_k = \left\| \chi_{\{\xi \geq 3\}} \langle \sigma \rangle^{7/8} \int_{\mathbb{R}^k \times (\lambda^{-1} \mathbb{Z})^k} \widehat{P_1 F}(\tau_1, \xi_1) \dots \widehat{P_1 F}(\tau_k, \xi_k) \widehat{P_{>1} W}(\tau_{k+1}, \xi_{k+1}) \right\|_{L_{\tau,\xi}^2}$$

where $\sigma = \tau + \xi|\xi|$ and $(\sum_{i=1}^{k+1} \tau_i, \sum_{i=1}^{k+1} \xi_i) = (\tau, \xi)$.

We divide $\mathbb{R}^{k+1} \times (\lambda^{-1}\mathbb{Z})^{k+1}$ in different regions.

- The region $|\sigma| \leq 2^{10}k$. In this region, clearly,

$$I_k \lesssim k \|P_{>1}W\|_{L_t^\infty L_\lambda^2} \|F\|_{A_\lambda}^k \lesssim k \|w\|_{M_{1,\lambda}^0} \|F\|_{A_\lambda}^k . \quad (82)$$

- The region $\{2^4k |\tau_{k+1} + \xi_{k+1}|\xi_{k+1}| \geq |\sigma| \text{ and } |\sigma| > 2^{10}k\}$. In this region it is easy to see that

$$I_k \lesssim k \|P_{>1}W\|_{X_\lambda^{7/8,0}} \|F\|_{A_\lambda}^k \lesssim k \|Q_1w\|_{X_\lambda^{7/8,-1}} \|F\|_{A_\lambda}^k \lesssim k \|w\|_{M_{1,\lambda}^0} \|F\|_{A_\lambda}^k . \quad (83)$$

- The region $\{\exists i \in \{1, \dots, k\}, \quad 2^4k |\tau_i + \xi_i|\xi_i| \geq \langle \sigma \rangle \text{ and } |\sigma| > 2^{10}k\}$. Then we have

$$\begin{aligned} I_k &\lesssim k \|P_1F\|_{\dot{X}_\lambda^{7/8,0}} \|P_1F\|_{A_\lambda}^{k-1} \|P_{>1}W\|_{A_\lambda} \\ &\lesssim k \|P_1F\|_{\dot{X}_\lambda^{1,0}} \|P_1F\|_{A_\lambda}^{k-1} \|P_{>1}W_x\|_{Z_\lambda^{0,0}} \\ &\lesssim k \|w\|_{M_{1,\lambda}^0} \|P_1F\|_{\dot{X}_\lambda^{1,0}} \|P_1F\|_{A_\lambda}^{k-1} \\ &\lesssim k \|w\|_{M_{1,\lambda}^0} (\|u\|_{N_{1,\lambda}} + \|u\|_{N_{1,\lambda}}^2) \|F\|_{A_\lambda}^{k-1} \end{aligned} \quad (84)$$

where we used (61) in the last step.

- The region $\{|\sigma| \geq 2^4k \max_{i=1, \dots, k+1} |\tau_i + \xi_i|\xi_i| \text{ and } |\sigma| > 2^{10}k\}$. In this region, since $\xi \geq 0$, we have

$$\langle \sigma \rangle \leq 2|\sigma| \leq 4 \left| \sigma - \sum_{i=1}^{k+1} (\tau_i - \xi_i|\xi_i|) \right| = \left| \left(\sum_{i=1}^{k+1} \xi_i \right)^2 - \sum_{i=1}^{k+1} \xi_i|\xi_i| \right| . \quad (85)$$

Let us denote by $|\xi_{i_1}| = \max |\xi_i|$ and $|\xi_{i_2}| = \max_{i \neq i_1} |\xi_i|$. We claim that (85) implies

$$\langle \sigma \rangle \leq 2^5k^2 |\xi_{i_1}| |\xi_{i_2}| . \quad (86)$$

Indeed, either $2k|\xi_{i_2}| \geq |\xi_{i_1}|$ and

$$\langle \sigma \rangle \leq 4 \left(\left(\sum_{i=1}^{k+1} |\xi_i| \right)^2 + \sum_{i=1}^{k+1} |\xi_i|^2 \right) \leq 2^5k^2 |\xi_{i_1}| |\xi_{i_2}|$$

or $|\xi_{i_1}| \geq 2k|\xi_{i_2}|$ and then ξ and ξ_{i_1} have the same sign so that

$$\langle \sigma \rangle \leq 4 \left(\sum_{i \neq i_1} |\xi_i|^2 + \left(\sum_{i \neq i_1} |\xi_i| \right)^2 + 2|\xi_{i_1}| \sum_{i \neq i_1} |\xi_i| \right) \leq 2^4k^2 |\xi_{i_1}| |\xi_{i_2}| .$$

From (86), we infer that in this region,

$$\begin{aligned}
I_k &\lesssim k \|P_1 F_x\|_{A_\lambda} \|P_{>1} W_x\|_{L^2_{t,\lambda}} \|F\|_{A_\lambda}^{k-1} + k(k-1) \|P_1 F_x\|_{A_\lambda}^2 \|P_{>1} W\|_{L^2_{t,\lambda}} \|F\|_{A_\lambda}^{k-2} \\
&\lesssim k^2 \|P_{>1} W_x\|_{L^2_{t,\lambda}} \|F\|_{A_\lambda}^{k-1} \\
&\lesssim k^2 \|w\|_{M^0_{1,\lambda}} \|F\|_{A_\lambda}^{k-1} .
\end{aligned} \tag{87}$$

ii) Estimate on $J_{k,l}$. Proceeding as in the treatment of the terms in (76), we can write $J_{k,l}$ as

$$J_{k,l} = \left\| P_{>3} \left((\psi_2 P_1 Q_{\frac{2}{k-1}} F)^2 P_{\leq 1} (F^l) G \right) \right\|_{X_\lambda^{7/8,0}}$$

where

$$G = \sum_{q=2}^k C_k^q (P_1 Q_{\frac{2}{k-1}} F)^{q-2} (P_{\frac{2}{k-1}} F)^{k-q} = \sum_{j=0}^{k-2} \frac{C_k^{j+2}}{C_{k-2}^j} C_{k-2}^j (P_1 Q_{\frac{2}{k-1}} F)^j (P_{\frac{2}{k-1}} F)^{k-2-j} .$$

Clearly

$$|\hat{G}| \lesssim k(k-1) |\widehat{P_1 F}| * \dots * |\widehat{P_1 F}| \tag{88}$$

and thus

$$\|G\|_{A_\lambda} \lesssim k^2 \|P_1 F\|_{A_\lambda}^{k-2} . \tag{89}$$

Note first that we can assume that $|\sigma| \geq 2^{10}(k+l)$ since otherwise obviously,

$$J_{k,l} \lesssim k^2 \|\psi_2 Q_{\frac{2}{k-1}} P_1 F\|_{L^4_{t,\lambda}}^2 \|F\|_{A_\lambda}^{k+l-2} \lesssim k^4 \|\psi_2 F_x\|_{L^4_{t,\lambda}}^2 \|F\|_{A_\lambda}^{k+l-2} \lesssim k^4 \|u\|_{N_{1,\lambda}}^2 \|F\|_{A_\lambda}^{k+l-2} \tag{90}$$

We have thus to estimate

$$\tilde{J}_{k,l} = \left\| \chi_{\{\xi \geq 3\}} \chi_{\{|\sigma| \geq 2^{10}k\}} \langle \sigma \rangle^{7/8} \mathcal{F}_{t,x} \left((\psi_2 Q_{\frac{2}{k-1}} P_1 F)^2 P_{\leq 1} (F^l) G \right) (\tau, \xi) \right\|_{L^2_{\tau,\xi}}$$

where $\sigma = \tau + \xi|\xi|$.

