Pulse Normalisation in Slow-Light Media

Bruno Macke, Bernard Ségard

HAL Id: hal-00016652
https://hal.science/hal-00016652

Pulse Normalisation in Slow-Light Media

Bruno Macke and Bernard Ségard[*]

Laboratoire de Physique des Lasers, Atomes et Molécules (PhLAM),

Centre d'Etudes et de Recherches Lasers et Applications (CERLA),

Université de Lille I, 59655 Villeneuve d'Ascq, France

(Dated: January 9, 2006)

Abstract

We analytically study the linear propagation of arbitrarily shaped light-pulses through an absorbing medium with a narrow transparency-window or through a resonant amplifying medium. We point out that, under certain general conditions, the pulse acquires a nearly Gaussian shape, irrespective of its initial shape and of the spectral profile of the line. We explicitly derive in this case the pulse parameters, including its skewness responsible for a deviation of the delay of the pulse maximum from the group delay. We illustrate our general results by analysing the slow-light experiments having demonstrated the largest fractional pulse-delays.

PACS numbers: 42.25.Bs, 42.50.Gy, 89.70.+c

---

[*]Electronic address: bernard.segard@univ-lille1.fr

I.   INTRODUCTION

The principle underlying most of the slow-light experiments is to exploit the steep normal dispersion of the refractive index associated with a pronounced peak in the transmission of the medium and the correlative reduction of the group velocity. The situation where the resulting time-delay of the light-pulse is large compared to its duration and can be controlled by an external laser field is of special importance for potential applications, especially in the domain of high-speed all-optical signal-processing. Harris and his co-workers [1, 2] opened the way to such experiments by exploiting the phenomenon of electromagnetically induced transparency (EIT) allowing one to create a narrow transparency-window in an otherwise optically thick atomic vapour. Using a true-shape detection of the pulses, they demonstrated propagation velocities as slow as $c/165$ and group delays τ_g as long as $4.6\,\tau_{in}$ where c and τ_{in} are respectively the velocity of the light in vacuum and the full width at half-maximum (FWHM) of the intensity-profile of the incident pulse. Much slower velocities have been attained in subsequent EIT experiments (for reviews see, e.g., [3, 4, 5]) and in experiments involving coherent population oscillations [6] or other processes to induce a transparency-window in an absorbing medium. It is however worth noticing that only few of these experiments, all using EIT, have succeeded in giving *direct* demonstrations of fractional delays τ_g/τ_{in} exceeding unity [2, 7, 8]. Theoretical discussions on the maximum time-delays attainable in such experiments can be found in [1, 9, 10]. A different way to achieve a system with a controllable transmission-peak is to optically induce a resonant gain in a transparent medium [11]. Initially proposed by Gauthier [12], the arrangement involving stimulated Brillouin scattering [13, 14, 15, 16] seems particularly attractive from the viewpoint of the above mentioned applications. The Brillouin gain is indeed directly implemented on an optical fibre and there are no severe constraints in the choice of the operating wavelength. The group delay τ_g has already been controlled on a range of $3.6\,\tau_{in}$ by this technique [15]. Note that preliminary experiments using a Raman fibre amplifier have also been achieved [17].

The purpose of our paper is to provide analytical results on the propagation of arbitrarily shaped pulses, the central frequency of which coincides with that of a pronounced maximum in the medium-transmission. We examine more specifically the case where the resulting time-delays of the pulses are large compared to their duration. Our study applies in particular but

not exclusively to the above mentioned systems. Our approach follows in part and extends that of Bukhman [18] with a special attention paid to the connection of the theoretical results with the experiments.

II. GENERAL ANALYSIS

We denote by $e_{in}(t)$ and $e_{out}(t)$ the slowly-varying envelopes of the incident and transmitted pulses and by $E_{in}(\Omega) = \int_{-\infty}^{\infty} e_{in}(t) \exp(-i\Omega t)dt$ and $E_{out}(\Omega)$ their Fourier transforms. The slow-light medium is characterised by its impulse response $h(t)$ or by its transfer function $H(\Omega)$, Fourier transform of $h(t)$. The input/output relation or transfer equation reads $e_{out}(t) = h(t) \otimes e_{in}(t)$ in the time-domain or $E_{out}(\Omega) = H(\Omega)E_{in}(\Omega)$ in the frequency-domain [19]. We assume that the incident pulse has a finite energy, that it is not chirped ($e_{in}(t)$ real and positive) and that $h(t)$ is also real. The local response of the medium is characterised by the complex gain-factor $\Gamma(\Omega) = \ln[H(\Omega)]$ whose real part $F(\Omega)$ and imaginary part $\Phi(\Omega)$ are respectively the logarithm of the medium amplitude-gain $|H(\Omega)|$ and the induced phase shift. The condition imposed to $h(t)$ implies that $H(-\Omega) = H^*(\Omega)$ and thus that $F(\Omega)$ and $\Phi(\Omega)$ are respectively even and odd functions of Ω. This has the advantage of eliminating the lowest-order pulse-distortions resulting from the gain-slope and from the group velocity dispersion at the frequency ω_0 of the optical carrier ($\Omega = 0$). Moreover the medium is then entirely characterised by the single real function $h(t)$. In order to have simple expressions we use for $e_{in}(t)$ a time origin located at the pulse centre-of-gravity and for $e_{out}(t)$ a time origin retarded by the transit time at the group velocity outside the frequency-domain of high-dispersion (local time picture). The time delays considered hereafter are thus only those originating in the high-dispersion region.

General properties of the transmitted pulse can be derived by Fourier analysis. Let $x(t)$ be any of the real functions $e_{in}(t)$, $h(t)$ or $e_{out}(t)$ and $X(\Omega)$ its Fourier transform. We remark that $X(0) = \int_{-\infty}^{\infty} x(t)dt$ and, following an usual procedure in probability theory [20], we characterise $X(\Omega)$ by its cumulants κ_n, such that

$$X(\Omega) = X(0) \exp\left(\sum_{n=1}^{\infty} \frac{\kappa_n}{n!}(-i\Omega)^n\right). \tag{1}$$

For $H(\Omega)$, we see that the cumulants are simply related to the coefficients of the series expansion of $\Gamma(\Omega)$ in powers of $-i\Omega$ and, in particular, that κ_1 coincides with the group

3

delay $\tau_g = -\frac{d\Phi}{d\Omega}\,|_{\Omega=0}$. We incidentally recall that, due to the causality principle, τ_g can be related to the gain profile [21, 22]. Within our assumptions, this relation reads

$$\tau_g = P \int_{-\infty}^{\infty} \frac{\ln H_0 - \ln|H(\Omega)|}{\pi\Omega^2}d\Omega \qquad (2)$$

where $H_0 = H(0)$ is the amplitude-gain of the medium at the frequency of the optical carrier. This confirms that large group delays are achieved when the gain $|H(\Omega)|$ has a pronounced maximum at $\Omega = 0$. We have then $\kappa_2 > 0$.

