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Stability of discontinuous diffusion coefficients and initial conditions in an inverse problem for the heat equation

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Abstract

We consider the heat equation with a discontinuous diffusion coefficient and give uniqueness and stability results for both the diffusion coefficient and the initial condition from a measurement of the solution on an arbitrary part of the boundary and at some arbitrary positive time. The key ingredient is the derivation of a Carleman-type estimate. The diffusion coefficient is assumed to be discontinuous across interfaces with a monotonicity condition and piecewise constant.

AMS 2000 subject classification: 35K05, 35R30.

0 Introduction

This paper is devoted to the question of the identification of a diffusion coefficient, c , for a heat transfer problem in a bounded domain, with the main particularity that c is discontinuous. Such regularity can be encountered in the case of embedded materials.

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set. The set $\bar{\Omega}$ is assumed to be a \mathcal{C}^2 submanifold with boundary in \mathbb{R}^n (see e.g. [11, Definition 1.2.1.2]). Let Ω_0 and Ω_1 be two non-empty open subsets of Ω such that

$$\Omega_0 \Subset \Omega, \text{ and } \Omega_1 = \Omega \setminus \bar{\Omega}_0.$$

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We denote by $S = \bar{\Omega}_0 \cap \bar{\Omega}_1$ the interface, which is assumed to be \mathcal{C}^2 .

Let $T > 0$. We shall use the following notations $\Omega' = \Omega_0 \cup \Omega_1$, $Q = \Omega \times (0, T)$, $Q' = \Omega' \times (0, T)$, $Q_i = \Omega_i \times (0, T)$, $i = 0, 1$, and $\Sigma = \Gamma \times (0, T)$. We consider the following transmission problem for the heat equation:

$$(0.1) \quad \begin{cases} \partial_t y - \nabla \cdot (c \nabla y) = 0 & \text{in } Q', \\ y(x, t) = h(t, x) & \text{on } \Sigma, \\ \text{transmission conditions (TC1)} & \text{on } S \times [0, T], \\ y(0, x) = y_0(x), & \text{in } \Omega, \end{cases}$$

with

$$(TC1) \quad \begin{aligned} y|_{S_0 \times [0, T]} &= y|_{S_1 \times [0, T]}, \\ c_0 \partial_n y|_{S_0 \times [0, T]} &= c_1 \partial_n y|_{S_1 \times [0, T]}, \end{aligned}$$

where

$$c = \begin{cases} c_0 & \text{in } \Omega_0, \\ c_1 & \text{in } \Omega_1, \end{cases} \quad \tilde{c} = \begin{cases} \tilde{c}_0 & \text{in } \Omega_0, \\ \tilde{c}_1 & \text{in } \Omega_1. \end{cases}$$

The boundary condition $h(t, x)$ shall be kept fixed. If we change the diffusion coefficient c into \tilde{c} we let \tilde{y} be the solution of (0.1) associated to \tilde{c} and \tilde{y}_0 for initial condition.

We assume that we can measure both the normal flux $\partial_n \partial_t y$ on $\gamma \subset \partial\Omega$ on the time interval (t_0, T) for some $t_0 \in (0, T)$ and Δy in Ω at time $T' \in (t_0, T)$. In the case of piecewise *constant* diffusion coefficients, i.e. $c|_{\Omega_i}$, $i = 0, 1$, is constant, our main results are (i) the injectivity of the map

$$\begin{aligned} L^\infty(\Omega) \times L^2(\Omega) &\rightarrow L^2((t_0, T) \times \gamma) \times L^2(\Omega), \\ (c, y_0) &\mapsto (\partial_n \partial_t y, \Delta y(T')), \end{aligned}$$

(uniqueness); (ii) the stability for the diffusion coefficient, c (Theorem 2.9): there exists $C > 0$ such that

$$|c - \tilde{c}|_{L^\infty(\Omega)}^2 \leq C |\partial_n (\partial_t y - \partial_t \tilde{y})|_{L^2((0, T) \times \gamma)}^2 + C |\Delta y(T', \cdot) - \Delta \tilde{y}(T', \cdot)|_{L^2(\Omega')}^2;$$

and (iii) the stability for the initial condition, y_0 (Theorem 4.5): there exists $C > 0$ such that

$$|y_0 - \tilde{y}_0|_{L^2(\Omega)} \leq C / \ln \left(|(y - \tilde{y})(T')|_{H^2(\Omega')} + |\partial_n (\partial_t y - \partial_t \tilde{y})|_{L^2((0, T) \times \gamma)} \right).$$

The key ingredient to these stability results is a *global* Carleman estimate for the operator $\partial_t - \nabla \cdot (c \nabla (\cdot))$ and the open set Ω .

The use of Carleman estimates to achieve uniqueness and stability results in inverse problems is now well-established. Some authors make use of *local* Carleman inequalities and deduce uniqueness and Hölder estimates (see [17],[16] and references cited therein). Others make use of *global* Carleman inequalities and deduce Lipschitz stability results (and hence uniqueness results). We shall follow this second approach. To our knowledge, this method was first used in [18] and then by others, e.g. [3]. For literature on Carleman estimates we refer

to [12, Chapter VIII], [13, Section: 28.2–3] for local estimates and [15] for the parabolic case. For global estimates we refer to [9] and [8].

Stability results for parabolic equations are recent, to our knowledge, (see [17],[7]). Apart from [18] there are few results on Lipschitz stability, even for linear cases.

One of the main difficulties in the present problem is to deal with discontinuous diffusion coefficients. Controllability for such parabolic equations has been studied by [6]. Null-controllability property is proved via an observability inequality for the adjoint system, which is deduced in [6] from a *global* Carleman estimate yet assuming a *monotonicity* on the coefficients c in connection to the observation location: roughly speaking, the observation zone has to be located in the region where the diffusion coefficient is the smallest. Here, to achieve a stability result we have to derive a Carleman estimate for the difference of the two solutions, y, \tilde{y} . This difference is solution of a non-homogeneous parabolic equation (with discontinuous coefficient); because of the discontinuity of the diffusion coefficients it does not satisfy the appropriate transmission conditions (TC1), on the interfaces S , defined above. For this reason, under the same monotonicity assumption as in [6], we derive a peculiar Carleman estimate which includes additional *interface terms* (see Theorem 1.3).

To obtain a stability result, one has to ‘manage’ the dependence of some constants with respect to (w.r.t.) the parameters, s and λ , that appear in the weight functions used in the Carleman estimate (see (1.4) in Section 1). The interface terms require some careful treatment. In particular, a stationary-phase argument is used to obtain a sufficiently sharp asymptotic estimates of these terms for s and λ large. Usually, stability estimate are obtained by letting the parameter s become large. Here we also make use of the second parameter λ (see Section 3).

The stability result we obtain is valid for one of the solutions, say \tilde{y} , in a particular class of solutions with some regularity and ‘positivity’ properties. To be complete, we prove that this class of function is not empty and give a possible choice of boundary condition $h(t, x)$ to achieve these assumptions.

As we are concerned with parabolic equations, we have to assume the observation of the solution occurs at some positive a time, $T' > 0$. We are convinced that this observation should be replaced by a more realistic one, or at least by an observation in a sub-domain of Ω . This is an open problem and this assumption appears in all papers deriving Lipschitz stability estimates from global Carleman inequalities (see the discussion in the introduction of [18]).

The paper is organized as follows. In Section 1 we derive a Carleman estimate adapted to our problem. In Section 2 we prove a stability result for the piecewise constant diffusion coefficient c when one of the solutions, say \tilde{y} , is in a particular class of solutions. In Section 3 we prove that this class is non empty. In Section 4 we prove a stability result for the initial condition under some additional assumptions, in particular on the initial condition itself. Section 5 is devoted to a generalization of the results to a more complicated geometry, for instance allowing for more than two embedded materials. Appendix A provides some basic regularity properties for the solutions to parabolic equation with non-smooth coefficients and provides a technical lemma.

We now give some notations and important assumptions. We denote by n the outward unit normal to Ω_1 on S and also the outward unit normal to Ω on Γ . Let S_0 (resp. S_1) be the side of the interface S corresponding to the positive (resp. negative) direction of the normal n .

Note that we do not assume that Ω_0 nor Ω_1 is a connected open set. We shall however assume that they are formed with a finite number of connected open sets, say $\Omega_{0,1}, \dots, \Omega_{0,p_0}$, and $\Omega_{1,1}, \dots, \Omega_{1,p_1}$, $p_0, p_1 \in \mathbb{N}$. We shall then denote by S_{ij} the interface (possibly empty) between $\Omega_{0,i}$ and $\Omega_{1,j}$.

We make the following assumption

Assumption 0.1. *The diffusion coefficient satisfies $c_i = c|_{\Omega_i} \in \mathcal{C}^1(\overline{\Omega_i})$, $i = 0, 1$.*

Assumption 0.2. *$c_{0|_S} \geq c_{1|_S}$ and $0 < c_{min} \leq c(x) \leq c_{max}$, $x \in \Omega'$.*

Remark 0.3. Assumption 0.1 will be significantly strengthened in Section 2 to obtain a stability result: namely the diffusion coefficients will be assumed to be piecewise constant. Yet, for some of the results such as the Carleman estimate proved in Section 1 and the regularity properties proved in Section 3, which can be of some use elsewhere, Assumption 0.1 is sufficient.

We let γ be a subset of the boundary Γ satisfying

Assumption 0.4. *The interior of γ is non-empty with respect to the topology on Γ induced by the Euclidean topology on \mathbb{R}^n . Each component of Ω_1 contains part of the interior of γ in its boundary.*

Examples of situations in which Assumption 0.4 is satisfied are given in Figure 1.

To obtain a Carleman estimate we introduce a geometric assumption, following [6].

Assumption 0.5. Geometric Condition (GC)

We assume that there exist two disjoint open subsets $\mathcal{O}^{(1)}, \mathcal{O}^{(2)} \Subset \Omega_0$ and two vector fields, $\zeta^{(i)} \in \mathcal{C}^1(\overline{\Omega_0}, \mathbb{R}^2)$, $i = 1, 2$, such that

$$\begin{aligned} \zeta^{(i)}(x) \cdot n(x) &> 0, \quad \forall x \in S, \quad i = 1, 2, \\ \zeta^{(i)}(x) \cdot n(x) &> 0, \quad \forall x \in \partial\mathcal{O}^{(i)}, \quad i = 1, 2, \\ \zeta^{(i)}(x) &\neq 0, \quad \forall x \in \Omega_0 \setminus \mathcal{O}^{(i)}, \quad i = 1, 2. \end{aligned}$$

Let $x^{(i)}$ be the integral curves of $\zeta^{(i)}$, i.e.

$$\begin{cases} \frac{dx^{(i)}(t)}{dt} = \zeta^{(i)}(x^{(i)}(t)), & t > 0, \\ x^{(i)}(0) = x_0, & x_0 \in S. \end{cases}$$

We also assume that there exists $\mathcal{T} > 0$ such that for all $x_0 \in S$, there exists $t^{(i)}(x_0) < \mathcal{T}$ satisfying

$$\begin{aligned} x^{(i)}(t) &\in \Omega_0 \setminus \mathcal{O}^{(i)}, \quad \text{for } 0 < t < t^{(i)}(x_0), \quad x_0 \in S, \quad i = 1, 2, \\ x^{(i)}(t^{(i)}(x_0)) &\in \partial\mathcal{O}^{(i)}, \quad \text{for } x_0 \in S, \quad i = 1, 2. \end{aligned}$$

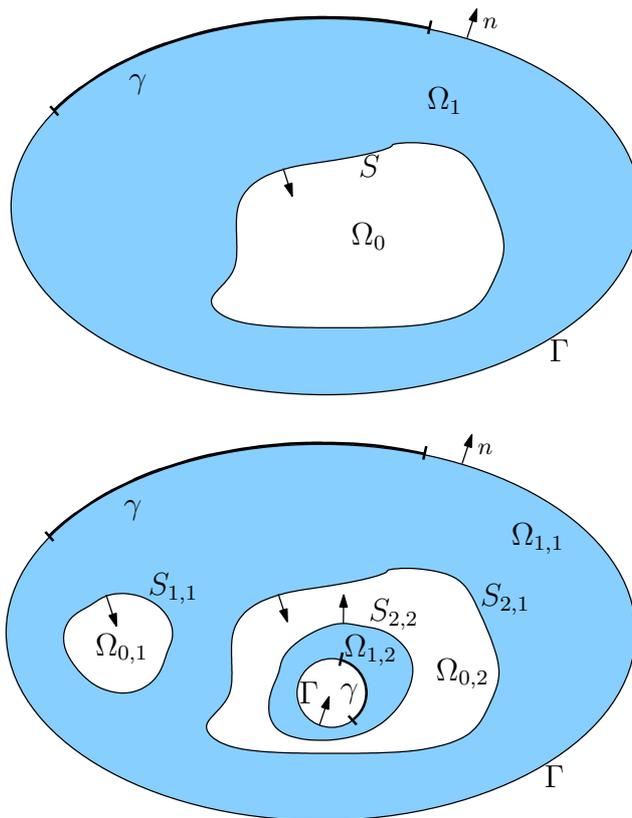


Figure 1: Geometric situations in which Assumption 0.4 and the geometric condition (GC) are satisfied. Shaded is Ω_1 . Arrows represent the normal unit vector n .

Note that in Assumption 0.5, there is no restriction to having Ω_0 composed with p_0 components. The examples given in Figure 1 satisfy Assumption 0.5.

We denote by $W^{m,p}(\Omega)$, $m \in \mathbb{N}$, $1 \leq p \leq \infty$, the usual Sobolev space defined by

$$W^{m,p} = \{u \in L^p(\Omega); \partial^\alpha u \in L^p(\Omega) \text{ for } |\alpha| \leq m\},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index and differentiation is to be understood in the weak sense. As usual we write $H^m(\Omega) = W^{m,2}(\Omega)$. For the definition of $W^{r,p}$, for $r \in \mathbb{R} \setminus \mathbb{N}$ we refer for instance to [1].

1 A Carleman estimate

We prove here a Carleman-type estimate with a boundary term on γ in the right-hand side of the estimate. For this purpose we shall first introduce a particular type of weight functions, which are constructed using the following lemma.

Lemma 1.1. *Assume that there exist two disjoint open subsets $\mathcal{O}^{(1)}, \mathcal{O}^{(2)} \Subset \Omega_0$ satisfying (GC). Let γ be a subset of $\Gamma = \partial\Omega$ satisfying Assumption 0.4 and $B^{(i)}$ and $\tilde{B}^{(i)}$, $i = 1, 2$ be open balls such that $B^{(1)} \Subset \tilde{B}^{(1)} \Subset \mathcal{O}^{(1)}$ and $B^{(2)} \Subset \tilde{B}^{(2)} \Subset \mathcal{O}^{(2)}$. Then there exists two functions $\tilde{\beta}^{(1)}$ and $\tilde{\beta}^{(2)}$ such that*

$$\tilde{\beta}^{(1)}(x) = \begin{cases} \tilde{\beta}_0^{(1)} & \text{in } \Omega_0, \\ \tilde{\beta}_1 & \text{in } \bar{\Omega}_1, \end{cases} \quad \tilde{\beta}^{(2)}(x) = \begin{cases} \tilde{\beta}_0^{(2)} & \text{in } \Omega_0, \\ \tilde{\beta}_1 & \text{in } \bar{\Omega}_1, \end{cases}$$

and the functions $\tilde{\beta}_0^{(1)}, \tilde{\beta}_0^{(2)}$, and $\tilde{\beta}_1$ satisfy the following properties: $\tilde{\beta}_1 \in \mathcal{C}^2(\bar{\Omega}_1)$, $\tilde{\beta}_1 > 0$ in Ω_1 , and

$$\tilde{\beta}_1 = 0 \text{ on } \Gamma \setminus \gamma, \quad \partial_n \tilde{\beta}_1 < 0 \text{ on } \Gamma \setminus \gamma,$$

$$\tilde{\beta}_1 = 2 \text{ on } S, \quad \partial_n \tilde{\beta}_1 < 0 \text{ on } S,$$

and

$$|\nabla \tilde{\beta}_1| > 0 \text{ in } \bar{\Omega}_1;$$

for $i = 1, 2$, $\tilde{\beta}_0^{(i)} \in \mathcal{C}^2(\bar{\Omega}_0)$, $\tilde{\beta}_0^{(i)} > 0$ in Ω_0 ,

$$\tilde{\beta}_0^{(i)} = \tilde{\beta}_1 = 2 \text{ on } S, \quad i = 1, 2,$$

$$c_0 \partial_n \tilde{\beta}_0^{(i)} = c_1 \partial_n \tilde{\beta}_1 \text{ on } S, \quad i = 1, 2,$$

$$(1.1) \quad \tilde{\beta}_0^{(1)} \geq 2\tilde{\beta}_0^{(2)} \text{ in } \tilde{B}^{(2)},$$

$$(1.2) \quad \tilde{\beta}_0^{(2)} \geq 2\tilde{\beta}_0^{(1)} \text{ in } \tilde{B}^{(1)},$$

and

$$(1.3) \quad |\nabla \tilde{\beta}_0^{(i)}| > 0 \text{ in } \bar{\Omega}_0 \setminus B^{(i)}, \quad i = 1, 2.$$

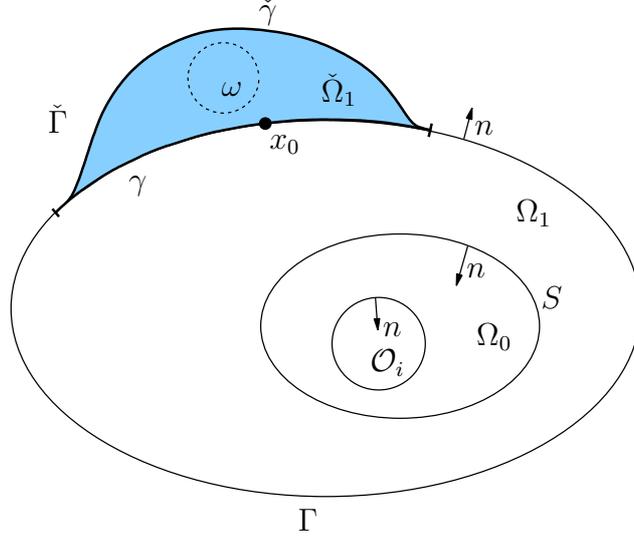


Figure 2: Geometrical situation for the proof Lemma 1.1. The shaded area represents $\check{\Omega}_1 \setminus \Omega_1 = \check{\Omega} \setminus \Omega$.