As in Lemma 4.4, one of the difficulties is that we do not know if $\mathcal{F}_{t,x}^{-1}(|\hat{F}|)$ belongs to $L^4_{t,\lambda}$. Using again the Littlewood-Paley decomposition it can be seen that for $l \geq 2$,

$$F^l = \sum_{i_1 \geq i_2 \geq 0} \Delta_{i_1}(F) \Delta_{i_2}(F) \sum_{0 \leq i_3, \dots, i_l \leq i_2} n(i_1, \dots, i_l) \prod_{j=3}^l \Delta_{i_j}(F) , \tag{91}$$

where $n(i_1, \dots, i_l)$ is an integer belonging to $\{1, \dots, l(l-1)\}$ (Note for instance that $n(i_1, \dots, i_l) = 1$ for $i_1 = \dots = i_l$ and $n(i_1, \dots, i_l) = l(l-1)$ for $i_1 \neq \dots \neq i_l$). We set

$$H_{j,q,l} = \Delta_j(F) \Delta_q(F) \sum_{0 \leq i_3, \dots, i_l \leq q} n(j, q, i_3, \dots, i_l) \prod_{m=3}^l \Delta_{i_m}(F) \quad .$$

It is clear that for $l \geq 2$,

$$\begin{aligned} \tilde{J}_{k,l} &\lesssim \sum_{j \geq q \geq 1} \left\| \chi_{\left\{ \begin{array}{l} \xi \geq 3 \\ |\sigma| \geq 2^{10k} \end{array} \right\}} \langle \sigma \rangle^{7/8} \mathcal{F}_{t,x} \left((\psi_2 Q_{\frac{2}{k-1}} P_1 F)^2 G P_{\leq 1}(H_{j,q,l}) \right) \right\|_{L_{\tau,\xi}^2} \\ &\quad + l \left\| \chi_{\left\{ \begin{array}{l} \xi \geq 3 \\ |\sigma| \geq 2^{10k} \end{array} \right\}} \langle \sigma \rangle^{7/8} \sum_{j \geq 0} \mathcal{F}_{t,x} \left((\psi_2 Q_{\frac{2}{k-1}} P_1 F)^2 G P_{\leq 1} \left(\Delta_j(F) \Delta_0(F)^{l-1} \right) \right) \right\|_{L_{\tau,\xi}^2} \\ &= \Lambda_{k,l} + \Gamma_{k,l} \quad . \end{aligned}$$

Let us write $\Lambda_{k,l}$ as the sum of two terms :

$$\begin{aligned} \Lambda_{k,l} &= \sum_{j \geq q \geq 1} \left\| \chi_{\{\xi \geq 3\}} \chi_{\{2^{10(k+l)} \leq |\sigma| < (k+l)^2 2^{8+j+q}\}} \langle \sigma \rangle^{7/8} \mathcal{F}_{t,x} \left((\psi_2 Q_{\frac{2}{k-1}} P_1 F)^2 G P_{\leq 1}(H_{j,q,l}) \right) \right\|_{L_{\tau,\xi}^2} \\ &\quad + \sum_{j \geq q \geq 1} \left\| \chi_{\left\{ \begin{array}{l} \xi \geq 3 \\ |\sigma| \geq \max(2^{10(k+l)}, (k+l)^2 2^{8+j+q}) \end{array} \right\}} \langle \sigma \rangle^{7/8} \mathcal{F}_{t,x} \left((\psi_2 Q_{\frac{2}{k-1}} P_1 F)^2 G P_{\leq 1}(H_{j,q,l}) \right) \right\|_{L_{\tau,\xi}^2} \\ &= \Lambda_{k,l}^1 + \Lambda_{k,l}^2 \quad . \end{aligned}$$

From the definition of $H_{j,q,l}$, (89) and (38) we infer that for $l \geq 2$,

$$\begin{aligned} \Lambda_{k,l}^1 &\lesssim k^2 \sum_{j=1}^{\infty} \sum_{q=1}^j 2^{7j/8} 2^{7q/8} \|F\|_{A_\lambda}^k \|\psi_2^2 H_{j,q,l}\|_{L_{\tau,\xi}^2} \\ &\lesssim (kl)^2 \|F\|_{A_\lambda}^{k+l-2} \left(\sum_{j=1}^{\infty} 2^{7j/8} \|\psi_2 \Delta_j F\|_{L_{t,\lambda}^4} \right)^2 \\ &\lesssim (kl)^2 \|F\|_{A_\lambda}^{k+l-2} \|\psi_2 F_x\|_{L_{t,\lambda}^4}^2 \lesssim (kl)^2 \|F\|_{A_\lambda}^{k+l-2} \|F_x\|_{N_\lambda}^2 \quad . \quad (92) \end{aligned}$$

On the other hand , using (88), it is easy to check that for $l \geq 2$,

$$\begin{aligned} \Lambda_{k,l}^2 &\lesssim (kl)^2 \left\| \chi_{\{\xi \geq 3\}} \langle \sigma \rangle^{7/8} \int_{\mathbb{R}^{k+l-1} \times (\lambda^{-1} \mathbb{Z})^{k+l-1}} \chi_{\{|\sigma| > 2^5(k+l)^2 |\xi_{i_1} \xi_{i_2}|\}} |Q_{\frac{2}{k-1}} \widehat{P_1 F}(\tau_1, \xi_1)| \right. \\ &\quad \left. |\mathcal{F}_{t,x} \left(\psi_2 Q_{\frac{2}{k-1}} P_1 F \right)(\tau_2, \xi_2)| |\widehat{P_1 F}(\tau_3, \xi_3)| |\widehat{P_1 F}(\tau_k, \xi_k)| |\widehat{F}(\tau_{k+1}, \xi_{k+1})| \dots |\widehat{F}(\tau_{k+l}, \xi_{k+l})| \right\|_{L_{\tau,\xi}^2} \end{aligned}$$

where $|\xi_{i_1}| = \max |\xi_i|$ and $|\xi_{i_2}| = \max_{i \neq i_1} |\xi_i|$. But according to (85)-(86), we must have $|\sigma| \leq 10(k+l) \max_{i=1, \dots, k+l} |\tau_i - \xi_i| |\xi_i|$ in the region of integration above. Therefore, according to Lemma 4.6,

$$\begin{aligned}
\Lambda_{k,l}^2 &\lesssim (kl)^2(k+l) \|\psi_2 Q_{\frac{2}{k-1}} P_1 F\|_{A_\lambda}^2 \|F\|_{A_\lambda}^{k+l-3} \|F\|_{\dot{X}_\lambda^{7/8,0}} \\
&\quad + (kl)^2(k+l) \|\psi_2 Q_{\frac{2}{k-1}} P_1 F\|_{A_\lambda} \|F\|_{A_\lambda}^{k+l-2} \|\psi_2 Q_{\frac{2}{k-1}} P_1 F\|_{X_\lambda^{7/8,0}} \\
&\lesssim (kl)^2(k+l)k \|F_x\|_{Z_\lambda^{0,0}} \|F\|_{A_\lambda}^{k+l-3} \\
&\quad \left(k \|F_x\|_{Z_\lambda^{0,0}} (\|P_3 F\|_{\dot{X}_\lambda^{1,0}} + \|Q_3 F\|_{X_\lambda^{7/8,0}}) + \|F\|_{A_\lambda} \|\psi_2 Q_{\frac{2}{k-1}} P_1 F\|_{X_\lambda^{7/8,0}} \right) \\
&\lesssim (k+l)^5 \|u\|_{N_{1,\lambda}}^2 (\|u\|_{N_{1,\lambda}} + \|F\|_{A_\lambda}) (1 + \|u\|_{N_{1,\lambda}}) \|F\|_{A_\lambda}^{k+l-3}. \tag{93}
\end{aligned}$$