Coming back to the general problem, we characterise the time function $x(t)$ by its area $S = \int_{-\infty}^{\infty} x(t)dt$ and its three lowest order moments, namely the mean value $\langle t \rangle = \frac{1}{S}\int_{-\infty}^{\infty} t\,x(t)dt$, the variance $\sigma^2 = \frac{1}{S}\int_{-\infty}^{\infty}(t - \langle t \rangle)^2 x(t)dt$ and the 3rd order centred moment $a = \frac{1}{S}\int_{-\infty}^{\infty}(t - \langle t \rangle)^3 x(t)dt$. We recognise in $\langle t \rangle$ (resp. σ) the location of the centre-of-gravity (resp. the *rms* duration) of the function $x(t)$. Its asymmetry may be characterised by the dimensionless parameter $\xi = a/\sigma^3$, the so-called skewness [20]. For a Gaussian function, $\sigma = \tau/(2\sqrt{\ln 2})$ where τ is the FWHM of the energy profile $x^2(t)$. An important result [20] is that the moments $\langle t \rangle$, σ^2 and a of $x(t)$ are equal to the cumulants κ_1, κ_2 and κ_3 of $X(\Omega)$. Moreover the transfer equation immediately leads to the relations $E_{out}(0) = H_0 E_{in}(0)$ and $\kappa_{n,out} = \kappa_{n,in} + \kappa_n$ where, as in all our paper, the indexes *in*, *out* and the absence of index respectively refer to the incident pulse, the transmitted pulse and the transfer-function or impulse response of the medium. With our choice of time origin, $\langle t_{in} \rangle = 0$. By combining the previous results, we finally obtain the four equations $S_{out} = H_0 S_{in}$, $\langle t_{out} \rangle = \tau_g$, $\sigma_{out}^2 = \sigma_{in}^2 + \sigma^2 > \sigma_{in}^2$ and $a_{out} = a_{in} + a$. In the studies of the linear pulse propagation, the first equation which relates the areas of the transmitted and incident pulses is known as the area theorem [23]. The second equation expresses that the time-delay of the pulse centre-of-gravity equals the group delay [22]. The two last ones specify how the *rms* duration and the asymmetry of the incident pulse are modified by the medium. All these results are valid provided that the involved moments are finite [19].

III.   ANALYTIC EXPRESSIONS OF THE MEDIUM IMPULSE-RESPONSE

In order to obtain a complete information on the shape and the amplitude of the transmitted pulse, we obviously have to specify the complex gain-factor $\Gamma(\Omega)$ of the medium. For

the medium with a resonant gain, it reads

$$\Gamma(\Omega) = p_N(\Omega)G/2 - A/2 \tag{3}$$

where $p_N(\Omega)$ is the normalised complex profile of the gain-line ($p_N(0) = 1$), G is the gain parameter for the intensity [14] and A stands for the attenuation introduced to reduce the effects of the amplified spontaneous emission [15] and/or to normalise the overall gain of the system. $G = g_0 L$ where g_0(resp.L) is the resonance gain-coefficient (resp. the thickness) of the medium. The previous expression of $\Gamma(\Omega)$ also holds for an absorbing medium with a transparency-window when the absorption background is assumed to be infinitely wide. We then get $\Gamma(\Omega) = -[1 - f\,p_N(\Omega)]\alpha_0 L/2$ where α_0 is the background absorption-coefficient, $f \leq 1$ specifies the depth of the transparency-window [9] and $p_N(\Omega)$ is the normalised complex profile of the line associated with the transparency-window. By putting $G = f\,\alpha_0 L$ and $A = \alpha_0 L$, we actually retrieve the expression of $\Gamma(\Omega)$ obtained for a gain medium. In both types of experiments, G and A are generally comparable in order that the resonance gain $H_0 = \mathrm{e}^{G/2-A/2}$ is close to 1 or, at least, does not differ too strongly from 1. Anyway the intensity-transmission on resonance exceeds its value far from resonance by the factor e^G.

To go beyond, it seems necessary to explicit the profile $p_N(\Omega)$. We first consider the reference case where $p_N(\Omega)$ is associated with a Lorentzian line. It then reads $p_N(\Omega) = 1/(1+i\Omega/\gamma)$ where γ is the half-width of the line [22] and we immediately get $\kappa_n = Gn!/(2\gamma^n)$ with in particular $\kappa_1 = \tau_g = G/(2\gamma)$, $\kappa_2 = \sigma^2 = G/\gamma^2$ and thus $\tau_g = \sigma\sqrt{G}/2$. The last relation shows that achieving substantial fractional delays τ_g/σ requires that $G \gg 1$. A quite remarkable property of the Lorentzian case is that the impulse response has an exact analytical expression. This result has been obtained by Crisp in a general study on the propagation of small-area pulses in absorbing and amplifying media [23] but it can easily be retrieved from $H(\Omega)$ by using standard procedures of Laplace transforms [20]. One get

$$h(t) = \mathrm{e}^{-A/2}\delta(t) + \mathrm{e}^{-A/2}\gamma G\,\frac{I_1(\sqrt{2G\gamma t})}{\sqrt{2G\gamma t}}\,\mathrm{e}^{-\gamma t}U(t) \tag{4}$$

where $\delta(t)$, $I_1(u)$ and $U(t)$ respectively designate the Dirac function, the 1st order modified Bessel function and the unit step function. The 1st term $h_i(t)$ in $h(t)$ results from the constant value $\mathrm{e}^{-A/2}$ of $H(\Omega)$ far from resonance. This part of the response is instantaneous in our local time picture and only the 2nd term $h_d(t)$, directly associated with the transmission peak, contribute to the delay. The areas of $h_i(t)$ and $h_d(t)$ are respectively $\mathrm{e}^{-A/2}$

5

and $H_0 - e^{-A/2}$, that is in a very small ratio ($\approx e^{-G/2}$) for the large values of G required to achieve substantial delays (see above). The effect of the instantaneous response then be-
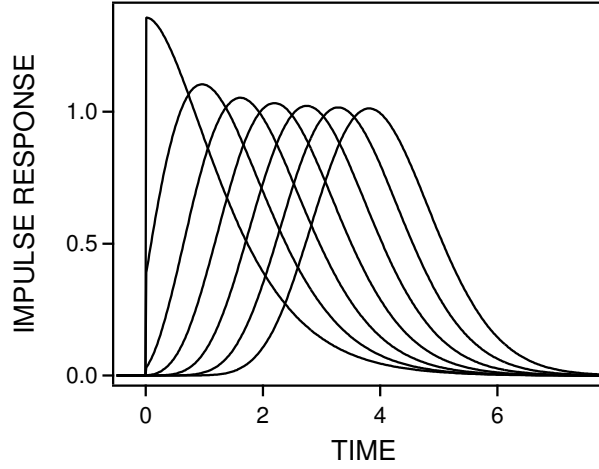

FIG. 1: Analytical form of the impulse response for a Lorentzian line. From the left to the right the gain parameter G (resp. the fractional delay τ_g/σ) is 4, 9, 16, 25, 36, 49 and 64 (resp. 1, 1.5, 2, 2.5, 3, 3.5 and 4). The horizontal (resp. vertical) unit is σ (resp. $H_0/\sigma\sqrt{2\pi}$).