Proof. For the construction of $\tilde{\beta}_0^{(i)}$, $i = 1, 2$, supported in the connected components of Ω_0 , we refer to [6, Lemma 3.2]. We now show how the function $\tilde{\beta}_1$ is constructed. In actuality, the procedure described here has to be performed in each connected component of Ω_1 , which is possible since each component contains part of the interior of γ in its boundary by Assumption 0.4.

Let x_0 be in the interior of γ . We can enlarge the open set Ω_1 locally around x_0 while preserving the \mathcal{C}^2 regularity of the boundary. Such a procedure is performed in a neighborhood U of x_0 such that $U \cap \Gamma \subset \gamma$. (This can be done by locally straightening out the boundary γ as $\bar{\Omega}$ is assumed to be a \mathcal{C}^2 submanifold with boundary in \mathbb{R}^n [11, Definition 1.2.1.2]). This enlarging procedure only affects γ and leaves $\Gamma \setminus \gamma$ untouched. We call the new boundary $\check{\Gamma}$. We denote $\check{\Omega}_1$ the extension of Ω_1 and $\check{\Omega}$ that of Ω ($\Omega_1 \subset \check{\Omega}_1$, $\Omega \subset \check{\Omega}$ and $\check{\Gamma} = \partial\check{\Omega}$). Let ω be an open subset such that $\omega \Subset \check{\Omega}_1 \setminus \Omega_1$. The geometry we describe here is illustrated in Figure 2.

There exists a function $\alpha \in \mathcal{C}^2(\bar{\Omega}_1)$ satisfying

$$\begin{aligned} \alpha &> 0 \text{ in } \check{\Omega}_1, & \alpha &= 0 \text{ on } \check{\Gamma}, & \alpha &= 2 \text{ on } S, \\ \partial_n \alpha|_{\check{\Gamma}} &< 0, & \partial_n \alpha|_S &< 0. \end{aligned}$$

Following [9, Lemma 1.1, Chapter 1] (see also [6, Lemma 3.1]), we apply Morse theorem [2] and find a so-called Morse function $\mu \in \mathcal{C}^2(\bar{\Omega}_1)$, i.e a \mathcal{C}^2 function with a finite number of critical points, such that

$$\begin{aligned} \mu &> 0 \text{ in } \check{\Omega}_1, & \mu &= 0 \text{ on } \check{\Gamma}, & \mu &= 2 \text{ on } S, \\ \partial_n \mu|_{\check{\Gamma}} &< 0, & \partial_n \mu|_S &< 0. \end{aligned}$$

In fact, there exists neighborhoods of $\tilde{\Gamma}$ and S in $\tilde{\Omega}_1$, say $V_\varepsilon(\tilde{\Gamma})$ and $V_\varepsilon(S)$, in which $\nabla\alpha \geq \alpha_0 > 0$. Since (\mathcal{C}^2) Morse functions are dense in $\mathcal{C}^1(\tilde{\Omega}_0)$ by Morse theorem, we can find a sequence of such functions, say $(\alpha_n)_{n \in \mathbb{N}}$, that converges to α in $\mathcal{C}^1(\tilde{\Omega}_0)$. We define χ in $\mathcal{C}^\infty(\tilde{\Omega}_0)$ such that χ vanishes outside $V_\varepsilon(\tilde{\Gamma})$ and $V_\varepsilon(S)$ and $\chi = 1$ in neighborhoods of $\tilde{\Gamma}$ and S . We then set

$$\mu_n = \chi\alpha + (1 - \chi)\alpha_n.$$

There exists n_0 , such that, for $n \geq n_0$, μ_n is a \mathcal{C}^2 Morse function and $\mu_n = 0$ on $\tilde{\Gamma}$ and $\mu = 2$ on S while $\nabla\mu \neq 0$ in $V_\varepsilon(\tilde{\Gamma}) \cup V_\varepsilon(S)$. Let $\mu = \mu_n$ for $n \geq n_0$.

We call y_i , $i = 1, \dots, p$, the critical points of μ . Since there is only a finite number of them, we can construct, as in [9, Lemma 1.1, Chapter 1], a diffeomorphism, φ , from $\tilde{\Omega}_1$ onto $\tilde{\Omega}_1$, leaving $V_\varepsilon(\tilde{\Gamma})$ and $V_\varepsilon(S)$ unchanged, such that $\varphi^{-1}(y_i) \in \omega$, $i = 1, \dots, p$. It follows that $\tilde{\mu} := \mu \circ \varphi$ has a finite number of critical points all located in ω . The function $\tilde{\beta}_0 := \tilde{\mu}|_{\tilde{\Omega}_1}$ satisfies the required properties. \square

Remark 1.2. Note that the extension of the open set Ω_1 , and thus the extension of Ω , performed in the proof of Lemma 1.1, is solely made for the sake of the construction of the function $\tilde{\beta}_1$. In the sequel all computations will be carried out on Ω .

Choosing two functions $\tilde{\beta}^{(i)}$, $i = 1, 2$, as in the previous lemma, we introduce $\beta^{(i)} = \tilde{\beta}^{(i)} + K^{(i)}$ with $K^{(i)} = m\|\tilde{\beta}^{(i)}\|_\infty$ and $m > 1$. For $\lambda > 0$ and $t \in (t_0, T)$, we define the following weight functions

$$(1.4) \quad \varphi^{(i)}(x, t) = \frac{e^{\lambda\beta^{(i)}(x)}}{(t - t_0)(T - t)}, \quad \eta^{(i)}(x, t) = \frac{e^{\lambda\tilde{\beta}^{(i)}} - e^{\lambda\beta^{(i)}(x)}}{(t - t_0)(T - t)}, \quad i = 1, 2,$$

with $\tilde{\beta}^{(i)} = 2m\|\tilde{\beta}^{(i)}\|_\infty$ (see [6],[8]).

We now prove a Carleman estimate adapted to our problem with an upper bound with a term integrated on a part γ of the boundary Γ of Ω . Let $g \in H^1([t_0, T], H^{\frac{1}{2}}(S))$. We introduce transmission conditions (TC2) on the interval $[t_0, T]$ which reads

$$(TC2) \quad \begin{aligned} q|_{S_0 \times [t_0, T]} &= q|_{S_1 \times [t_0, T]}, \\ c_0 \partial_n q|_{S_0 \times [t_0, T]} &= c_1 \partial_n q|_{S_1 \times [t_0, T]} + g(x, t), \end{aligned}$$

for a function q which is H^2 in each open set Ω_i , $i = 0, 1$. These transmission conditions is to be found in Section 2 for the difference of the solutions $y - \tilde{y}$ to problem (0.1) with c and \tilde{c} , respectively, as diffusion coefficients.

We introduce

$$\mathfrak{N}_g = \left\{ q \in H^1(t_0, T, H_0^1(\Omega)); q|_{(t_0, T) \times \Omega_i} \in L^2(t_0, T, H^2(\Omega_i)), \quad i = 0, 1, \right. \\ \left. q|_\Sigma = 0 \text{ and } q \text{ satisfies (TC2) a.e. w.r.t. } t \right\}.$$

Theorem 1.3. *Let γ be a subset of the boundary Γ of an open set Ω of \mathbb{R}^n that satisfies Condition (GC), and γ satisfies Assumption 0.4. Assume further that $c_0|_S - c_1|_S \geq \Delta > 0$. Let $g \in H^1(t_0, T, H^{\frac{1}{2}}(S))$. There exists*

$\lambda_1 = \lambda_1(\Omega, \gamma, \mathcal{O}^{(1)}, \mathcal{O}^{(2)}, c_{min}, c_{max}, \Delta) > 0$, $s_1 = s_1(\lambda_1) > 0$ and a positive constant $C = C(\Omega, \gamma, \mathcal{O}^{(1)}, \mathcal{O}^{(2)}, c_{min}, c_{max}, \Delta)$ so that the following estimate holds

$$\begin{aligned}
(1.5) \quad & |M_1^{(1)}(e^{-s\eta^{(1)}} q)|_{L^2(Q')}^2 + |M_1^{(2)}(e^{-s\eta^{(2)}} q)|_{L^2(Q')}^2 \\
& + |M_2^{(1)}(e^{-s\eta^{(1)}} q)|_{L^2(Q')}^2 + |M_2^{(2)}(e^{-s\eta^{(2)}} q)|_{L^2(Q')}^2 \\
& + s\lambda^2 \iint_Q (e^{-2s\eta^{(1)}} \varphi^{(1)} + e^{-2s\eta^{(2)}} \varphi^{(2)}) |\nabla q|^2 dx dt \\
& + s^3 \lambda^4 \iint_Q (e^{-2s\eta^{(1)}} \varphi^{(1)3} + e^{-2s\eta^{(2)}} \varphi^{(2)3}) |q|^2 dx dt \\
& \leq C \left[s\lambda \int_{t_0}^T \int_\gamma (e^{-2s\eta^{(1)}} \varphi^{(1)} + e^{-2s\eta^{(2)}} \varphi^{(2)}) |\partial_n q|^2 d\sigma dt \right. \\
& + \iint_{Q'} (e^{-2s\eta^{(1)}} + e^{-2s\eta^{(2)}}) |\partial_t q - \nabla \cdot (c\nabla q)|^2 dx dt \\
& + s\lambda \int_{t_0}^T \int_S (e^{-2s\eta^{(1)}} \varphi^{(1)} + e^{-2s\eta^{(2)}} \varphi^{(2)}) |g|^2 d\sigma dt \\
& + \int_{t_0}^T \int_S (e^{-2s\eta^{(1)}} \varphi^{(1)4} + e^{-2s\eta^{(2)}} \varphi^{(2)4}) |g|^2 d\sigma dt \\
& \left. + s^{-2} \int_{t_0}^T \int_S (e^{-2s\eta^{(1)}} + e^{-2s\eta^{(2)}}) |\partial_t g|^2 d\sigma dt \right],
\end{aligned}$$

for $s \geq s_1$, $\lambda \geq \lambda_1$ and for all $q \in \mathfrak{N}_g$, with M_1 and M_2 to be defined below.

For a function ρ with a trace on the interface S , from both sides, defined in some sense, we shall denote ρ_i the trace of $\rho|_{\Omega_i}$ on S , $i = 0, 1$, when there is no ambiguity; in the case $\rho_0 = \rho_1$ we shall simply write ρ . We shall use the notation $[\rho]_S = \rho_0 - \rho_1$ for the jump of ρ across the interface S . We shall adopt Einstein's summation convention for repeated indices.

Proof. We consider $s > 0$ and $q \in \mathfrak{N}_g$. Let us set $f = \partial_t q - \nabla \cdot (c\nabla q)$. Then $f \in L^2(Q')$.

In the first part of the proof we shall write η , φ , M_1 , etc, in place of $\eta^{(i)}$, $\varphi^{(i)}$, $M_1^{(i)}$, etc, $i = 1, 2$, and treat the two cases at a time. We set $\psi = e^{-s\eta} q$. We observe that $\psi(t_0) = \psi(T) = 0$ and since q satisfies transmission conditions (TC2) (and $q(t, \cdot)|_{\Omega_i} \in H^2(\Omega_i)$ a.e. w.r.t. t), we have (a.e. w.r.t. t)

$$(1.6) \quad c_0 \partial_n \psi_{0|_S}(t, \cdot) = c_1 \partial_n \psi_{1|_S}(t, \cdot) + g_s(t, \cdot),$$

$$(1.7) \quad \nabla_\tau \psi_{0|_S}(t, \cdot) = \nabla_\tau \psi_{1|_S},$$

$$(1.8) \quad \psi_{0|_S}(t, \cdot) = \psi_{1|_S}(t, \cdot),$$

where $g_s = e^{-s\eta} g$ and ∇_τ denotes the component of the gradient that is tangential to S .

The function ψ satisfies in each Ω_i , $i = 0, 1$,

$$M_1 \psi + M_2 \psi = f_s$$

with

$$(1.9) \quad M_1\psi = \nabla \cdot (c\nabla\psi) + s^2\lambda^2\varphi^2|\nabla\beta|^2c\psi + s(\partial_t\eta)\psi,$$

$$(1.10) \quad M_2\psi = \partial_t\psi - 2s\lambda\varphi c\nabla\beta \cdot \nabla\psi - 2s\lambda^2\varphi c|\nabla\beta|^2\psi,$$

$$(1.11) \quad f_s = e^{-s\eta}f + s\lambda\varphi\nabla \cdot (c\nabla\beta)\psi - s\lambda^2\varphi c|\nabla\beta|^2\psi.$$

We have

$$|M_1\psi|_{L^2(Q')}^2 + |M_2\psi|_{L^2(Q')}^2 + 2\operatorname{Re}(M_1\psi, M_2\psi)_{L^2(Q')} = |f_s|_{L^2(Q')}^2$$

With the same notations as in [6, Theorem 3.3], we write $\operatorname{Re}(M_1\psi, M_2\psi)_{L^2(Q')}$ as a sum of 9 terms I_{ij} , $1 \leq i, j \leq 3$, where I_{ij} is the inner product of the i^{th} term in the expression of $M_1\psi$ and the j^{th} term in the expression of $M_2\psi$.

As compared to the proof of the Carleman estimate in [6, Theorem 3.3] we only need to adjust the computation of I_{11} , I_{12} and I_{13} to the present case. In fact the other terms do not involve transmission conditions (1.6) in their computation and thus remain unchanged from the terms obtained in [6].

The term I_{11} follows as

$$\begin{aligned} I_{11} &= \operatorname{Re} \iint_{Q'} \nabla \cdot (c\nabla\psi) \partial_t\bar{\psi} \, dxdt \\ &= -\operatorname{Re} \iint_{Q'} c\nabla\psi \cdot \partial_t(\nabla\bar{\psi}) \, dxdt + \sum_{i=0,1} (-1)^{i+1} \operatorname{Re} \int_{t_0}^T \int_S c_i \partial_n \psi_i \partial_t\bar{\psi} \, d\sigma dt, \end{aligned}$$

by integration by parts; the surface integral on Γ vanishes since $\partial_t\psi = 0$ there. Noting that $\operatorname{Re}\nabla\psi \cdot \partial_t(\nabla\bar{\psi}) = \frac{1}{2}\partial_t(|\nabla\psi|^2)$, the first term vanishes since ψ , and thus $\nabla\psi$, vanish at $t = t_0$ and $t = T$. For the remaining surface terms we use (1.6), which yields

$$I_{11} = -\operatorname{Re} \int_{t_0}^T \int_S g_s \partial_t\bar{\psi} \, d\sigma dt = \operatorname{Re} \int_{t_0}^T \int_S \partial_t(g_s) \bar{\psi} \, d\sigma dt,$$

since $g_s \in H^1(t_0, T, H^{\frac{1}{2}}(S))$.

The term I_{12} is given by

$$\begin{aligned} I_{12} &= -2s\lambda\operatorname{Re} \iint_{Q'} \varphi\nabla \cdot (c\nabla\psi)c\nabla\beta \cdot \nabla\bar{\psi} \, dxdt \\ &= 2s\lambda\operatorname{Re} \iint_{Q'} c\nabla\psi \cdot \nabla(\varphi c\nabla\beta \cdot \nabla\bar{\psi}) \, dxdt - 2s\lambda\operatorname{Re} \int_{t_0}^T \int_{\Gamma} \varphi c^2(\nabla\beta \cdot \nabla\bar{\psi})(\partial_n\psi) \, d\sigma dt \\ &\quad + 2s\lambda\operatorname{Re} \sum_{i=0,1} (-1)^i \int_{t_0}^T \int_S \varphi c_i^2(\nabla\beta_i \cdot \nabla\bar{\psi}_i)(\partial_n\psi_i) \, d\sigma dt. \end{aligned}$$

This integration by parts is justified since $\psi(t, \cdot)$ is in H^2 in each Ω_i , $i = 0, 1$. Denoting by I'_{12} the remaining volume integral, we obtain

$$\begin{aligned} I'_{12} &= 2s\lambda^2 \iint_{Q'} \varphi c^2|\nabla\psi \cdot \nabla\beta|^2 \, dxdt + 2s\lambda\operatorname{Re} \iint_{Q'} \varphi c\partial_{x_i}(c\partial_{x_j}\beta)\partial_{x_i}\psi\partial_{x_j}\bar{\psi} \, dxdt \\ &\quad + s\lambda \iint_{Q'} \varphi c^2\partial_{x_j}\beta\partial_{x_j}|\nabla\psi|^2 \, dxdt. \end{aligned}$$

We further compute the last volume integral, denoted by I''_{12} . Observe that $|\nabla\psi|_{|\Omega_i}^2$ is in $W^{1,1}(\Omega_i)$ since $\psi|_{\Omega_i}(t, \cdot) \in H^2(\Omega_i)$, $i = 0, 1$. This allows to further integrate by parts, since $(c^2\varphi\partial_{x_j}\beta)|_{\Omega_i} \in \mathcal{C}^1(\Omega_i)$, $i = 0, 1$, and yields

$$\begin{aligned} I''_{12} = & -s\lambda \iint_{Q'} \partial_{x_j}(\varphi c^2 \partial_{x_j} \beta) |\nabla\psi|^2 dxdt + s\lambda \int_{t_0}^T \int_{\Gamma} \varphi c^2 \partial_n \beta |\nabla\psi|^2 d\sigma dt \\ & + s\lambda \sum_{i=0,1} (-1)^{i+1} \int_{t_0}^T \int_S \varphi c_i^2 \partial_n \beta_i |\nabla\psi_i|^2 d\sigma dt. \end{aligned}$$

The remaining volume integral can be further expanded into

$$I'''_{12} = -s\lambda \iint_{Q'} \varphi \partial_{x_j} (c^2 \partial_{x_j} \beta) |\nabla\psi|^2 dxdt - s\lambda^2 \iint_{Q'} \varphi c^2 |\nabla\beta|^2 |\nabla\psi|^2 dxdt.$$

Collecting the surface integrals in a term denoted by J_{12} we find

$$\begin{aligned} I_{12} = & -s\lambda^2 \iint_{Q'} \varphi c^2 |\nabla\beta|^2 |\nabla\psi|^2 dxdt + 2s\lambda^2 \iint_{Q'} c^2 \varphi |\nabla\psi \cdot \nabla\beta|^2 dxdt \\ & + X_1 + J_{12}, \end{aligned}$$

where

$$\begin{aligned} X_1 = & 2s\lambda \operatorname{Re} \iint_{Q'} \varphi c \partial_{x_i} (c \partial_{x_j} \beta) \partial_{x_i} \psi \partial_{x_j} \bar{\psi} dxdt \\ & - s\lambda \iint_{Q'} \varphi \partial_{x_j} (c^2 \partial_{x_j} \beta) |\nabla\psi|^2 dxdt. \end{aligned}$$

