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A SOLUTION TO DILWORTH'S CONGRUENCE LATTICE PROBLEM

FRIEDRICH WEHRUNG

ABSTRACT. We construct a distributive algebraic lattice D that is not isomorphic to the congruence lattice of any lattice. This solves a long-standing open problem traditionally attributed to R. P. Dilworth from the forties. The lattice D has compact top element and $\aleph_{\omega+1}$ compact elements. If we restrict our attention to lattices with m -permutable congruences, then we may take D with \aleph_{2^m} compact elements. Our results extend to all algebras possessing a polynomially definable structure of bounded semilattice.

1. INTRODUCTION

For an *algebra* L (i.e., a nonempty set with a collection of operations from finite powers of L to L), a *congruence* of L is an equivalence relation on L compatible with all operations of L . For elements $x, y \in L$, we denote by $\Theta_L(x, y)$ the least congruence that identifies x with y , and we call the finite joins of such congruences *finitely generated*. We denote by $\text{Con } L$ (resp., $\text{Con}_c L$) the lattice (resp., $(\vee, 0)$ -semilattice) of all congruences (resp., finitely generated congruences) of L under inclusion. A *polynomial* of L is a composition of operations of L , allowing elements of L as parameters. A homomorphism of join-semilattices $\mu: S \rightarrow T$ is *weakly distributive* at an element \mathbf{x} of S , if for all $\mathbf{y}_0, \mathbf{y}_1 \in T$ such that $\mu(\mathbf{x}) \leq \mathbf{y}_0 \vee \mathbf{y}_1$, there are $\mathbf{x}_0, \mathbf{x}_1 \in S$ such that $\mathbf{x} \leq \mathbf{x}_0 \vee \mathbf{x}_1$ and $\mu(\mathbf{x}_i) \leq \mathbf{y}_i$, for all $i < 2$. We say that μ is *weakly distributive*, if it is weakly distributive at every element of S . (For S and T distributive, this is equivalent to the definition presented in [34]. Moreover, it extends the original definition given by Schmidt [28, 29].)

In the present paper we prove the following result (cf. Theorem 7.1).

Theorem. *There exists a distributive $(\vee, 0, 1)$ -semilattice S such that for any algebra L with a polynomially definable structure of bounded semilattice, there exists no $(\vee, 0, 1)$ -homomorphism $\mu: \text{Con}_c L \rightarrow S$ which is weakly distributive at the largest congruence of L . Furthermore, S has $\aleph_{\omega+1}$ elements.*

It follows that the semilattice S is not isomorphic to $\text{Con}_c L$, for any lattice L . Hence the ideal lattice of S is not isomorphic to the congruence lattice of any lattice. Furthermore, our proof constructs a certain semilattice-theoretical statement, belonging to the infinitary language $\mathcal{L}_{(2^\kappa)^+, \kappa^+}$ (using model-theoretical notation), where $\kappa = \aleph_{\omega+1}$, which holds in all weakly distributive images of $\text{Con}_c L$ for a bounded semilattice L , but which fails in S .

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We shall now give some background on the problem solved by our theorem. Funayama and Nakayama [5] proved in 1942 that $\text{Con } L$ is *distributive*, for any lattice (L, \vee, \wedge) . Dilworth proved soon after that conversely, every finite distributive lattice is isomorphic to the congruence lattice of some finite lattice (see [3, pp. 455–456] and [8]). Birkhoff and Frink [2] proved in 1948 that the congruence lattice of any algebra is what is nowadays called an *algebraic* lattice, that is, it is *complete* and every element is a join of *compact* elements (see [9]). The question whether every distributive algebraic lattice is isomorphic to $\text{Con } L$ for some lattice L , often referred to as CLP (‘Congruence Lattice Problem’), is one of the most intriguing and longest-standing open problems of lattice theory. The first published occurrence of this problem, as well as the first published proof of Dilworth’s abovesited result, seems to appear in Grätzer and Schmidt’s paper [12]. However, it seems that the earliest attempts at CLP were made by Dilworth himself, see [3, pp. 455–456].

