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Friedrich Wehrung

A SOLUTION TO DILWORTH'S CONGRUENCE LATTICE PROBLEM

FRIEDRICH WEHRUNG

Dedicated to George Grätzer and Tamás Schmidt

Abstract. We construct an algebraic distributive lattice D that is not isomorphic to the congruence lattice of any lattice. This solves a long-standing open problem, traditionally attributed to R. P. Dilworth, from the forties. The lattice D has compact top element and $\aleph_{\omega+1}$ compact elements. Our results extend to any algebra possessing a polynomially definable structure of a join-semilattice with largest element.

1. Introduction

For an *algebra* L (i.e., a nonempty set with a collection of operations from finite powers of L to L), a *congruence* of L is an equivalence relation on L compatible with all operations of L. For elements $x, y \in L$, we denote by $\Theta_L(x, y)$ the least congruence that identifies x with y, and we call the finite joins of such congruences *finitely generated*. We denote by $\operatorname{Con} L$ (resp., $\operatorname{Con_c} L$) the lattice (resp., $(\vee, 0)$-semilattice) of all congruences (resp., finitely generated congruences) of L under inclusion. A *polynomial* of L is a composition of operations of L, allowing elements of L as parameters. A homomorphism of join-semilattices $\mu\colon S \to T$ is *weakly distributive* at an element $\boldsymbol{x}$ of S, if for all $\boldsymbol{y}_0, \boldsymbol{y}_1 \in T$ such that $\mu(\boldsymbol{x}) \leq \boldsymbol{y}_0 \vee \boldsymbol{y}_1$, there are $\boldsymbol{x}_0, \boldsymbol{x}_1 \in S$ such that $\boldsymbol{x} \leq \boldsymbol{x}_0 \vee \boldsymbol{x}_1$ and $\mu(\boldsymbol{x}_i) \leq \boldsymbol{y}_i$, for all $i < 2$. We say that μ is *weakly distributive*, if it is weakly distributive at every element of S. (For S and T distributive, this is equivalent to the definition presented in [35]. Moreover, it extends the original definition given by Schmidt [29, 30].)

In the present paper we prove the following result (cf. Theorem 6.1).

Theorem. *There exists a distributive $(\vee, 0, 1)$-semilattice S such that for any algebra L with a polynomially definable structure of $(\vee, 1)$-semilattice, there exists no weakly distributive $(\vee, 0)$-homomorphism $\mu\colon \operatorname{Con_c} L \to S$ with 1 in its range. Furthermore, S has $\aleph_{\omega+1}$ elements.*

It follows that the semilattice S is not isomorphic to $\operatorname{Con_c} L$, for any lattice L. Hence the ideal lattice of S is not isomorphic to the congruence lattice of any lattice.

We shall now give some background on the problem solved by our theorem. Funayama and Nakayama [5] proved in 1942 that $\operatorname{Con} L$ is *distributive*, for any lattice $(L, \vee, \wedge)$. Dilworth proved soon after that conversely, every finite distributive

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lattice is isomorphic to the congruence lattice of some finite lattice (see [3, pp. 455–456] and [8]). Birkhoff and Frink [2] proved in 1948 that the congruence lattice of any algebra is what is nowadays called an *algebraic* lattice, that is, it is *complete* and every element is a join of *compact* elements (see [9]). The question whether every algebraic distributive lattice is isomorphic to $\mathrm{Con}\, L$ for some lattice L, often referred to as CLP ('Congruence Lattice Problem'), is one of the most intriguing and longest-standing open problems of lattice theory. The first published occurrence of this problem, as well as the first published proof of Dilworth's abovecited result, seems to appear in Grätzer and Schmidt's paper [12]. However, it seems that the earliest attempts at CLP were made by Dilworth himself, see [3, pp. 455–456].

This problem has generated an enormous amount of work since then, in a somewhat complex pattern of interconnected waves. Grätzer and Schmidt proved in 1963 that every algebraic lattice is isomorphic to the congruence lattice of some algebra [13]. The reader can find in Schmidt's monograph [30] a survey about congruence lattice representations of algebras. The surveys by Grätzer and Schmidt [14, 15] and Grätzer's monograph [10] are focused on congruence lattices of (mainly finite) lattices, while the survey by Tůma and Wehrung [32] is more focused on congruence lattices of infinite lattices. The main connection between the finite case and the infinite case originates in Pudlák's idea [25] of lifting, with respect to the $\mathrm{Con_c}$ functor, *diagrams* of finite distributive $(\vee, 0)$-semilattices. Růžička, Tůma, and Wehrung prove in [28] that there are bounded lattices of cardinality $\aleph_2$ whose congruence lattices are isomorphic neither to the normal subgroup lattice of any group, nor to the submodule lattice of any module; furthermore, the bound $\aleph_2$ is optimal. Some of the more recent works emphasize close connections between congruence lattices of lattices, ideal lattices of rings, dimension theory of lattices, and nonstable K-theory of rings, see for example [1, 6, 7, 26, 34, 35, 36].

Distributive algebraic lattices are ideal lattices of distributive $(\vee, 0)$-semilattices (see Section 2), and for a lattice L, $\mathrm{Con}\, L$ is isomorphic to the ideal lattice of $\mathrm{Con_c}\, L$. We obtain the following more convenient equivalent formulation of CLP (see [32] for details):

CLP (semilattice formulation). *Is every distributive* $(\vee, 0)$*-semilattice* representable*, that is, isomorphic to* $\mathrm{Con_c}\, L$*, for some lattice* L*?*

In particular, the semilattice S of our theorem provides a counterexample to CLP.

Among the classical positive partial results are the following:

(1) Every distributive $(\vee, 0)$-semilattice S of cardinality at most $\aleph_1$ is representable, see Huhn [16, 17].
(2) Every distributive lattice with zero is representable, see Schmidt [29].

Further works extended the class of all representable distributive $(\vee, 0)$-semilattices, for example to all $(\vee, 0)$-direct limits of sequences of distributive lattices with zero, see [37]. Moreover, the representing lattice L can be taken *relatively complemented with zero*. This also holds for case (2) above. However, the latter result has been extended further by Růžička [26], who proved that the representing lattice can be taken relatively complemented, modular, and locally finite. This is not possible for (1) above, as, for $|S| \leq \aleph_1$, one can take L relatively complemented modular [36], relatively complemented and locally finite [11], but not necessarily both [38].

On the negative side, the works in [24, 31, 34, 35] show that lattices with permutable congruences are not sufficient to solve CLP. More precisely, *there exists a representable distributive* $(\vee, 0, 1)$*-semilattice of cardinality* $\aleph_2$ *that is not isomorphic to* $\mathrm{Con_c}\, L$ *for any lattice* L *with permutable congruences.* The finite combinatorial reason for this lies in the impossibility to prove certain 'congruence amalgamation properties'. The infinite combinatorial reason for this is Kuratowski's free set Theorem (see Section 2). The latter is used to prove that certain infinitary statements called 'uniform refinement properties' fail in certain distributive semilattices.

Our proof carries a flavor of (a semilattice-theoretical version of) commutator theory, essentially because of Lemma 5.1, the *Erosion Lemma*. A precedent of this sort of situation occurs with Bill Lampe's wonderful trick used in [4] to prove that certain algebraic lattices require, for their congruence representations, algebras with many operations: namely, the term condition used in commutator theory in, say, congruence-modular varieties (or larger, as considered in [20, 33]).

2. Basic concepts

A $(\vee, 0)$-semilattice S is *distributive*, if $\boldsymbol{c} \leq \boldsymbol{a} \vee \boldsymbol{b}$ in S implies that there are $\boldsymbol{x} \leq \boldsymbol{a}$ and $\boldsymbol{y} \leq \boldsymbol{b}$ in S such that $\boldsymbol{x} \leq \boldsymbol{a}$, $\boldsymbol{y} \leq \boldsymbol{b}$, and $\boldsymbol{c} = \boldsymbol{x} \vee \boldsymbol{y}$. Equivalently, the ideal lattice of S is a distributive lattice, see [9, Section II.5].