It remains to estimate $\Gamma_{k,l}$ for $l \geq 2$. We notice that

$$\begin{aligned}
\Gamma_{k,l} &\lesssim k^2 l \left\| \int_{\mathbb{R}^{k+l-1} \times (\lambda^{-1} \mathbb{Z})^{k+l-1}} |\mathcal{F}_{t,x}(\psi_2 Q_{\frac{2}{k-1}} P_1 F)(\tau_1, \xi_1)| |\mathcal{F}_{t,x}(\psi_2 Q_{\frac{2}{k-1}} P_1 F)(\tau_2, \xi_2)| \right. \\
&\quad \left. \prod_{i=3}^k |\widehat{P_1 F}(\tau_i, \xi_i)| |\widehat{F}(\tau_{k+1}, \xi_{l+1})| \prod_{i=k+2}^{k+l} |\widehat{P_3 F}(\tau_i, \xi_i)| \right\|_{L_{\tau,\xi}^2}
\end{aligned}$$

which can be estimated in the same way we did for I_k . More precisely, in the region, $2^4(k+l) \max_{i=1, \dots, k+l} |\tau_i - \xi_i| |\xi_i| \geq |\sigma|$ we easily get as above

$$\Gamma_{k,l} \lesssim (k+l)^5 \|u\|_{N_{1,\lambda}}^2 (\|u\|_{N_{1,\lambda}} + \|F\|_{A_\lambda}) (1 + \|u\|_{N_{1,\lambda}}) \|F\|_{A_\lambda}^{k+l-3}, \tag{94}$$

and in the region $|\sigma| \geq 2^4(k+l) \max_{i=1, \dots, k+l} |\tau_i - \xi_i| |\xi_i|$ we infer from (86) that

$$\tilde{J}_{k,l} \lesssim (k+l)^5 \|u\|_{N_{1,\lambda}}^2 \|F\|_{A_\lambda}^{k+l-2}. \tag{95}$$

Finally, we notice that $\tilde{J}_{k,0}$ and $\tilde{J}_{k,1}$ with $k \geq 3$ can be estimated exactly in the same way.

iii) Estimate on L_k This term can be treated in the same way as the preceding one and is even much simpler. Since $k \geq 2$ we can decompose $Q_1(F)^k$ as we did for F^l in (91) and then proceed exactly in the same way as for $J_{k,l}$. We get

$$L_k \lesssim k^5 \|u\|_{N_{1,\lambda}}^k (1 + \|u\|_{N_{1,\lambda}}^k) \tag{96}$$

Gathering (81)-(84), (87)-(90) and (92)-(96), we finally deduce that

$$\|P_{>3} F_x^*\|_{X_\lambda^{7/8,-1}} \lesssim \left(\|w\|_{M_{1,\lambda}^0} (1 + \|u\|_{N_{1,\lambda}}) + \|u\|_{N_{1,\lambda}}^2 \right) e^{\tilde{K}} \tag{97}$$

which ends the proof of (23).

6 Uniform estimates and Lipschitz bound for small initial data

We will now prove a uniform estimate for small initial data. More precisely, for any given real number $K > 0$, we will prove a uniform estimate for initial data in $\dot{H}_\lambda^s \cap B_{K,\lambda}$, where $B_{K,\lambda}$ is the small closed ball of \dot{L}_λ^2 defined by

$$B_{K,\lambda} = \left\{ \varphi \in \dot{L}_\lambda^2, \|\widehat{\partial_x^{-1} \varphi}\|_{L_\xi^1} \lesssim K \text{ and } \|\varphi\|_{L_\lambda^2} \lesssim \varepsilon^2 \right\} \quad (98)$$

where $0 < \varepsilon = \varepsilon(K) \ll 1$ only depends on K and the implicit constant contained in the above estimates. At this stage, it worth recalling that these implicit constants do not depend on the period λ .

6.1 Uniform estimate for small initial data

For $K > 0$ given, let u_0 belonging to $\dot{H}_\lambda^\infty \cap B_{K,\lambda}$. We want first to show that the emanating solution $u \in C(\mathbb{R}; \dot{H}_\lambda^\infty)$, given by the classical well-posedness results (cf. [1], [13]), satisfies

$$\|u\|_{N_{1,\lambda}} \lesssim \varepsilon^2 \quad \text{and} \quad \|w\|_{M_{1,\lambda}^0} \lesssim \varepsilon^2 \quad . \quad (99)$$

Clearly, since u satisfies the equation, u belongs in fact to $C^\infty(\mathbb{R}; H_\lambda^\infty)$ and thus u and w belong to $M_{1,\lambda}^\infty \cap N_{1,\lambda}$. We are going to implement a bootstrap argument. Since we have chosen to take $T = 1$ we can not use any continuity argument in time but as in [6] we will apply a continuity argument on the space period. Recall that if $u(t, x)$ is a $2\lambda\pi$ -periodic solution of (BO) on $[0, T]$ with initial data u_0 then $u_\beta(t, x) = \beta^{-1}u(\beta^{-2}t, \beta^{-1}x)$ is a $(2\pi\lambda\beta)$ -periodic solution of (BO) on $[0, \beta^2 T]$ emanating from $u_{0,\beta} = \beta^{-1}u_0(\beta^{-1}x)$. Moreover, denoting by w_β the gauge transform of u_β , it is worth noticing that

$$w_\beta(t, x) = \beta^{-1}w(\beta^{-2}t, \beta^{-1}x) \quad . \quad (100)$$

Straightforward computations give

$$\|u_{0,\beta}\|_{L_{\lambda\beta}^2} = \beta^{-1/2}\|u_0\|_{L_\lambda^2} \quad \text{and} \quad \|\widehat{\partial_x^{-1} u_{0,\beta}}\|_{L_\xi^1} = \|\widehat{\partial_x^{-1} u_0}\|_{L_\xi^1}$$

and in the same way one can easily check that the $N_{1,\lambda}$ -norm of u_β and the $M_{1,\lambda}^0$ norm of w_β tend to 0 as β tends to infinity. Hence, for β large enough u_β, w_β satisfy

$$\|u_\beta\|_{N_{1,\lambda\beta}} + \|w_\beta\|_{M_{1,\lambda\beta}^0} \lesssim \varepsilon \quad .$$