comes negligible. Fig.1 shows the delayed response obtained for increasing values of the gain and thus of the fractional group-delay. We see that the curves, first strongly asymmetric, become more and more symmetric as G increases and that the location of their maximum then approaches the group delay. They have a discontinuity $H_0 e^{-G/2}\gamma G/2$ at $t = 0$, the relative amplitude of which becomes negligible when $G \gg 1$. From the asymptotic behaviour of $I_1(u)$ [20], we then get

$$h_d(t) \approx \frac{H_0}{\sigma\sqrt{2\pi}}\left(1 - \frac{3\theta}{4\tau_g}\right)\exp\left(-\frac{\theta^2}{2\sigma^2}\right) \tag{5}$$

with $\theta = t - \tau_g$. The maximum of $h_d(t)$ occurs at the instant $\tau_g - \Delta t$ with $\Delta t \approx 3\sigma^2/(4\tau_g)$. When $G \to \infty$,

$$h(t) \to h^{(2)}(t) = \frac{H_0}{\sigma\sqrt{2\pi}}\exp\left(-\frac{\theta^2}{2\sigma^2}\right). \tag{6}$$

This Gaussian form is that of the normal distribution derived by means of the central limit theorem in probability theory. This theorem can also be used for an approximate evaluation of the convolution of n deterministic functions [19]. It applies to our case by splitting the medium in n cascaded sections, $h(t)$ being then the convolution of the impulse responses of each section. According to this analysis, one may expect that the normal form $h^{(2)}(t)$

6

is universal. From the frequency viewpoint, it originates in the fact that, when $G \gg 1$, the transmission peak is roughly $\sqrt{G}$ times narrower than the line. In the region where the relative gain $|H(\Omega)|/H_0$ is not negligible, the curves $\Phi(\Omega)$ vs Ω (phase-shift) and $F(\Omega)$ vs Ω (line-profile) are well approximated respectively by a straight line and a parabola (Fig.2). This means that only the first two cumulants $\kappa_1 = \tau_g$ and $\kappa_2 = \sigma^2$ play a significant role.
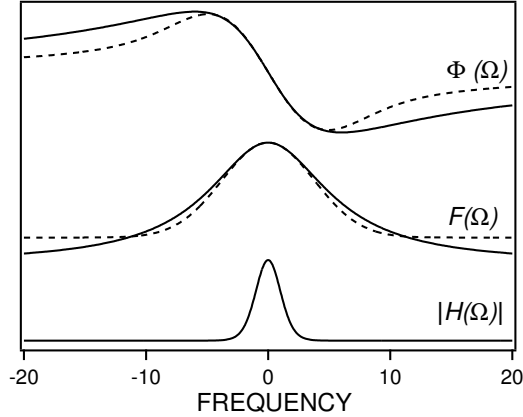

FIG. 2: $\Phi(\Omega)$ and $F(\Omega)$ associated with a Lorentzian line (full line) and a Gaussian line (dashed line). The physical parameters are chosen in order that $H_0 = 0.6$ and $\tau_g/\sigma = 3$ in both cases. The resulting gain profiles $|H(\Omega)|$ are indistinguishable at the figure scale. The frequency unit is $1/\sigma$ (angular frequency).

We then get $H(\Omega) \approx H^{(2)}(\Omega) = H_0 \exp(-i\Omega\tau_g - \sigma^2\Omega^2/2)$ and thus $h(t) \approx h^{(2)}(t)$ irrespective of the line-profile. This confirms the universality of the normal form of $h(t)$ when $G \to \infty$. In fact, $h^{(2)}(t)$ is a good approximation of the exact impulse-response for the gains currently achieved in the experiments. Fig.3 shows the result obtained with Lorentzian and Gaussian line-profiles when $\tau_g = 3\sigma$. In the second case, $p_N(\Omega) = \exp(-\Omega^2/\gamma^2) - 2iD(\Omega/\gamma)/\sqrt{\pi}$ with $D(u) = e^{-u^2}\int_0^u e^{v^2}dv$ [24] and the first cumulants read $\kappa_1 = G/(\gamma\sqrt{\pi})$, $\kappa_2 = G/\gamma^2$ and $\kappa_3 = 4G/(\gamma^3\sqrt{\pi})$. The parameters A, G and γ are chosen such that H_0, τ_g and σ have the same values in both cases. Though the line-profiles are quite different (see Fig.2), the impulse responses are both close to the normal form $h^{(2)}(t)$. Similar results (not shown) are obtained with other line-profiles, including the EIT profile (see hereafter).

When the gain parameter is large but not very large, a better approximation of the impulse response is obtained by considering the effect of the 3rd cumulant κ_3, equal to the asymmetry parameter a of $h(t)$. Provided that this effect may be considered as a

7

small perturbation, $H(\Omega) \approx H^{(3)}(\Omega) \approx (1 + i\kappa_3\Omega^3/3!) H^{(2)}(\Omega)$. From the correspondence $i\Omega \leftrightarrow d/dt$, we finally get

$$h(t) \approx h^{(3)}(t) \approx \left(1 - \frac{a\theta}{2\sigma^4}\right) h^{(2)}(t). \tag{7}$$

This result generalises that obtained in the Lorentzian case where $a = 3G/\gamma^3$ and $a/(2\sigma^4) = 3/(4\tau_g)$. In the Gaussian case, we find $a/(2\sigma^4) = 2/(\pi\tau_g)$, a value not far from the previous one. This explains why the two impulse responses are very close (see Fig.3). Anyway they are very well approximated by $h^{(3)}(t)$ in each case. Quite generally the maximum of $h^{(3)}(t)$ occurs at $\tau_g - \Delta t$ with $\Delta t \approx a/(2\sigma^2) = \xi\sigma/2$ where ξ is the skewness of $h(t)$. In all the above calculations we have implicitly assumed that Δt is small compared to σ. This implies that $|\xi| \ll 2$ but we checked that $h^{(3)}(t)$ keeps a fairly good approximation of $h(t)$ for skewness up to 1.
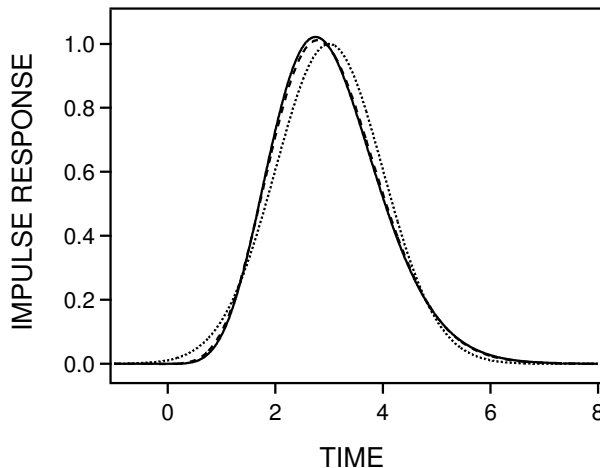

FIG. 3: Comparison of the exact impulse-responses for the Lorentzian profile (full line) and the Gaussian profile (dashed line) to the normal form $h^{(2)}(t)$ (dotted line). The fit by the improved forms $h^{(3)}(t)$ (not shown for clarity) is nearly perfect. Parameters as in Fig.2. Units as in Fig.1.