The assignment $L \mapsto \mathrm{Con_c}\, L$ is extended the usual way to a functor from algebras with homomorphisms to $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms. For a positive integer m, an algebra L has $(m+1)$*-permutable congruences*, if $\boldsymbol{a} \vee \boldsymbol{b} = \boldsymbol{c}_0 \circ \boldsymbol{c}_1 \circ \cdots \circ \boldsymbol{c}_m$ where $\boldsymbol{c}_i$ equals $\boldsymbol{a}$ if i is even and $\boldsymbol{b}$ if i is odd, for all congruences $\boldsymbol{a}$ and $\boldsymbol{b}$ of L (the symbol $\circ$ denotes, as usual, composition of relations).

For an algebra L endowed with a structure of semilattice, with join operation denoted by $\vee$, we put $\Theta_L^+(x, y) = \Theta_L(y, x \vee y)$, for all $x, y \in L$. We say that the semilattice structure on L is *polynomially definable*, if $\vee$ is a polynomial on L. In such a case, any congruence of L is also a $\vee$-congruence, and thus $\Theta_L^+(x, z) \subseteq \Theta_L^+(x, y) \vee \Theta_L^+(y, z)$, for all $x, y, z \in L$.

For partially ordered sets P and Q, a map $f \colon P \to Q$ is *isotone*, if $x \leq y$ implies that $f(x) \leq f(y)$, for all $x, y \in P$.

We shall also use standard set-theoretical notation and terminology, referring the reader to [18] for further information. We shall denote by $\mathfrak{P}(X)$ the powerset of a set X, by $[X]^{<\omega}$ the set of all finite subsets of X, and by $[X]^n$ (for $n < \omega$) the set of all n-element subsets of X. For a map $\Phi \colon [X]^n \to [X]^{<\omega}$, we say that a $(n+1)$-element subset U of X is *free with respect to* Φ, if $x \notin \Phi(U \setminus \{x\})$ for all $x \in U$. The following statement of infinite combinatorics is one direction of a theorem due to Kuratowski [21].

Kuratowski's free set Theorem. *Let* n *be a natural number and let* X *be a set with* $|X| \geq \aleph_n$*. For every map* $\Phi \colon [X]^n \to [X]^{<\omega}$*, there exists a* $(n+1)$*-element free subset of* X *with respect to* Φ*.*

We identify every natural number n with the set $\{0, 1, \ldots, n-1\}$, and we denote by ω the set of all natural numbers, which is also the first limit ordinal. We shall usually denote elements in semilattices by bold math characters $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, ...

3. Free distributive extension of a $(\vee, 0)$-semilattice

As in [39], we shall use the construction of a "free distributive extension" $\mathcal{R}(S)$ of a $(\vee, 0)$-semilattice S given by Ploščica and Tůma in [23, Section 2]. The larger semilattice $\mathcal{R}(S)$ is constructed by adding new elements $\bowtie(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$, for $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in S$ such that $\boldsymbol{c} \leq \boldsymbol{a} \vee \boldsymbol{b}$, subjected to the only relations $\boldsymbol{c} = \bowtie(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \vee \bowtie(\boldsymbol{b}, \boldsymbol{a}, \boldsymbol{c})$ and $\bowtie(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \leq \boldsymbol{a}$. It is a semilattice version of the dimension group construction $\mathbf{I}_K(E)$ presented in [34, Section 1]. For convenience, we present an equivalent formulation here.

For a $(\vee, 0)$-semilattice S, we shall put $\mathcal{C}(S) = \{(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in S^3 \mid \boldsymbol{w} \leq \boldsymbol{u} \vee \boldsymbol{v}\}$. A *finite* subset $\boldsymbol{x}$ of $\mathcal{C}(S)$ is *projectable* (resp., *reduced*), if it satisfies condition (1) (resp., (1)–(3)) below:

(1) $\boldsymbol{x}$ contains exactly one diagonal triple, that is, a triple of the form $(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u})$; we put $\boldsymbol{u} = \pi(\boldsymbol{x})$.
(2) $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{x}$ and $(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w}) \in \boldsymbol{x}$ implies that $\boldsymbol{u} = \boldsymbol{v} = \boldsymbol{w}$, for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in S$.
(3) $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{x} \setminus \{(\pi(\boldsymbol{x}), \pi(\boldsymbol{x}), \pi(\boldsymbol{x}))\}$ implies that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \nleq \pi(\boldsymbol{x})$, for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in S$.

In particular, observe that if $\boldsymbol{x}$ is reduced, $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{x}$, and $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ is non-diagonal, then $\boldsymbol{u} \neq \boldsymbol{v}$ and the elements $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are nonzero.

We denote by $\overline{\mathcal{R}}(S)$ (resp., $\mathcal{R}(S)$) the set of all projectable (resp., reduced) subsets of $\mathcal{C}(S)$, endowed with the binary relation $\leq$ defined by

$$\boldsymbol{x} \leq \boldsymbol{y} \iff \forall(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{x} \setminus \boldsymbol{y}, \text{ either } \boldsymbol{u} \leq \pi(\boldsymbol{y}) \text{ or } \boldsymbol{w} \leq \pi(\boldsymbol{y}). \tag{3.1}$$

We call π the *canonical projection* from $\mathcal{R}(S)$ onto S. Observe that in general, π is not a join-homomorphism (however, see Remark 3.3). It is straightforward to verify that $\leq$ is a partial ordering on $\overline{\mathcal{R}}(S)$ (and thus on the subset $\mathcal{R}(S)$). Now we shall present, in terms of rewriting rules, the steps (i)–(iv) of the algorithm stated in [23, Lemma 2.1].

For finite subsets $\boldsymbol{x}$ and $\boldsymbol{y}$ of $\mathcal{C}(S)$, let $\boldsymbol{x} \to_1 \boldsymbol{y}$ hold, if there exists a non-diagonal $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \boldsymbol{x}$ such that $(\boldsymbol{b}, \boldsymbol{a}, \boldsymbol{c}) \in \boldsymbol{x}$ and $\boldsymbol{y} = (\boldsymbol{x} \setminus \{(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}), (\boldsymbol{b}, \boldsymbol{a}, \boldsymbol{c})\}) \cup \{(\boldsymbol{c}, \boldsymbol{c}, \boldsymbol{c})\}$. Denote by $\to_1^*$ the reflexive and transitive closure of $\to_1$ on finite subsets of $\mathcal{C}(S)$, and denote by $\mathcal{R}_1(S)$ the set of all finite $\boldsymbol{x} \subseteq \mathcal{C}(S)$ such that $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \boldsymbol{x}$ and $(\boldsymbol{b}, \boldsymbol{a}, \boldsymbol{c}) \in \boldsymbol{x}$ implies that $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{c}$, for all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in S$. Put $\overline{\mathcal{R}}_1(S) = \overline{\mathcal{R}}(S) \cap \mathcal{R}_1(S)$. For a finite subset $\boldsymbol{x}$ of $\mathcal{C}(S)$, we put

$$\varphi(\boldsymbol{x}) = (\boldsymbol{x} \setminus \{(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}) \mid \boldsymbol{u} \in X\}) \cup \left\{\left(\bigvee X, \bigvee X, \bigvee X\right)\right\},$$
$$\text{where } X = \{\boldsymbol{u} \in S \mid (\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}) \in \boldsymbol{x}\}.$$

For $\boldsymbol{x} \in \overline{\mathcal{R}}(S)$ and a finite subset $\boldsymbol{y}$ of $\mathcal{C}(S)$, let $\boldsymbol{x} \to_2 \boldsymbol{y}$ hold, if there exists a non-diagonal $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \boldsymbol{x}$ such that $\boldsymbol{b} \leq \pi(\boldsymbol{x})$ and

$$\boldsymbol{y} = (\boldsymbol{x} \setminus \{(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}), (\pi(\boldsymbol{x}), \pi(\boldsymbol{x}), \pi(\boldsymbol{x}))\}) \cup \{(\boldsymbol{c} \vee \pi(\boldsymbol{x}), \boldsymbol{c} \vee \pi(\boldsymbol{x}), \boldsymbol{c} \vee \pi(\boldsymbol{x}))\}.$$