We now observe that since β is constant on S we have

$$(\nabla\beta \cdot \nabla\psi_i)|_S = (\partial_n \beta \partial_n \psi_i)|_S, \quad i = 0, 1.$$

Writing $|\nabla\psi|^2 = |\nabla_\tau\psi|^2 + |\partial_n\psi|^2$ we find

$$\begin{aligned} J_{12} = & s\lambda \sum_{i=0,1} (-1)^i \int_{t_0}^T \int_S \varphi c_i^2 \partial_n \beta_i |\partial_n \psi_i|^2 d\sigma dt \\ & - s\lambda \sum_{i=0,1} (-1)^i \int_{t_0}^T \int_S \varphi c_i^2 \partial_n \beta_i |\nabla_\tau \psi_i|^2 d\sigma dt - s\lambda \int_{t_0}^T \int_{\Gamma} \varphi c^2 \partial_n \beta |\partial_n \psi|^2 d\sigma dt \end{aligned}$$

where we have used that $\psi|_\Sigma$ is constant. Recall that $\nabla_\tau\psi_0 = \nabla_\tau\psi_1$, and that $c_0\partial_n\beta_0 = c_1\partial_n\beta_1$. From transmission conditions (TC2) we have

$$|c_0\partial_n\psi_0|^2 = |c_1\partial_n\psi_1|^2 + |g_s|^2 + 2\operatorname{Re}(c_1(\partial_n\bar{\psi}_1)g_s).$$

We thus obtain

$$\begin{aligned} J_{12} = & s\lambda \int_{t_0}^T \int_S \varphi [\partial_n \beta]_S |c_1 \partial_n \psi_1|^2 d\sigma dt \\ & + s\lambda \int_{t_0}^T \int_S \varphi \partial_n \beta_0 |g_s|^2 d\sigma dt - s\lambda \int_{t_0}^T \int_S \varphi [c]_S (c \partial_n \beta) |\nabla_\tau \psi|^2 d\sigma dt \\ & - s\lambda \int_{t_0}^T \int_{\Gamma} \varphi c^2 \partial_n \beta |\partial_n \psi|^2 d\sigma dt + Y_1 \end{aligned}$$

with

$$(1.12) \quad Y_1 = 2s\lambda \operatorname{Re} \int_{t_0}^T \int_S \varphi c_1 \partial_n \bar{\psi}_1 \partial_n \beta_0 g_s \, d\sigma dt.$$

We thus have

$$\begin{aligned} I_{12} = & -s\lambda \iint_{Q'} \partial_{x_j} (c^2 \varphi \partial_{x_j} \beta) |\nabla \psi|^2 \, dx dt + 2s\lambda^2 \iint_{Q'} c^2 \varphi |\nabla \psi \cdot \nabla \beta|^2 \, dx dt \\ & + s\lambda \int_{t_0}^T \int_S \varphi [\partial_n \beta]_S |c_1 \partial_n \psi_1|^2 \, d\sigma dt + s\lambda \int_{t_0}^T \int_S \varphi \partial_n \beta_0 |g_s|^2 \, d\sigma dt \\ & - s\lambda \int_{t_0}^T \int_S \varphi [c]_S (c \partial_n \beta) |\nabla_\tau \psi|^2 \, d\sigma dt - s\lambda \int_{t_0}^T \int_\Gamma \varphi c^2 \partial_n \beta |\partial_n \psi|^2 \, d\sigma dt \\ & + X_1 + Y_1. \end{aligned}$$

The term I_{13} is given by

$$\begin{aligned} I_{13} = & -2s\lambda^2 \operatorname{Re} \iint_{Q'} \varphi \nabla \cdot (c \nabla \psi) c |\nabla \beta|^2 \bar{\psi} \, dx dt \\ & = 2s\lambda^2 \operatorname{Re} \iint_{Q'} c \nabla \psi \cdot \nabla (\varphi c |\nabla \beta|^2 \bar{\psi}) \, dx dt \\ & + 2s\lambda^2 \operatorname{Re} \sum_{i=0,1} (-1)^i \int_{t_0}^T \int_S \varphi (c_i \partial_n \psi_i) c_i |\nabla \beta_i|^2 \bar{\psi} \, d\sigma dt, \end{aligned}$$

where we have used that $\psi|_\Gamma = 0$. Expanding the integrand in the volume integral and using (TC2) in the surface term we obtain

$$I_{13} = 2s\lambda^2 \iint_{Q'} \varphi c^2 |\nabla \beta|^2 |\nabla \psi|^2 \, dx dt + X_2 + Y_2$$

where

$$\begin{aligned} X_2 = & 2s\lambda^2 \operatorname{Re} \iint_{Q'} \varphi c \nabla \psi \cdot \nabla (c |\nabla \beta|^2) \bar{\psi} \, dx dt \\ & + 2s\lambda^3 \operatorname{Re} \iint_{Q'} \varphi c^2 \nabla \psi \cdot \nabla \beta |\nabla \beta|^2 \bar{\psi} \, dx dt \\ & + 2s\lambda^2 \operatorname{Re} \int_{t_0}^T \int_S \varphi (c \partial_n \beta) [\partial_n \beta]_S (c_1 \partial_n \psi_1) \bar{\psi} \, d\sigma dt, \end{aligned}$$

since $\nabla_\tau \beta|_S = 0$ and

$$Y_2 = 2s\lambda^2 \operatorname{Re} \int_{t_0}^T \int_S \varphi c_0 (\partial_n \beta_0)^2 g_s \bar{\psi} \, d\sigma dt.$$

Following the proof of Theorem 3.3 in [6] we find

$$I_{21} = \frac{1}{2} s^2 \lambda^2 \iint_{Q'} \varphi^2 c |\nabla \beta|^2 \partial_t |\psi|^2 \, dx dt = -\frac{1}{2} s^2 \lambda^2 \iint_{Q'} \partial_t (\varphi^2) c |\nabla \beta|^2 |\psi|^2 \, dx dt,$$

and

$$\begin{aligned} I_{22} &= -s^3 \lambda^3 \iint_{Q'} \varphi^3 c^2 |\nabla \beta|^2 \nabla \beta \cdot \nabla (|\psi|^2) dxdt \\ &= 3s^3 \lambda^4 \iint_{Q'} \varphi^3 c^2 |\nabla \beta|^4 |\psi|^2 dxdt + s^3 \lambda^3 \int_{t_0}^T \int_S \varphi^3 |c \partial_n \beta|^2 [\partial_n \beta]_S |\psi|^2 d\sigma dt + X_3, \end{aligned}$$

with X_3 given by

$$X_3 = s^3 \lambda^3 \iint_{Q'} \varphi^3 \nabla \cdot (c^2 |\nabla \beta|^2 \nabla \beta) |\psi|^2 dxdt.$$

The terms I_{23} , I_{31} are given by

$$I_{23} = -2s^3 \lambda^4 \iint_{Q'} \varphi^3 c^2 |\nabla \beta|^4 |\psi|^2 dxdt,$$

$$I_{31} = \frac{1}{2} s \iint_{Q'} \partial_t \eta \partial_t (|\psi|^2) dxdt = -\frac{1}{2} s \iint_{Q'} \partial_t^2 \eta |\psi|^2 dxdt$$

The term I_{32} is given by

$$\begin{aligned} I_{32} &= -s^2 \lambda \iint_{Q'} \varphi (\partial_t \eta) c \nabla \beta \cdot \nabla (|\psi|^2) dxdt = s^2 \lambda^2 \iint_{Q'} \varphi (\partial_t \eta) c |\nabla \beta|^2 |\psi|^2 dxdt \\ &\quad + s^2 \lambda \iint_{Q'} \varphi \nabla \cdot ((\partial_t \eta) c \nabla \beta) |\psi|^2 dxdt, \end{aligned}$$

since $\psi_0 = \psi_1$. Finally the term I_{33} is given by

$$I_{33} = -2s^2 \lambda^2 \iint_{Q'} \varphi c (\partial_t \eta) |\nabla \beta|^2 |\psi|^2 dxdt.$$

Collection the terms I_{ij} just computed we obtain

$$\begin{aligned} (1.13) \quad & |M_1 \psi|_{L^2(Q')}^2 + |M_2 \psi|_{L^2(Q')}^2 + 4s \lambda^2 \iint_{Q'} c^2 \varphi |\nabla \psi \cdot \nabla \beta|^2 dxdt \\ & + 2s \lambda^2 \iint_{Q'} \varphi c^2 |\nabla \beta|^2 |\nabla \psi|^2 dxdt + 2s^3 \lambda^4 \iint_{Q'} \varphi^3 c^2 |\nabla \beta|^4 |\psi|^2 dxdt \\ & + 2s \lambda \int_{t_0}^T \int_S \varphi [\partial_n \beta]_S |c_1 \partial_n \psi_1|^2 d\sigma dt - 2s \lambda \int_{t_0}^T \int_\Gamma \varphi c^2 \partial_n \beta |\partial_n \psi|^2 d\sigma dt \\ & - 2s \lambda \int_{t_0}^T \int_S \varphi [c]_S (c \partial_n \beta) |\nabla_\tau \psi|^2 d\sigma dt + 2s^3 \lambda^3 \int_{t_0}^T \int_S \varphi^3 |c \partial_n \beta|^2 [\partial_n \beta]_S |\psi|^2 d\sigma dt \\ & \quad + 2s \lambda \int_{t_0}^T \int_S \varphi \partial_n \beta_0 |g_s|^2 d\sigma dt \\ & = |f_s|_{L^2(Q')}^2 - 2(I_{11} + X_1 + Y_1 + X_2 + Y_2 + I_{21} + X_3 + I_{31} + I_{32} + I_{33}). \end{aligned}$$

We now consider the surface terms I_{11} , Y_1 , Y_2 involving the function g_s and write

$$(1.14) \quad |I_{11}| = \left| \operatorname{Re} \int_{t_0}^T \int_S \partial_t (g_s) \psi d\sigma dt \right| \leq C s^{-2} \int_{t_0}^T \int_S |\partial_t g_s|^2 d\sigma dt + C s^2 \int_{t_0}^T \int_S |\psi|^2 d\sigma dt.$$

In the proof of Lemma 1.1 we are free to choose β such that $\partial_n \beta_1 / c_0 \leq -1$. Since we assume $c_0 - c_1 \geq \Delta$ we obtain

$$(1.15) \quad [\partial_n \beta]_S = \partial_n \beta_0 - \partial_n \beta_1 = \frac{\partial_n \beta_1}{c_0} (c_1 - c_0) \geq \Delta > 0.$$

The second term in (1.14) can thus be absorbed by the term

$$2s^3 \lambda^3 \int_{t_0}^T \int_S \varphi^3 |c \partial_n \beta|^2 [\partial_n \beta]_S |\psi|^2 d\sigma dt$$

in (1.13) for s sufficiently large.

The term Y_1 in (1.12) can be estimated by

$$(1.16) \quad |Y_1| = \left| 2s\lambda \operatorname{Re} \int_{t_0}^T \int_S \varphi c_1 \partial_n \bar{\psi}_1 \partial_n \beta_0 g_s d\sigma dt \right| \\ \leq C_\varepsilon s \lambda \int_{t_0}^T \int_S \varphi |g_s|^2 d\sigma dt + \varepsilon s \lambda \int_{t_0}^T \int_S \varphi |c_1 \partial_n \psi_1|^2 (\partial_n \beta_0)^2 d\sigma dt, \quad \varepsilon > 0.$$

For ε sufficiently small, the second surface term in (1.16) can be ‘absorbed’ by the term

$$2s\lambda \int_{t_0}^T \int_S \varphi [\partial_n \beta]_S |c_1 \partial_n \psi_1|^2 d\sigma dt$$

in (1.13) by (1.15).

The term Y_2 can be estimated by

$$|Y_2| = \left| 2s\lambda^2 \operatorname{Re} \int_{t_0}^T \int_S \varphi c_0 (\partial_n \beta_0)^2 g_s \bar{\psi} d\sigma dt \right| \leq C s \lambda \int_{t_0}^T \int_S \varphi |g_s|^2 d\sigma dt \\ + C s \lambda^3 \int_{t_0}^T \int_S \varphi c_0^2 (\partial_n \beta_0)^4 |\psi|^2 d\sigma dt.$$

Observing that $\varphi \leq CT^4 \varphi^3$, the second surface term can be ‘absorbed’ by the term

$$2s^3 \lambda^3 \int_{t_0}^T \int_S \varphi^3 |c \partial_n \beta|^2 [\partial_n \beta]_S |\psi|^2 d\sigma dt$$

in (1.13) for s sufficiently large by (1.15). The two previous estimates are the points in the proof where the hypothesis $c_0 - c_1 \geq \Delta > 0$ is needed.

Note also that

$$s^{-2} |\partial_t g_s|^2 \leq C s^{-2} e^{-2s\eta} |\partial_t g|^2 + C (\partial_t \eta)^2 e^{-2s\eta} |g|^2 \\ \leq C s^{-2} e^{-2s\eta} |\partial_t g|^2 + CT^2 \varphi^4 e^{-2s\eta} |g|^2,$$

where we have used that $|\partial_t \eta| \leq CT\varphi^2$ [6, equation (90)] (which makes use of the particular choices made above for $K^{(i)}$ and $\bar{\beta}^{(i)}$, $i = 1, 2$ which implies that $\bar{\beta} \leq 2\beta$).

Applying the technique presented in the proof of Theorem 3.3 in [6], the previous observations yield the following Carleman estimate (we use the notation $\eta^{(i)}$ instead of η)

$$\begin{aligned}
(1.17) \quad & |M_1^{(i)}(e^{-s\eta^{(i)}} q)|_{L^2(Q')}^2 + |M_2^{(i)}(e^{-s\eta^{(i)}} q)|_{L^2(Q')}^2 \\
& + s\lambda^2 \iint_Q e^{-2s\eta^{(i)}} \varphi^{(i)} |\nabla q|^2 dx dt + s^3 \lambda^4 \iint_Q e^{-2s\eta^{(i)}} \varphi^{(i)3} |q|^2 dx dt \\
\leq C & \left[s\lambda \int_{t_0}^T \int_\gamma e^{-2s\eta^{(i)}} \varphi^{(i)} |\partial_n q|^2 d\sigma dt + s^3 \lambda^4 \int_{t_0}^T \int_{\tilde{B}_i} e^{-2s\eta^{(i)}} \varphi^{(i)3} |q|^2 dx dt \right. \\
& + \iint_{Q'} e^{-2s\eta^{(i)}} |\partial_t q - \nabla \cdot (c\nabla q)|^2 dx dt + s^{-2} \int_{t_0}^T \int_S e^{-2s\eta^{(i)}} |\partial_t g|^2 d\sigma dt \\
& \left. + \int_{t_0}^T \int_S e^{-2s\eta^{(i)}} \varphi^{(i)4} |g|^2 d\sigma dt + s\lambda \int_{t_0}^T \int_S e^{-2s\eta^{(i)}} \varphi^{(i)} |g|^2 d\sigma dt \right],
\end{aligned}$$

for $i = 1, 2$. Note that the condition $[c]_s \geq 0$ is needed to obtain the previous estimate.

Adding (1.17) for $i = 1, 2$, we deduce (1.5) with the same argumentation as in the proof of Theorem 3.4 in [6]. The terms integrated over $\tilde{B}_i \times (t_0, T)$ are absorbed by other terms using properties (1.1)–(1.3) of $\tilde{\beta}^{(i)}$, $i = 1, 2$. \square

Remark 1.4. The Carleman estimate that was just derived is peculiar because of the presence of terms integrated on the interface S . In particular, two terms involve the function g with different powers for the parameters s and λ and for the weight functions $\varphi^{(i)}$, $i = 1, 2$. This Carleman estimate is the key ingredient in the subsequent analysis. The interface terms will require some special treatment. The two parameters s and λ will also have an important role to play in the next section.

Remark 1.5. In the case $g = 0$, the previous Carleman estimate simplifies. By inspection of the proof of Theorem 1.3, observe that in the case $g = 0$, there is no need to assume $c_{0|_S} - c_{1|_S} \geq \Delta > 0$ to obtain the Carleman estimate.

2 Uniqueness and stability estimate for the diffusion coefficients

In this section we establish a uniqueness result for the discontinuous diffusion coefficient c as well as a stability inequality. This inequality estimates the discrepancy in the coefficients c and \tilde{c} of two materials (with the same geometry) with an upper bound given by some Sobolev norms of the difference between the solutions y and \tilde{y} to

$$(2.1) \quad \begin{cases} \partial_t \tilde{y} - \nabla \cdot (\tilde{c} \nabla \tilde{y}) = 0, & \text{in } Q, \\ \tilde{y}(t, x) = h(t, x), & \text{on } \Sigma, \\ \text{transmission conditions (TC1),} & \text{on } S \times [0, T], \\ \tilde{y}(0) = \tilde{y}_0, & \end{cases}$$

and

$$(2.2) \quad \begin{cases} \partial_t y - \nabla \cdot (c \nabla y) = 0, & \text{in } Q, \\ y(t, x) = h(t, x), & \text{on } \Sigma, \\ \text{transmission conditions (TC1),} & \text{on } S \times [0, T], \\ y(0) = y_0, & \end{cases}$$

The Carleman estimate proved in the previous section will be the key ingredient in the proof of such a stability estimate.

We introduce

$$\xi = c - \tilde{c} = \begin{cases} \xi_0 = c_0 - \tilde{c}_0 & \text{in } \Omega_0, \\ \xi_1 = c_1 - \tilde{c}_1 & \text{in } \Omega_1. \end{cases}$$

We set $u = y - \tilde{y}$ and $v = \partial_t u$. Then v solution to the following problem

$$(2.3) \quad \begin{cases} \partial_t v - \nabla \cdot (c \nabla v) = \nabla \cdot (\xi \nabla \partial_t \tilde{y}), & \text{in } Q', \\ v = 0, & \text{on } \Sigma, \\ \text{transmission conditions (TC2),} & \text{on } S \times [0, T], \end{cases}$$

with

$$(TC2) \quad \begin{cases} v|_{S_0 \times [0, T]} = v|_{S_1 \times [0, T]}, \\ c_0 \partial_n v|_{S_0 \times [0, T]} = c_1 \partial_n v|_{S_1 \times [0, T]} + g(x, t), \end{cases}$$

where

$$g(x, t) = \xi_1 \partial_n \partial_t \tilde{y}|_{S_1 \times [0, T]} - \xi_0 \partial_n \partial_t \tilde{y}|_{S_0 \times [0, T]} = \alpha \partial_n \partial_t \tilde{y}|_{S_0 \times [0, T]},$$

with $\alpha = (\xi_1 \frac{\tilde{c}_0}{c_1} - \xi_0)|_S$.