(22) then clearly ensures that $\|w_\beta\|_{M_{1,\lambda\beta}^0} \lesssim (1 + \|u_{0,\beta}\|_{L_{\lambda\beta}^2})\|u_{0,\beta}\|_{L_{\lambda\beta}^2}$ and (23)-(25) ensure that

$$\|u_\beta\|_{N_{1,\lambda\beta}} \lesssim (1 + \|u_{0,\beta}\|_{L_{\lambda\beta}^2})\|u_{0,\beta}\|_{L_{\lambda\beta}^2} \quad .$$

Therefore, by (99), we finally get

$$\|u_\beta\|_{N_{1,\lambda\beta}} + \|w_\beta\|_{M_{1,\lambda\beta}^0} \lesssim (1 + \|u_{0,\beta}\|_{L_{\lambda\beta}^2})\|u_{0,\beta}\|_{L_{\lambda\beta}^2} \lesssim \beta^{-1/2}\varepsilon^2 \quad .$$

By a continuity argument in β we can thus take $\beta = 1$ and (99) is established. Finally, we notice that (99) together with (22) and (24) proves that for $0 \leq s \leq 1/2$,

$$\|u\|_{L_1^\infty H_\lambda^s} + \|w\|_{M_{1,\lambda}^s} \lesssim (1 + \|u_0\|_{L_\lambda^2})\|u_0\|_{H_\lambda^s} \quad . \quad (101)$$

6.2 Lipschitz bound

To prove the continuity of the solution as well as the continuity the flow-map we will derive a Lipschitz bound on the solution-map $u_0 \mapsto u$ for small solutions of (BO) (Note that up to now this map is only defined on H_λ^∞).

Let u_1 and u_2 be two solutions of (BO) in $N_{1,\lambda} \cap C([0, T]; H^s \lambda)$ associated with initial data φ_1 and φ_2 in $B_{K,\lambda} \cap H_\lambda^s$ such that their gauge transforms w_1 and w_2 belong to $M_{1,\lambda}^s$. We assume that they satisfy

$$\|u_i\|_{N_{1,\lambda}} + \|w_i\|_{M_{1,\lambda}^0} \lesssim \varepsilon^2, \quad i = 1, 2 \quad . \quad (102)$$

where $0 < \varepsilon \ll 1$ is taken as above.

We set $W_i = P_+(e^{-iF_i/2})$ with $F_i = \partial_x^{-1}u_i$, $w_i = \partial_x W_i$, $v = u_1 - u_2$, $Z = W_1 - W_2$ and $z = Z_x$.

It is easy to check that

$$\begin{aligned} v &= 2ie^{iF_1/2} \left[z + \partial_x P_- \left(e^{-iF_1/2} - e^{-iF_2/2} \right) \right] \\ &\quad + 2i(e^{iF_1/2} - e^{iF_2/2}) \left(w_2 + \partial_x P_- (e^{-iF_2/2}) \right) \end{aligned} \quad (103)$$

and that z satisfies

$$\begin{aligned} z_t - iz_{xx} &= -\partial_x P_+ \left[W_1 \partial_x P_- (v) \right] - \partial_x P_+ \left[Z P_- (\partial_x u_2) \right] \\ &\quad + \frac{i}{4} \left(P_0(u_1^2)z + P_0(u_1^2 - u_2^2)w_2 \right) . \end{aligned} \quad (104)$$

As in the obtention of (20), we substitute (103) in (104) to get

$$\begin{aligned}
z_t - iz_{xx} = & 2i\partial_x P_+ \left[W_1 \partial_x P_- (e^{-iF_1/2} \bar{z} + (e^{-iF_1/2} - e^{-iF_1/2}) \overline{w_2}) \right] \\
& + 2i\partial_x P_+ \left[W_1 \partial_x P_- \left(e^{-iF_1/2} \partial_x P_+ (e^{iF_1/2} - e^{iF_2/2}) \right) \right] \\
& + 2i\partial_x P_+ \left[W_1 \partial_x P_- \left((e^{iF_1/2} - e^{iF_2/2}) \partial_x P_+ (e^{iF_2/2}) \right) \right] + 2i\partial_x P_+ \left(Z \partial_x P_- (e^{-iF_2/2} \overline{w_2}) \right) \\
& + 2i\partial_x P_+ \left[Z \partial_x P_- \left(e^{-iF_2/2} \partial_x P_+ (e^{iF_2/2}) \right) \right] + \frac{i}{4} \left(P_0(u_1^2)z + P_0(u_1^2 - u_2^2)w_2 \right).
\end{aligned}$$

This expression seems somewhat complicated but actually each term can be treated as in Section 4. We extend the functions w_i and F_i in the same way as in Section 4.2. To deal with the difference $e^{i\tilde{F}_1/2} - e^{i\tilde{F}_2/2}$ we use that formally

$$e^{i\tilde{F}_1/2} - e^{i\tilde{F}_2/2} = \sum_{k \in \mathbb{N}} \frac{(i/2)^k}{k!} (\tilde{F}_1^k - \tilde{F}_2^k) = \sum_{k \geq 1} \frac{(i/2)^k}{k!} (\tilde{F}_2 - \tilde{F}_1) \left(\sum_{j=0}^{k-1} \tilde{F}_1^j \tilde{F}_2^{k-1-j} \right)$$

Moreover, as in (60) we have

$$\|P_3(\tilde{F}_1 - \tilde{F}_2)\|_{A_\lambda} \lesssim \left\| \mathcal{F}_x^{-1} \left(\partial_x^{-1} (u_1(0) - u_2(0)) \right) \right\|_{L_\xi^1} + \|\tilde{u}_1 - \tilde{u}_2\|_{N_\lambda} (\|\tilde{u}_1\|_{N_\lambda} + \|\tilde{u}_2\|_{N_\lambda})$$

and thus

$$\|\tilde{F}_1 - \tilde{F}_2\|_{A_\lambda} \lesssim \left\| \mathcal{F}_x^{-1} \left(\partial_x^{-1} (u_1(0) - u_2(0)) \right) \right\|_{L_\xi^1} + \|\tilde{u}_1 - \tilde{u}_2\|_{N_\lambda} (1 + \|\tilde{u}_1\|_{N_\lambda} + \|\tilde{u}_2\|_{N_\lambda}).$$

Therefore, on account of Lemmas 2.1-2.2, 4.3-4.4 and (55), we infer that, for $0 \leq s \leq 1/2$,

$$\begin{aligned}
\|z\|_{M_{1,\lambda}^s} & \lesssim \|z(0)\|_{H_\lambda^s} + e^{\tilde{K}_1 + \tilde{K}_2} \left[\|w_1\|_{X_{1,\lambda}^{1/2,s}} \left(\|z\|_{X_{1,\lambda}^{1/2,0}} \|u_1\|_{N_{1,\lambda}} \right. \right. \\
& \quad \left. \left. + (\|\widehat{\partial_x^{-1} v_0}\|_{L_\xi^1} + \|v\|_{N_{1,\lambda}} + \|v\|_{N_{1,\lambda}}^2) (\|w_2\|_{X_{1,\lambda}^{1/2,0}} + \|u_1\|_{N_{1,\lambda}} + \|u_2\|_{N_{1,\lambda}}) \right) \right. \\
& \quad \left. + \|z\|_{X_{1,\lambda}^{1/2,s}} (\|w_2\|_{X_{1,\lambda}^{1/2,0}} + \|u_2\|_{N_{1,\lambda}} + \|u_2\|_{N_{1,\lambda}}^2) + \|z\|_{X_{1,\lambda}^{1/2,0}} \|u_1\|_{N_{1,\lambda}}^2 \right. \\
& \quad \left. + \|v\|_{N_{1,\lambda}} \|u_2\|_{N_{1,\lambda}} \|w_2\|_{X_{1,\lambda}^{1/2,0}} \right].
\end{aligned}$$

where

$$\tilde{K}_1 + \tilde{K}_2 = C \left(\|\widehat{\partial_x^{-1} u_1(0)}\|_{L_\xi^1} + \|\widehat{\partial_x^{-1} u_2(0)}\|_{L_\xi^1} + \|u_1\|_{N_{1,\lambda}}^2 + \|u_2\|_{N_{1,\lambda}}^2 \right).$$