IV. NORMAL FORM OF THE TRANSMITTED PULSE

The impulse response being known, the envelope of the transmitted pulse is given by the relation $e_{out}(t) = h(t) \otimes e_{in}(t)$ and will generally differ from that of the incident pulse. However the distortion will be negligible if the duration of $h(t)$ is small compared to that of

the pulse. We then get $h(t) \approx \delta(t - \tau_g) \int_{-\infty}^{+\infty} h(t)dt$ and thus $e_{out}(t) \approx H_0 \, e_{in}(t - \tau_g)$. Since we are interested in the situations where the time delay is large compared to the pulse duration, we obtain the double condition $\sigma \ll \sigma_{in} \ll \tau_g$ which can only be met with extremely large gain parameters. Taking for example $\sigma = \sigma_{in}/7$ and $\tau_g = 7\sigma_{in}$ that is $\tau_g = 49\sigma$, we get $G \approx 9600$ in the Lorentzian case. Fig.4 shows the results obtained with these parameters. As expected the pulse distortion is small, even in the sensitive case of a square-shaped pulse. Note that the gain parameter considered is not unrealistic. It is comparable to that used by Harris and co-workers in their pioneering EIT experiment where $G \approx A \approx 6000$ [2].
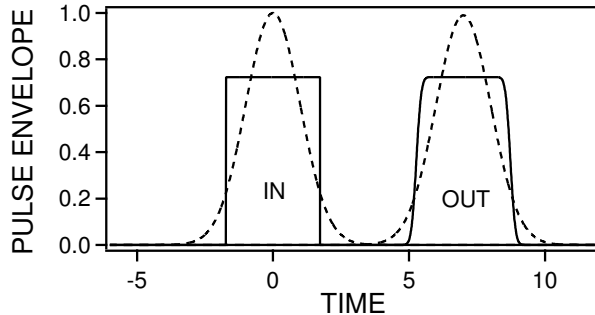

FIG. 4: Propagation of a square-shaped (full line) and a Gaussian-shaped (dashed line) light-pulse with large delay and low distortion. The parameters are chosen in order that $\sigma = \sigma_{in}/7$, $\tau_g = 7\sigma_{in}$ and $H_0 = 1$. The envelopes of the input pulses are given for reference. The time unit is the common rms duration σ_{in} of the two incident pulses. As expected, the distortion of the Gaussian-shaped pulse is negligible and there is only a slight softening of the rise and of the fall of the square-shaped pulse.

However most of the direct demonstrations of large fractional pulse delays have been achieved with smaller gain parameters, typically ranging from 10 to 100, and with incident pulses whose rms duration σ_{in} is comparable to and often smaller than σ. Substantial pulse-reshaping is then expected. Suppose first that the normal form $h^{(2)}(t)$ provides a good approximation of the impulse response and that the incident pulse is Gaussian-shaped. We have then $e_{in}(t) = \exp(-t^2/2\sigma^2)$ and $e_{out}(t)$, convolution of two Gaussian functions, is itself Gaussian. It reads $e_{out}(t) = H_0(\sigma_{in}/\sigma_{out}) \exp(-\theta^2/2\sigma_{out}^2)$, where $\sigma_{out}^2 = \sigma_{in}^2 + \sigma^2$. The effect of the medium on the light-pulse is simply to delay its maximum exactly by the group delay, to broaden it by the factor $\sqrt{1 + \sigma^2/\sigma_{in}^2}$ [1, 9] and to modify its amplitude accordingly in order to respect the area theorem [23]. Since this point is often overlooked, we stress that

9

the broadening mechanism radically differs from that occurring in standard optical fibres [25]. It originates in the 2nd order gain-dispersion instead of in the group-velocity dispersion and the pulse envelope keeps real (no phase modulation or frequency chirping). In fact, provided that σ_{in} be smaller than or comparable to σ and that $|\xi_{in}| < 1$, $e_{out}(t)$ is well approximated by a Gaussian function whatever the shape of the incident pulse is. This is again a consequence of the central limit theorem, the response $e_{out}(t)$ being obtained by an extra convolution added to those used to build $h(t)$. The conditions on σ_{in} and $|\xi_{in}|$ originate in the requirement that all the terms to convolute should have moments of the same order of magnitude. We then obtain $e_{out}(t) \approx e_{out}^{(2)}(t)$ where $e_{out}^{(2)}(t)$ has the normal (Gaussian) form

$$e_{out}^{(2)}(t) = H_0 \frac{S_{in}}{\sigma_{out}\sqrt{2\pi}} \exp\left(-\frac{\theta^2}{2\sigma_{out}^2}\right). \tag{8}$$

This result extends the previous one and shows that incident pulses having different shapes but the same area S_{in} and the same variance σ_{in} are reshaped in the medium to give approximately the same Gaussian-shaped pulse (Fig.5). From an experimental viewpoint, the dramatic reshaping of a square-shaped pulse has been clearly demonstrated (but not commented on) by Turukhin et al. [8] in their EIT experiment in a solid (see their figure 2c for 0 probe detuning). Pulse reshaping is also apparent in the Brillouin scattering experiment by Song et al. [13] where a flat-topped pulse is actually transformed in a gaussian-like pulse (see their figure 4 and compare the shapes obtained for gains 0dB and 30dB).

A more precise approximation $e_{out}^{(3)}(t)$ of $e_{out}(t)$ can be obtained by taking into account the effect of the 3rd order cumulants. Using the approach already used to determine $h^{(3)}(t)$, we get

$$e_{out}^{(3)}(t) \approx \left(1 - \frac{a_{out}\theta}{2\sigma_{out}^4}\right) e_{out}^{(2)}(t) \tag{9}$$

with $a_{out} = a_{in} + a$. When the incident pulse is symmetric ($a_{in} = 0$) as in most experiments, the skewness ξ_{out} of the transmitted pulse reads $\xi_{out} = a/\left(\sigma^2 + \sigma_{in}^2\right)^{3/2}$ and the pulse maximum occurs at $\tau_g - \Delta t_{out}$ with $\Delta t_{out} = a/\left[2\left(\sigma^2 + \sigma_{in}^2\right)\right]$. Since $|\xi_{out}| < |\xi|$, the transmitted pulse is closer to a normal form than the impulse response of the medium. The previous results hold without restriction to the value of σ_{in} when $e_{in}(t)$ is Gaussian. In the case of a Lorentzian line-profile, we easily get

$$\Delta t_{out} = \frac{3}{2\gamma\left(1 + \sigma_{in}^2\gamma^2/G\right)}. \tag{10}$$

We have compared the theoretical delay of the pulse maximum, namely $\tau_g - \Delta t_{out}$, with the delay actually observed by Okawachi et al. in their Brillouin scattering experiment
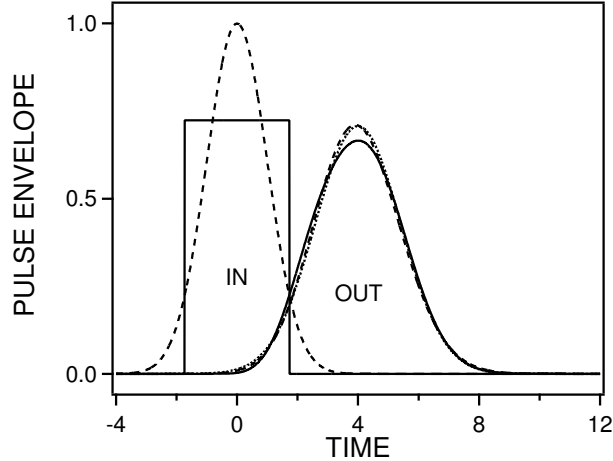
FIG. 5: Example of pulse reshaping and broadening. The square-shaped (full line) and Gaussian-shaped (dashed line) incident pulses originate nearly identical transmitted pulses, respectively close and very close to the normal form $e_{out}^{(2)}(t)$ (dotted line). The time unit is σ_{in} and the parameters are such that $\sigma = \sigma_{in}$, $\tau_g/\sigma = 4$ and $H_0 = 1$. With this choice of H_0 and σ, the transmitted pulses have the same area that the incident ones (area theorem) and a *rms* duration $\sqrt{2}$ times larger.