Observe that necessarily, $\boldsymbol{y}$ belongs to $\overline{\mathcal{R}}(S)$ as well, and denote by $\to_2^*$ the reflexive and transitive closure of $\to_2$ on $\overline{\mathcal{R}}(S)$. Denote by $\mathcal{R}_2(S)$ the set of all $\boldsymbol{x} \in \overline{\mathcal{R}}_1(S)$ such that for all non-diagonal $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \boldsymbol{x}$, the inequality $\boldsymbol{b} \nleq \pi(\boldsymbol{x})$ holds. For any $\boldsymbol{x} \in \overline{\mathcal{R}}(S)$, we put

$$\psi(\boldsymbol{x}) = \boldsymbol{x} \setminus \{(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \boldsymbol{x} \text{ non-diagonal } \mid \text{ either } \boldsymbol{a} \leq \pi(\boldsymbol{x}) \text{ or } \boldsymbol{c} \leq \pi(\boldsymbol{x})\}.$$

The correspondence with the algorithm stated in [23, Lemma 2.1] is as follows: the relation $\to_1$ corresponds to step (i); the function φ corresponds to step (ii); the relation $\to_2$ corresponds to step (iii); the function ψ corresponds to step (iv). The following lemma is a reformulation, in terms of $\to_1$, $\to_2$, φ, and ψ, of [23, Lemma 2.1].

Lemma 3.1. *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}(S)$. Then there exists $(\boldsymbol{z}_1, \boldsymbol{z}_2) \in \mathcal{R}_1(S) \times \mathcal{R}_2(S)$ such that $\boldsymbol{x} \cup \boldsymbol{y} \to_1^* \boldsymbol{z}_1$ and $\varphi(\boldsymbol{z}_1) \to_2^* \boldsymbol{z}_2$. Furthermore, for any such pair $(\boldsymbol{z}_1, \boldsymbol{z}_2)$, $\varphi(\boldsymbol{z}_1)$ belongs to $\overline{\mathcal{R}}_1(S)$ and $\psi(\boldsymbol{z}_2)$ is the join, in $\mathcal{R}(S)$, of $\boldsymbol{x}$ and $\boldsymbol{y}$.*

Corollary 3.2. *The set $\mathcal{R}(S)$ is a $(\vee, 0)$-semilattice under the partial ordering defined in* (3.1)*. Furthermore, the map $j_S \colon S \to \mathcal{R}(S)$, $\boldsymbol{x} \mapsto \{(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x})\}$ is a $(\vee, 0)$-embedding.*

Remark 3.3. We shall identify $\boldsymbol{x}$ with the element $\{(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x})\}$ of $\mathcal{R}(S)$, for all $\boldsymbol{x} \in S$. Then observe that the canonical map $\pi \colon \mathcal{R}(S) \twoheadrightarrow S$ is *isotone* and that the restriction of π to S is the identity. The following is an easy consequence of (3.1).

$$\boldsymbol{x} \leq \boldsymbol{y} \iff \boldsymbol{x} \leq \pi(\boldsymbol{y}), \qquad \text{for all } (\boldsymbol{x}, \boldsymbol{y}) \in S \times \mathcal{R}(S). \tag{3.2}$$

Now the elements of $\mathcal{R}(S) \setminus S$ are exactly those subsets $\boldsymbol{x}$ of $\mathcal{C}(S) \cup S$ (disjoint union) containing exactly one element of S, denoted by $\pi(\boldsymbol{x})$, while $\boldsymbol{x} \setminus \{\pi(\boldsymbol{x})\}$ is nonempty and all its elements are triples $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \mathcal{C}(S)$ such that $(\boldsymbol{b}, \boldsymbol{a}, \boldsymbol{c}) \notin \boldsymbol{x}$ and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \nleq \pi(\boldsymbol{x})$.

We shall use the symbol $\bowtie_S$, or $\bowtie$ if S is understood, to denote the elements of $\mathcal{R}(S)$ defined as

$$\bowtie_S(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \begin{cases} \boldsymbol{w}, & \text{if either } \boldsymbol{u} = \boldsymbol{v} \text{ or } \boldsymbol{v} = 0 \text{ or } \boldsymbol{w} = 0, \\ 0, & \text{if } \boldsymbol{u} = 0, \\ \{(0,0,0), (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})\}, & \text{otherwise,} \end{cases}$$

for all $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in \mathcal{C}(S)$. Then one can prove easily the formula

$$\boldsymbol{x} = \bigvee(\bowtie_S(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \mid (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \boldsymbol{x}), \quad \text{for all } \boldsymbol{x} \in \mathcal{R}(S). \tag{3.3}$$

The following is a slight strengthening of [23, Theorem 2.3], with the same proof. The uniqueness statement follows from (3.3).

Lemma 3.4. *Let S and T be $(\vee, 0)$-semilattices and let $f \colon S \to T$ be a $(\vee, 0)$-homomorphism. Furthermore, let $\imath \colon \mathcal{C}(\operatorname{im} f) \to T$ be a map such that $\imath(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \vee \imath(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{z}) = \boldsymbol{z}$ and $\imath(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \leq \boldsymbol{x}$, for all $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \mathcal{C}(\operatorname{im} f)$. Then there exists a unique map $f_{(\imath)} \colon \mathcal{R}(S) \to T$ such that $f_{(\imath)}(\bowtie_S(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})) = \imath(f(\boldsymbol{x}), f(\boldsymbol{y}), f(\boldsymbol{y}))$, for all $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \mathcal{C}(S)$.*

By applying Lemma 3.4 to the map $j_T \circ f$ and defining $\imath$ as the restriction of $\bowtie_T$ to $\mathcal{C}(\operatorname{im} f)$, we obtain item (1) of the following result. Item (2) follows easily.

Proposition 3.5.

(1) *For $(\vee, 0)$-semilattices S and T, every $(\vee, 0)$-homomorphism $f \colon S \to T$ extends to a unique $(\vee, 0)$-homomorphism $\mathcal{R}(f) \colon \mathcal{R}(S) \to \mathcal{R}(T)$ such that $\mathcal{R}(f)(\bowtie_S(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})) = \bowtie_T(f(\boldsymbol{u}), f(\boldsymbol{v}), f(\boldsymbol{w}))$, for all $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in \mathcal{C}(S)$.*

(2) *The assignment $S \mapsto \mathcal{R}(S)$, $f \mapsto \mathcal{R}(f)$ is a functor.*

Putting $\mathcal{R}^0(S) = S$ and $\mathcal{R}^{n+1}(S) = \mathcal{R}(\mathcal{R}^n(S))$ for each n, the increasing union $\mathcal{D}(S) = \bigcup(\mathcal{R}^n(S) \mid n < \omega)$ is a distributive $(\vee, 0)$-semilattice extending S. Furthermore, putting $\mathcal{D}(f) = \bigcup(\mathcal{R}^n(f) \mid n < \omega)$ for each $(\vee, 0)$-homomorphism f, we obtain that $\mathcal{D}$ is a functor. The proof of the following lemma is straightforward.

Lemma 3.6. *Let S be a $(\vee, 0)$-semilattice and let $(S_i \mid i \in I)$ be a family of $(\vee, 0)$-subsemilattices of S. The following statements hold:*

(1) $\mathcal{R}\left(\bigcap_{i\in I} S_i\right) = \bigcap_{i\in I} \mathcal{R}(S_i)$ *and* $\mathcal{D}\left(\bigcap_{i\in I} S_i\right) = \bigcap_{i\in I} \mathcal{D}(S_i)$.

(2) *If I is a nonempty upward directed partially ordered set and $(S_i \mid i \in I)$ is isotone, then* $\mathcal{R}\left(\bigcup_{i\in I} S_i\right) = \bigcup_{i\in I} \mathcal{R}(S_i)$ *and* $\mathcal{D}\left(\bigcup_{i\in I} S_i\right) = \bigcup_{i\in I} \mathcal{D}(S_i)$.

Definition 3.7. For a $(\vee, 0)$-semilattice S and an element $\boldsymbol{x} \in \mathcal{D}(S)$, we define the *rank* of $\boldsymbol{x}$, denoted by $\operatorname{rk} \boldsymbol{x}$, as the least natural number n such that $\boldsymbol{x} \in \mathcal{R}^n(S)$.