Thanks to (102) we thus obtain that

$$\begin{aligned} \|z\|_{M_{1,\lambda}^s} &\lesssim \left(1 + \|\varphi_2\|_{L_\lambda^2} + \|\varphi_1\|_{L_\lambda^2}(1 + \lambda^{1/2})\right) \|\varphi_1 - \varphi_2\|_{H_\lambda^s} \\ &\quad + \varepsilon^2 e^K \left[\|w_1\|_{X_{1,\lambda}^{1/2,s}} (\|z\|_{X_{1,\lambda}^{1/2,0}} + \|\widehat{\partial_x^{-1}v(0)}\|_{L_\xi^1} + \|v\|_{N_{1,\lambda}}) \right. \\ &\quad \left. + \|z\|_{X_{1,\lambda}^{1/2,s}} + \|v\|_{N_{1,\lambda}} \right] , \end{aligned} \quad (105)$$

since, by Lemma 4.1, it can be easily seen that

$$\begin{aligned} \|z(0)\|_{H_\lambda^s} &\lesssim \|\varphi_1 - \varphi_2\|_{H_\lambda^s} \left(1 + \|\varphi_1\|_{L_\lambda^2} + \|\varphi_2\|_{L_\lambda^2}\right) \\ &\quad + \|e^{-iF_1(0)} - e^{-iF_2(0)}\|_{L_\lambda^\infty} \|\varphi_1\|_{H_\lambda^s} (1 + \|\varphi_1\|_{L_\lambda^2}) \end{aligned}$$

with

$$\|e^{-iF_1(0)} - e^{-iF_2(0)}\|_{L_\lambda^\infty} \lesssim \|\partial_x^{-1}(\varphi_1 - \varphi_2)\|_{L_\lambda^\infty} \lesssim \lambda^{1/2} \|\varphi_1 - \varphi_2\|_{L_\lambda^2} .$$

On the other hand, proceeding as in Section 5 and using (102), one can check that

$$\|v\|_{N_{1,\lambda}} \lesssim \|v(0)\|_{L_\lambda^2} + \varepsilon^2 \left(\|z\|_{M_{1,\lambda}^0} + \|\widehat{\partial_x^{-1}v(0)}\|_{L_\xi^1} + \|v\|_{N_{1,\lambda}} \right) e^{K_1+K_2} . \quad (106)$$

Noticing that by Cauchy-Schwarz in ξ ,

$$\|\widehat{\partial_x^{-1}v(0)}\|_{L_\xi^1} \lesssim \lambda^{1/2} \|\widehat{v(0)}\|_{L_\xi^2} \sim \lambda^{1/2} \|v(0)\|_{L_\lambda^2}$$

and gathering (105) and (106) we obtain

$$\|v\|_{N_{1,\lambda}} + \|z\|_{M_{1,\lambda}^0} \lesssim (1 + \varepsilon^2 \lambda^{1/2}) \|\varphi_1 - \varphi_2\|_{L_\lambda^2} . \quad (107)$$

Coming back to (105) this leads to

$$\|z\|_{M_{1,\lambda}^s} \lesssim \left(1 + \varepsilon^2 \lambda^{1/2}\right) \|\varphi_1 - \varphi_2\|_{H_\lambda^s} . \quad (108)$$

Now, proceeding as in (18), we infer that

$$\begin{aligned} v &= \partial_x F_1 - \partial_x F_2 \\ &= 2ie^{iF_1/2} \left[z + \partial_x P_- \left(e^{-iF_1/2} - e^{-iF_2/2} \right) \right] + 2i \left(e^{iF_1/2} - e^{iF_2/2} \right) \left(w_2 + \partial_x P_- \left(e^{-iF_2/2} \right) \right) \end{aligned}$$

and thus

$$\begin{aligned} P_{>1}v &= 2iP_{>1} \left(e^{iF_1/2} z \right) + 2iP_{>1} \left[P_{>1} \left(e^{iF_1/2} \right) \partial_x P_- \left(e^{-iF_1/2} - e^{-iF_2/2} \right) \right] \\ &+ 2iP_{>1} \left[\left(e^{iF_1/2} - e^{iF_2/2} \right) w_2 \right] + 2iP_{>1} \left[P_{>1} \left(e^{iF_1/2} - e^{iF_2/2} \right) \partial_x P_- \left(e^{-iF_2/2} \right) \right] \end{aligned} \quad (109)$$

Therefore, by Lemmas 4.1-4.2, (102) and (25)

$$\begin{aligned} \|J_x^s Q_1 v\|_{L_1^\infty L_\lambda^2} &\lesssim \left(\|z\|_{Y_{1,\lambda}^s} + \varepsilon^2 (\|v\|_{L_1^\infty L_\lambda^2} + \varepsilon^2 \|\partial_x^{-1} v\|_{L_1^\infty L_\lambda^\infty}) \right) e^{\tilde{K}} \\ &\lesssim \left(\|z\|_{Y_{1,\lambda}^s} + \varepsilon^2 (1 + \varepsilon^2 \lambda^{1/2}) \|v\|_{L_1^\infty L_\lambda^2} \right) e^{\tilde{K}} \end{aligned}$$

Since on the other hand (see (72)),

$$\|P_1 v\|_{L_1^\infty L_\lambda^2} \lesssim \|\varphi_1 - \varphi_2\|_{L_\lambda^2} + \|v\|_{L_{1,\lambda}^4} \left(\|u_1\|_{L_{1,\lambda}^4} + \|u_2\|_{L_{1,\lambda}^4} \right), \quad (110)$$

we finally deduce from (106)-(107) that

$$\|J_x^s v\|_{L_1^\infty L_\lambda^2} \lesssim (1 + \varepsilon^2 \lambda^{1/2})^2 \|\varphi_1 - \varphi_2\|_{H_\lambda^s} \quad . \quad (111)$$

7 Proof of Theorem 1.1

We will first prove the local well-posedness result for small data, the result for arbitrary large data will then follow from scaling arguments.