[14]. Fig.6 shows this delay as a function of the gain parameter G for two values of the pulse duration, respectively $\tau_{in} = 63\,ns$ ($\sigma_{in} \approx 38\,ns$) and $\tau_{in} = 15\,ns$ ($\sigma_{in} \approx 9\,ns$), with $\gamma = 0.22\,ns^{-1}$ (γ is the half of the full Brillouin linewidth Γ_B). Without any adjustment of parameters, our analytical results satisfactorily fit the observations. Note that the shifts Δt_{out} are negligible for the longer pulse but significant for the shorter one.

V.   EFFECT OF THE TRANSMISSION BACKGROUND

In the previous calculations, we have not taken into account the effect of the instantaneous part $h_i(t) = e^{-A/2}\delta(t)$ of the impulse response, arguing that its area is small compared to that of the delayed part. As a matter of fact, $h_i(t)$ originates a contribution $e^{-A/2}e_{in}(t)$ to $e_{out}(t)$, the amplitude of which is roughly $e^{G/2}\sigma_{in}/\sigma_{out}$ times smaller than that of the main part and is thus actually negligible in every case of substantial delay ($G \gg 1$, σ_{out} and σ_{in} of the same order of magnitude). We should however remark that this result lies on the assumption that the transmission peak is put on an uniform background.

We will now examine the case where the transmission background is not uniform. This
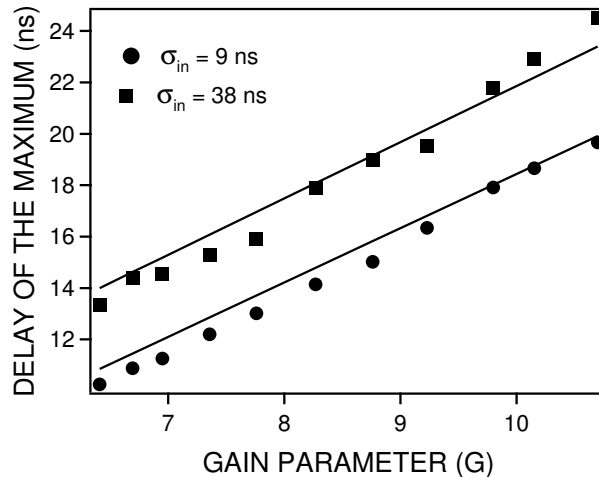
FIG. 6: Comparison of the delays of the maximum of the transmitted pulse observed in a Brillouin scattering experiment [14] (filled squares and circles) with our analytical predictions (full lines).

happens in all the experiments where a transparency-window is induced in a absorption-profile of finite width, in particular in the EIT experiments. As an illustrative example we consider the simplest Λ arrangement with a resonant control field. From the results given in [4], we easily get

$$\Gamma(\Omega) = -\frac{\gamma_{ba}\left(i\Omega + \gamma_{ca}\right)A/2}{\left(i\Omega + \gamma_{ba}\right)\left(i\Omega + \gamma_{ca}\right) + \Omega_s^2/4} \tag{11}$$

where γ_{ba} (resp. $\gamma_{ca} \ll \gamma_{ba}$) is the coherence relaxation-rate for the probe transition (resp. for the forbidden transition), Ω_s is the modulus of the Rabi frequency associated with the control field and $A = \alpha_0 L \gg 1$ is the resonance optical thickness in the absence of control field. The control field makes the resonance gain rise from $\mathrm{e}^{-A/2} \approx 0$ to $H_0 = \exp\left(-A\gamma_{ba}\gamma_{ca}/\left[2\left(\gamma_{ba}\gamma_{ca} + \Omega_s^2/4\right)\right]\right)$ and a good transparency is induced when Ω_s is larger than or comparable to $\gamma_{ba}\gamma_{ca}A$. The width of the transparency-window ($\propto \Omega_s/\sqrt{A}$) is then much smaller than that of the absorption background ($\propto \gamma_{ba}\sqrt{A}$).

Without any approximation, the partial fraction decomposition of $\Gamma(\Omega)$ allows us to write the transfer-function of the medium as a product of simpler functions, namely $H(\Omega) = H_1(\Omega)H_2(\Omega)$ with $H_j(\Omega) = \exp\left[C_j/\left(i\Omega + \gamma_j\right)\right]$. According to the control power, the parameters C_1, C_2, γ_1 and γ_2 are real or complex. When $\Omega_s < \left(\gamma_{ba} - \gamma_{ca}\right)$, γ_1 and γ_2 are real and positive whereas C_1 and C_2 are also real but of opposite sign. The EIT medium is then equivalent to a medium with two Lorentzian lines both centred at $\Omega = 0$, respectively an absorption-line and a narrower gain-line. It is also equivalent to a cascade

12

of a gain medium and an absorbing medium. When $\Omega_s > (\gamma_{ba} - \gamma_{ca})$, all the parameters are complex with $\gamma_2 = \gamma_1^*$ and $C_2 = C_1^*$. The two lines are now located at $\Omega = \pm\mathrm{Im}(\gamma_1)$. They have the same intensity and the same width, but they are hybrid in the sense that, due to the complex nature of C_1 and C_2, their absorption and dispersion profiles are both the sum of an absorption-like and a dispersion-like profile. We incidentally note that the parameters used to obtain the figure 8 in [4] correspond to such a situation. In all cases, the impulse responses associated to $H_1(\Omega)$ and $H_2(\Omega)$ have analytical expressions [23, 26] and the impulse response $h(t)$ of the medium is their convolution product. This general analysis is satisfactory from a formal viewpoint. It provides some physical insight into the EIT mechanisms but is not really operational to determine the shape of the transmitted pulse. From this viewpoint, a fruitful approach consists in exploiting the fact that the medium is opaque except in the narrow region of induced transparency and in the far wings of the background absorption-line (Fig.7). In the first region, $H(\Omega)$ is well approximated by the forms $H^{(2)}(\Omega)$
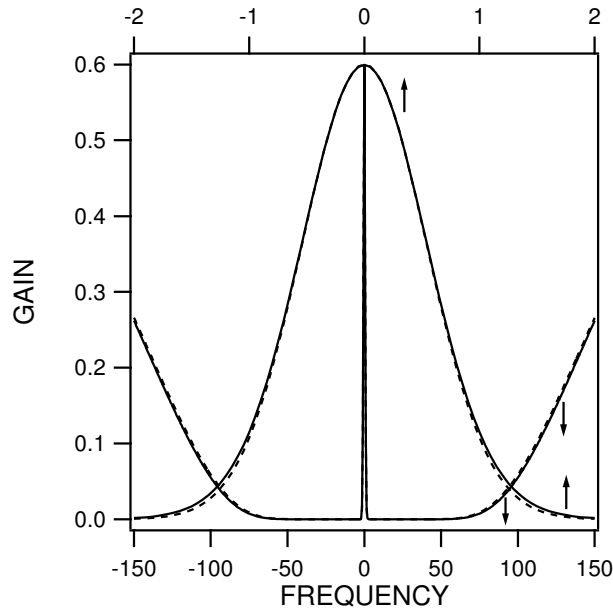