Let $T' = \frac{1}{2}(T + t_0)$. We make the following assumption.

Assumption 2.1. *The solutions \tilde{y} and y belong to $H^2(t_0, T, H^1(\Omega))$ and are such that $y|_{\Omega_i} \in H^1(t_0, T, H^2(\Omega_i))$, $\tilde{y}|_{\Omega_i} \in H^2(t_0, T, H^2(\Omega_i))$, $i = 0, 1$. Furthermore, \tilde{y} satisfies*

1. *Let $r > 0$. The solution \tilde{y} is such that $|\Delta \tilde{y}(T')| \geq r > 0$ in Ω' ;*
2. *$\tilde{y}|_{\Omega_i}$ are in a bounded domains of $W^{2, \infty}(t_0, T, H^2(\Omega_i))$, $i = 0, 1$: there exists $M > 0$ such that*

$$|\tilde{y}|_{\Omega_i}(t, \cdot)|_{H^2(\Omega_i)}^2 + |\partial_t \tilde{y}|_{\Omega_i}(t, \cdot)|_{H^2(\Omega_i)}^2 + |\partial_t^2 \tilde{y}|_{\Omega_i}(t, \cdot)|_{H^2(\Omega_i)}^2 \leq M, \quad i = 0, 1,$$

a.e. for $t \in (t_0, T)$;

3. *$\Delta \partial_t \tilde{y}|_{\Omega_i}$ are in a bounded domain of $L^2(t_0, T, L^\infty(\Omega_i))$, $i = 0, 1$: there exists $K > 0$ such that*

$$\int_{t_0}^T |\Delta \partial_t \tilde{y}|_{\Omega_i}(t, \cdot)|_{L^\infty(\Omega_i)}^2 dt \leq K^2, \quad i = 0, 1.$$

In Section 3 we shall show that for *any* initial conditions y_0, \tilde{y}_0 in $L^2(\Omega)$ we can achieve the properties listed in Assumption 2.1 by using some particular boundary conditions $h(t, x)$.

From Assumption 2.1, the functions \tilde{y} and v are such that $\tilde{y}|_{\Omega_i}, v|_{\Omega_i} \in H^2(\Omega_i)$, $i = 0, 1$. Then $g \in H^1(t_0, T, H^{\frac{1}{2}}(S))$. The second equality in condition transmission (TC2) thus takes place in the space $H^{\frac{1}{2}}(S)$. Observe that $v = \partial_t(y - \tilde{y}) \in \mathfrak{N}_g$ from the above assumption. We can thus apply Carleman estimate (1.5) to v .

We shall use the notations of the proof of Theorem 1.3. We set $\psi^{(i)} = e^{-s\eta^{(i)}} v$, $i = 1, 2$. With the operator $M_2^{(i)}$ defined in (1.10) we introduce, following [3],

$$I^{(i)} = \operatorname{Re} \int_{t_0}^T \int_{\Omega'} M_2^{(i)} \psi^{(i)} \varphi^{(i)\frac{3}{2}} \overline{\psi^{(i)}} dx dt, \quad i = 1, 2, \quad \text{and } I = \frac{1}{2}(I^{(1)} + I^{(2)}).$$

Note the additional $\varphi^{(i)\frac{3}{2}}$ factor as compared to [3]. This will be of importance below.

We have the following estimates.

Lemma 2.2. *Let $\lambda \geq \lambda_1$ and $s \geq s_1$ then*

$$\begin{aligned} |I| \leq C s^{-3/2} \lambda^{-2} & \left[s \lambda \int_{t_0}^T \int_{\gamma} (e^{-2s\eta^{(1)}} \varphi^{(1)} + e^{-2s\eta^{(2)}} \varphi^{(2)}) |\partial_n v|^2 d\sigma dt \right. \\ & + \iint_{Q'} (e^{-2s\eta^{(1)}} + e^{-2s\eta^{(2)}}) |\partial_t v - \nabla \cdot (c \nabla v)|^2 dx dt \\ & + s \lambda \int_{t_0}^T \int_S (e^{-2s\eta^{(1)}} \varphi^{(1)} + e^{-2s\eta^{(2)}} \varphi^{(2)}) |g|^2 d\sigma dt \\ & + \int_{t_0}^T \int_S (e^{-2s\eta^{(1)}} \varphi^{(1)4} + e^{-2s\eta^{(2)}} \varphi^{(2)4}) |g|^2 d\sigma dt \\ & \left. + s^{-2} \int_{t_0}^T \int_S (e^{-2s\eta^{(1)}} + e^{-2s\eta^{(2)}}) |\partial_t g|^2 d\sigma dt \right]. \end{aligned}$$

Proof. Observe that

$$|I^{(i)}| \leq \frac{1}{2} s^{-3/2} \lambda^{-2} \left(|M_2^{(i)} \psi^{(i)}|_{L^2(Q')}^2 + s^3 \lambda^4 \iint_Q \varphi^{(i)3} e^{-2s\eta^{(i)}} |v|^2 dx dt \right), \quad i = 1, 2.$$

Thus

$$\begin{aligned} |I| \leq \frac{1}{2} s^{-3/2} \lambda^{-2} & \left(|M_2^{(1)} \psi^{(1)}|_{L^2(Q')}^2 + |M_2^{(2)} \psi^{(2)}|_{L^2(Q')}^2 \right. \\ & \left. + s^3 \lambda^4 \iint_Q (e^{-2s\eta^{(1)}} \varphi^{(1)3} + e^{-2s\eta^{(2)}} \varphi^{(2)3}) |v|^2 dx dt \right), \end{aligned}$$

which yields the result from Carleman estimate (1.5). \square

Lemma 2.3. *Let $\lambda \geq \lambda_1$ and $s \geq s_1$ then*

$$\begin{aligned}
& \int_{\Omega'} \left(e^{-2s\eta^{(1)}} \varphi^{(1)\frac{3}{2}} + e^{-2s\eta^{(2)}} \varphi^{(2)\frac{3}{2}} \right) (T', x) |v(T', \cdot)|^2 dx \\
& \leq C s^{-3/2} \lambda^{-2} \left[s\lambda \int_{t_0}^T \int_{\gamma} (e^{-2s\eta^{(1)}} \varphi^{(1)} + e^{-2s\eta^{(2)}} \varphi^{(2)}) |\partial_n v|^2 d\sigma dt \right. \\
& \quad + \iint_{Q'} (e^{-2s\eta^{(1)}} + e^{-2s\eta^{(2)}}) |\nabla \cdot (\xi \nabla \partial_t \tilde{y})|^2 dx dt \\
& \quad + s\lambda \int_{t_0}^T \int_S (e^{-2s\eta^{(1)}} \varphi^{(1)} + e^{-2s\eta^{(2)}} \varphi^{(2)}) |g|^2 d\sigma dt \\
& \quad + \int_{t_0}^T \int_S (e^{-2s\eta^{(1)}} \varphi^{(1)4} + e^{-2s\eta^{(2)}} \varphi^{(2)4}) |g|^2 d\sigma dt \\
& \quad \left. + s^{-2} \int_{t_0}^T \int_S (e^{-2s\eta^{(1)}} + e^{-2s\eta^{(2)}}) |\partial_t g|^2 d\sigma dt \right].
\end{aligned}$$

Proof. We evaluate integral $I^{(i)}$, $i = 1, 2$, using (1.10)

$$\begin{aligned}
I^{(i)} &= \operatorname{Re} \int_{t_0}^{T'} \int_{\Omega'} \left(\partial_t \psi^{(i)} - 2s\lambda \varphi^{(i)} c \nabla \beta^{(i)} \cdot \nabla \psi^{(i)} \right. \\
& \quad \left. - 2s\lambda^2 \varphi^{(i)} c |\nabla \beta^{(i)}|^2 \psi^{(i)} \right) \varphi^{(i)\frac{3}{2}} \overline{\psi^{(i)}} dx dt \\
&= \frac{1}{2} \int_{t_0}^{T'} \int_{\Omega'} \varphi^{(i)\frac{3}{2}} \partial_t |\psi^{(i)}|^2 dx dt - s\lambda \int_{t_0}^{T'} \int_{\Omega'} \varphi^{(i)\frac{5}{2}} c \nabla \beta^{(i)} \cdot \nabla |\psi^{(i)}|^2 dx dt \\
& \quad - 2s\lambda^2 \int_{t_0}^{T'} \int_{\Omega'} \varphi^{(i)\frac{5}{2}} c |\nabla \beta^{(i)}|^2 |\psi^{(i)}|^2 dx dt \\
&= \frac{1}{2} \int_{t_0}^{T'} \int_{\Omega'} \varphi^{(i)\frac{3}{2}} \partial_t |\psi^{(i)}|^2 dx dt + s\lambda \int_{t_0}^{T'} \int_{\Omega'} \nabla \cdot (\varphi^{(i)\frac{5}{2}} c \nabla \beta^{(i)}) |\psi^{(i)}|^2 dx dt \\
& \quad - 2s\lambda^2 \int_{t_0}^{T'} \int_{\Omega'} \varphi^{(i)\frac{5}{2}} c |\nabla \beta^{(i)}|^2 |\psi^{(i)}|^2 dx dt,
\end{aligned}$$

by integration by parts, without any remaining integral over $S \times (t_0, T')$ by condition transmission (1.8). With an integration by parts w.r.t. t in the first integral, we then obtain

$$\begin{aligned}
(2.4) \quad & \frac{1}{2} \int_{\Omega'} \varphi^{(i)\frac{3}{2}} |\psi^{(i)}(T', \cdot)|^2 dx = I^{(i)} + \frac{5}{2} s\lambda^2 \int_{t_0}^{T'} \int_{\Omega'} \varphi^{(i)\frac{5}{2}} c |\nabla \beta^{(i)}|^2 |\psi^{(i)}|^2 dx dt \\
& - s\lambda \int_{t_0}^{T'} \int_{\Omega'} \varphi^{(i)\frac{5}{2}} \nabla \cdot (c \nabla \beta^{(i)}) |\psi^{(i)}|^2 dx dt + \frac{3}{4} \int_{t_0}^{T'} \int_{\Omega'} (\partial_t \varphi^{(i)}) \varphi^{(i)\frac{1}{2}} |\psi^{(i)}|^2 dx dt, \\
& \qquad \qquad \qquad i = 1, 2,
\end{aligned}$$

since $\varphi^{(i)\frac{3}{2}}\psi^{(i)}(t_0) = 0$. Adding (2.4) for $i = 1, 2$ we obtain

$$(2.5) \quad \int_{\Omega'} \left(e^{-2s\eta^{(1)}} \varphi^{(1)\frac{3}{2}} + e^{-2s\eta^{(2)}} \varphi^{(2)\frac{3}{2}} \right) (T', x) |v(T', \cdot)|^2 dx \leq 4|I| \\ + C(s\lambda^2 + s\lambda + 1) \int_{t_0}^{T'} \int_{\Omega'} \left(e^{-2s\eta^{(1)}(t,x)} \varphi^{(1)\frac{5}{2}} + e^{-2s\eta^{(2)}(t,x)} \varphi^{(2)\frac{5}{2}} \right) |v|^2 dx dt,$$

observing that $|\partial_t \varphi^{(i)}| \leq CT\varphi^{(i)2}$, $i = 1, 2$. We use Carleman estimate (1.5) to obtain an upper-bound for the last term in (2.5) which yields the result by Lemma 2.2. \square

We shall now assume:

Assumption 2.4. *The diffusion coefficients c and \tilde{c} are piecewise constant, in the sense that $c|_{\Omega_i}$, resp. $\tilde{c}|_{\Omega_i}$, are constant in each connected component of Ω_i , $i = 0, 1$. We define*

$$c_{0,j} = c|_{\Omega_{0,j}}, \quad j = 1, \dots, p_0, \\ c_{1,j} = c|_{\Omega_{1,j}}, \quad j = 1, \dots, p_1.$$

with similar notations for \tilde{c} and ξ .

In this case observe that, in Ω' ,

$$v(T', x) = c\Delta u(T', x) + \xi\Delta\tilde{y}(T', x) \\ = \sum_{j=1}^{p_0} c_{0,j}\Delta u(T', x)\chi_{\Omega_{0,j}} + \sum_{j=1}^{p_1} c_{1,j}\Delta u(T', x)\chi_{\Omega_{1,j}} \\ + \sum_{j=1}^{p_0} \xi_{0,j}\Delta\tilde{y}(T', x)\chi_{\Omega_{0,j}} + \sum_{j=1}^{p_1} \xi_{1,j}\Delta\tilde{y}(T', x)\chi_{\Omega_{1,j}}.$$

From Lemma 2.3, we obtain

$$\int_{\Omega'} \left(e^{-2s\eta^{(1)}} \varphi^{(1)\frac{3}{2}} + e^{-2s\eta^{(2)}} \varphi^{(2)\frac{3}{2}} \right) (T', x) |\xi\Delta\tilde{y}(T', x)|^2 dx \\ \leq C \int_{\Omega'} \left(e^{-2s\eta^{(1)}} \varphi^{(1)\frac{3}{2}} + e^{-2s\eta^{(2)}} \varphi^{(2)\frac{3}{2}} \right) (T', x) |c\Delta u(T', x)|^2 dx \\ + Cs^{-3/2}\lambda^{-2} \left[s\lambda \int_{t_0}^T \int_{\gamma} (e^{-2s\eta^{(1)}} \varphi^{(1)} + e^{-2s\eta^{(2)}} \varphi^{(2)}) |\partial_n v|^2 d\sigma dt \right. \\ + \iint_{Q'} (e^{-2s\eta^{(1)}} + e^{-2s\eta^{(2)}}) |\nabla \cdot (\xi\nabla\partial_t\tilde{y})|^2 dx dt \\ + s\lambda \int_{t_0}^T \int_S (e^{-2s\eta^{(1)}} \varphi^{(1)} + e^{-2s\eta^{(2)}} \varphi^{(2)}) |g|^2 d\sigma dt \\ + \int_{t_0}^T \int_S (e^{-2s\eta^{(1)}} \varphi^{(1)4} + e^{-2s\eta^{(2)}} \varphi^{(2)4}) |g|^2 d\sigma dt \\ \left. + s^{-2} \int_{t_0}^T \int_S (e^{-2s\eta^{(1)}} + e^{-2s\eta^{(2)}}) |\partial_t g|^2 d\sigma dt \right].$$

From Assumption 2.1 we find that

$$|\Delta \partial_t \tilde{y}(t, x)|^2 \leq k^2(t) |\Delta \tilde{y}(T', x)|^2, \text{ in each } (t_0, T) \times \Omega_i, \quad i = 0, 1,$$

for

$$k(t) = \frac{1}{r} \sup_{i=0,1} |\Delta \partial_t \tilde{y}|_{\Omega_i}(t, \cdot)|_{L^\infty(\Omega_i)}.$$

From Assumption 2.1, $k \in L^2(t_0, T)$ and $|k|_{L^2(t_0, T)} \leq K' = \frac{1}{r} K$. These observations yield

$$\begin{aligned} & \int_{t_0}^T \int_{\Omega_i} (e^{-2s\eta^{(1)}} + e^{-2s\eta^{(2)}}) |\xi \Delta \partial_t \tilde{y}|^2 dx dt \\ & \leq K'^2 \sum_{i=0,1} \sum_{j=1}^{p_i} |\xi_{i,j}|^2 \int_{\Omega_{i,j}} \left(e^{-2s\eta^{(1)}(T', x)} + e^{-2s\eta^{(2)}(T', x)} \right) |\Delta \tilde{y}(T', x)|^2 dx \end{aligned}$$

where we have used that

$$e^{-2s\eta^{(i)}(t,x)} \leq e^{-2s\eta^{(i)}(T',x)}, \quad x \in \Omega, \quad t \in (t_0, T), \quad i = 1, 2.$$

Observing that $0 < C \leq \varphi^{(i)}$, $i = 1, 2$ since $\beta^{(i)} \geq 0$ and $\frac{1}{(T-t)(t-t_0)} \geq C > 0$, we obtain

$$\begin{aligned} & \int_{t_0}^T \int_{\Omega_i} (e^{-2s\eta^{(1)}} + e^{-2s\eta^{(2)}}) |\xi \Delta \partial_t \tilde{y}|^2 dx dt \\ & \leq K'^2 \sum_{i=0,1} \sum_{j=1}^{p_i} |\xi_{i,j}|^2 \int_{\Omega_{i,j}} \left(e^{-2s\eta^{(1)}} \varphi^{(1)\frac{3}{2}} \right. \\ & \quad \left. + e^{-2s\eta^{(2)}(T', x)} \varphi^{(2)\frac{3}{2}} \right) (T', x) |\Delta \tilde{y}(T', x)|^2 dx. \end{aligned}$$

We now treat the interface terms that appear in the right-hand-side of the Carleman estimate. Recall that

$$\begin{aligned} g(x, t) &= \alpha \partial_n \partial_t \tilde{y}|_{S_0 \times [0, T]}, \\ \alpha &= \xi_1 \frac{\tilde{c}_0}{\tilde{c}_1} - \xi_0. \end{aligned}$$