4. The functors $\mathcal{L}$ and $\mathcal{G}$

For a set Ω, we denote by $\mathcal{L}(\Omega)$ the $(\vee, 0)$-semilattice defined by generators 1 and $\boldsymbol{a}_0^\xi$, $\boldsymbol{a}_1^\xi$ (for $\xi \in \Omega$), subjected to the relations

$$\boldsymbol{a}_0^\xi \vee \boldsymbol{a}_1^\xi = 1, \quad \text{for all } \xi \in \Omega. \tag{4.1}$$

Hence $\mathcal{L}(\Omega)$ is the same semilattice as the one presented in [23, Section 3]. It is a semilattice version of the dimension group $\mathbf{E}_K(\Omega)$ presented in [34, Section 2]. It can be 'concretely' represented as the (semi)lattice of all pairs $(X, Y) \in \mathfrak{P}(\Omega) \times \mathfrak{P}(\Omega)$ such that either X and Y are finite and disjoint or $X = Y = \Omega$, with

$$\boldsymbol{a}_0^\xi = (\{\xi\}, \varnothing) \text{ and } \boldsymbol{a}_1^\xi = (\varnothing, \{\xi\}), \quad \text{for all } \xi \in \Omega.$$

We shall identify $\mathcal{L}(X)$ with the $(\vee, 0, 1)$-subsemilattice of $\mathcal{L}(\Omega)$ generated by the subset $\{\boldsymbol{a}_i^\xi \mid \xi \in X \text{ and } i < 2\}$, for all $X \subseteq \Omega$. For sets X and Y, any map $f\colon X \to Y$ gives rise to a unique $(\vee, 0, 1)$-homomorphism $\mathcal{L}(f)\colon \mathcal{L}(X) \to \mathcal{L}(Y)$ such that $\mathcal{L}(f)(\boldsymbol{a}_i^\xi) = \boldsymbol{a}_i^{f(\xi)}$, for all $(\xi, i) \in X \times \{0, 1\}$. Of course, the assignment $X \mapsto \mathcal{L}(X)$, $f \mapsto \mathcal{L}(f)$ is a functor from the category of sets with maps to the category of $(\vee, 0, 1)$-semilattices and $(\vee, 0, 1)$-homomorphisms.

Next, we put $\mathcal{G} = \mathcal{D} \circ \mathcal{L}$, the composition of the two functors $\mathcal{D}$ and $\mathcal{L}$. Hence, for a set Ω, the semilattice $\mathcal{G}(\Omega)$ may be loosely described as a 'free distributive $(\vee, 0)$-semilattice defined by generators $\boldsymbol{a}_i^\xi$, for $\xi \in \Omega$ and $i < 2$, and relations (4.1)'. It is a distributive $(\vee, 0, 1)$-semilattice, of the same cardinality as Ω in case Ω is infinite.

The proof of the following lemma is straightforward (see Lemma 3.6).

Lemma 4.1. *Let Ω be a set and let $(X_i \mid i \in I)$ be a family of subsets of Ω. The following statements hold:*

(1) $\mathcal{L}\left(\bigcap_{i\in I} X_i\right) = \bigcap_{i\in I} \mathcal{L}(X_i)$ *and* $\mathcal{G}\left(\bigcap_{i\in I} X_i\right) = \bigcap_{i\in I} \mathcal{G}(X_i)$.

(2) *If I is a nonempty upward directed partially ordered set and the family $(X_i \mid i \in I)$ is isotone, then* $\mathcal{L}\left(\bigcup_{i\in I} X_i\right) = \bigcup_{i\in I} \mathcal{L}(X_i)$ *and* $\mathcal{G}\left(\bigcup_{i\in I} X_i\right) = \bigcup_{i\in I} \mathcal{G}(X_i)$.

Corollary 4.2. *For any set Ω and any $\boldsymbol{x} \in \mathcal{G}(\Omega)$, there exists a least (finite) subset X of Ω such that $\boldsymbol{x} \in \mathcal{G}(X)$.*

We shall call the subset X of Corollary 4.2 the *support* of $\boldsymbol{x}$, and denote it by $\operatorname{supp}(\boldsymbol{x})$.

Lemma 4.3. *Let Ω be a set, let $\alpha \in \Omega$, and let $i < 2$. Then $\boldsymbol{x} \leq \boldsymbol{y} \vee \boldsymbol{a}_i^\alpha$ implies that $\boldsymbol{x} \leq \boldsymbol{y}$, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{G}(\Omega \setminus \{\alpha\})$.*

Proof. There exists a unique retraction $r \colon \mathcal{L}(\Omega) \twoheadrightarrow \mathcal{L}(\Omega \setminus \{\alpha\})$ such that $r(\boldsymbol{a}_i^\alpha) = 0$. Put $s = \mathcal{D}(r)$, and observe that $s(\boldsymbol{x}) = \boldsymbol{x}$, $s(\boldsymbol{y}) = \boldsymbol{y}$, and $s(\boldsymbol{a}_i^\alpha) = 0$. By applying s to the inequality $\boldsymbol{x} \leq \boldsymbol{y} \vee \boldsymbol{a}_i^\alpha$, we get the conclusion. □

The following crucial lemma describes an 'evaporation process' in $\mathcal{G}(\Omega)$.

Lemma 4.4 (Evaporation)**.** *Let α, β, δ be distinct elements in a set Ω, let $i, j < 2$, $\boldsymbol{x} \in \mathcal{G}(\Omega \setminus \{\beta\})$, $\boldsymbol{y} \in \mathcal{G}(\Omega \setminus \{\alpha\})$, and $\boldsymbol{z} \in \mathcal{G}(\Omega \setminus \{\delta\})$. Then*

$$\boldsymbol{z} \leq \boldsymbol{x} \vee \boldsymbol{y}, \quad \boldsymbol{x} \leq \boldsymbol{a}_0^\delta, \boldsymbol{a}_i^\alpha, \quad \textit{and} \quad \boldsymbol{y} \leq \boldsymbol{a}_1^\delta, \boldsymbol{a}_j^\beta$$

implies that $\boldsymbol{z} = 0$.

Proof. For $s \in \omega$ and $\boldsymbol{u} \in \mathcal{R}^{s+1}\mathcal{L}(\Omega) \setminus \mathcal{R}^s\mathcal{L}(\Omega)$, we shall denote by $\pi(\boldsymbol{u})$ the image of $\boldsymbol{u}$ under the canonical projection from $\mathcal{R}^{s+1}\mathcal{L}(\Omega)$ to $\mathcal{R}^s\mathcal{L}(\Omega)$. Put $m = \operatorname{rk} \boldsymbol{x}$, $n = \operatorname{rk} \boldsymbol{y}$, and $k = \operatorname{rk} \boldsymbol{z}$. We argue by induction on $m + n + k$. If $\boldsymbol{z} \leq \boldsymbol{x}$, then $\boldsymbol{z} \leq \boldsymbol{a}_0^\delta$, thus, as $\boldsymbol{z} \in \mathcal{G}(\Omega \setminus \{\delta\})$, it follows from Lemma 4.3 that $\boldsymbol{z} = 0$ so we are done. The conclusion is similar in case $\boldsymbol{z} \leq \boldsymbol{y}$. So suppose that $\boldsymbol{z} \nleq \boldsymbol{x}, \boldsymbol{y}$. If $m = 0$, then, as $\boldsymbol{x} \in \mathcal{L}(\Omega)$ and $\boldsymbol{x} \leq \boldsymbol{a}_0^\delta, \boldsymbol{a}_i^\alpha$ with $\alpha \neq \delta$, we get $\boldsymbol{x} = 0$, so $\boldsymbol{z} \leq \boldsymbol{y}$, a contradiction; hence $m > 0$. Similarly, $n > 0$. Put $l = \max\{m, n\}$, $\boldsymbol{x}^* = \boldsymbol{x} \setminus \{\pi(\boldsymbol{x})\}$, and $\boldsymbol{y}^* = \boldsymbol{y} \setminus \{\pi(\boldsymbol{y})\}$ (see Remark 3.3). Furthermore, we define (using again Remark 3.3) a finite subset $\boldsymbol{w}$ of $\mathcal{C}\mathcal{R}^{l-1}\mathcal{L}(\Omega)$ as

$$\boldsymbol{w} = \begin{cases} \boldsymbol{x}^* \cup \boldsymbol{y}^* \cup \{\pi(\boldsymbol{x}) \vee \pi(\boldsymbol{y})\}, & \text{if } m = n, \\ \boldsymbol{y}^* \cup \{\boldsymbol{x} \vee \pi(\boldsymbol{y})\}, & \text{if } m < n, \\ \boldsymbol{x}^* \cup \{\pi(\boldsymbol{x}) \vee \boldsymbol{y}\}, & \text{if } m > n. \end{cases} \tag{4.2}$$