7.1 Well-posedness for small initial data

For any $K > 0$ and $\lambda \geq 1$ given, let $u_0 \in B_{K,\lambda} \cap H_\lambda^s$ with $0 \leq s \leq 1/2$ and let $\{u_0^n\} \subset \dot{H}^\infty(\mathbb{T}) \cap B_{K,\lambda}$ converging to u_0 in $H^s(\mathbb{T})$. We denote by u_n the solution of (BO) emanating from u_0^n . From standard existence theorems (see for instance [1], [13]), $u_n \in C(\mathbb{R}; \dot{H}_\lambda^\infty)$. According to (99) and (101), for all $n \in \mathbb{N}^*$,

$$\|u_n\|_{N_{1,\lambda}} + \|w_n\|_{M_{1,\lambda}^0} \lesssim \varepsilon^2 \quad (112)$$

and

$$\|u_n\|_{L_1^\infty H_\lambda^s} + \|w_n\|_{M_{1,\lambda}^s} \lesssim (1 + \|u_0\|_{L_\lambda^2}) \|u_0\|_{H_\lambda^s}, \quad (113)$$

where $w_n = \partial_x P_+(e^{-iF_n/2})$ is the gauge transform of u_n . Note that this uniform bound would enable to prove the local existence for $s > 0$ by using weak convergences. On the other hand, for $s = 0$, weak convergences would not be sufficient to pass to the limit on the nonlinear term u^2 . Actually, with (107) and (111) in hand, we observe that the approximative sequence u^n constructed for the local existence result is a Cauchy sequence in $C([0, 1]; \dot{H}_\lambda^s) \cap N_{1,\lambda}$ since the u_n satisfy (99)-(101) and $u_{0,n}$ converges to u_0 in \dot{H}_λ^s . Hence, u_n converges strongly to some u in $C([0, 1]; \dot{H}_\lambda^s) \cap N_{1,\lambda}$. This strong convergence permits to pass easily to the limit on the nonlinear term and thus u is a solution of (BO). Moreover, from (101) and (108) it follows

that the sequence of gauge transforms w_n of u_n is a Cauchy sequence in $M_{1,\lambda}^s$. Hence $w_n = \partial_x P_+(e^{-iF_n/2})$ converges toward some function w in $Y_{1,\lambda}^s$ and from the strong convergence of u it is easy to check that $w = P_+(e^{-iF/2})$ with $F = \partial_x^{-1}u$.

Now let u^1 and u^2 be two solutions emanating from u_0 belonging to $N_{1,\lambda}$ such that their associated gauge functions belong to $X_\lambda^{1/2,0}$. According to (22), the gauge functions belong in fact to $M_{1,\lambda}^0$ and by the same dilation argument we use to prove the uniform boundness of the solution, we can show that for β large enough and $i = 1, 2$,

$$\|u_\beta^i\|_{N_{1,\lambda\beta}^0} + \|w_\beta^i\|_{M_{1,\lambda\beta}^0} \lesssim (1 + \|u_{0,\beta}\|_{L_{\lambda\beta}^2}) \|u_{0,\beta}\|_{L_{\lambda\beta}^2} \lesssim \beta^{-1/2} \varepsilon(K)^2$$

with $K = \|\widehat{\partial_x^{-1}u_{0,\beta}}\|_{L_\xi^1} = \|\widehat{\partial_x^{-1}u_0}\|_{L_\xi^1}$. Therefore (u_β^i, w_β^i) satisfy the smallness condition (102) with $\varepsilon = \varepsilon(K, \lambda\beta)$ and $u_{0\beta} \in B_{K,\lambda\beta}$. It then follows from (107) that $u_\beta^1 \equiv u_\beta^2$ on $[0, 1]$ and thus $u^1 \equiv u^2$ on $[0, 1/\beta^2]$. This proves the uniqueness result for initial data belonging to $B_{K,\lambda}$. Moreover, (111) clearly ensures that the flow-map is Lipschitz from $B_{K,\lambda} \cap H_\lambda^s$ into $C([0, 1]; \dot{H}_\lambda^s)$.

7.2 The case of arbitrary large initial data

We use again the dilation invariance of (BO) to extend the result for arbitrary large data. Recall that if $u(t, x)$ is a 2π -periodic solution of (BO) on $[0, T]$ with initial data u_0 then $u_\lambda(t, x) = \lambda^{-1}u(\lambda^{-2}t, \lambda^{-1}x)$ is a $(2\pi\lambda)$ -periodic solution of (BO) on $[0, \lambda^2T]$ emanating from $u_{0,\lambda} = \lambda^{-1}u_0(\lambda^{-1}x)$. Recall also that the associated gauge functions satisfy $w_\lambda(t, x) = \lambda^{-1}w(\lambda^{-2}t, \lambda^{-1}x)$. Let $u_0 \in \dot{H}^s(\mathbb{T})$ with $0 \leq s \leq 1/2$. Note that

$$\|\widehat{\partial_x^{-1}u_0}\|_{L_\xi^1} \lesssim \|u_0\|_{L^2} \quad .$$

We thus set $K = \|u_0\|_{L^2}$ and take

$$\lambda = \max\left(1, \varepsilon(K)^{-4} \|u_0\|_{L^2}^2\right) \geq 1$$

so that

$$\|u_{0,\lambda}\|_{L_\lambda^2} \leq \lambda^{-1/2} \|u_0\|_{L^2} \leq \varepsilon(K)^2 \quad .$$

It follows that $u_{0,\lambda}$ belongs to $B_{K,\lambda}$ and so we are reduced to the case of small initial data. Therefore, there exists a unique solution $u_\lambda \in C([0, 1]; \dot{H}_\lambda^s) \cap$

$N_{1,\lambda}$ of (BO) with $w_\lambda \in M_{1,\lambda}^s$. This proves the existence and uniqueness of the solution u of (BO) in the class

$$u \in C([0, T]; \dot{L}^2(\mathbb{T})) \cap N_T, \quad w \in M_T^0$$

emanating from u_0 with $T = T(\|u_0\|_{L^2})$ and $T \rightarrow +\infty$ as $\|u_0\|_{L^2} \rightarrow 0$. The fact that the flow-map is Lipschitz on every bounded set of $\dot{H}^s(\mathbb{T})$ follows as well since λ only depends on $\|u_0\|_{L^2}$.

Note that the change of unknown (15) preserves the continuity of the solution and the continuity of the flow-map in $H^s(\mathbb{T})$. Moreover, the Lipschitz property (on bounded sets) of the flow-map is also preserved on the hyperplans of $H^s(\mathbb{T})$ of functions with fixed mean value. Finally, the global well-posedness result follows directly by combining the conservation of the L^2 -norm and the local well-posedness result.

8 Proof of Theorem 1.2

8.1 Analyticity of the flow-map

Let us prove the analyticity of the solution-map $\Psi : u_0 \mapsto u$ from $\dot{H}^s(\mathbb{T})$ to $C([0, 1]; H^s(\mathbb{T}))$ at the origin. Note that the other points of $\dot{H}^s(\mathbb{T})$ could be handle in the same way. Also we restrict ourself to the case $0 \leq s \leq 1/2$ but the case $s \geq 1/2$ can be treated in a similar way (in fact easier) by using the results of [18].

The analyticity of the flow-map will be a direct consequence the three following ingredients :

- The Lipschitz property of Ψ proven in Section 6.
- The fact that it appears only polynomial or analytic function of u in the equations we deal with.
- We have an absolute convergence, in the norms we are interested in, of the serie obtained by replacing the analytic functions of u by their associated entire series.