Note that $\eta(t, \cdot)$ is constant on S . We denote this constant by $\eta(t, S)$. More generally we shall denote $\rho(S)$ the value on S of a function ρ which is constant on S . We obtain

$$\begin{aligned} & \int_{t_0}^T \int_S (e^{-2s\eta^{(1)}} + e^{-2s\eta^{(2)}}) |\partial_t g|^2 d\sigma dt \\ & \leq \int_{t_0}^T (e^{-2s\eta^{(1)}(t, S)} + e^{-2s\eta^{(2)}(t, S)}) \int_S |\partial_t g|^2 d\sigma dt \\ & \leq M' |\xi|^2 \int_{t_0}^T (e^{-2s\eta^{(1)}(t, S)} + e^{-2s\eta^{(2)}(t, S)}) dt, \end{aligned}$$

from trace inequalities and from Assumption 2.1, for $M' = C_{Tr}(1 + \frac{c_{max}}{c_{min}})^2 M$, where $|\xi| = \sqrt{\xi_0^2 + \xi_1^2}$, since $|\alpha| \leq (1 + \frac{c_{max}}{c_{min}})|\xi|$ from Assumption 0.2. The constant C_{Tr} is the constant found in the trace estimates

$$\int_S |\partial_n \tilde{\rho}|^2 \leq C_{Tr} |\rho|_{H^2(\Omega_i)}^2, \quad i = 0, 1,$$

if $\rho|_{\Omega_i} \in H^2(\Omega_i)$, $i = 0, 1$. Similarly, since $\varphi^{(i)}$, $i = 1, 2$, are constant on S , we have

$$\begin{aligned} & \int_{t_0}^T \int_S (e^{-2s\eta^{(1)}} \varphi^{(1)j} + e^{-2s\eta^{(2)}} \varphi^{(2)j}) |g|^2 d\sigma dt \\ & \leq M' |\xi|^2 \int_{t_0}^T (e^{-2s\eta^{(1)}} \varphi^{(1)j} + e^{-2s\eta^{(2)}} \varphi^{(2)j})(t, S) dt, \quad j \in \mathbb{N}. \end{aligned}$$

With

$$w_k(s, \lambda) := \int_{t_0}^T (e^{-2s\eta^{(1)}} \varphi^{(1)k} + e^{-2s\eta^{(2)}} \varphi^{(2)k})(t, S) dt, \quad k \in \mathbb{N},$$

and

$$\begin{aligned} W_{i,j}(s, \lambda) &:= \int_{\Omega_{i,j}} (e^{-2s\eta^{(1)}} \varphi^{(1)\frac{3}{2}} + e^{-2s\eta^{(2)}} \varphi^{(2)\frac{3}{2}})(T', x) |\Delta \tilde{y}(T', x)|^2 dx, \\ & \quad i = 0, 1, j = 1, \dots, p_i, \end{aligned}$$

we thus obtain, for $\lambda \geq \lambda_1$ and $s \geq s_1$,

$$\begin{aligned} (2.6) \quad & \sum_{i=0,1} \sum_{j=1}^{p_i} |\xi_{i,j}|^2 \left\{ (1 - CK'^2 s^{-\frac{3}{2}} \lambda^{-2}) W_{i,j}(s, \lambda) \right. \\ & \quad \left. - CM' [s^{-\frac{1}{2}} \lambda^{-1} w_1(s, \lambda) + s^{-\frac{3}{2}} \lambda^{-2} w_4(s, \lambda) + s^{-\frac{7}{2}} \lambda^{-2} w_0(s, \lambda)] \right\} \\ & \leq C \int_{\Omega'} \left(e^{-2s\eta^{(1)}} \varphi^{(1)\frac{3}{2}} + e^{-2s\eta^{(2)}} \varphi^{(2)\frac{3}{2}} \right) (T', x) |c \Delta u(T', \cdot)|^2 dx \\ & \quad + Cs^{-\frac{1}{2}} \lambda^{-1} \int_{t_0}^T \int_{\gamma} (e^{-2s\eta^{(1)}} \varphi^{(1)} + e^{-2s\eta^{(2)}} \varphi^{(2)}) |\partial_n v|^2 d\sigma dt. \end{aligned}$$

To obtain a stability result we need to prove that the coefficients for $|\xi_{i,j}|^2$, $i = 0, 1, j = 1, \dots, p_i$, can be made positive. To do so we need to understand the behavior of the integral over the interface S as s and λ become large.

We first establish the asymptotic behavior of $w_k(s, \lambda)$. We set

$$w_k^{(i)}(s, \lambda) := \int_{t_0}^T e^{-2s\eta^{(i)}(t,S)} \varphi^{(i)k}(t, S) dt, \quad k \in \mathbb{N}.$$

Lemma 2.5. *The following estimates holds*

$$w_k^{(i)}(s, \lambda) = e^{-2s\eta^{(i)}(T', S)} \varphi^{(i)k}(T', S) \left\{ \frac{\sqrt{\pi} s^{-\frac{1}{2}}}{\sqrt{\phi''(T')} \sqrt{e^{\lambda\bar{\beta}^{(i)}} - e^{\lambda\beta^{(i)}(S)}}} + \mathcal{O}\left(\frac{s^{-\frac{3}{2}}}{(e^{\lambda\bar{\beta}^{(i)}} - e^{\lambda\beta^{(i)}(S)})^{\frac{3}{2}}}\right) \right\}$$

with $\phi(t) = \frac{1}{(T-t)(t-t_0)}$.

Proof. We write

$$\begin{aligned} w_k^{(i)}(s, \lambda) &:= e^{k\lambda\beta^{(i)}(S)} \int_{t_0}^T e^{-2\tau(s, S)\phi(t)} \phi^k(t) dt \\ &= e^{k\lambda\beta^{(i)}(S)} e^{-2\tau(s, S)\phi(T')} \int_{t_0}^T e^{-2\tau(s, S)(\phi(t) - \phi(T'))} \phi^k(t) dt \quad k \in \mathbb{N} \end{aligned}$$

with $\tau(s, S) = s(e^{\lambda\bar{\beta}^{(i)}} - e^{\lambda\beta^{(i)}(S)})$. We then apply the stationary phase formula [14, Theorem 7.7.5] for the complex phase $f(t) = i(\phi(t) - \phi(T'))$; note that $\text{Im}(f) \geq 0$ and note also that $\phi''(T') > 0$. In fact, apply formula (7.7.12) in [14] with $k = 2$ to obtain the power $\frac{3}{2}$ in the remaining term. Rigorously, one should first use a cut-off function $\chi \in \mathcal{C}_c^\infty([t_0, T])$ such that $\chi = 1$ in a neighborhood of T' and treat the integral

$$\int_{t_0}^T e^{-2\tau(s, S)(\phi(t) - \phi(T'))} \chi(t) \phi^k(t) dt$$

with formula (7.7.12) in [14]; the two additional terms are then found to be $\mathcal{O}(\tau(s, S)^{-j})$ for all $j \in \mathbb{N}$ by integration by parts (see also Theorem 7.7.1 in [14]). Finally, observe that the \mathcal{O} function is independent of the parameters s and λ . \square

To achieve our goal we also need an estimation from below for the terms $W_{i,j}(s, \lambda)$. We set

$$W_{k,j}^{(i)}(s, \lambda) := \int_{\Omega_{k,j}} e^{-2s\eta^{(i)}(T', x)} \varphi^{(i)\frac{3}{2}}(T', x) |\Delta\tilde{y}(T', x)|^2 dx, \quad k = 0, 1, j = 1, \dots, p_k, i = 1, 2.$$

For the terms $W_{0,j}^{(i)}(s, \lambda)$, we have the following

Lemma 2.6. *Let $\varepsilon > 0$. We have*

$$W_{0,j}^{(i)}(s, \lambda) \geq C_{s_1, i, j} \frac{r^2 |S_j|}{s\lambda} e^{-2s\eta^{(i)}(T', S)} (\varphi^{(i)}(T', S))^{\frac{1}{2}} e^{-\lambda\varepsilon}, \quad i = 1, 2, j = 1, \dots, p_0,$$

for $s \geq s_1$ and where $S_j = \bigcup_{k=1, \dots, p_1} S_{jk}$.

Proof. In the proof, we shall write β , etc, in place of $\beta^{(i)}$, etc. Taking δ sufficiently small, we start by choosing a small neighborhood W of S_j in $\bar{\Omega}_0$ globally parameterized by $(\sigma, y) \in [0, \delta] \times S_j$ (see the proof of Lemma A.7 in Appendix A). In fact, we can choose the coordinates and the small neighborhood of S_j such that $\sigma = cst$ corresponds to level sets for the function β (use $\nabla\beta$ for the vector field v in proof of Lemma A.7). Note that in the neighborhood W the function β decreases with σ .

Estimating from below the Jacobian¹ originating from the change of variable and observing that the integrand is constant w.r.t. y we obtain

$$\begin{aligned} W_{0,j}(s, \lambda) &\geq Cr^2|S_j| \int_0^\delta e^{-2s\eta(T', \sigma)} \varphi^{\frac{3}{2}}(T', \sigma) d\sigma \\ &= Cr^2|S_j| e^{-2s\eta(T', S)} \int_0^\delta e^{-2s(\eta(T', \sigma) - \eta(T', S))} \varphi^{\frac{3}{2}}(T', \sigma) d\sigma. \end{aligned}$$

We now use the change of variables $\sigma' = \eta(T', \sigma) - \eta(T', S) \geq 0$ which yields

$$W_{0,j}(s, \lambda) \geq Cr^2|S_j| \lambda^{-1} e^{-2s\eta(T', S)} \int_0^{\delta'} e^{-2s\sigma'} \varphi^{\frac{1}{2}}(T', \sigma) |\partial_\sigma \beta|^{-1} d\sigma',$$

where $\delta' = \eta(T', \delta) - \eta(T', S)$. We can find in W a positive lower bound for $(\partial_\sigma \beta)^{-1}$ independent of δ , i.e. the size of W . We thus obtain

$$\begin{aligned} W_{0,j}(s, \lambda) &\geq Cr^2|S_j| \lambda^{-1} e^{-2s\eta(T', S)} \varphi^{\frac{1}{2}}(T', \delta) \int_0^{\delta'} e^{-2s\sigma'} d\sigma' \\ &\geq Cr^2|S_j| s^{-1} \lambda^{-1} e^{-2s\eta(T', S)} \varphi^{\frac{1}{2}}(T', \delta) \int_0^{s\delta'} e^{-2\sigma'} d\sigma' \\ &\geq C'(s) r^2 |S_j| s^{-1} \lambda^{-1} e^{-2s\eta(T', S)} \varphi^{\frac{1}{2}}(T', \delta), \end{aligned}$$

with $C'(s)$ increasing with s . Observe now that

$$\begin{aligned} \varphi(T', \delta) &= \frac{e^{\lambda\beta(\delta)}}{(T - T')(T' - t_0)} = \frac{e^{\lambda\beta(S)}}{(T - T')(T' - t_0)} e^{\lambda(\beta(\delta) - \beta(S))} \\ &= \varphi(T', S) e^{\lambda(\beta(\delta) - \beta(S))}. \end{aligned}$$

Choosing δ sufficiently small such that $\frac{1}{2}(\beta(S) - \beta(\delta)) \leq \varepsilon$ thus yields the result. \square

With the previous lemmas we can now prove that the coefficient of $|\xi_{0,j}|^2$, $j = 1, \dots, p_0$, in (2.6) can be made positive. This requires taking *both* λ and s sufficiently large.

Proposition 2.7. *Let $1 \leq j \leq p_0$. There exists $\lambda_{2,j} \geq \lambda_1$ such that if $\lambda \geq \lambda_{2,j}$ then for s sufficiently large*

$$\begin{aligned} A_{0,j} &= (1 - CK'^2 s^{-\frac{3}{2}} \lambda^{-2}) W_{0,j}(s, \lambda) - CM' [s^{-\frac{1}{2}} \lambda^{-1} w_1(s, \lambda) \\ &\quad + s^{-\frac{3}{2}} \lambda^{-2} w_4(s, \lambda) + s^{-\frac{7}{2}} \lambda^{-2} w_0(s, \lambda)] \geq C(s, \lambda) > 0. \end{aligned}$$

with $C(s, \lambda) = C(s, \lambda, r, K, M, c_{max}, c_{min}, j)$.

¹Note that the estimation from below of the Jacobian is independent from the size of the neighborhood W .

Proof. It suffices to prove the result for $w_k^{(i)}(s, \lambda)$ and $W_{0,j}^{(i)}(s, \lambda)$. We shall write β , etc, in place of $\beta^{(i)}$, etc. We take s sufficiently large such that $(1 - CK'^2 s^{-\frac{3}{2}} \lambda^{-2}) \geq c_0 > 0$.

From Lemmas 2.5 and 2.6, for $\varepsilon > 0$ we obtain

$$(2.7) \quad A_{0,j} \geq C_1 e^{-2s\eta(T',S)} \left(\frac{1}{s} \rho(\lambda) - \nu(\lambda) \mathcal{O}(s^{-\frac{3}{2}}) \right) \\ + \frac{1}{s\lambda} e^{-2s\eta(T',S)} \left[\frac{C_0}{2} C_\varepsilon r^2 |S_j| (\varphi(T',S))^{\frac{1}{2}} e^{-\lambda\varepsilon} \right. \\ \left. - CM' \frac{\sqrt{\pi} \varphi(T',S)}{\sqrt{\phi''(T')} \sqrt{e^{\lambda\bar{\beta}} - e^{\lambda\beta(S)}}} \right],$$

where ρ and ν are some continuous functions. We first treat the second term in the previous expression. Note that this term originates from the estimate from below for $W_{0,j}^{(i)}$ and the estimate of $s^{-\frac{1}{2}} \lambda^{-1} w_1^{(i)}$. The other terms in $A_{0,j}$ are lumped in the first term of (2.7).

Choose now $\varepsilon < \frac{1}{2}(\bar{\beta} - \beta(S))$. Then since $\bar{\beta} > \beta(S)$ we have

$$\frac{\varphi(T',S)}{\sqrt{e^{\lambda\bar{\beta}} - e^{\lambda\beta(S)}}} = o((\varphi(T',S))^{\frac{1}{2}} e^{-\lambda\varepsilon})$$

for λ large. Thus the second term can be made positive for λ , say $\lambda = \lambda_{2,j}$, sufficiently large.

Once λ is fixed larger than $\lambda_{2,j}$, the first term in (2.7) can be made positive by taking s sufficiently large. \square

We now prove that the coefficient of $|\xi_{1,j}|^2$, $j = 1, \dots, p_1$, in (2.6) can be made positive. Here the parameter λ is not of use.

Proposition 2.8. *Let $1 \leq j \leq p_1$. Let $\lambda \geq \lambda_1$. Then for s sufficiently large*

$$A_{1,j} = (1 - CK'^2 s^{-\frac{3}{2}} \lambda^{-2}) W_{1,j}(s, \lambda) - CM' [s^{-\frac{1}{2}} \lambda^{-1} w_1(s, \lambda) \\ + s^{-\frac{3}{2}} \lambda^{-2} w_4(s, \lambda) + s^{-\frac{7}{2}} \lambda^{-2} w_0(s, \lambda)] \geq C(s, \lambda) > 0.$$

with $C(s, \lambda) = C(s, \lambda, r, K, M, c_{max}, c_{min})$.

Proof. It suffices to prove the result for $w_k^{(i)}(s, \lambda)$ and $W_{1,j}^{(i)}(s, \lambda)$. We shall write β , etc, in place of $\beta^{(i)}$, etc. We take s sufficiently large such that $(1 - CK'^2 s^{-\frac{3}{2}} \lambda^{-2}) \geq C_0 > 0$.

We first write

$$e^{-2s\eta(t,S)} = e^{-2s\eta(T',S)} e^{-2s(\eta(t,S) - \eta(T',S))},$$

and observe that for $s \geq s_0 > 0$

$$\int_{t_0}^T e^{-2s(\eta(t,S) - \eta(T',S))} \varphi^k(t, S) dt \leq L(s_0, \lambda, k),$$

for some positive $L(s_0, \lambda, k)$. From Lemma 1.1, there exists $\omega \Subset \Omega_{1,j}$ such that $\inf_{x \in \omega} \beta > \beta(S)$. Then with

$$\eta_{max}^{T', \omega} = \sup_{x \in \omega} \frac{e^{2\lambda K^{(i)}} - e^{\lambda \beta^{(i)}(x)}}{(T' - t_0)(T - T')}$$

we have $-\eta(T', x) \geq -\eta_{max}^{T', \omega} > -\eta(T', S)$, for $x \in \omega$, and $s > 0$. These observations yield

$$W_{1,j}(s) \geq r^2 \int_{\omega} e^{-2s\eta(T', x)} \varphi^{\frac{3}{2}}(T', x) dx \geq C(\lambda) r^2 |\omega| e^{-2s\eta_{max}^{T', \omega}},$$

and

$$w_k(s) \leq L(s_0, \lambda, k) e^{-2s\eta(T', S)},$$

which implies the result. \square

With (2.6) and Propositions 2.7 and 2.8, recalling that $v = u_t = \partial_t(y - \tilde{y})$, We have thus obtained the following stability result.

Theorem 2.9. *Let γ be a subset of the boundary Γ of an open set Ω of \mathbb{R}^n that satisfies Condition (GC), and γ satisfies Assumption 0.4. We assume that the diffusion coefficients c and \tilde{c} satisfy Assumptions 0.2 and 2.4 and $c_0 - c_1 \geq \Delta > 0$. Let y_0, \tilde{y}_0 in $L^2(\Omega)$ and let y, \tilde{y} be solutions to (2.1)–(2.2) satisfying Assumption 2.1. Then there exists a constant C*

$$C = C(\Omega, T, t_0, \gamma, S, \mathcal{O}^{(1)}, \mathcal{O}^{(2)}, M, K, r, c_{min}, c_{max}, \Delta)$$

such that

$$(2.8) \quad |c - \tilde{c}|_{L^\infty(\Omega)}^2 \leq C |\partial_n(\partial_t y - \partial_t \tilde{y})|_{L^2((0,T) \times \gamma)}^2 + C |\Delta y(T', \cdot) - \Delta \tilde{y}(T', \cdot)|_{L^2(\Omega')}^2.$$

We shall see in Proposition 3.5, below, that we can achieve the regularity properties and estimates of Assumption 2.1.

Remark 2.10. Observe than in the statement of Theorem 2.9 the initial condition y_0 and \tilde{y}_0 need not be equal (see systems (2.1)–(2.2)).

Remark 2.11. Note that if we assume that $y(T', \cdot) = \tilde{y}(T', \cdot)$ then the stability estimate becomes

$$|c - \tilde{c}|_{L^\infty(\Omega)}^2 \leq C |\partial_n(\partial_t y - \partial_t \tilde{y})|_{L^2((0,T) \times \gamma)}^2.$$

Such an additional assumption is sometimes made, e.g. in [18].

With Theorem 2.9 we have the following uniqueness result

Corollary 2.12. *Under the same assumptions as Theorem 2.9 and if*

$$\begin{aligned} \partial_n(\partial_t(y - \tilde{y}))(t, x) &= 0 \text{ in } \gamma \times (t_0, T), \\ \Delta y(T', x) - \Delta \tilde{y}(T', x) &= 0, \text{ in } \Omega', \end{aligned}$$

then $c = \tilde{c}$. Furthermore $y_0 = \tilde{y}_0$.