Claim. *The set $\boldsymbol{w}$ belongs to $\mathcal{R}^l\mathcal{L}(\Omega)$, and $\boldsymbol{x}, \boldsymbol{y} \leq \boldsymbol{w}$.*

Proof of Claim. We need to verify that $\boldsymbol{w}$ is a reduced subset of $\mathcal{C}\mathcal{R}^{l-1}\mathcal{L}(\Omega)$, modulo the identification of elements with diagonal triples (see Remark 3.3). It is obvious that there exists exactly one element in $\boldsymbol{w} \cap \mathcal{R}^{l-1}\mathcal{L}(\Omega)$, namely,

$$\pi(\boldsymbol{w}) = \begin{cases} \pi(\boldsymbol{x}) \vee \pi(\boldsymbol{y}), & \text{if } m = n, \\ \boldsymbol{x} \vee \pi(\boldsymbol{y}), & \text{if } m < n, \\ \pi(\boldsymbol{x}) \vee \boldsymbol{y}, & \text{if } m > n. \end{cases}$$

This settles item (1) of the definition of a reduced set.

Now suppose that there exists a non-diagonal triple $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ of elements of $\mathcal{R}^{l-1}\mathcal{L}(\Omega)$ such that $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \boldsymbol{w}$ and $(\boldsymbol{b}, \boldsymbol{a}, \boldsymbol{c}) \in \boldsymbol{w}$. As both $\boldsymbol{x}$ and $\boldsymbol{y}$ are reduced sets, the only possibility is $m = n$ and, say, $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \boldsymbol{x}$ and $(\boldsymbol{b}, \boldsymbol{a}, \boldsymbol{c}) \in \boldsymbol{y}$. As $\boldsymbol{x} \in \mathcal{G}(\Omega \setminus \{\beta\})$ and $\boldsymbol{y} \in \mathcal{G}(\Omega \setminus \{\alpha\})$, all elements $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ belong to $\mathcal{G}(\Omega \setminus \{\alpha, \beta\})$ (see Lemma 4.1). As $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \boldsymbol{x}$ and $\boldsymbol{x} \leq \boldsymbol{a}_i^\alpha$, it follows from (3.1) and the assumption that $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is non-diagonal that either $\boldsymbol{a} \leq \boldsymbol{a}_i^\alpha$ or $\boldsymbol{c} \leq \boldsymbol{a}_i^\alpha$. As $\boldsymbol{a}, \boldsymbol{c} \in \mathcal{G}(\Omega \setminus \{\alpha\})$, it follows from Lemma 4.3 that either $\boldsymbol{a} = 0$ or $\boldsymbol{c} = 0$, a contradiction. This settles item (2) of the definition of a reduced set.

Finally, let $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \boldsymbol{w}$ be a non-diagonal triple of elements of $\mathcal{R}^{l-1}\mathcal{L}(\Omega)$, we must verify that $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \nleq \pi(\boldsymbol{w})$. Suppose, for example, that $\boldsymbol{a} \leq \pi(\boldsymbol{w})$. If $m = n$, then $\boldsymbol{a} \leq \pi(\boldsymbol{x}) \vee \pi(\boldsymbol{y})$ and, say, $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \boldsymbol{x}^*$. From $\pi(\boldsymbol{y}) \leq \boldsymbol{y} \leq \boldsymbol{a}_j^\beta$ it follows that $\boldsymbol{a} \leq \pi(\boldsymbol{x}) \vee \boldsymbol{a}_j^\beta$. As $\boldsymbol{a}, \pi(\boldsymbol{x}) \in \mathcal{G}(\Omega \setminus \{\beta\})$ and by Lemma 4.3, it follows that

$\boldsymbol{a} \leq \pi(\boldsymbol{x})$, which contradicts the assumption that $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ is a non-diagonal triple in $\boldsymbol{x}$. If $m < n$, then $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \boldsymbol{y}^*$ and $\boldsymbol{a} \leq \boldsymbol{x} \vee \pi(\boldsymbol{y})$, so $\boldsymbol{a} \leq \boldsymbol{a}_i^\alpha \vee \pi(\boldsymbol{y})$, and so, as $\boldsymbol{a}, \pi(\boldsymbol{y}) \in \mathcal{G}(\Omega \setminus \{\alpha\})$ and by Lemma 4.3, it follows that $\boldsymbol{a} \leq \pi(\boldsymbol{y})$, which contradicts the assumption that $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ is a non-diagonal triple in $\boldsymbol{y}$. The proof for the case $m > n$ is similar. So we have proved that $\boldsymbol{a} \nleq \pi(\boldsymbol{w})$. The proofs for $\boldsymbol{b}$ and $\boldsymbol{c}$ are similar. This settles item (3) of the definition of a reduced set.

The verification of the inequalities $\boldsymbol{x}, \boldsymbol{y} \leq \boldsymbol{w}$ (see (3.1)) is straightforward. In fact, it is not hard to verify, using Lemma 3.1, that $\boldsymbol{w} = \boldsymbol{x} \vee \boldsymbol{y}$. □ Claim.

Now we complete the proof of Lemma 4.4. From the claim above it follows that $\boldsymbol{z} \leq \boldsymbol{w}$. If $k < l$ then $\boldsymbol{z} \leq \pi(\boldsymbol{w})$, hence, as $\pi(\boldsymbol{w}) \in \{\pi(\boldsymbol{x}) \vee \pi(\boldsymbol{y}), \boldsymbol{x} \vee \pi(\boldsymbol{y}), \pi(\boldsymbol{x}) \vee \boldsymbol{y}\}$ and by the induction hypothesis, $\boldsymbol{z} = 0$. So suppose from now on that $k \geq l$; in particular, $k > 0$. As $\pi(\boldsymbol{z}) \leq \boldsymbol{z} \leq \boldsymbol{x} \vee \boldsymbol{y}$, it follows from the induction hypothesis that $\pi(\boldsymbol{z}) = 0$. Hence, if $\boldsymbol{z} \neq 0$, then there exists a non-diagonal triple $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \boldsymbol{z} \cap \mathcal{CR}^{l-1}\mathcal{L}(\Omega)$. As $\boldsymbol{z} \leq \boldsymbol{w}$, we obtain that either $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \boldsymbol{w}$ or $\boldsymbol{a} \leq \boldsymbol{w}$ or $\boldsymbol{c} \leq \boldsymbol{w}$. In the first case, say, $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \boldsymbol{x}$, we get $\bowtie(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \leq \boldsymbol{x} \leq \boldsymbol{a}_0^\delta$ with $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \in \mathcal{G}(\Omega \setminus \{\delta\})$ (because $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \boldsymbol{z}$), so $\bowtie(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) = 0$ by Lemma 4.3, a contradiction. If either $\boldsymbol{a} \leq \boldsymbol{w}$ or $\boldsymbol{c} \leq \boldsymbol{w}$, then, by the induction hypothesis, either $\boldsymbol{a} = 0$ or $\boldsymbol{c} = 0$, a contradiction. Therefore, $\boldsymbol{z} = 0$. □

5. The Erosion Lemma

The proofs of our negative results are based on the conflict between a *non-structure theorem* on the semilattices $\mathcal{G}(\Omega)$, here the 'evaporation lemma' (Lemma 4.4), and a *structure theorem* on arbitrary bounded semilattices, Lemma 5.1, that we shall now introduce. This lemma, the *Erosion Lemma*, contains, despite its extreme simplicity, the gist of the present paper. Moreover, further extensions of our methods seem to use the same formulation of the Erosion Lemma, while there seem to be many different 'evaporation lemmas' (such as Lemma 4.4).