FIG. 7: Gain profile in an EIT experiment and expanded view of its central part (upper scale). In the wings as in the transparency window, the exact gain $|H(\Omega)|$ (full line) is scarcely distinguishable from the approximate form $|H^{(2)}(\Omega) + H_{off}(\Omega)|$ (dashed line). Remind that $|H(\Omega)| \to 1$ in the far wings. The parameters are $A = 63$, $\gamma_{ba}/2\pi = 5$ MHz, $\gamma_{ca}/\gamma_{ba} = 2.2\ 10^{-3}$ and $\Omega_s/\gamma_{ba} = 0.73$. The frequency unit is 10^6 Rd/s (angular frequency).

or, if necessary, $H^{(3)}(\Omega)$, obtained by keeping only the 2 or 3 first cumulants of $H(\Omega)$ as

in the case of an uniform background. We only give here the simplified expressions of these cumulants when $\gamma_{ca} \ll \gamma_{ba}$ and $\Omega_s^2 \gg \gamma_{ca}\gamma_{ba}$ (conditions of good induced transparency). We then get $\kappa_1 = \tau_g \approx 2A\gamma_{ba}/\Omega_s^2$, $\kappa_2 = \sigma^2 \approx 16A\gamma_{ba}^2/\Omega_s^4$ and $\kappa_3 = a \approx 48A\gamma_{ba}(4\gamma_{ba}^2 - \Omega_s^2)/\Omega_s^6$ with $H_0 \approx \exp\left(-2A\gamma_{ba}\gamma_{ca}/\Omega_s^2\right)$. In the far wings $|\Omega| \gg \Omega_s$ and $H(\Omega) \approx H_{off}(\Omega)$, where $H_{off}(\Omega) = \exp\left(-A/[2(1 + i\Omega/\gamma_{ba})]\right)$ is the transfer-function when the control field is off. Finally we get the relation $H(\Omega) \approx H^{(p)}(\Omega) + H_{off}(\Omega)$ with $p = 2$ or 3, valid at every frequency. Fig.7, obtained for typical physical parameters, shows that $\left|H^{(2)}(\Omega) + H_{off}(\Omega)\right|$ already provides a good approximation of the exact gain. Now reduced to a simple sum instead of a convolution product, the medium impulse-response reads $h(t) = h^{(p)}(t) + h_{off}(t)$ where $h_{off}(t)$, associated with a Lorentzian absorption-line, has an analytical expression [23]. As in the case of a gain-line, this expression can be retrieved from $H_{off}(\Omega)$ by using standard procedures of Laplace transforms [20]. It reads

$$h_{off}(t) = \delta(t) - \gamma_{ba}A\frac{J_1(\sqrt{2A\gamma_{ba}t})}{\sqrt{2A\gamma_{ba}t}}\exp\left(-\gamma_{ba}t\right)U(t) \qquad (12)$$

where $J_1(u)$ designates the ordinary 1st order Bessel function. The envelope $e_{out}(t)$ of the transmitted pulse will be thus the sum of two terms. The first one is the approximate solution $e_{out}^{(p)}(t)$ obtained by the cumulants procedure. The second one reads $e_{off}(t) = h_{off}(t) \otimes e_{in}(t)$. It is worth remarking that $h_{off}(t)$ is rapidly oscillating (characteristic time $\propto 1/A\gamma_{ba}$) and that its area is very small ($\int_{-\infty}^{+\infty} h_{off}(t)dt = H_{off}(0) = \mathrm{e}^{-A/2}$). This entails that $e_{off}(t)$ will have a negligible amplitude ($\propto e^{-A/2}$) when $e_{in}(t)$ is smooth enough so that the far wings of its Fourier spectrum do not overlap those of the absorption-line. Fig.8 shows the result obtained with a Gaussian-shaped incident-pulse. A, γ_{ba} and σ_{in} being given, we have chosen the other parameters in order to reproduce the location and the amplitude of the maximum of the transmitted pulse in the celebrated experiment by Hau *et al.* [7]. We then get $\sigma_{out}/\sigma_{in} \approx 1.6$, a broadening consistent with the observations, and $\xi_{out} \approx 0.16$. The asymmetry being very slight, the delay of the maximum is very close to τ_g and $e_{out}(t)$ is well fitted by the normal (Gaussian) form $e_{out}^{(2)}(t)$. A perfect fit is obtained by using the improved form $e_{out}^{(3)}(t)$. The contribution $e_{off}(t)$ associated with $h_{off}(t)$ is actually too small to be visible. Conversely $h_{off}(t)$ will be responsible for the generation of short transients when $e_{in}(t)$ comprises localised defects. As expected and recently discussed about the EIT experiments [27], the front of these transients will propagate at the velocity c (instantaneously in our local time picture). Their peak amplitude will be especially large
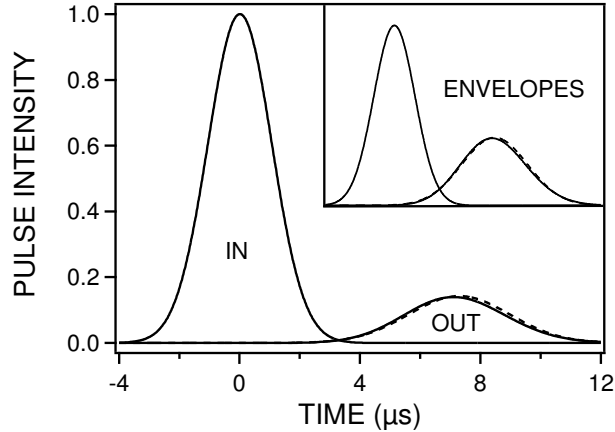
14

FIG. 8: Propagation of a Gaussian-shaped pulse in the EIT experiment. The parameters are as in Fig.7 with $\sigma_{in} = 1.5\,\mu s$. The full and dashed lines respectively are the exact intensity-profile of the pulse and the normal form. The location of the two maximums differ by $0.2\,\mu s$ in agreement with the relation $\Delta t_{out} = a/2\sigma_{out}^2$. Inset : the corresponding pulse-envelopes for $-5\,\mu s \le t \le 15\,\mu s$.