Proof. The second assertion remains to be proved. If $c = \tilde{c}$ then $u = y - \tilde{y} \in \mathcal{D}_A$, with $A = \nabla \cdot (c\nabla(\cdot))$ (see Appendix A), is solution to

$$\begin{cases} \partial_t u - \nabla \cdot (c\nabla u) = 0 & \text{in } Q, \\ u = 0, & \text{on } \Sigma, \\ u(0, x) = u_0(x), & \text{in } \Omega, \end{cases}$$

with $u_0 = y_0 - \tilde{y}_0$. Thus $u = S(t)u_0$. We have $\Delta(u)(T') = 0$ in Ω' . Thus $\nabla \cdot (c\nabla u)|_{\Omega'}(T') = 0$. Since $u(T') \in \mathcal{D}_A$ we have $u(T') = 0$. Since the semigroup $S(t)$ generated by $-\nabla \cdot (c\nabla(\cdot))$ is analytic by Proposition A.2 we obtain that $S(t)u_0 = 0$ for all $t > 0$. The continuity in $t = 0^+$ yields $u_0 = 0$. \square

If we make further assumptions on the initial conditions y_0 and \tilde{y}_0 we can in fact obtain a stability result for these initial conditions as well. This is the subject of Section 4.

Remark 2.13. In the stability result obtained here, we have made the choice to make some of the measurements on part of the boundary Γ . Derivation of a Carleman estimate, as in [6], with a right-hand-side with an ‘observation’ in an inner volume ω of Ω_1 would yield a stability estimate like (2.8) with $|\partial_t y - \partial_t \tilde{y}|_{L^2(\omega \times (t_0, T))}$ in the right-hand-side.

3 Existence of solutions y, \tilde{y} satisfying Assumption 2.1

We propose a possible choice of boundary condition h and of initial condition \tilde{y}_0 to achieve the particular properties for the solutions y and \tilde{y} listed in Assumption 2.1 needed in the proof of Theorem 2.9 in Section 2.

We shall denote $S(t)$ (resp. $\tilde{S}(t)$) the analytic semi-group generated by unbounded operator A (resp. \tilde{A}) formally defined by $-\nabla \cdot (c\nabla(\cdot))$ (resp. $-\nabla \cdot (\tilde{c}\nabla(\cdot))$) on $L^2(\Omega)$ with domain (See appendix A)

$$\begin{aligned} \mathcal{D}_A &= \{u \in H_0^1(\Omega); c\nabla u \in H(\text{div}, \Omega)\}, \\ \text{(resp. } \mathcal{D}_{\tilde{A}} &= \{u \in H_0^1(\Omega); \tilde{c}\nabla u \in H(\text{div}, \Omega)\}). \end{aligned}$$

The convention we use here is ‘ $S(t) = e^{-tA}$ ’.

Observe that, for a diffusion coefficient satisfying Assumption 0.1, if a function $p \in H_0^1(\Omega)$ then $p \in \mathcal{D}_A$ if and only if $p|_{\Omega_i} \in H^2(\Omega_i)$, $i = 0, 1$ and p satisfies condition transmission (TC1).

Lemma 3.1. *Let $r > 0$ and let $\tilde{c} \in L^\infty(\Omega)$. There exists $\tilde{y}_0 \in \mathcal{D}_{\tilde{A}}$ and $\chi : [0, T] \rightarrow \mathbb{R}$ such that the solution to*

$$(3.1) \quad \begin{cases} \partial_t \tilde{y} - \nabla \cdot (\tilde{c}\nabla \tilde{y}) = 0, & \text{in } Q, \\ \tilde{y}(t, x) = \chi(t), & \text{on } \Sigma, \\ \tilde{y}(t, \cdot) - \chi(t) \in \mathcal{D}_{\tilde{A}}, & 0 < t \leq T, \\ \tilde{y}(0) = \tilde{y}_0, \end{cases}$$

satisfies $|\nabla \cdot (\tilde{c}\nabla\tilde{y})(T')| \geq r > 0$ a.e.. The function χ can be chosen such that χ' is a positive constant.

Proof. Observe that $p(t, x) = \tilde{y}(t, x) - \chi(t)$ is solution to

$$(3.2) \quad \begin{cases} \partial_t p - \nabla \cdot (\tilde{c}\nabla p) = -\chi'(t), & \text{in } Q, \\ p(t, x) = 0, & \text{on } \Sigma, \\ p(t, \cdot) \in \mathcal{D}_{\tilde{A}}, & 0 < t \leq T, \\ p(0, x) = \tilde{y}_0(x) - \chi(0) = p_0 \in L^2(\Omega). \end{cases}$$

and is thus given by Duhamel's formula [20]

$$(3.3) \quad p(t) = \tilde{S}(t)p_0 - \int_0^t \tilde{S}(t-s)\chi'(s)ds.$$

In fact, we choose χ of the form $\chi(t) = -\rho t$, where ρ is a negative constant. We also choose \tilde{y}_0 such that $p_0 = \tilde{y}_0 \in \mathcal{D}_{\tilde{A}}$ and $\nabla \cdot (\tilde{c}\nabla\tilde{y}_0) \geq r_0 > r$ a.e. in Ω (choose $f \in L^2(\Omega)$, such that $f > r_0$, and solve the elliptic problem $\nabla \cdot (\tilde{c}\nabla\tilde{y}_0) = f$ for \tilde{y}_0 in $H_0^1(\Omega)$). We choose ρ such that $-r_0 < \rho \leq -r < 0$.

The solution p to (3.2) is unique in $\mathcal{C}^1([0, T], L^2(\Omega)) \cap \mathcal{C}^0([0, T], \mathcal{D}_{\tilde{A}})$ and given by (3.3) [5, Theorem 3 and following Remark 2, Section XVII B.1]. Denoting by $\mathbf{1}$ the function identically equal to 1 on Ω we find

$$p(t) = \tilde{S}(t)p_0 + \rho \int_0^t \tilde{S}(s)\mathbf{1}ds,$$

which yields $q := -\tilde{A}p + \rho\mathbf{1} := \tilde{S}(t)(\nabla \cdot (\tilde{c}\nabla p_0) + \rho\mathbf{1})$ [20, Theorem 1.2.4]. Hence q is the solution to

$$\begin{cases} \partial_t q - \nabla \cdot (\tilde{c}\nabla q) = 0, & \text{in } Q, \\ q(t, x) = 0, & \text{on } \Sigma, \\ q(t, \cdot) \in \mathcal{D}_{\tilde{A}}, & 0 < t \leq T, \\ q(0, x) = q_0(x) := \nabla \cdot (\tilde{c}\nabla p_0) + \rho\mathbf{1}. \end{cases}$$

We now apply the maximum principle (which is valid for L^∞ diffusion coefficients) [4, proof of Theorem IX.3] which reads for the time interval $[0, T']$

$$\text{ess inf}_{\overline{Q_{T'}}} q \geq \min(0, \text{ess inf}_{\Omega} q_0) = 0, \quad Q_{T'} = \Omega \times (0, T').$$

This yields $\nabla \cdot (\tilde{c}\nabla p)(T', x) \geq -\rho \geq r > 0$ a.e.. \square

Lemma 3.2. *Let $m > n/2$ and $m \geq 2$. Let $\tilde{c} \in L^\infty(\Omega)$ be such that $\tilde{c}|_{\Omega_i}$ is $\mathcal{C}^{m-1}(\overline{\Omega}_i)$, $i = 0, 1$. Let Ω be such that S and $\partial\Omega$ are of class \mathcal{C}^m . Let $\tilde{y}_0 \in \mathcal{D}_{\tilde{A}}$ and the function $\chi : [0, T] \rightarrow \mathbb{R}$, such that χ' is constant, be both chosen according to Lemma 3.1. Then $\nabla \cdot (\tilde{c}\nabla\tilde{y})|_{\Omega_i} \in \mathcal{C}^k((0, T], L^\infty(\Omega_i))$, $i = 0, 1$, for all $k \in \mathbb{N}$. Let $\varepsilon > 0$ then $\nabla \cdot (\tilde{c}\nabla\tilde{y})|_{\Omega_i}$, $i = 0, 1$, remain in a bounded domain of $\mathcal{C}^k([\varepsilon, T], L^\infty(\Omega_i))$ for all $k \in \mathbb{N}$, uniformly w.r.t. \tilde{c} and \tilde{y}_0 , for $0 < c_{\min} \leq \tilde{c} \leq c_{\max}$ and $\nabla \cdot (\tilde{c}\nabla\tilde{y}_0)$ in a bounded domain of $L^2(\Omega)$.*

Proof. We use the notations of the proof of Lemma 3.1. We set $p(t, x) = \tilde{y}(t, x) - \chi(t)$ and observe that $q := -\tilde{A}p + \rho\mathbf{1}$ is the solution to

$$\begin{cases} \partial_t q - \nabla \cdot (\tilde{c}\nabla q) = 0, & \text{in } Q, \\ q(t, x) = 0, & \text{on } \Sigma, \\ q(t, \cdot) \in \mathcal{D}_{\tilde{A}}, & 0 < t \leq T, \\ q(0, x) = q_0(x) := \nabla \cdot (\tilde{c}\nabla p_0) + \rho\mathbf{1}. \end{cases}$$

From Corollary A.5 we have that $q|_{(0, T] \times \Omega_i} \in \mathcal{C}^k((0, T]; H^m(\Omega_i))$, $i = 0, 1$, for all $k \in \mathbb{N}$. Since $m > n/2$, the space $H^m(\Omega_i)$ is continuously embedded in $L^\infty(\Omega_i)$ which yields the result. The last statement follows from Remarks A.4 and A.6. \square

Remark 3.3. In the case of $n=2,3$, which concerns most of the applications, we choose $m = 2$. The condition on S , $\partial\Omega$ and the coefficients $\tilde{c}|_{\Omega_i}$ in the previous lemma are then the default ones assumed in the introduction.

Let the function χ , such that χ' is constant, be chosen according to Lemma 3.1. We then have the following regularity property.

Lemma 3.4. *Let $c, \tilde{c} \in L^\infty(\Omega)$ be such that $c|_{\Omega_i}, \tilde{c}|_{\Omega_i}$ is $\mathcal{C}^1(\overline{\Omega_i})$, $i = 0, 1$ and $y_0, \tilde{y}_0 \in L^2(\Omega)$. The solutions \tilde{y} and y to*

$$\begin{cases} \partial_t \tilde{y} - \nabla \cdot (\tilde{c}\nabla \tilde{y}) = 0, & \text{in } Q, \\ \tilde{y}(t, x) = \chi(t), & \text{on } \Sigma, \\ \text{trans. condition (TC1)}, \\ \tilde{y}(0) = \tilde{y}_0, \end{cases} \quad \begin{cases} \partial_t y - \nabla \cdot (c\nabla y) = 0, & \text{in } Q, \\ y(t, x) = \chi(t), & \text{on } \Sigma, \\ \text{trans. condition (TC1)}, \\ y(0) = y_0, \end{cases}$$

belong to $\mathcal{C}^k((0, T], L^2(\Omega)) \cap \mathcal{C}^k((0, T], H^1(\Omega))$ and are such that $\tilde{y}|_{\Omega_i}, y|_{\Omega_i} \in \mathcal{C}^k((0, T], H^2(\Omega_i))$, $i = 0, 1$, for all $k \in \mathbb{N}$. Let $\varepsilon > 0$, then for all $k \in \mathbb{N}$, $\tilde{y}|_{\Omega_i}, y|_{\Omega_i}$ remain in a bounded domain of $\mathcal{C}^k((\varepsilon, T], H^2(\Omega_i))$, $i=0,1$, uniformly w.r.t. \tilde{c}, y_0 , and \tilde{y}_0 if $0 < c_{\min} \leq c, \tilde{c} \leq c_{\max}$ and y_0 , and \tilde{y}_0 remain in a bounded domain of $L^2(\Omega)$.

Proof. We work out the proof for y . Set $\rho = -\chi'$ and define $p(t, x) = y(t, x) - \chi(t)$. The function p is solution to

$$(3.4) \quad \begin{cases} \partial_t p - \nabla \cdot (c\nabla p) = \rho, & \text{in } Q, \\ p(t, x) = 0, & \text{on } \Sigma, \\ p(t, \cdot) \in \mathcal{D}_A, & 0 < t \leq T, \\ p(0) = p_0 = y_0 - \chi(0) \in L^2(\Omega). \end{cases}$$

It suffices to prove the result for p . Since ρ is constant, the (mild) solution to (3.4) is a classical solution [20, Theorem 4.3.2]. We prove below that $p \in \mathcal{C}^k((0, T], \mathcal{D}_A)$, $k > 0$. Thus $p \in \mathcal{C}^k((0, T], L^2(\Omega))$. Since $D_A \subset H_0^1$ with continuous injection then $p \in \mathcal{C}^k((0, T], H_0^1(\Omega))$. By Proposition A.3 the maps $p \mapsto p|_{\Omega_i}$, $i = 0, 1$, are continuous from \mathcal{D}_A into $H^2(\Omega_i)$. Thus $p|_{\Omega_i} \in \mathcal{C}^k((0, T], H^2(\Omega_i))$.

The solution p is given by

$$p(t) = S(t)p_0 + \rho \int_0^t S(s)\mathbf{1}ds,$$

where $\mathbf{1}$ is the function identically equal to 1 on Ω . The first term $p_1 = S(t)p_0$ in $\mathcal{C}^k((0, T], \mathcal{D}_{A^l})$ for all $k, l > 0$ by Proposition A.1. For the second term $p_2 = \rho \int_0^t S(s)\mathbf{1}ds$ we have [20, Theorem 1.2.4] $-Ap_2 = \rho(S(t)\mathbf{1} - \mathbf{1})$. Thus $Ap_2 \in \mathcal{C}^k((0, T], L^2(\Omega))$, i.e. $p_2 \in \mathcal{C}^k((0, T], \mathcal{D}_A)$, for all $k > 0$. The boundedness statement follows from Remarks A.4 and A.6. \square

With the proposed initial condition \tilde{y}_0 and boundary condition $h(t, x) = \chi(t)$ we have thus obtained the following regularity and boundedness properties.

Proposition 3.5. *Let $r > 0$. Let $y_0 \in L^2(\Omega)$. Let Ω be such that S and $\Gamma = \partial\Omega$ are of class \mathcal{C}^m and $\tilde{c} \in L^\infty(\Omega)$ be such that $\tilde{c}|_{\Omega_i}$ is $\mathcal{C}^{m-1}(\overline{\Omega}_i)$, $i = 0, 1$, with $m > n/2$, $m \geq 2$. There exists $h(t, x) \in \mathcal{C}([0, T] \times \Gamma)$ and an initial condition $\tilde{y}_0 \in \mathcal{D}_{\tilde{A}}$ such that the solutions y, \tilde{y} to systems (2.1)–(2.2) satisfy*

1. $\nabla \cdot (\tilde{c}\nabla\tilde{y})(T') \geq r > 0$;
2. $\nabla \cdot (\tilde{c}\nabla\tilde{y})|_{\Omega_i} \in \mathcal{C}^k([t_0, T], L^\infty(\Omega_i))$, $i = 0, 1$, for all $k \in \mathbb{N}$;
3. $y, \tilde{y} \in \mathcal{C}^k([t_0, T], L^2(\Omega)) \cap \mathcal{C}^k((t_0, T], H^1(\Omega))$, for all $k \in \mathbb{N}$;
4. $y|_{\Omega_i}, \tilde{y}|_{\Omega_i} \in \mathcal{C}^k([t_0, T], H^2(\Omega_i))$, $i = 0, 1$, for all $k \in \mathbb{N}$.

The restrictions $\tilde{y}|_{\Omega_i}$ remain in a bounded domain of $\mathcal{C}^k([t_0, T], H^2(\Omega_i))$ and $\nabla \cdot (\tilde{c}\nabla\tilde{y})|_{\Omega_i}$ in a bounded domain of $\mathcal{C}^k([t_0, T], L^\infty(\Omega_i))$, uniformly w.r.t. \tilde{c} and \tilde{y}_0 if $0 < c_{\min} \leq \tilde{c} \leq c_{\max}$ and $\nabla \cdot (\tilde{c}\nabla\tilde{y}_0)$ remain in a bounded domain of $L^2(\Omega)$.

With Proposition 3.5 we observe that Assumption 2.1 in Section 2 can be fulfilled in the framework of Assumption 2.4 when $c_{\min} \leq \tilde{c} \leq c_{\max}$ and $\tilde{y}_0 \in \mathcal{D}_{\tilde{A}}$ such that $\nabla \cdot (\tilde{c}\nabla\tilde{y}_0)$ remain in a bounded domain of $L^2(\Omega)$ for properly chosen boundary conditions $h(t, x)$.

Remark 3.6. Observe that we could simply assume that $\tilde{y}_0 \in L^2(\Omega)$ and design the boundary condition $h(t, x)$ to reach a proper state in $\mathcal{D}_{\tilde{A}}$ in a finite time $t_1 < t_0$. This can be achieved as the parabolic equation we study here is null-controllable, i.e. exactly controllable to the trajectories [6].

4 Uniqueness and stability estimate for the initial conditions

In this section we closely follow the method of [21]. We shall assume

Assumption 4.1. *Let $r_0 > 0$. The initial conditions y_0 and \tilde{y}_0 satisfy*

1. y_0 is in a bounded domain of \mathcal{D}_A ;

2. \tilde{y}_0 is in a bounded domain of $\mathcal{D}_{\tilde{A}}$;
3. $\nabla \cdot (c\nabla \tilde{y}_0) \geq r_0$,
4. $\tilde{y} \in \mathcal{C}^1([0, T], L^2(\Omega))$.