From now on, we shall denote by ε the 'parity function' on the natural numbers, defined by the rule

$$\varepsilon(n) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd,} \end{cases} \qquad \text{for every natural number } n. \tag{5.1}$$

Throughout this section, we let L be an algebra possessing a polynomially definable structure of semilattice $(L, \vee)$. We put

$$U \vee V = \{u \vee v \mid (u,v) \in U \times V\}, \quad \text{for all } U, V \subseteq L,$$

and we denote by $\mathrm{Con}_{\mathrm{c}}^U L$ the $(\vee, 0)$-subsemilattice of $\mathrm{Con}_{\mathrm{c}} L$ generated by all principal congruences $\Theta_L(u,v)$, where $(u,v) \in U \times U$.

Lemma 5.1 (The Erosion Lemma). *Let $x_0, x_1 \in L$, and let $Z = \{z_i \mid 0 \leq i \leq n\}$, with $n \in \omega \setminus \{0\}$, be a finite subset of L with $\bigvee_{i<n} z_i \leq z_n$. Put*

$$\boldsymbol{a}_j = \bigvee (\Theta_L(z_i, z_{i+1}) \mid i < n,\ \varepsilon(i) = j), \text{ for all } j < 2.$$

Then there are congruences $\boldsymbol{u}_j \in \mathrm{Con}_{\mathrm{c}}^{\{x_j\} \vee Z} L$, for $j < 2$, such that

$$z_0 \vee x_0 \vee x_1 \equiv z_n \vee x_0 \vee x_1 \pmod{\boldsymbol{u}_0 \vee \boldsymbol{u}_1} \quad \textit{and} \quad \boldsymbol{u}_j \subseteq \boldsymbol{a}_j \cap \Theta_L^+(z_n, x_j), \textit{ for all } j < 2.$$

Proof. Put $\boldsymbol{v}_i = \Theta_L(z_i \vee x_{\varepsilon(i)}, z_{i+1} \vee x_{\varepsilon(i)})$, for all $i < n$. Observe that $\boldsymbol{v}_i$ belongs to $\mathrm{Con}_{\mathrm{c}}^{\{x_{\varepsilon(i)}\}\cup Z} L$. From $z_n \leq x_{\varepsilon(i)} \pmod{\Theta_L^+(z_n, x_{\varepsilon(i)})}$ and $z_i \equiv z_{i+1} \pmod{\boldsymbol{a}_{\varepsilon(i)}}$ it follows, respectively (and using $z_i \vee z_n = z_{i+1} \vee z_n$ in the first case), that

$$\boldsymbol{v}_i \subseteq \Theta_L^+(z_n, x_{\varepsilon(i)}) \quad \text{and} \quad \boldsymbol{v}_i \subseteq \boldsymbol{a}_{\varepsilon(i)}. \tag{5.2}$$

Now we put

$$\boldsymbol{u}_j = \bigvee(\boldsymbol{v}_i \mid i < n,\ \varepsilon(i) = j), \quad \text{for all } j < 2.$$

Hence $\boldsymbol{u}_j \in \mathrm{Con}_{\mathrm{c}}^{\{x_j\}\vee Z} L$, for all $j < 2$. Furthermore, from (5.2) it follows that $\boldsymbol{u}_j \subseteq \boldsymbol{a}_j \cap \Theta_L^+(z_n, x_j)$. Finally, from $z_i \vee x_{\varepsilon(i)} \equiv z_{i+1} \vee x_{\varepsilon(i)} \pmod{\boldsymbol{v}_i}$, for all $i < n$, it follows that $z_i \vee x_0 \vee x_1 \equiv z_{i+1} \vee x_0 \vee x_1 \pmod{\boldsymbol{u}_0 \vee \boldsymbol{u}_1}$. Therefore, $z_0 \vee x_0 \vee x_1 \equiv z_n \vee x_0 \vee x_1 \pmod{\boldsymbol{u}_0 \vee \boldsymbol{u}_1}$. □

6. The proof

Our main theorem is the following.

Theorem 6.1. *Let Ω be a set of cardinality at least $\aleph_{\omega+1}$ and let L be an algebra. If L has a polynomially definable structure of $(\vee, 1)$-semilattice, then there is no weakly distributive $(\vee, 0, 1)$-homomorphism from $\mathrm{Con}_{\mathrm{c}} L$ to $\mathcal{G}(\Omega)$ with 1 in its range.*

The remainder of this section will be devoted to a proof of Theorem 6.1. Suppose, to the contrary, that L and $\mu\colon \mathrm{Con}_{\mathrm{c}} L \to \mathcal{G}(\Omega)$ are as above. We fix a polynomially definable structure of $(\vee, 1)$-semilattice on L. There are a positive integer m and elements $t_0, \ldots, t_{m-1}$ in L such that

$$\bigvee_{r<m} \mu\Theta_L(t_r, 1) = 1. \tag{6.1}$$

For each $\xi \in \Omega$, as $\mu\Theta_L(t_r, 1) \leq 1 = \boldsymbol{a}_0^\xi \vee \boldsymbol{a}_1^\xi$ holds for each $r < m$, we obtain, by using the weak distributivity of μ at $\Theta_L(t_r, 1)$, an integer $n_\xi \geq 2$ and elements $z_{r,i}^\xi \in L$, for $0 \leq r < m$ and $0 \leq i \leq n_\xi$, such that $z_{r,0}^\xi = t_r$, $z_{r,n_\xi}^\xi = 1$, and

$$\mu\Theta_L(z_{r,i}^\xi, z_{r,i+1}^\xi) \leq \boldsymbol{a}_{\varepsilon(i)}^\xi, \quad \text{for all } r < m \text{ and } i < n_\xi. \tag{6.2}$$

(We recall that ε is the parity function defined in (5.1).) After replacing $z_{r,i}^\xi$ by $t_r \vee z_{r,i}^\xi$, we may also assume that $t_r \leq z_{r,i}^\xi$ holds, for all $r < m$, $i \leq n_\xi$, and $\xi \in \Omega$. As $|\Omega| \geq \aleph_{\omega+1}$ and $\aleph_{\omega+1}$ is a regular cardinal (this is the reason why $\aleph_\omega$ would not work *a priori*), there are a positive integer n and $\Omega' \subseteq \Omega$ such that $|\Omega'| = \aleph_{\omega+1}$ and $n_\xi = n$ for all $\xi \in \Omega'$. Pick any retraction $\rho\colon \Omega \twoheadrightarrow \Omega'$ and replace μ by $\mathcal{G}(\rho) \circ \mu$. We might lose the weak distributivity of μ, but we keep the elements $z_{r,i}^\xi$ and the statements (6.2), which are all that matters. Furthermore, after replacing L by L/θ where $(x, y) \in \theta$ iff $\mu\Theta_L(x, y) = 0$ (for all $x, y \in L$), we may assume that μ *separates zero*, that is, $\mu^{-1}\{0\} = \{0\}$.

Hence we shall assume, from now on, that μ separates zero and $n_\xi = n$ for all $\xi \in \Omega$. For every finite subset X of Ω, we shall denote by $S(X)$ the join-subsemilattice of L generated by $\{z_{r,i}^\xi \mid 0 \leq r < m,\ 0 \leq i \leq n, \text{ and } \xi \in X\}$. As $S(X)$ is finite, $\Phi(X) = \bigcup(\operatorname{supp} \mu\Theta_L(x, y) \mid x, y \in S(X))$ is a finite subset of Ω.

As $|\Omega| \geq \aleph_{2^n}$, it follows from Kuratowski's free set Theorem that there exists a $(2^n + 1)$-element subset U of Ω which is free with respect to the restriction of Φ to 2^n-elements subsets of Ω.