when the defects consist in discontinuities. Consider again a square-shaped incident-pulse, the total duration $2\tau_p$ and the amplitude η of which are such that its area and its variance equal those of the Gaussian-shaped pulse (Fig.9). $e_{off}(t)$ is then easily derived from $h_{off}(t)$. It reads

$$e_{off}(t) = \eta\left[f(t+\tau_p) - f(t-\tau_p)\right] \tag{13}$$

with

$$f(t') = U(t') - \gamma_{ba}A \int_0^{t'} \frac{J_1(\sqrt{2A\gamma_{ba}x})}{\sqrt{2A\gamma_{ba}x}} \exp\left(-\gamma_{ba}x\right)dx. \tag{14}$$

Each discontinuity in $e_{in}(t)$ actually originates a large transient. Its initial amplitude is equal to that of the incident pulse and its successive maximums of intensity occur at the instants $j_{1n}^2/2A\gamma_{ba}$ later, j_{1n} being the nth zero of $J_1(u)$ [28]. Note that the amplitude of the transients exceeds that of the smooth part of $e_{out}(t)$. In a real experiment however the finite values of the rise and fall times of the incident pulse and of the detection bandwidth will generally limit the importance of the transients. By a deliberate reduction of the detection bandwidth, it is even possible to bring their intensity to a very low level without significantly affecting the delayed Gaussian-like part (see inset of Fig.9).

The results obtained on the model EIT-arrangement hold for an extended class of systems having a transparency-window in a wide absorption profile. They only lie on three
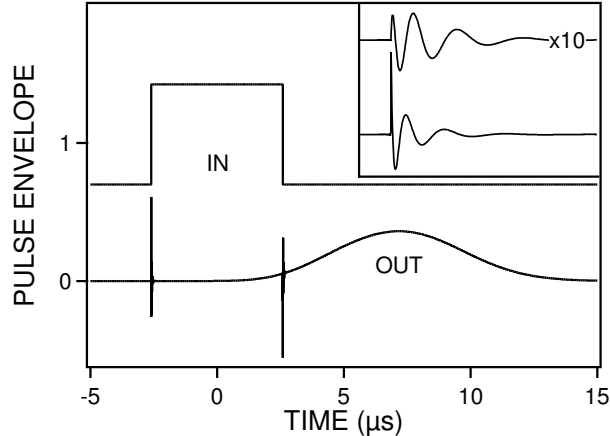
15

FIG. 9: Propagation of a square-shaped pulse in the EIT experiment. Parameters as in the two previous figures. The upper and lower curves are the envelopes of the incident and transmitted pulses. The peak amplitude of the transients is slightly smaller than its theoretical value due to the finite time-resolution of the computations (0.4 ns). Inset : the first transient expanded on a $0.2\mu s$ time-interval (bottom) and the same ($\times 10$) after passage through a 1st order filter of time-constant $\sigma_{out}/100$ (top). Such a filtering does not significantly affect the smooth part of the pulse which keeps very close to that obtained with an incident Gaussian-shaped pulse but reduces the amplitude (resp. the intensity) of the transient by a factor of about 30 (resp. 900).

assumptions: (i) $\Gamma(\Omega)$ is Lorentzian in the far wings of the absorption profile (ii) the opaque regions are much wider that the transparency-window (iii) the transfer-function does not significantly deviate from the normal form in the transparency-window. The first condition (i) is generally met even when $\Gamma(\Omega)$ is not Lorentzian in its central part. Anyway, it is not essential. If it is not met, the detailed shape of the transients is modified but not their main features (instantaneous transmission, duration proportional to the inverse of the spectral width of the opaque regions). The conditions (ii) and (iii), which are closely related, are met in the EIT experiments when the medium transmission is good at the frequency of the optical carrier (see before) but this is not always sufficient. As a counter-example we consider the experiment achieved by Tanaka *et al.* in an atomic vapour with a natural transparency-window between two strong absorption lines [29]. The complex gain-factor reads

$$\Gamma(\Omega) = -\frac{A}{2}\left[\frac{1}{1+i(\Omega+\Delta)/\gamma} + \frac{1}{1+i(\Omega-\Delta)/\gamma}\right] \tag{15}$$

16

where 2Δ is the doublet splitting. Despite an apparent similarity, the associated transfer-function dramatically differs from that obtained in EIT when $\Omega_s > (\gamma_{ba} - \gamma_{ca})$. Indeed the two involved lines are here purely Lorentzian (not hybrid) and a good transparency at $\Omega = 0$ is achieved only if $\Delta \gg \gamma$. We then get $H_0 \approx \exp\left(-A\gamma^2/\Delta^2\right)$, $\tau_g \approx A\gamma/\Delta^2$, $\sigma^2 \approx 6A\gamma^2/\Delta^4$, $a \approx -6A\gamma/\Delta^4$ and $\xi \approx -\Delta^2/(\gamma^2\sqrt{6A})$. Choosing the physical parameters such that H_0, τ_g and σ^2 equal their values in the EIT experiment , we actually obtain a quite different gain profile with opaque regions whose width is smaller than that of the transparency-window (Fig.10). This entails that the approximation $|H(\Omega)| \approx \left|H^{(2)}(\Omega)\right|$
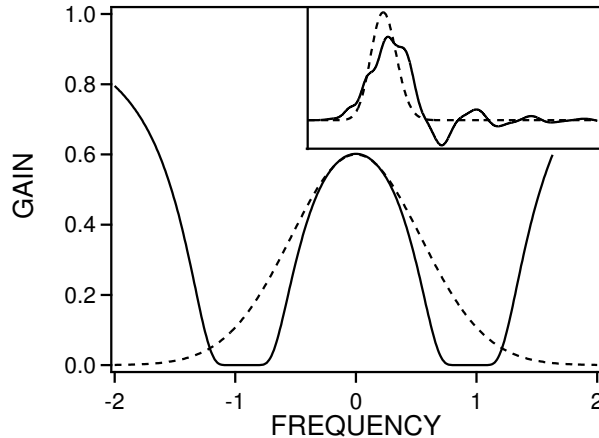

FIG. 10: . Gain-profile $|H(\Omega)|$ for the two absorption-lines arrangement (full line) compared to the corresponding normal gain-profile $|H^{2(}(\Omega)|$ (dotted line). H_0, τ_g, σ and frequency unit as in Fig.7. Inset : envelope of the transmitted pulse for a Gaussian-shaped incident pulse (full line) compared to that obtained with the EIT-arrangement (dotted line). $\sigma_{in} = 1.5\,\mu s$

only works in the immediate vicinity of $\Omega = 0$. It is the same for the law $\Phi(\Omega) \propto \Omega$, the skewness being very large ($\xi \approx -7.6$). In such conditions, even a Gaussian-shaped pulse is strongly distorted (see inset of Fig.10). In fact the two absorption-lines arrangement allows one to attain large fractional delays τ_g/σ_{in} with moderate distortion and broadening but this requires to involve much larger absorption-parameters and to accept a lower transmission. For example, Tanaka *et al.* succeeded in obtaining $\tau_g/\sigma_{in} \approx 13$ with Gaussian-like pulses (see Fig.4c in [29]) but the peak-intensity of the transmitted pulse was 75 times smaller than that of the incident pulse. Their results are well reproduced with our two-lines model by taking $A = 2.6 \times 10^4$ and $\Delta/\gamma = 110$. We then get $\sigma^2/\sigma_{in}^2 \approx 1/25 \ll 1$ and we actually are in a case of low distortion as previously discussed (see Fig.4). However, pulses with discontinuities

17

are excluded. Since $H(\infty) \gg H_0$, the resulting transients (not delayed) would indeed be much larger than the delayed part of the transmitted pulse and would obscure it. Moreover, due to the narrowness of the opaque regions, the time scales of the transients and of the delayed part do not considerably differ and it is thus impossible to filter out the former without denaturing the latter.