We denote $\tilde{z} = \partial_t \tilde{y} \in \mathcal{C}([0, T], L^2(\Omega))$ and thus $\tilde{z}(0)$ is well defined in $L^2(\Omega)$. We introduce w the solution to

$$(4.1) \quad \begin{cases} \partial_t w - \nabla \cdot (c\nabla w) = 0, & \text{in } Q, \\ w(t, x) = \partial_t h(t, x), & \text{on } \Sigma, \\ \text{transmission conditions (TC1),} & \text{on } S \times [0, T], \\ w(0) = \tilde{z}(0), & \end{cases}$$

and we further assume

Assumption 4.2. *The functions \tilde{z}, w are in a bounded domain of $L^2(0, T, H^1(\Omega))$ and $\tilde{y}|_{\Omega_i}, y|_{\Omega_i}$ are in a bounded domain of $\mathcal{C}([0, T], H^2(\Omega_i))$, $i = 0, 1$.*

Observe that if we choose the boundary condition $h(t, x) = \chi(t) = -\rho t$ for $0 < t \leq T$ according to the proof of Lemma 3.1 (with $0 < r < r_0$), then the results of Section 3 thus show that Assumption 2.1 is then fulfilled. In addition, item 4 in Assumption 4.1 and Assumption 4.2 are fulfilled:

Lemma 4.3. *If $h(t, x) = \chi(t) = -\rho t$ then the solutions y, \tilde{y} to (2.1)–(2.2) and w to (4.1) satisfy item 4 in Assumption 4.1 and Assumption 4.2.*

Proof. We prove $w \in L^2(0, T, H^1(\Omega))$. The proof is the same for \tilde{z} . Let $p(t, x) = w(t, x) - \partial_t h(t, x) = w(t, x) + \rho$. Then p satisfies

$$\begin{cases} \partial_t p - \nabla \cdot (c\nabla p) = 0, & \text{in } Q, \\ p(t, x) = 0, & \text{on } \Sigma, \\ \text{transmission conditions (TC1),} & \text{on } S \times [0, T], \\ p(0) = \tilde{z}(0) + \rho \in L^2(\Omega), & \end{cases}$$

We thus have the usual energy estimate

$$\frac{1}{2}|p(t)|_{L^2(\Omega)}^2 + \int_0^t \int_{\Omega} c|\nabla p|^2 dt dx = \frac{1}{2}|p(0)|_{L^2(\Omega)}^2,$$

and p , and thus w , is in $L^2(0, T, H^1(\Omega))$ and remains in a bounded domain of this space if $c_{\min} \leq c \leq c_{\max}$ and \tilde{y}_0 remains in a bounded domain of $\mathcal{D}_{\tilde{A}}$.

We now prove $y|_{\Omega_i} \in \mathcal{C}([0, T], H^2(\Omega_i))$. The proof is the same for $\tilde{y}|_{\Omega_i}$. We set $p(t, x) = y(t, x) - \chi(t)$ and observe that $q := -Ap + \rho \mathbf{1}$ is the solution to

$$\begin{cases} \partial_t q - \nabla \cdot (c\nabla q) = 0, & \text{in } Q, \\ q(t, x) = 0, & \text{on } \Sigma, \\ q(t, \cdot) \in \mathcal{D}_A, & 0 < t \leq T, \\ q(0, x) = q_0(x) := \nabla \cdot (c\nabla y_0) + \rho \mathbf{1} \in L^2(\Omega). & \end{cases}$$

Thus $q \in \mathcal{C}([0, T], L^2(\Omega))$ and thus $p \in \mathcal{C}([0, T], \mathcal{D}_A)$. We conclude with Proposition A.3 and Remark A.6 and Assumption 4.1. Uniform boundedness is obtained as above.

With $p \in \mathcal{C}([0, T], \mathcal{D}_A)$ then $p \in \mathcal{C}^1([0, T], L^2(\Omega))$ and thus $y \in \mathcal{C}^1([0, T], L^2(\Omega))$. \square

As in Section 2 we introduce $u = y - \tilde{y}$ and $v = \partial_t u$ and note that

$$(4.2) \quad v = \nabla \cdot (c \nabla u) + \nabla \cdot (\xi \cdot \nabla \tilde{y}) = c(\Delta y - \Delta \tilde{y}) + \xi \Delta \tilde{y}, \quad \text{in } \Omega'.$$

From Lemma 4.3, $v \in \mathcal{C}([0, T], L^2(\Omega'))$, $i = 0, 1$, and from Assumption 4.1, there exists $K > 0$ such that $|v(0)|_{L^2(\Omega')} \leq K$. Note that v is the unique solution to problem (2.3) that satisfies (4.2) at $t = 0$.

Define v_1 and v_2 that satisfy

$$(4.3) \quad \begin{cases} \partial_t v_1 - \nabla \cdot (c \nabla v_1) = \nabla \cdot (\xi \nabla \partial_t \tilde{y}), & \text{in } Q', \\ v_1 = 0, & \text{on } \Sigma, \\ \text{transmission conditions (TC2)}, & \text{on } S \times [0, T], \\ v_1(0) = 0, & \end{cases}$$

and

$$(4.4) \quad \begin{cases} \partial_t v_2 - \nabla \cdot (c \nabla v_2) = 0, & \text{in } Q', \\ v_2 = 0, & \text{on } \Sigma, \\ \text{transmission conditions (TC1)}, & \text{on } S \times [0, T], \\ v_2(0) = v(0). & \end{cases}$$

Then $v = v_1 + v_2$.

With an argument of logarithmic convexity we have [19]

$$|v_2(t)|_{L^2(\Omega')} \leq K^{1-t/T'} |v_2(T')|_{L^2(\Omega')}^{t/T'}.$$

We now prove

Lemma 4.4. *There exists $C > 0$, such that*

$$|v_1(t)|_{L^2(\Omega')} \leq C |c - \tilde{c}|_{L^\infty(\Omega)}^{\frac{1}{2}}, \quad 0 \leq t \leq T.$$

Note that v_1 satisfies transmission condition (TC2) and thus does not belong to \mathcal{D}_A . We thus cannot use some argument of regularity w.r.t. the source term from parabolic theory for v_1 .

Proof. First observe that $v_1 = w - \tilde{z}$. From Assumption 4.2, w and \tilde{z} remain in a bounded domain of $L^2(0, T, H^1(\Omega))$. Now $\partial_t(w - \tilde{z}) - \nabla \cdot (c \nabla w - \tilde{c} \nabla \tilde{z}) = 0$, which after multiplication by $\bar{w} - \tilde{z}$, integration over Ω and an integration by parts, yields

$$\begin{aligned} 0 &= \frac{1}{2} \partial_t \int_{\Omega} |w - \tilde{z}|^2 dx + \operatorname{Re} \int_{\Omega} (c \nabla w - \tilde{c} \nabla \tilde{z}) \cdot (\overline{\nabla w} - \overline{\nabla \tilde{z}}) dx \\ &= \frac{1}{2} \partial_t \int_{\Omega} |w - \tilde{z}|^2 dx + \operatorname{Re} \int_{\Omega} (c - \tilde{c}) \nabla w \cdot (\overline{\nabla w} - \overline{\nabla \tilde{z}}) dx + \int_{\Omega} \tilde{c} |\nabla w - \nabla \tilde{z}|^2 dx \end{aligned}$$

since $c\nabla w - \tilde{c}\nabla\tilde{z} \in H(\operatorname{div}, \Omega)$ by the definitions of \mathcal{D}_A and $\mathcal{D}_{\tilde{A}}$ (see Appendix A). We thus obtain

$$\begin{aligned} \frac{1}{2}\partial_t \int_{\Omega} |w - \tilde{z}|^2 dx &\leq |\operatorname{Re} \int_{\Omega} (c - \tilde{c})\nabla w \cdot (\overline{\nabla w} - \overline{\nabla\tilde{z}}) dx| \\ &\leq |c - \tilde{c}|_{L^\infty(\Omega)} |\nabla w|_{L^2(\Omega)} |\nabla w - \nabla\tilde{z}|_{L^2(\Omega)} \end{aligned}$$

Integrating over $(0, t)$ yields the result. \square

We thus have

$$|v(t)|_{L^2(\Omega')} \leq |v_1(t)|_{L^2(\Omega')} + |v_2(t)|_{L^2(\Omega')} \leq C(|c - \tilde{c}|_{L^\infty(\Omega)}^{\frac{1}{2}} + |v_2(T')|_{L^2(\Omega')}^{t/T'})$$

$$0 \leq t \leq T,$$

and

$$\begin{aligned} |v_2(T')|_{L^2(\Omega')} &\leq |v(T')|_{L^2(\Omega')} + |v_1(T')|_{L^2(\Omega')} \\ &\leq C(|(\Delta y - \Delta\tilde{y})(T')|_{L^2(\Omega')} + |c - \tilde{c}|_{L^\infty(\Omega)} + |c - \tilde{c}|_{L^\infty(\Omega)}^{\frac{1}{2}}), \end{aligned}$$

which yields

$$|v(t)|_{L^2(\Omega')} \leq C \left(|c - \tilde{c}|_{L^\infty(\Omega)}^{\frac{1}{2}} + (|(\Delta y - \Delta\tilde{y})(T')|_{L^2(\Omega')} + |c - \tilde{c}|_{L^\infty(\Omega)} + |c - \tilde{c}|_{L^\infty(\Omega)}^{\frac{1}{2}})^{t/T'} \right).$$

Because of the regularity of v w.r.t. time t we now have

$$\begin{aligned} |y_0 - \tilde{y}_0|_{L^2(\Omega)} = |u_0|_{L^2(\Omega)} &= \left| \int_0^{T'} v(t) dt - u(T') \right|_{L^2(\Omega')} \\ &\leq \int_0^{T'} |v(t)|_{L^2(\Omega')} dt + |y(T') - \tilde{y}(T')|_{L^2(\Omega)}, \end{aligned}$$

which gives

$$|y_0 - \tilde{y}_0|_{L^2(\Omega)} \leq C \left(T' \frac{\alpha - 1}{\ln(\alpha)} + T' |c - \tilde{c}|_{L^\infty(\Omega)}^{\frac{1}{2}} + |y(T') - \tilde{y}(T')|_{L^2(\Omega)} \right),$$

with $\alpha = |(\Delta y - \Delta\tilde{y})(T')|_{L^\infty(\Omega')} + |c - \tilde{c}|_{L^\infty(\Omega)} + |c - \tilde{c}|_{L^\infty(\Omega)}^{\frac{1}{2}}$.

With Theorem 2.9 we obtain the following stability theorem for the initial conditions.

Theorem 4.5. *Under the hypothesis of Theorem 2.9 in addition to Assumptions 4.1 and 4.2 there exists a constant $C > 0$*

$$C = C(\Omega, T, t_0, \gamma, S, \mathcal{O}^{(1)}, \mathcal{O}^{(2)}, M, K, r, c_{\min}, c_{\max}, \Delta)$$

such that

$$|y_0 - \tilde{y}_0|_{L^2(\Omega)} \leq C / \ln \left(|(y - \tilde{y})(T')|_{H^2(\Omega')} + |\partial_n(\partial_t y - \partial_t \tilde{y})|_{L^2((0, T) \times \gamma)}^2 \right)$$

for $|(y - \tilde{y})(T')|_{H^2(\Omega')} + |\partial_n(\partial_t y - \partial_t \tilde{y})|_{L^2((0, T) \times \gamma)}^2 < 1$.

5 A generalization to embedded materials

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set. As before $\bar{\Omega}$ is assumed to be a \mathcal{C}^2 submanifold with boundary in \mathbb{R}^n . We consider some open sets V_0, \dots, V_m such that

$$V_0 \Subset V_1 \Subset \dots \Subset V_m = \Omega.$$

We then set $\Omega_i := V_i \setminus \bar{V}_{i-1}$, $i = 1, \dots, m$ and $\Omega_0 = V_0$. Each open set Ω_i , $i = 0, \dots, m$, is assumed to have p_i connected components, $\Omega_{i,j}$, $j = 1, \dots, p_i$. We let $S_{i-1}^i = \bar{\Omega}_{i-1} \cap \bar{\Omega}_i$ be the interface between Ω_{i-1} and Ω_i , $i = 1, \dots, m$. The interface S_{i-1}^i is assumed to be \mathcal{C}^2 . Similarly, we denote by $S_{i-1,j}^{i,k}$ the interface (possibly empty) between $\Omega_{i-1,j}$ and $\Omega_{i,k}$. We also set

$$S_{i,j} = \bigcup_{k=1, \dots, p_{i+1}} S_{i,j}^{i+1,k}, \text{ for } i = 0, \dots, m-1,$$

$$S^{i,j} = \bigcup_{k=1, \dots, p_{i-1}} S_{i-1,k}^{i,j}, \text{ for } i = 1, \dots, m,$$

and $S = \bigcup_{i=0, \dots, m-1} S_i^{i+1}$. On an interface S_{i-1}^i , we denote by n the outward unit normal vector to Ω_i and the outward unit normal vector to Ω_m on Γ . We set $\Omega' = \Omega_0 \cup \dots \cup \Omega_m$.

We introduce the following geometric condition.

Assumption 5.1. Geometric Condition (GC_m)

Let $\Omega_{i,j}$ be a component of Ω_i . We assume that either

Case 1 $i = 0$ or $\Omega_{i,j}$ has no interface in common with any component of Ω_{i-1} , i.e. $S^{i,j} = \emptyset$.

Case 2 $\Omega_{i,j}$ has an interface with only one component of Ω_{i-1} .

In **case 1** we further assume that there exists two disjoint open subsets $\mathcal{O}_{i,j}^{(1)}$, $\mathcal{O}_{i,j}^{(2)} \Subset \Omega_{i,j}$ and two vector fields, $\zeta_{i,j}^{(k)} \in \mathcal{C}^1(\bar{\Omega}_{i,j}, \mathbb{R}^2)$, $k = 1, 2$, such that

$$\zeta_{i,j}^{(k)}(x) \cdot n(x) > 0, \quad \forall x \in S_{i,j}, \quad k = 1, 2,$$

$$\zeta_{i,j}^{(k)}(x) \cdot n(x) > 0, \quad \forall x \in \partial \mathcal{O}_{i,j}^{(k)}, \quad k = 1, 2,$$

$$\zeta_{i,j}^{(k)}(x) \neq 0, \quad \forall x \in \Omega_{i,j} \setminus \mathcal{O}_{i,j}^{(k)}, \quad i = 1, 2.$$

Let $x_{i,j}^{(k)}$ be the integral curves of $\zeta_{i,j}^{(k)}$, i.e.

$$\begin{cases} \frac{dx_{i,j}^{(k)}(t)}{dt} = \zeta_{i,j}^{(k)}(x_{i,j}^{(k)}(t)), & t > 0, \\ x_{i,j}^{(k)}(0) = x_0, & x_0 \in S_{i,j}. \end{cases}$$

We also assume that there exists $\mathcal{T}_{i,j} > 0$ such that for all $x_0 \in S_{i,j}$, there exists $t_{i,j}^{(k)}(x_0) < \mathcal{T}_{i,j}$ satisfying

$$x_{i,j}^{(k)}(t) \in \Omega_{i,j} \setminus \mathcal{O}_{i,j}^{(k)}, \quad \text{for } 0 < t < t_{i,j}^{(k)}(x_0), \quad x_0 \in S_{i,j}, \quad k = 1, 2,$$

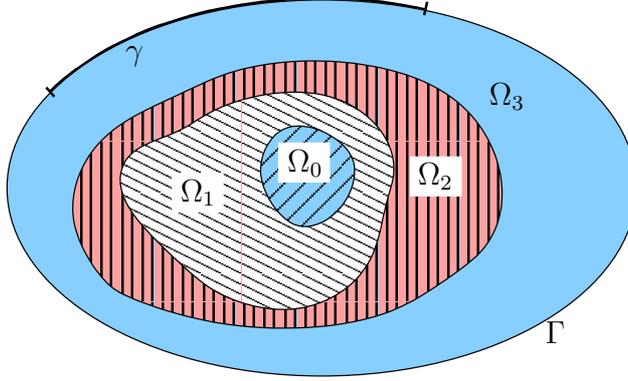


Figure 3: A geometric situation in which Assumption 5.1 is satisfied in the case $m = 3$.

$$x_{i,j}^{(k)}(t_{i,j}^{(k)}(x_0)) \in \partial\mathcal{O}_{i,j}^{(k)}, \quad \text{for } x_0 \in S_{i,j}, \quad k = 1, 2.$$

In case 2, with an interface with say $\Omega_{i-1,k}$, we further assume that there exists a vector field $\zeta_{i,j} \in \mathcal{C}^1(\bar{\Omega}_{i,j}, \mathbb{R}^2)$, such that

$$\zeta_{i,j}(x) \cdot n(x) > 0, \quad \forall x \in S_{i,j}$$

$$\zeta_{i,j}(x) \cdot n(x) > 0, \quad \forall x \in S_{i-1,k}^{i,j}$$

$$\zeta_{i,j}(x) \neq 0, \quad \forall x \in \Omega_{i,j}$$

Let $x_{i,j}$ be the integral curves of $\zeta_{i,j}$, i.e.

$$\begin{cases} \frac{dx_{i,j}(t)}{dt} = \zeta_{i,j}(x_{i,j}(t)), & t > 0, \\ x_{i,j}(0) = x_0, & x_0 \in S_{i,j}. \end{cases}$$

We also assume that there exists $\mathcal{T}_{i,j} > 0$ such that for all $x_0 \in S_{i,j}$, there exists $t_{i,j}(x_0) < \mathcal{T}_{i,j}$ satisfying

$$x_{i,j}(t) \in \Omega_{i,j}, \quad \text{for } 0 < t < t_{i,j}(x_0), \quad x_0 \in S_{i,j},$$

$$x_{i,j}(t_{i,j}(x_0)) \in S_{i-1,k}^{i,j}, \quad \text{for } x_0 \in S_{i,j}.$$

Remark 5.2. In the simple case illustrated in Figure 3 where each open set Ω_i is connected, then only Ω_0 falls in the case 1 and $\Omega_1, \dots, \Omega_m$ satisfy case 2.

We let $A_1 = \cup_{i,j} \Omega_{i,j}$ where the union is performed over the i, j such that the sets $\Omega_{i,j}$ satisfy case 1 and $A_2 = \cup_{i,j} \Omega_{i,j}$ where the union is performed over the i, j such that the sets $\Omega_{i,j}$ satisfy case 2. Figures 3–6 illustrate cases where Assumption 5.1 is, or is not, satisfied.

We now assume

Assumption 5.3. The diffusion coefficient satisfies $c_i := c|_{\Omega_i} \in \mathcal{C}^1(\bar{\Omega}_i)$, $i = 0, \dots, m$.