This problem has generated an enormous amount of work since then, in a somewhat complex pattern of interconnected waves. Grätzer and Schmidt proved in 1963 that every algebraic lattice is isomorphic to the congruence lattice of some algebra [13]. The reader can find in Schmidt’s monograph [29] a survey about congruence lattice representations of algebras. The surveys by Grätzer and Schmidt [14, 15] and Grätzer’s monograph [10] are focused on congruence lattices of (mainly finite) lattices, while the survey by Tůma and Wehrung [31] is more focused on congruence lattices of infinite lattices. The main connection between the finite case and the infinite case originates in Pudlák’s idea [25] of lifting, with respect to the Con_c functor, *diagrams* of finite distributive $(\vee, 0)$ -semilattices. Růžička, Tůma, and Wehrung prove in [27] that there are bounded lattices of cardinality \aleph_2 whose congruence lattices are isomorphic neither to the normal subgroup lattice of any group, nor to the submodule lattice of any module; furthermore, the bound \aleph_2 is optimal. Some of the more recent works emphasize close connections between congruence lattices of lattices, ideal lattices of rings, dimension theory of lattices, and nonstable K-theory of rings, see for example [1, 6, 7, 26, 33, 34, 35, 37].

Distributive algebraic lattices are ideal lattices of distributive $(\vee, 0)$ -semilattices (see Section 2), and for a lattice L , $\text{Con } L$ is isomorphic to the ideal lattice of $\text{Con}_c L$. We obtain the following more convenient equivalent formulation of CLP (see [31] for details):

CLP (semilattice formulation). *Is every distributive $(\vee, 0)$ -semilattice representable, that is, isomorphic to $\text{Con}_c L$, for some lattice L ?*

In particular, the semilattice S of our theorem provides a counterexample to CLP. In case we deal only with lattices with m -permutable congruences (where m is a positive integer), we may take S of cardinality \aleph_{2^m} .

Among the classical positive partial results are the following:

- (1) Every distributive $(\vee, 0)$ -semilattice S of cardinality at most \aleph_1 is representable, see Huhn [16, 17].
- (2) Every distributive lattice with zero is representable, see Schmidt [28].

Further works extended the class of all representable distributive $(\vee, 0)$ -semilattices, for example to all $(\vee, 0)$ -direct limits of sequences of distributive lattices with zero, see [36]. Moreover, the representing lattice L can be taken *relatively complemented with zero*. This also holds for case (2) above. However, the latter result has been extended further by Růžička [26], who proved that the representing

lattice can be taken relatively complemented, modular, and locally finite. This is not possible for (1) above, as, for $|S| \leq \aleph_1$, one can take L relatively complemented modular [35], relatively complemented and locally finite [11], but not necessarily both [37].

On the negative side, the works in [24, 30, 33, 34] show that lattices with permutable congruences are not sufficient to solve CLP. More precisely, *there exists a representable distributive $(\vee, 0, 1)$ -semilattice of cardinality \aleph_2 that is not isomorphic to $\text{Con}_c L$ for any lattice L with permutable congruences*. The finite combinatorial reason for this lies in the impossibility to prove certain ‘congruence amalgamation properties’. The infinite combinatorial reason for this is Kuratowski’s free set Theorem (see Section 2). The latter is used to prove that certain infinitary statements called ‘uniform refinement properties’ fail in certain distributive semilattices.

Hindsight proves that it is often more useful, for future applications, to state such results in the language of certain infinitary axioms called ‘uniform refinement properties’ [31, 34]. Hence we shall use this approach. The axiom obtained here, denoted by CLR (see Section 5), is an infinitary semilattice-theoretical statement that holds in $\text{Con}_c L$, for any lattice L , but not in all distributive $(\vee, 0, 1)$ -semilattices. In fact, CLR *holds in every weakly distributive image of an algebra with a polynomially definable $(\vee, 0, 1)$ -semilattice structure* (see Lemma 5.5 and Theorem 5.6), hence our results extend far beyond the world of lattices. The step from bounded lattices to arbitrary lattices is performed *via* the easy Proposition 5.1.