So, let $\varphi \in \dot{H}^s(\mathbb{T})$ with $\|\varphi\|_{H^s} = 1$ and let $\varepsilon > 0$ be a small real number to be fixed later. Taking $u_0 = \varepsilon\varphi$ we know from (107), (108) and (111) that, for ε small enough, there exists $c_1 > 0$ such that the corresponding solution u and its gauge transform w verify

$$\|u\|_{N_1} + \|u\|_{L_1^\infty H^s} + \|w\|_{M_1^s} \leq C_1 \varepsilon \quad , \quad (114)$$

Now let $C > 0$ be a universal constant we take very large constant (We can take for example $C > 0$ to be the exponential of the sum of all the implicit constants interfering in our estimates in Sections 4-5). According to (66) and (67), we get

$$\|P_3 F - \varepsilon V(t) P_3 \varphi\|_{N_1} \leq C(c_1 \varepsilon)^2 \quad .$$

On the other hand, since $\partial_x^{-1} \varphi \in H^{s+1}$ which is an algebra, it holds in H^{s+1}

$$W(0) = P_+(e^{-i\partial_x^{-1} \varphi/2}) = 1 - \frac{i}{2} \varepsilon P_+(\partial_x^{-1} \varphi) + \sum_{k \geq 2} \left(\frac{-i\varepsilon}{2}\right)^k \frac{1}{k!} P_+ \left((\partial_x^{-1} \varphi)^k \right) \quad .$$

and thus

$$w(0) = -\frac{i}{2} \varepsilon P_+(\varphi) + \Lambda_\varepsilon \text{ with } \|\Lambda_\varepsilon\|_{H^s} \leq 4\varepsilon \quad .$$

Consequently,

$$V(t)w(0) = -\frac{i}{2} \varepsilon V(t) P_+ \varphi + V(t) \Lambda_\varepsilon \text{ with } \|V(t) \Lambda_\varepsilon\|_{M_1^s} \leq C(4\varepsilon)^2 \quad .$$

Now according to (28), (31), (48) and (55), we infer that $\|w - V(t)w(0)\|_{M_1^s} \leq C(c_1 \varepsilon)^2$ and thus

$$\|w + \frac{i}{2} \varepsilon V(t) P_+ \varphi\|_{M_1^s} \leq 2C(c_1 \varepsilon)^2 \quad . \quad (115)$$

It then follows from (21)-(71), (73), (78)-(80) and (97) that

$$P_{>3}(u) = 2i P_{>3} w + \tilde{\Lambda}_\varepsilon = \varepsilon V(t) P_{>3}(\varphi) + \tilde{\Lambda}_\varepsilon$$

for some function $\tilde{\Lambda}_\varepsilon$ satisfying $\|\tilde{\Lambda}_\varepsilon\|_{N_1} + \|\tilde{\Lambda}_\varepsilon\|_{L_1^\infty H^s} \leq 3C(c_1 \varepsilon)^2$.

We thus finally get,

$$\|u - \varepsilon V(t) \varphi\|_{N_{1,\lambda}} + \|u - \varepsilon V(t) \varphi\|_{L_1^\infty H^s} + \|w + \frac{i}{2} \varepsilon V(t) P_{>3} \varphi\|_{M_{1,\lambda}^s} \leq 6C(c_1 \varepsilon)^2 \quad . \quad (116)$$

In the same way, according to (20), expanding $e^{-iF/2}$ and $e^{iF/2}$ as in Section 4, with (114)-(116) in hand, we get

$$w = -\frac{i}{2} \varepsilon V(t) P_+(\varphi) - \varepsilon^2 \left[\frac{1}{4} V(t) P_+(\varphi \partial_x^{-1} \varphi) + 2i \int_0^t V(t-t') \partial_x P_+(W_1 \partial_x P_-(\overline{w_1})) \right] + \Lambda_\varepsilon \quad .$$

where

$$u_1 = V(t) \varphi, \quad W_1 = -\frac{i}{2} V(t) P_+(\partial_x^{-1} \varphi), \quad w_1 = \partial_x W_1$$

and $\|\Lambda_\varepsilon\|_{M_1^s} \lesssim 6C^2 (c_1\varepsilon)^3$ and so on ...

Itering this process we obtain that there exists $\varepsilon_0 > 0$ such that the following asymptotic expansion of u in term of φ holds absolutely in $C([0, 1]; H^s)$ for $0 < \varepsilon \leq \varepsilon_0$,

$$u = \sum_{k \geq 1} \varepsilon^k A_k(\varphi) \quad . \quad (117)$$

Here, $A_1(\varphi) = t \mapsto V(t)\varphi$ and more generally A_k is a continuous k -linear operator from $\dot{H}^s(\mathbb{T})$ to $C([0, 1]; H^s)$. Therefore u is real-analytic and in particular C^∞ at the origin of $\dot{H}^s(\mathbb{T})$. Moreover, since

$$u(t, \cdot) = \varepsilon U(t)\varphi + \frac{1}{2} \int_0^t V(t-t') \partial_x u^2(t') dt' ,$$

by identification we infer that

$$A_k(\varphi) = t \mapsto \frac{1}{2} \sum_{\substack{k_1, k_2 \geq 1 \\ k_1 + k_2 = k}} \int_0^t V(t-t') \partial_x \left(A_{k_1}(\varphi) A_{k_2}(\varphi) \right) (t') dt' \quad (118)$$

8.2 Non smoothness of the flow-map in $H^s(\mathbb{T})$, $s < 0$.

Let us start by computing $A_k(t, \lambda \cos(Nx))$ for $k=1,2,3$. Of course,

$$A_1(t, \cos(Nx)) = \cos(Nx - N^2t) \quad .$$

Since $\partial_x \left(A_1(t, \cos(Nx)) \right)^2 = -N \sin(2Nx - 2N^2t)$ we infer that

$$\begin{aligned} A_2(t, \cos(Nx)) &= \frac{1}{2} \int_0^t V(t-t') \partial_x \left(A_1(t, \cos(Nx)) \right)^2 (t') dt \\ &= -\frac{N}{2} \int_0^t \sin(2Nx - 2N^2t' - 4N^2(t-t')) dt' \\ &= \frac{1}{4N} \left[\cos(2Nx - 2N^2t) - \cos(2Nx - 4N^2t) \right] \end{aligned}$$

In the same way,

$$\begin{aligned} \partial_x \left(A_1(1, \cos(Nx)) A_2(t, \cos(Nx)) \right) &= -\frac{1}{8} \left[\sin(Nx - N^2t) - \sin(Nx - 3N^2t) \right] \\ &\quad -\frac{3}{8} \left[\sin(3Nx - 3N^2t) - \sin(3Nx - 5N^2t) \right] \end{aligned}$$

and thus

$$\begin{aligned}
A_3(t, \cos(Nx)) &= \int_0^t V(t-t') \partial_x \left(A_1(t, \cos(Nx)) A_2(t, \cos(Nx)) \right) (t') dt \\
&= -\frac{1}{8} \int_0^t \left[\sin(Nx - N^2 t) - \sin(Nx - 3N^2 t' - N^2(t-t')) \right] dt' \\
&\quad -\frac{3}{8} \int_0^t \left[\sin(3Nx - 3N^2 t' - 9N^2(t-t')) - \sin(Nx - 5N^2 t' - N^2(t-t')) \right] dt' \\
&= -\frac{t}{8} \sin(Nx - N^2 t) \\
&\quad + \frac{1}{16N^2} \left[\cos(Nx - 3N^2 t) - \cos(Nx - N^2 t) \right] \\
&\quad + \frac{1}{16N^2} \left[\cos(3Nx - 3N^2 t) - \cos(3Nx - 9N^2 t) \right] \\
&\quad - \frac{3}{32N^2} \left[\cos(3Nx - 5N^2 t) - \cos(3Nx - 9N^2 t) \right]
\end{aligned}$$