For all natural numbers k, l with $k \leq n-1$ and $l \leq 2^k$, let $P(k,l)$ hold, if for all $r < m$ and all disjoint $X, Y \subseteq U$ with $|X| = 2^k - l$ and $|Y| = 2l$, the following equality $E_r(X,Y)$ holds:

$$\bigvee(z^{\xi}_{r,n-k} \mid \xi \in X) \vee \bigvee(z^{\eta}_{r,n-k-1} \mid \eta \in Y) = 1. \qquad (E_r(X,Y))$$

The method used to prove Lemma 6.2 below could be described as 'the erosion method': namely, prove, using the Erosion Lemma, that joins of larger and larger subsets of L of the form $\{z^{\xi}_{r,n-k} \mid \xi \in X\} \cup \{z^{\eta}_{r,n-k-1} \mid \eta \in Y\}$, with k larger and larger, remain equal to 1. For large enough k, this will lead naturally to $t_r = 1$.

Lemma 6.2 (Descent)**.** *The statement $P(k,l)$ holds, for all natural numbers k,l such that $k \leq n-1$ and $l \leq 2^k$.*

Proof. We argue by induction on $2^k + l$. Obviously, $P(0,0)$ holds. Assuming that $P(k,l)$ holds, we shall establish $P(k',l')$ for the next value (k',l'). As $P(k,2^k)$ is equivalent to $P(k+1,0)$, we may assume that $l < 2^k$, so $k' = k$ and $l' = l+1$. So let $X, Y \subseteq U$ disjoint with $|X| = 2^k - l - 1$ and $|Y| = 2l+2$. As $|X|+|Y| = 2^k+l+1 \leq 2^n$ and $|U| = 2^n + 1$, there exists an element $\delta \in U \setminus (X \cup Y)$. Pick $r < m$ and distinct elements $\eta_0, \eta_1 \in Y$, set $Y' = Y \setminus \{\eta_0, \eta_1\}$ and

$$x_j = \bigvee(z^{\xi}_{r,n-k} \mid \xi \in X) \vee \bigvee(z^{\eta}_{r,n-k-1} \mid \eta \in Y' \cup \{\eta_j\}), \quad \text{for all } j < 2. \qquad (6.3)$$

It follows from the induction hypothesis that

$$\bigvee(z^{\xi}_{r,n-k} \mid \xi \in X \cup \{\eta_j\}) \vee \bigvee(z^{\eta}_{r,n-k-1} \mid \eta \in Y') = 1, \quad \text{for all } j < 2. \qquad (6.4)$$

Now recall that, by (6.2),

$$\mu\Theta_L(z^{\eta_j}_{r,n-k}, z^{\eta_j}_{r,n-k-1}) \leq \boldsymbol{a}^{\eta_j}_{\varepsilon(n-k-1)}, \quad \text{for all } j < 2.$$

Using (6.3) and (6.4), it follows that $\mu\Theta_L(x_j, 1) \leq \boldsymbol{a}^{\eta_j}_{\varepsilon(n-k-1)}$, for all $j < 2$. Therefore, using Lemma 5.1 with $z^{\delta}_{r,i}$ in place of z_i, for $0 \leq i \leq n$, and observing that $t_r \leq x_0 \vee x_1$ (because $t_r \leq z^{\xi}_{r,i}$ everywhere), we obtain congruences $\boldsymbol{u}_j \in \mathrm{Con}^{S(X \cup Y' \cup \{\eta_j, \delta\})}_{\mathrm{c}} L$, for $j < 2$, such that

$$\Theta_L(x_0 \vee x_1, 1) \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1 \quad \text{and} \quad \mu(\boldsymbol{u}_j) \leq \boldsymbol{a}^{\eta_j}_{\varepsilon(n-k-1)}, \boldsymbol{a}^{\delta}_j, \quad \text{for all } j < 2. \qquad (6.5)$$

It follows from the definition of Φ that $\mu(\boldsymbol{u}_j) \in \mathcal{G}\Phi(X \cup Y' \cup \{\eta_j, \delta\})$ and $\mu\Theta_L(x_0 \vee x_1, 1) \in \mathcal{G}\Phi(X \cup Y)$. Using the monotonicity of Φ and the freeness of U with respect to the restriction of Φ to 2^n-element subsets, we obtain

$$\Phi(X \cup Y) \subseteq \Omega \setminus \{\delta\},$$
$$\Phi(X \cup Y' \cup \{\eta_j, \delta\}) \subseteq \Omega \setminus \{\eta_{1-j}\}, \quad \text{for all } j < 2.$$

As $\mu\Theta_L(x_0 \vee x_1, 1)$ belongs to $\mathcal{G}\Phi(X \cup Y)$ and by using (6.5) together with Lemma 4.4, we obtain that $\mu\Theta_L(x_0 \vee x_1, 1) = 0$, that is, since μ separates zero, $x_0 \vee x_1 = 1$, which completes the proof of the equality $E_r(X,Y)$. $\square$

Now pick $\delta \in U$ and put $Y = U \setminus \{\delta\}$, so $|Y| = 2^n$. By applying Lemma 6.2 to $k = n-1$ and $l = 2^{n-1}$, we obtain the equality $\bigvee(z^{\eta}_{r,0} \mid \eta \in Y) = 1$, that is, $t_r = 1$. But this holds for all $r < m$, which contradicts (6.1). This completes the proof of Theorem 6.1.

Remark 6.3. In the assumptions of Theorem 6.1, it is sufficient to restrict the weak distributivity assumption of μ to congruences $\Theta_L(t_r, 1)$, for $r < m$, satisfying (6.1).

7. Consequences on congruence lattices of lattices

Observe that Theorem 6.1 applies to L a *lattice with largest element*. We now extend this result to arbitrary lattices.

Theorem 7.1. *For any set Ω and any algebra L with a polynomially definable lattice structure, if $|\Omega| \geq \aleph_{\omega+1}$, then there exists no weakly distributive $(\vee, 0)$-homomorphism $\mu\colon \operatorname{Con_c} L \to \mathcal{G}(\Omega)$ with 1 in its range.*

Proof. Let $\mu\colon \operatorname{Con_c} L \to \mathcal{G}(\Omega)$ be a weakly distributive $(\vee, 0)$-homomorphism with 1 in its range, where $|\Omega| \geq \aleph_{\omega+1}$. Pick a polynomially definable lattice structure on L. As $1 = \bigvee_{i<n} \mu\Theta_L(u_i, v_i)$, for a positive integer n and elements $u_i \leq v_i$ of L, for $i < n$, we get $1 = \mu\Theta_L(u, v)$, where $u = \bigwedge_{i<n} u_i$ and $v = \bigvee_{i<n} v_i$. Put $K = [u, v]$. It follows from [35, Proposition 1.2] that the canonical homomorphism $\jmath\colon \operatorname{Con_c} K \to \operatorname{Con_c} L$ is weakly distributive. Hence $\mu \circ \jmath$ is a weakly distributive homomorphism from $\operatorname{Con_c} K$ to $\mathcal{G}(\Omega)$ with 1 in its range, with K a *bounded* lattice. This contradicts Theorem 6.1. □

In particular, we obtain a negative solution to CLP.

Corollary 7.2. *Let Ω be a set. If $|\Omega| \geq \aleph_{\omega+1}$, then there exists no lattice L with $\operatorname{Con_c} L \cong \mathcal{G}(\Omega)$.*

By contrast, Lampe proved in [22] that *every $(\vee, 0, 1)$-semilattice is isomorphic to $\operatorname{Con_c} G$ for some groupoid G with 4-permutable congruences.* In particular, $\mathcal{G}(\aleph_{\omega+1}) \cong \operatorname{Con_c} G$ for some groupoid G with 4-permutable congruences, while there is no lattice L such that $\mathcal{G}(\aleph_{\omega+1}) \cong \operatorname{Con_c} L$. This shows a critical discrepancy between general algebras and lattices.

8. Discussion

8.1. **A new uniform refinement property.** In many works such as [24, 28, 31, 32, 35, 38, 39], the classes of semilattices that are representable with respect to various functors are separated from the corresponding counterexamples by infinitary statements called *uniform refinement properties.* We shall now discuss briefly how this can also be done here. As the proofs do not seem to add much to the already existing results, we shall omit the details.