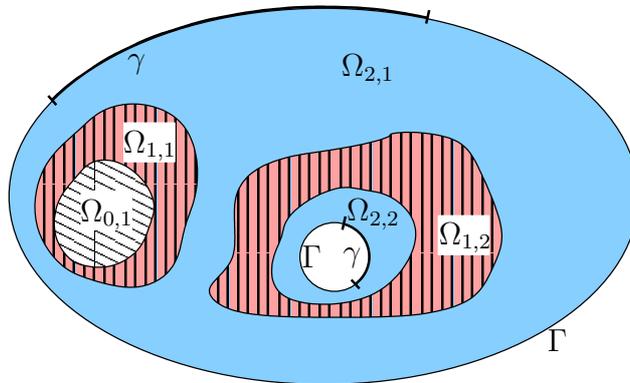


Figure 4: A geometric situation in which Assumption 5.1 is satisfied in the case $m = 2$.

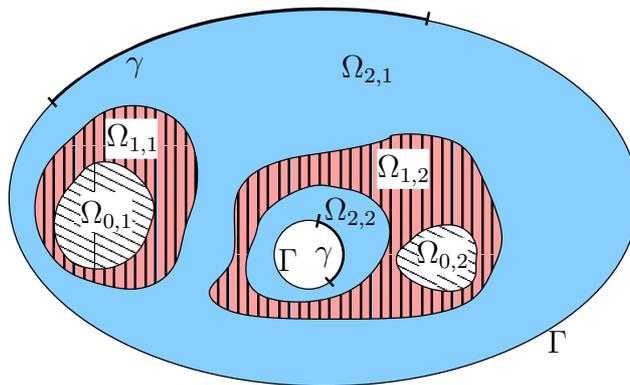


Figure 5: A geometric situation in which Assumption 5.1 is *not* satisfied.

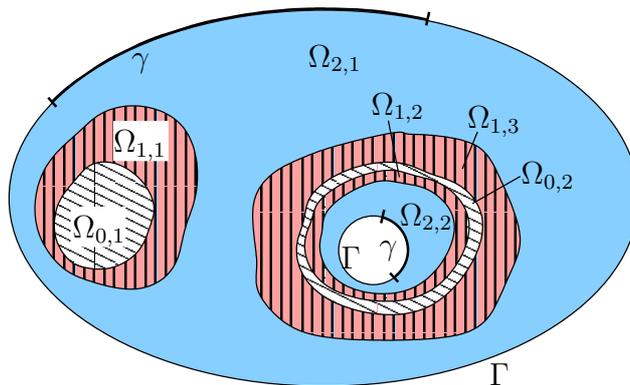


Figure 6: A geometric situation in which Assumption 5.1 is satisfied.

Assumption 5.4. We have $0 < c_{min} \leq c(x) \leq c_{max}$, $x \in \Omega'$. If $R = S_{i,j}^{i+1,k} \neq \emptyset$, for $i = 0, \dots, m-1$, $1 \leq j \leq p_i$, and $1 \leq k \leq p_{i+1}$, then $c_{i|R} \geq c_{i+1|R}$.

The geometric condition (GC_m) thus assumes that different materials are embedded in each other and the previous assumption thus states that the diffusion coefficient increases at the interfaces when one goes from the outer surface Γ to the inner parts of the material.

We can construct weight functions for a Carleman estimate with the following lemma.

Lemma 5.5. Let the Geometric Condition (GC_m) be satisfied and let γ be a subset of $\Gamma = \partial\Omega$ satisfying Assumption 0.4. For the components $\Omega_{i,j}$ of Ω_i , $i = 1, \dots, m$, satisfying case 1 in (GC_m) we let $B_{i,j}^{(k)}$ and $\tilde{B}_{i,j}^{(k)}$, $k = 1, 2$ be open balls such that $B_{i,j}^{(k)} \Subset \tilde{B}_{i,j}^{(k)} \Subset \mathcal{O}_{i,j}^{(k)}$, $k = 1, 2$. Then, there exists two functions $\tilde{\beta}^{(1)}$ and $\tilde{\beta}^{(2)}$ in $\mathcal{C}^0(\bar{\Omega})$ such that

$$\tilde{\beta}^{(k)}(x) = \begin{cases} \tilde{\beta}_1^{(k)} & \text{in } A_1, \\ \tilde{\beta}_2 & \text{in } \bar{A}_2, \end{cases}$$

and the functions $\tilde{\beta}_1^{(1)}$, $\tilde{\beta}_1^{(2)}$, and $\tilde{\beta}_2$ satisfy the following properties:

$$\tilde{\beta}_2 = 0 \text{ on } \Gamma \setminus \gamma, \quad \partial_n \tilde{\beta}_2 < 0 \text{ on } \Gamma \setminus \gamma.$$

For $\bar{\Omega}_{i,j}$ satisfying case 2, $\tilde{\beta}_2|_{\Omega_{i,j}} \in \mathcal{C}^2(\bar{\Omega}_{i,j})$, $\tilde{\beta}_2 > 0$ in $\Omega_{i,j}$, and

$$\tilde{\beta}_2 = i + 2 \text{ on } S_{i,j}, \quad \partial_n \tilde{\beta}_2 < 0 \text{ on } S_{i,j}, \text{ if } i < m,$$

and

$$|\nabla \tilde{\beta}_2| > 0 \text{ on } \bar{\Omega}_{i,j}.$$

For $\bar{\Omega}_{i,j}$ satisfying case 1, for $k = 1, 2$, $\tilde{\beta}_1^{(k)}|_{\Omega_{i,j}} \in \mathcal{C}^2(\bar{\Omega}_{i,j})$, $\tilde{\beta}_1^{(k)} > 0$ in $\Omega_{i,j}$,

$$\tilde{\beta}_1^{(k)} = 2 \text{ on } S_{i,j}, \quad k = 1, 2,$$

$$\tilde{\beta}_1^{(1)} \geq 2\tilde{\beta}_1^{(2)} \text{ in } \tilde{B}_{i,j}^{(2)},$$

$$\tilde{\beta}_1^{(2)} \geq 2\tilde{\beta}_1^{(1)} \text{ in } \tilde{B}_{i,j}^{(1)},$$

and

$$|\nabla \tilde{\beta}_1^{(k)}| > 0 \text{ in } \bar{\Omega}_{i,j} \setminus B_{i,j}^{(k)}, \quad k = 1, 2.$$

Finally on an (non-empty) interface $R = S_{i,j}^{i+1,k}$

$$(c_i \partial_n \tilde{\beta}^{(k)})|_{\Omega_{i,j}} = (c_{i+1} \partial_n \tilde{\beta}^{(k)})|_{\Omega_{i+1,k}} \text{ on } R, \quad k = 1, 2.$$

We set $\mathcal{O}^{(k)} = \bigcup_{i,j} \mathcal{O}_{i,j}^{(k)}$, $k = 1, 2$, where the union is performed over the i, j such that the sets $\Omega_{i,j}$ satisfy case 1 in Assumption 5.1.

We introduce the functions $\beta^{(k)}$ and the weight functions $\varphi^{(k)}, \eta^{(k)}, k = 1, 2$, following section 1. Let $g \in H^1([t_0, T], H^{\frac{1}{2}}(S))$. We introduce transmission condition (TC3) on the interval $[t_0, T]$: for each (non-empty) interface $R = S_{i,j}^{i+1,k}$, we have

$$(TC3) \quad \begin{aligned} q|_{\Omega_{i,j} \times [t_0, T]} &= q|_{\Omega_{i+1,k} \times [t_0, T]}, \text{ on } R \\ (c_i \partial_n q)|_{\Omega_{i,j} \times [t_0, T]} &= (c_{i+1} \partial_n q)|_{\Omega_{i+1,k} \times [t_0, T]} + g(x, t), \text{ on } R \end{aligned}$$

for a function q which is H^2 in each open set $\Omega_{i,j}, i = 0, \dots, m, j = 1, \dots, p_i$. Following Section 1 we introduce

$$\mathfrak{N}_g = \left\{ q \in H^1(t_0, T, H_0^1(\Omega)); q|_{(t_0, T) \times \Omega_i} \in L^2(t_0, T, H^2(\Omega_i)), i = 0, 1, \right. \\ \left. q|_{\Sigma} = 0 \text{ and } q \text{ satisfies (TC3) a.e. w.r.t. } t \right\}.$$

We then obtain

Theorem 5.6. *Let γ be a subset of the boundary Γ of an open set Ω of \mathbb{R}^n that satisfies Condition (GC_m) , and γ satisfies Assumption 0.4. Assume further that there exists $\Delta > 0$ such that, for each (non-empty) interface $R = S_{i,j}^{i+1,k}$, we have $c_{i|R} - c_{i+1|R} \geq \Delta > 0$. Let $g \in H^1(t_0, T, H^{\frac{1}{2}}(S))$. There exists $\lambda_1 = \lambda_1(\Omega, \gamma, \mathcal{O}^{(1)}, \mathcal{O}^{(2)}, c_{min}, c_{max}, \Delta) > 0, s_1 = s_1(\lambda_1) > 0$ and a positive constant $C = C(\Omega, \gamma, \mathcal{O}^{(1)}, \mathcal{O}^{(2)}, c_{min}, c_{max}, \Delta)$ so that Carleman estimate (1.5) holds for $s \geq s_1, \lambda \geq \lambda_1$ and for all $q \in \mathfrak{N}_g$.*

Following the method of Section 2 we thus obtain the following stability result.

Theorem 5.7. *Let γ be a subset of the boundary Γ of an open set Ω of \mathbb{R}^n that satisfies Condition (GC_m) , and γ satisfies Assumption 0.4. We assume that the diffusion coefficients c and \tilde{c} satisfy Assumptions 5.4 and that c and \tilde{c} are piecewise constant, in the sense that $c|_{\Omega_i}, \text{ resp. } \tilde{c}|_{\Omega_i},$ are constant in each connected component of $\Omega_i, i = 0, \dots, m$ and that furthermore there exists $\Delta > 0$ such that, for each (non-empty) interface $R = S_{i,j}^{i+1,k}$, we have $c_{i|R} - c_{i+1|R} \geq \Delta > 0$. Let y_0, \tilde{y}_0 in $L^2(\Omega)$ and let y, \tilde{y} be solutions to (2.1)–(2.2) satisfying Assumption 2.1. Then there exists a constant C*

$$C = C(\Omega, T, t_0, \gamma, S, \mathcal{O}^{(1)}, \mathcal{O}^{(2)}, M, K, r, c_{min}, c_{max}, \Delta)$$

such that

$$|c - \tilde{c}|_{L^\infty(\Omega)}^2 \leq C |\partial_n(\partial_t y - \partial_t \tilde{y})|_{L^2((0, T) \times \gamma)}^2 + C |\Delta y(T', \cdot) - \Delta \tilde{y}(T', \cdot)|_{L^2(\Omega')}^2.$$

Similarly, following Section 4 we obtain

Theorem 5.8. *Under the hypothesis of Theorem 5.7 in addition to Assumption 4.1 there exists a constant $C > 0$*

$$C = C(\Omega, T, t_0, \gamma, S, \mathcal{O}^{(1)}, \mathcal{O}^{(2)}, M, K, r, c_{min}, c_{max}, \Delta)$$

such that

$$|y_0 - \tilde{y}_0|_{L^2(\Omega)} \leq C / \ln \left(|(y - \tilde{y})(T')|_{H^2(\Omega')} + |\partial_n(\partial_t y - \partial_t \tilde{y})|_{L^2((0, T) \times \gamma)}^2 \right)$$

for $|(y - \tilde{y})(T')|_{H^2(\Omega')} + |\partial_n(\partial_t y - \partial_t \tilde{y})|_{L^2((0, T) \times \gamma)}^2 < 1$.

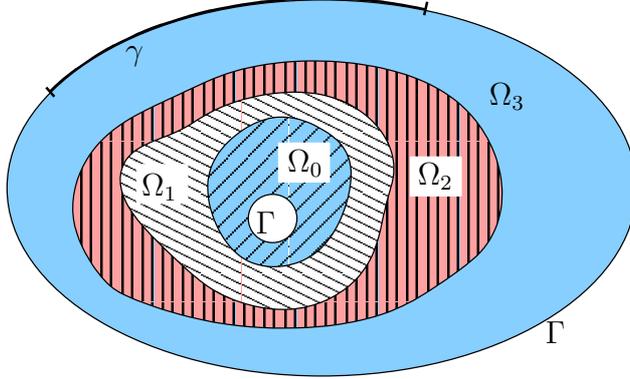


Figure 7: A geometrical situation for which only on set of weight functions is need to obtain a Carleman estimate.

Remark 5.9. In the generalization proposed here we assume some monotonicity for the diffusion coefficient c at the inner interfaces. Note that if some of these assumptions are not satisfied we may still obtain stability results by introducing additional measurements in some $\omega \times (t_0, T)$ for $\omega \subset \Omega$. Each geometrical configuration requires the derivation of a particular Carleman estimate to apply the method exposed here.

Remark 5.10. Here we have assumed

$$V_0 \in V_1 \in \dots \in V_m = \Omega.$$

Observe that this is not true in the case illustrated in Figure 7. then ∂_Ω contains a component of Γ . In such a case we can however obtain a Carleman estimate by using a single set of weight functions η and φ instead of two (see (1.4) and (1.5)). Stability results follow similarly in such a geometrical configuration.

A Basic regularity properties

Let \mathcal{A} be formally defined by $-\nabla \cdot (c\nabla(\cdot))$ on $L^2(\Omega)$. The diffusion coefficient c is first assumed to be in $L^\infty(\Omega)$ and such that $c(x) \geq \alpha > 0$ for all $x \in \Omega$. We denote by A the unbounded operator with domain

$$\mathcal{D}_A = \{u \in H_0^1(\Omega); c\nabla u \in H(\text{div}, \Omega)\}.$$

defined by $A(u) = -\nabla \cdot (c\nabla(u))$ for $u \in \mathcal{D}_A$. We recall that $v : \Omega \rightarrow \mathbb{R}^n$ is in $H(\text{div}, \Omega)$ if $\nabla \cdot v \in L^2(\Omega)$ (see e.g. [10]).

Proposition A.1. *Let $u_0 \in L^2(\Omega)$. There exists a unique u such that*

$$u \in \mathcal{C}([0, T]; L^2(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega)) \cap \mathcal{C}([0, T]; \mathcal{D}_A),$$

and

$$(A.1) \quad \begin{cases} \partial_t u - \nabla \cdot (c\nabla(u)) = 0, & t \in]0, T], \\ u(0) = u_0. \end{cases}$$

for $T > 0$. (T can be chosen to be ∞). Furthermore $u \in L^2(0, T; H_0^1(\Omega))$ and $u \in \mathcal{C}^k([0, T]; \mathcal{D}_A)$ for all $k, l \in \mathbb{N}$.

If $u_0 \in \mathcal{D}_A$ then

$$u \in \mathcal{C}^1([0, T]; L^2(\Omega)) \cap \mathcal{C}([0, T]; \mathcal{D}_A).$$

Proposition A.2. *The semigroup $S(t)$ generated by the unbounded operator A on $L^2(\Omega)$ is analytic.*

We now give further regularity properties when placed in the geometrical situation studied in this paper, that is if the diffusion coefficient c is piecewise \mathcal{C}^1 and discontinuous across some \mathcal{C}^2 interface S . We use the notations set in the main text of the paper.

Proposition A.3. *Let the diffusion coefficient, c , be such that $c|_{\Omega_i} \in \mathcal{C}^1(\overline{\Omega}_i)$, $i = 0, 1$. If $p \in \mathcal{D}_A$ then $p|_{\Omega_i} \in H^2(\Omega_i)$, $i = 0, 1$. Furthermore $|p|_{\Omega_i}|_{H^2(\Omega_i)} \leq C|\nabla \cdot (c\nabla p)|_{L^2(\Omega)}$.*

Remark A.4. Observe that the constant in the inequalities of the previous result can be chosen uniform w.r.t. c for $c_{min} \leq c \leq c_{max}$.

Corollary A.5. *Let $m \in \mathbb{N}$ and let $c|_{\Omega_i} \in \mathcal{C}^{m+1}(\Omega_i)$, $i = 0, 1$, and S and $\partial\Omega$ be of class \mathcal{C}^{m+2} . Then if $u_0 \in L^2(\Omega)$ the solution u to (A.1) is such that $u|_{\Omega_i} \in \mathcal{C}^k((0, T], H^{m+2}(\Omega_i))$, $i = 0, 1$, for all $k \in \mathbb{N}$.*

Remark A.6. Let $\varepsilon > 0$. With the notations of the above corollary, observe that the map

$$\begin{aligned} L_{\varepsilon, i} : L^2(\Omega) &\rightarrow \mathcal{C}^k([\varepsilon, T], H^{m+2}(\Omega_i)), \\ u_0 &\mapsto (t \mapsto u|_{\Omega_i}(t)), \end{aligned}$$

is continuous for $i = 0, 1$, since [4, Theorem VII.7]

$$\begin{aligned} |u(t)|_{L^2(\Omega)} &\leq |u_0|_{L^2(\Omega)}, \\ |\partial_t u(t)|_{L^2(\Omega)} &= |\nabla \cdot (c\nabla u(t))|_{L^2(\Omega)} \leq \left| \frac{1}{t} u_0 \right|_{L^2(\Omega)}. \end{aligned}$$

Note that the operator norm of $L_{\varepsilon, i}$ can be bounded uniformly w.r.t. c for $c_{min} \leq c \leq c_{max}$.

We finish with the following lemma which is needed in Section 2.

Lemma A.7. *There exists a neighborhood W of S globally parametrized with $(\sigma, y) \in]-\varepsilon, \varepsilon[\times S$.*

Proof. In a small neighborhood of S we can extend the unit normal vector n to S into a \mathcal{C}^2 vector field v . In an even smaller neighborhood U of S we can assume that v is such that $|v| \geq a > 0$. If we integrate this vector field we find that there is $\varepsilon > 0$ such that the flow χ_σ of this \mathcal{C}^2 vector field over the interval $]-\varepsilon, \varepsilon[$ is confined in U , since S is compact. For $y \in S$, the orientation of the unit normal (see above) is such that $\chi_\sigma(y) \in \Omega_1$ for $\sigma \in]-\varepsilon, 0[$, $\chi_0(y) = y$ and $\chi_\sigma(y) \in \Omega_0$ for $\sigma \in]0, \varepsilon[$. Set now

$$W = \{\chi_\sigma(y); y \in S \text{ and } \sigma \in]-\varepsilon, \varepsilon[\}$$

which is an open neighborhood of S . Note that if x in W then there exists a unique $y \in S$ and a unique $\sigma \in]-\varepsilon, \varepsilon[$ such that $x = \chi_\sigma(y)$. \square

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