Furthermore, our proof also carries a flavor of (a semilattice-theoretical version of) commutator theory, essentially because of Lemma 6.2. A precedent of this sort of situation occurs with Bill Lampe’s wonderful trick used in [4] to prove that certain algebraic lattices require, for their congruence representations, algebras with many operations: namely, the two term condition used in commutator theory in, say, congruence-modular varieties (or larger, as considered in [20, 32]).

2. BASIC CONCEPTS

A $(\vee, 0)$ -semilattice S is *distributive*, if $\mathbf{c} \leq \mathbf{a} \vee \mathbf{b}$ in S implies that there are $\mathbf{x} \leq \mathbf{a}$ and $\mathbf{y} \leq \mathbf{b}$ in S such that $\mathbf{x} \leq \mathbf{a}$, $\mathbf{y} \leq \mathbf{b}$, and $\mathbf{c} = \mathbf{x} \vee \mathbf{y}$. Equivalently, the ideal lattice of S is a distributive lattice, see [9, Section II.5].

The assignment $L \mapsto \text{Con}_c L$ is extended the usual way to a functor from algebras with homomorphisms to $(\vee, 0)$ -semilattices with $(\vee, 0)$ -homomorphisms. For a positive integer m , an algebra L has *m -permutable congruences*, if $\mathbf{a} \vee \mathbf{b} = \mathbf{c}_0 \circ \mathbf{c}_1 \circ \dots \circ \mathbf{c}_m$ where \mathbf{c}_i equals \mathbf{a} if i is even and \mathbf{b} if i is odd, for all congruences \mathbf{a} and \mathbf{b} of A (the symbol \circ denotes, as usual, the composition of relations).

For an algebra L endowed with a structure of join-semilattice, with join operation denoted by \vee , we put $\Theta_L^+(x, y) = \Theta_L(y, x \vee y)$, for all $x, y \in L$. We say that the join-semilattice structure on L is *polynomially definable*, if \vee is a polynomial on L . In such a case, any congruence of L is also a \vee -congruence, and thus $\Theta_L^+(x, z) \subseteq \Theta_L^+(x, y) \vee \Theta_L^+(y, z)$, for all $x, y, z \in L$.

For partially ordered sets P and Q , a map $f: P \rightarrow Q$ is *isotone*, if $x \leq y$ implies that $f(x) \leq f(y)$, for all $x, y \in P$.

We shall also use standard set-theoretical notation and terminology, referring the reader to [18] for further information. We shall denote by $\mathfrak{P}(X)$ the powerset of a set X , by $[X]^{<\omega}$ the set of all finite subsets of X , and by $[X]^n$ (for $n < \omega$)

the set of all n -element subsets of X . For a map $\Phi: [X]^n \rightarrow [X]^{<\omega}$, we say that a $(n+1)$ -element subset U of X is *free with respect to Φ* , if $x \notin \Phi(U \setminus \{x\})$ for all $x \in U$. The following statement of infinite combinatorics is one direction of a theorem due to Kuratowski [21].

Kuratowski's Theorem. *Let n be a natural number and let X be a set with $|X| \geq \aleph_n$. For every map $\Phi: [X]^n \rightarrow [X]^{<\omega}$, there exists a $(n+1)$ -element free subset of X with respect to Φ .*

We identify every natural number n with the set $\{0, 1, \dots, n-1\}$, and we denote by ω the set of all natural numbers, which is also the first limit ordinal. We shall usually denote elements in semilattices by bold math characters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$.