Therefore, setting $\Psi_N = N^{-s} \cos(Nx)$ it follows that

$$\|A_3(t, \Psi_N)\|_{H^s} \gtrsim t N^{-2s} \|\Psi_N\|_{H^s}^3$$

and from standard considerations (cf. [5]) the flow-map cannot be of class C^3 at the origin from $\dot{H}^s(\mathbb{T})$ into $\dot{H}^s(T)$ as soon as $s < 0$. Moreover, by a direct induction argument it is not too hard to check that for any $k \geq 4$,

$$\|A_k(t, \cos(Nx))\|_{H^s} \leq \tilde{C}_k N^s \quad .$$

Therefore, for any fixed integer $K \geq 4$,

$$\left\| \sum_{k=4}^{K+2} A_k(t, \cos(Nx)) \right\|_{H^s} \leq C_K \varepsilon^4 N^s \quad .$$

Now, taking as initial data $\varphi_N = \varepsilon_N \cos(Nx)$ with $0 < \varepsilon_N \leq \varepsilon_0/2$, we know from (117) that the associated solution u_N can be written in $L^2(\mathbb{T})$ as

$$u_N(t, \cdot) = \sum_{k \geq 1} \varepsilon_N^k A_k(t, \cos(Nx)) \quad .$$

For N large enough and $s \leq 0$, we thus have

$$\begin{aligned}
\|u_N(t, \cdot) - V(t)\varphi_N\|_{H^s} &\gtrsim t \varepsilon_N^3 \|\sin(Nx - N^2 t)\|_{H^s} - C_K \varepsilon^4 N^s \\
&\quad - \tilde{C} \sum_{k=K+3}^{\infty} \left(\frac{\varepsilon_N}{\varepsilon_0} \right)^k \|A_k(t, \varepsilon_0 \cos(Nx))\|_{L^2} \\
&\gtrsim t \varepsilon_N^3 \|\sin(Nx - N^2 t)\|_{H^s} - C_K \varepsilon_N^4 N^s - C \varepsilon_N^{K+3} \\
&\gtrsim \varepsilon_N^3 N^s \left(t - C_K \varepsilon_N - C \varepsilon_N^K N^{-s} \right) \quad .
\end{aligned}$$

For any $0 < \alpha < 1$ and $s < 0$ fixed, we take $K > 0$ such that

$$\frac{4}{K} < \alpha \quad .$$

Setting

$$\varepsilon_N = \min\left(\frac{\varepsilon_0}{2}, \frac{t}{4C_K}, \left(\frac{tN^s}{4C}\right)^{\frac{1}{K}}\right)$$

we infer that

$$\begin{aligned} \|u_N(t, \cdot) - V(t)\varphi_N\|_{H^s} &\gtrsim t\varepsilon_N^3 N^s \\ &\gtrsim t\varepsilon_N^{2-\alpha} N^{-\alpha s} \|\varphi_N\|_{H^s}^{1+\alpha} \\ &\gtrsim tN^{-\frac{\alpha s}{2}} \|\varphi_N\|_{H^s}^{1+\alpha} \quad . \end{aligned}$$

It follows that the flow-map (if it coincides with the standard flow-map on $H^\infty(\mathbb{T})$) cannot be of class $C^{1+\alpha}$ at the origin from $\dot{H}^s(\mathbb{T})$ into $\dot{H}^s(\mathbb{T})$.

9 Appendix

9.1 Proof of Lemma 4.1

We separate the low and the high modes of h . To treat the high modes part, we observe that by Leibniz rule for fractional derivatives (cf. [15]) and Sobolev inequality,

$$\begin{aligned} \left\| J_x^\alpha \left(Q_1(h)g \right) \right\|_{L_\lambda^q} &\lesssim \left\| J_x^\alpha Q_1(h) \right\|_{L_\lambda^{4/\alpha}} \|g\|_{L_\lambda^{\frac{4q}{4-4\alpha q}}} + \|h\|_{L_\lambda^\infty} \|J_x^\alpha g\|_{L_\lambda^q} \\ &\lesssim \left\| J_x^{\alpha+1/2-\alpha/4} Q_1(h) \right\|_{L_\lambda^2} \|J_x^{\alpha/4} g\|_{L_\lambda^q} + \|h\|_{L_\lambda^\infty} \|J_x^\alpha g\|_{L_\lambda^q} \\ &\lesssim (\|\partial_x h\|_{L_\lambda^2} + \|h\|_{L_\lambda^\infty}) \|J_x^\alpha g\|_{L_\lambda^q} \quad . \end{aligned}$$

On the other hand, one can easily check that

$$\left\| J_x^1 \left(P_1(h)g \right) \right\|_{L_\lambda^q} \lesssim (\|h\|_{L_\lambda^\infty} + \|\partial_x h\|_{L_\lambda^2}) \|J_x^1 g\|_{L_\lambda^q} \quad \text{and} \quad \|P_1(h)g\|_{L_\lambda^q} \lesssim \|h\|_{L_\lambda^\infty} \|g\|_{L_\lambda^q} \quad .$$

Interpolating between this two estimates we obtain the desired estimate on the low modes part.

9.2 Proof of Lemma 4.2

Clearly the low modes part of zv can be estimated directly by an Holder inequality. Now, using the nonhomogeneous Littewood-paley decomposition, we get for $q \geq 8$,

$$\begin{aligned} \Delta_q(zv) &= \sum_{|i| \leq 2} \Delta_q \left(\Delta_{q-i}(z) \sum_{j=0}^{q-i-2} \Delta_j(v) \right) \\ &\quad + \sum_{|i| \leq 2} \Delta_q \left(\Delta_{q-i}(v) \sum_{j=0}^{q-i-2} \Delta_j(z) \right) + \Delta_q \left(\sum_{i \geq q-2} \sum_{|j| \leq 1} \Delta_{i-j}(v) \Delta_i(z) \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{q \geq 8} \|\Delta_q(zv)\|_{L_{t,\lambda}^4}^2 &\lesssim \|z\|_{L_{t,\lambda}^\infty}^2 \sum_{q \geq 4} \|\Delta_q(v)\|_{L_{t,\lambda}^4}^2 \\ &\quad + \|v\|_{L_{t,\lambda}^4} \left(\sum_{q \geq 4} \|\Delta_q(z)\|_{L_{t,\lambda}^\infty}^2 + \sum_{q \geq 4} \sum_{k \geq q-2} \|\Delta_k(z)\|_{L_{t,\lambda}^\infty}^2 \right). \end{aligned} \quad (119)$$

The desired result follows since for $k \geq 2$,

$$\|\Delta_k(z)\|_{L_{t,\lambda}^\infty} \lesssim 2^{-k/4} \|z_x\|_{L_t^\infty L_\lambda^2}.$$

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