For a positive integer m and a nonempty set Ω, denote by $\operatorname{Sem}(m, \Omega)$ the join-semilattice defined by generators $\bar{0}$, $\bar{1}$, and $k \cdot \dot{\xi}$ for $0 \leq k \leq m+1$ and $\xi \in \Omega$, subjected to the relations

$$\bar{0} = 0 \cdot \dot{\xi} \leq 1 \cdot \dot{\xi} \leq \cdots \leq m \cdot \dot{\xi} \leq (m+1) \cdot \dot{\xi} = \bar{1}, \quad \text{for } \xi \in \Omega.$$

Definition 8.1. For an element $\boldsymbol{e}$ in a $(\vee, 0)$-semilattice S, we say that S satisfies $\operatorname{CLR}(\boldsymbol{e})$, if for every nonempty set Ω and every family $(\boldsymbol{a}_i^{\xi} \mid (\xi, i) \in \Omega \times \{0, 1\})$ with entries in S such that $\boldsymbol{e} \leq \boldsymbol{a}_0^{\xi} \vee \boldsymbol{a}_1^{\xi}$ for all $\xi \in \Omega$, there are a decomposition $\Omega = \bigcup(\Omega_m \mid m \in \omega \setminus \{0\})$ and families $\boldsymbol{c}_m\colon \operatorname{Sem}(m, \Omega_m) \times \operatorname{Sem}(m, \Omega_m) \to S$, for $m \in \omega \setminus \{0\}$, such that the following statements hold for every positive integer m:

1. $p \leq q$ implies that $\boldsymbol{c}_m(p, q) = 0$, for all $p, q \in \operatorname{Sem}(m, \Omega_m)$;
2. $\boldsymbol{c}_m(p, r) \leq \boldsymbol{c}_m(p, q) \vee \boldsymbol{c}_m(q, r)$, for all $p, q, r \in \operatorname{Sem}(m, \Omega_m)$;
3. $\boldsymbol{c}_m(p \vee q, r) = \boldsymbol{c}_m(p, r) \vee \boldsymbol{c}_m(q, r)$, for all $p, q, r \in \operatorname{Sem}(m, \Omega_m)$;
4. $\boldsymbol{c}_m(\bar{1}, \bar{0}) = \boldsymbol{e}$;

(5) The inequality $\boldsymbol{c}_m((k+1)\cdot\dot{\xi}, k\cdot\dot{\xi}) \leq \boldsymbol{a}^{\xi}_{\varepsilon(k)}$ holds, for all $\xi \in \Omega_m$ and all $k \leq m$.

If, for a fixed $m \in \omega \setminus \{0\}$, we can always take $\Omega_m = \Omega$ while $\Omega_n = \varnothing$ for all $n \neq m$, we say that S satisfies $\mathrm{CLR}_m(\boldsymbol{e})$.

The statement $\mathrm{CLR}(\boldsymbol{e})$ is an analogue, for arbitrary lattices, of the 'uniform refinement property' introduced in [35], denoted by 'URP^- at $\boldsymbol{e}$' in [32]. It is easy to verify that for any $(\vee,0)$-semilattices S and T, any $\boldsymbol{e} \in S$, and any weakly distributive $(\vee,0)$-homomorphism $\mu\colon S \to T$, if S satisfies $\mathrm{CLR}(\boldsymbol{e})$, then T satisfies $\mathrm{CLR}(\mu(\boldsymbol{e}))$. A similar observation applies to CLR_m. Furthermore, a straightforward, although somewhat tedious, modification of the proof of Theorem 6.1, gives, for example, the following result.

Theorem 8.2. *Let L be a lattice and let $\mathbf{e}$ be a principal congruence of L. Then* $\mathrm{Con_c}\, L$ *satisfies* $\mathrm{CLR}(\boldsymbol{e})$. *Furthermore, if L has $(m+1)$-permutable congruences (where m is a given positive integer), then* $\mathrm{Con_c}\, L$ *satisfies* $\mathrm{CLR}_m(\boldsymbol{e})$. *On the other hand,* $\mathcal{G}(\aleph_{\omega+1})$ *(resp.,* $\mathcal{G}(\aleph_{2^m})$*) does not satisfy* $\mathrm{CLR}(1)$ *(resp.,* $\mathrm{CLR}_m(1)$*).*

8.2. **Open problems.** The most obvious problem suggested by the present paper is to fill the cardinality gap between $\aleph_2$ and $\aleph_\omega$. In the meantime, this problem has been solved by Pavel Růžička [27], who introduced a strengthening of Kuratowski's free set Theorem that made it possible to prove, by using the original Erosion Lemma (Lemma 5.1) and modifications of both the Evaporation Lemma (Lemma 4.4) and the Descent Lemma (Lemma 6.2) the following result: *For any set Ω such that $|\Omega| \geq \aleph_2$, there are no algebra L with a polynomially definable structure of bounded semilattice and no weakly distributive $(\vee,0,1)$-homomorphism $\mu\colon \mathrm{Con_c}\, L \to \mathcal{G}(\Omega)$.* In fact, it is not hard to modify Růžička's proof to establish that for $|\Omega| \geq \aleph_2$, the semilattice $\mathcal{G}(\Omega)$ does not satisfy $\mathrm{CLR}(1)$ (cf. Subsection 8.1).

The discussion in Subsection 8.1 about CLR and CLR_m also suggests the following problem.

Problem 1. Prove that there exists a lattice K such that for every positive integer m, there is no lattice L with m-permutable congruences such that $\mathrm{Con}\, K \cong \mathrm{Con}\, L$.

Now that we know that the answer to CLP is negative, a natural question is the corresponding one for congruence-distributive varieties.

Problem 2. Is every algebraic distributive lattice isomorphic to the congruence lattice of some algebra generating a congruence-distributive variety?

Recall the classical open problem asking whether every algebraic distributive lattice is isomorphic to the congruence lattice of some algebra with finitely many operations. In view of Theorem 6.1, we may try to find the algebra with a polynomially definable structure of $(\vee,0)$-semilattice (but not $(\vee,1)$-semilattice).

Kearnes proves in [19] that there exists an algebraic lattice that is not isomorphic to the congruence lattice of any locally finite algebra. In light of this result, one may ask the following question.

Problem 3. Does there exist a lattice L such that $\mathrm{Con}\, L$ is not isomorphic to the congruence lattice of any locally finite lattice (resp., algebra)?

In [31], infinite semilattices considered earlier in [35, 24, 34] are approximated by finite semilattices, yielding, in particular, a $\{0,1\}^3$-indexed diagram of finite

Boolean semilattices that cannot be lifted, with respect to the $\mathrm{Con_c}$ functor, by congruence-permutable lattices. The methods used in the present paper suggest that those works could be extended to find a $\{0,1\}^{2^m+1}$-indexed diagram of finite Boolean semilattices that cannot be lifted, with respect to the $\mathrm{Con_c}$ functor, by lattices with $(m+1)$-permutable congruences.

Tůma and Wehrung prove in [33] that there exists a diagram of finite Boolean semilattices, indexed by a finite *partially ordered set*, that cannot be lifted, with respect to the $\mathrm{Con_c}$ functor, by any diagram of lattices (or even algebras in any variety satisfying a nontrivial congruence lattice identity). This leaves open the following problem.

Problem 4. Prove that any diagram of finite distributive $(\vee, 0)$-semilattices and $(\vee, 0)$-homomorphisms, indexed by a finite lattice, can be lifted, with respect to the $\mathrm{Con_c}$ functor, by a diagram of (finite?) lattices and lattice homomorphisms.

We conclude with the following problem, which also appears, with a slightly different formulation, as [10, Problem 10.6].

Problem 5. Prove that there exists a lattice K such that there is no *modular* lattice M with $\operatorname{Con} K \cong \operatorname{Con} M$.

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LMNO, CNRS UMR 6139, Département de Mathématiques, BP 5186, Université de Caen, Campus 2, 14032 Caen cedex, France

E-mail address: wehrung@math.unicaen.fr

URL: http://www.math.unicaen.fr/~wehrung