3. FREE DISTRIBUTIVE EXTENSION OF A $(\vee, 0)$ -SEMILATTICE

As in [38], we shall use the construction of a “free distributive extension” $\mathcal{R}(S)$ of a $(\vee, 0)$ -semilattice S given by Ploščica and Tůma in [23, Section 2]. The larger semilattice $\mathcal{R}(S)$ is constructed by adding new elements $\bowtie(\mathbf{a}, \mathbf{b}, \mathbf{c})$, for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in S$ such that $\mathbf{c} \leq \mathbf{a} \vee \mathbf{b}$, subjected to the only relations $\mathbf{c} = \bowtie(\mathbf{a}, \mathbf{b}, \mathbf{c}) \vee \bowtie(\mathbf{b}, \mathbf{a}, \mathbf{c})$ and $\bowtie(\mathbf{a}, \mathbf{b}, \mathbf{c}) \leq \mathbf{a}$. It is a semilattice version of the dimension group construction $\mathbf{I}_K(E)$ presented in [33, Section 1]. For convenience, we present an equivalent formulation here.

For a $(\vee, 0)$ -semilattice S , we shall put $\mathcal{C}(S) = \{(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in S^3 \mid \mathbf{w} \leq \mathbf{u} \vee \mathbf{v}\}$. A finite subset \mathbf{x} of $\mathcal{C}(S)$ is *projectable* (resp., *reduced*), if it satisfies condition (1) (resp., (1)–(3)) below:

- (1) \mathbf{x} contains exactly one diagonal triple, that is, a triple of the form $(\mathbf{u}, \mathbf{u}, \mathbf{u})$; we put $\mathbf{u} = \pi(\mathbf{x})$.
- (2) $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbf{x}$ and $(\mathbf{v}, \mathbf{u}, \mathbf{w}) \in \mathbf{x}$ implies that $\mathbf{u} = \mathbf{v} = \mathbf{w}$, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in S$.
- (3) $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbf{x} \setminus \{(\pi(\mathbf{x}), \pi(\mathbf{x}), \pi(\mathbf{x}))\}$ implies that $\mathbf{u}, \mathbf{v}, \mathbf{w} \not\leq \pi(\mathbf{x})$, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in S$.

In particular, observe that if \mathbf{x} is reduced, $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbf{x}$, and $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is non-diagonal, then $\mathbf{u} \neq \mathbf{v}$ and the elements \mathbf{u}, \mathbf{v} , and \mathbf{w} are nonzero.

We denote by $\overline{\mathcal{R}}(S)$ (resp., $\mathcal{R}(S)$) the set of all projectable (resp., reduced) subsets of $\mathcal{C}(S)$, endowed with the binary relation \leq defined by

$$\mathbf{x} \leq \mathbf{y} \iff \forall (\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbf{x} \setminus \mathbf{y}, \text{ either } \mathbf{u} \leq \pi(\mathbf{y}) \text{ or } \mathbf{w} \leq \pi(\mathbf{y}). \quad (3.1)$$

We call π the *canonical projection* from $\mathcal{R}(S)$ onto S . Observe that in general, π is not a join-homomorphism (however, see Remark 3.3). It is straightforward to verify that \leq is a partial ordering on $\overline{\mathcal{R}}(S)$ (and thus on the subset $\mathcal{R}(S)$). Now we shall present, in terms of rewriting rules, the steps (i)–(iv) of the algorithm stated in [23, Lemma 2.1].