VI.   SUMMARY AND DISCUSSION

Privileging the time-domain analysis, we have studied the linear propagation of light pulses, the frequency of which coincides with that of a pronounced maximum in the transmission of the medium. An important point is that substantial pulse-delays are only attained when the corresponding transmission exceeds the minimum one by a very large factor C (contrast). The impulse response of the medium then tends to a normal (Gaussian) form, irrespective of the line profile associated with the transmission peak. The propagation of arbitrarily shaped light-pulses with significant delays and low distortion is possible when the *rms* duration σ of the medium impulse-response is small compared to that of the incident pulse (σ_{in}), which should itself be small compared to the group delay τ_g. The fulfilment of this double condition requires systems where C is extremely large, typically several tens of thousands of dB in a logarithmic scale. Systems with such contrast have actually been used [2] but, in most slow-light experiments, C ranges from 30 to 600 dB. Significant fractional time-delays τ_g/σ_{in} keep attainable with such values of C by using incident pulses, the duration of which is comparable to or smaller than σ. As the medium impulse-response, the transmitted pulse then tends to acquire a normal (Gaussian) shape whatever its initial shape is. This reshaping is particularly striking when the incident pulse is square-shaped but is reduced to a simple broadening when the latter is itself Gaussian-shaped. Despite its asymptotic character, the normal form generally provides a good approximation of the shape of the transmitted pulse. More precise shapes are obtained by a perturbation method, allowing us in particular to specify how much the delay of the pulse-maximum deviates from the group delay.

All these results have been first established by assuming that the transmission peak is put on an uniform background. We have shown that they also apply when the transmission peak is associated with a transparency window in an absorption-profile of finite width.

18

This however requires that the nearly opaque regions flanking the transparency window be considerably wider than the latter. Other things being equal, there are then no differences between the cases of uniform and non uniform transmission-backgrounds, at least when the envelope of the incident pulse is smooth. Conversely localised defects in this envelope will be responsible for the generation of very short transients which complement the normal (Gaussian) part of the signal. The front of the transients is instantaneously propagated in our local time picture (that is at the velocity c in a dilute sample). In extreme cases, their amplitude may be comparable to that of the delayed signal but, due to their location and their duration, they can easily be eliminated without altering the latter.

The slow and fast light experiments have a common feature. In both cases, the observation of significant effects requires media with a very large contrast between the maximum and the minimum of transmission. This results from the causality principle and implies severe limits to the effects attainable in fast-light experiments, whatever the involved system is [30]. From this viewpoint the slow-light case is obviously less pathologic and the constraints, although real, are much softer.

VII.   ACKNOWLEDGEMENTS

[1] S.E. Harris, J.E. Field, and A. Kasapi, Phys. Rev. A **46**, R29 (1992).

[2] A. Kasapi, M. Jain, G.Y. Yin, and S.E. Harris, Phys. Rev. Lett. **74**, 2447 (1995).

[3] A.B. Matsko, O. Kocharovskaya, Y. Rostovtsev, G. R. Welch, A. S. Zibrov, and M. O. Scully, Adv. At. Mol. Opt. Phys. **46**, 191 (2001).

[4] R.W. Boyd and D.J. Gauthier in *Progress in Optics*, E. Wolf, ed. (Elsevier, 2002), Vol. **43**, p.497.

[5] P.W. Milonni, *Fast Light, Slow Light and Left-Handed Light* (IOP, 2005).

[6] M.S. Bigelow, N.N. Lepeshkin, and R.W. Boyd, Phys. Rev. Lett. **90**, 113903 (2003).

[7] L.V. Hau, S.E. Harris, Z. Dutton, and C.H. Behrooz, Nature **397**, 594 (1999).

[8] A.V. Turukhin, V.S. Sudarshanam, M.S. Shahriar, J.A. Musser, B.S. Ham, and P.R. Hemmer, Phys. Rev. Lett. **88**, 023602 (2002)

[9] R. W. Boyd, D. J. Gauthier, A. L. Gaeta, and A. E. Willner, Phys. Rev. A **71**, 023801 (2005).

[10] A.B. Matsko, D.V. Strekalov, and L. Maleki, Optics Express **13**, 2210 (2005)

[11] K. Lee and N.M. Lawandy, Appl. Phys. Lett. **78**, 703 (2001).

[12] D.J. Gauthier, *2nd Annual Summer School, Fitzpatrick Center for Photonics and Communication*, Duke University, Durham, N.C. (2004).

[13] K.Y Song, M. González-Herráez, and L. Thévenaz, Optics Express **13**, 82 (2005)

[14] Y. Okawachi, M.S. Bigelow, J.E. Sharping, Z. Zhu, A. Schweinsberg, D.J. Gauthier, R.W. Boyd, and A.L. Gaeta, Phys. Rev. Lett. **94**, 153902 (2005).

[15] K.Y Song, M. González-Herráez, and L. Thévenaz, Opt. Lett. **30**, 1782 (2005).

[16] M. González-Herráez, K.Y. Song, and L. Thévenaz, Appl. Phys. Lett. **87**, 081113 (2005).

[17] J.E. Sharping, Y. Okawachi, and A.L. Gaeta, Optics Express **13**, 6092 (2005).

[18] N.S. Bukhman, Quant. Electron. **34**, 299 (2004).

[19] A.Papoulis, *The Fourier Integral and its Applications* (McGraw-Hill 1987).

[20] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical functions* (Dover, 1972).

[21] G. Wunsch, Nachrichtentechnik **6**, 244 (1956).

[22] B. Macke and B. Ségard, Eur. Phys. J. D **23**, 121 (2003).

[23] M. D. Crisp, Phys. Rev. A **1**, 1604 (1970).

[24] L. Casperson and A. Yariv, Phys. Rev. Lett. **26**, 293 (1971).

[25] G.P. Agrawal, *Nonlinear Fiber Optics, 3rd Ed.* (Academic Press, 2001).

[26] H.J. Hartmann and A. Laubereau, Opt. Commun. **47**, 117 (1983).

[27] M.D. Stenner, D.J. Gauthier, and M.A. Neifeld, Phys. Rev. Lett. **94**, 053902 (2005).

[28] For an experimental demonstration of these transients, see, e.g., B. Ségard, J. Zemmouri, and B. Macke, Europhys. Lett. **4**, 47 (1987).

[29] H. Tanaka, H. Niwa, K. Hayami, S. Furue, K. Nakayama,T. Kohmoto, M. Kunitomo, and Y. Fukuda, Phys. Rev. A **68**, 053801 (2003).

[30] B. Macke, B Ségard, and F. Wielonsky Phys. Rev. E **72**, 035601(R) (2005).