For finite subsets \mathbf{x} and \mathbf{y} of $\mathcal{C}(S)$, let $\mathbf{x} \rightarrow_1 \mathbf{y}$ hold, if there exists a non-diagonal $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{x}$ such that $(\mathbf{b}, \mathbf{a}, \mathbf{c}) \in \mathbf{x}$ and $\mathbf{y} = (\mathbf{x} \setminus \{(\mathbf{a}, \mathbf{b}, \mathbf{c}), (\mathbf{b}, \mathbf{a}, \mathbf{c})\}) \cup \{(\mathbf{c}, \mathbf{c}, \mathbf{c})\}$. Denote by \rightarrow_1^* the reflexive and transitive closure of \rightarrow_1 on finite subsets of $\mathcal{C}(S)$, and denote by $\mathcal{R}_1(S)$ the set of all finite $\mathbf{x} \subseteq \mathcal{C}(S)$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{x}$ and $(\mathbf{b}, \mathbf{a}, \mathbf{c}) \in \mathbf{x}$ implies that $\mathbf{a} = \mathbf{b} = \mathbf{c}$, for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in S$. Put $\overline{\mathcal{R}}_1(S) = \overline{\mathcal{R}}(S) \cap \mathcal{R}_1(S)$. For a finite subset \mathbf{x} of $\mathcal{C}(S)$, we put

$$\varphi(\mathbf{x}) = (\mathbf{x} \setminus \{(\mathbf{u}, \mathbf{u}, \mathbf{u}) \mid \mathbf{u} \in X\}) \cup \left\{ \left(\bigvee X, \bigvee X, \bigvee X \right) \right\},$$

where $X = \{\mathbf{u} \in S \mid (\mathbf{u}, \mathbf{u}, \mathbf{u}) \in \mathbf{x}\}$.

For $\mathbf{x} \in \overline{\mathcal{R}}(S)$ and a finite subset \mathbf{y} of $\mathcal{C}(S)$, let $\mathbf{x} \rightarrow_2 \mathbf{y}$ hold, if there exists a non-diagonal $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{x}$ such that $\mathbf{b} \leq \pi(\mathbf{x})$ and

$$\mathbf{y} = (\mathbf{x} \setminus \{(\mathbf{a}, \mathbf{b}, \mathbf{c}), (\pi(\mathbf{x}), \pi(\mathbf{x}), \pi(\mathbf{x}))\}) \cup \{(\mathbf{c} \vee \pi(\mathbf{x}), \mathbf{c} \vee \pi(\mathbf{x}), \mathbf{c} \vee \pi(\mathbf{x}))\}.$$

Observe that necessarily, \mathbf{y} belongs to $\overline{\mathcal{R}}(S)$ as well, and denote by \rightarrow_2^* the reflexive and transitive closure of \rightarrow_2 on $\overline{\mathcal{R}}(S)$. Denote by $\mathcal{R}_2(S)$ the set of all $\mathbf{x} \in \overline{\mathcal{R}}_1(S)$ such that for all non-diagonal $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{x}$, the inequality $\mathbf{b} \not\leq \pi(\mathbf{x})$ holds. For any $\mathbf{x} \in \overline{\mathcal{R}}(S)$, we put

$$\psi(\mathbf{x}) = \mathbf{x} \setminus \{(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{x} \text{ non-diagonal} \mid \text{either } \mathbf{a} \leq \pi(\mathbf{x}) \text{ or } \mathbf{c} \leq \pi(\mathbf{x})\}.$$

The correspondence with the algorithm stated in [23, Lemma 2.1] is as follows: the relation \rightarrow_1 corresponds to step (i); the function φ corresponds to step (ii); the relation \rightarrow_2 corresponds to step (iii); the function ψ corresponds to step (iv). The following lemma is a reformulation, in terms of \rightarrow_1 , \rightarrow_2 , φ , and ψ , of [23, Lemma 2.1].

Lemma 3.1. *Let $\mathbf{x}, \mathbf{y} \in \mathcal{R}(S)$. Then there exists $(\mathbf{z}_1, \mathbf{z}_2) \in \mathcal{R}_1(S) \times \mathcal{R}_2(S)$ such that $\mathbf{x} \cup \mathbf{y} \rightarrow_1^* \mathbf{z}_1$ and $\varphi(\mathbf{z}_1) \rightarrow_2^* \mathbf{z}_2$. Furthermore, for any such pair $(\mathbf{z}_1, \mathbf{z}_2)$, $\varphi(\mathbf{z}_1)$ belongs to $\overline{\mathcal{R}}_1(S)$ and $\psi(\mathbf{z}_2)$ is the join, in $\mathcal{R}(S)$, of \mathbf{x} and \mathbf{y} .*

Corollary 3.2. *The set $\mathcal{R}(S)$ is a $(\vee, 0)$ -semilattice under the partial ordering defined in (3.1). Furthermore, the map $j_S: S \rightarrow \mathcal{R}(S)$, $\mathbf{x} \mapsto \{(\mathbf{x}, \mathbf{x}, \mathbf{x})\}$ is a $(\vee, 0)$ -embedding.*

Remark 3.3. We shall identify \mathbf{x} with the element $\{(\mathbf{x}, \mathbf{x}, \mathbf{x})\}$ of $\mathcal{R}(S)$, for all $\mathbf{x} \in S$. Then observe that the canonical map $\pi: \mathcal{R}(S) \rightarrow S$ is *isotone* and that the restriction of π to S is the identity. The following is an easy consequence of (3.1).

$$\mathbf{x} \leq \mathbf{y} \iff \mathbf{x} \leq \pi(\mathbf{y}), \quad \text{for all } (\mathbf{x}, \mathbf{y}) \in S \times \mathcal{R}(S). \quad (3.2)$$

Now the elements of $\mathcal{R}(S) \setminus S$ are exactly those subsets \mathbf{x} of $\mathcal{C}(S) \cup S$ (disjoint union) containing exactly one element of S , denoted by $\pi(\mathbf{x})$, while $\mathbf{x} \setminus \{\pi(\mathbf{x})\}$ is nonempty and all its elements are triples $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathcal{C}(S)$ such that $(\mathbf{b}, \mathbf{a}, \mathbf{c}) \notin \mathbf{x}$ and $\mathbf{a}, \mathbf{b}, \mathbf{c} \not\leq \pi(\mathbf{x})$.

We shall use the symbol \bowtie_S , or \bowtie if S is understood, to denote the elements of $\mathcal{R}(S)$ defined as

$$\bowtie_S(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \begin{cases} \mathbf{w}, & \text{if either } \mathbf{u} = \mathbf{v} \text{ or } \mathbf{v} = 0 \text{ or } \mathbf{w} = 0, \\ 0, & \text{if } \mathbf{u} = 0, \\ \{(0, 0, 0), (\mathbf{u}, \mathbf{v}, \mathbf{w})\}, & \text{otherwise,} \end{cases}$$

for all $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathcal{C}(S)$. Then one can prove easily the formula

$$\mathbf{x} = \bigvee (\bowtie_S(\mathbf{a}, \mathbf{b}, \mathbf{c}) \mid (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{x}), \quad \text{for all } \mathbf{x} \in \mathcal{R}(S). \quad (3.3)$$

The following is a slight strengthening of [23, Theorem 2.3], with the same proof. The uniqueness statement follows from (3.3).

Lemma 3.4. *Let S and T be $(\vee, 0)$ -semilattices and let $f: S \rightarrow T$ be a $(\vee, 0)$ -homomorphism. Furthermore, let $\iota: \mathcal{C}(\text{im } f) \rightarrow T$ be a map such that $\iota(\mathbf{x}, \mathbf{y}, \mathbf{z}) \vee \iota(\mathbf{y}, \mathbf{x}, \mathbf{z}) = \mathbf{z}$ and $\iota(\mathbf{x}, \mathbf{y}, \mathbf{z}) \leq \mathbf{x}$, for all $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathcal{C}(\text{im } f)$. Then there exists a unique map $f_{(\iota)}: \mathcal{R}(S) \rightarrow T$ such that $f_{(\iota)}(\bowtie_S(\mathbf{x}, \mathbf{y}, \mathbf{z})) = \iota(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c}))$, for all $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathcal{C}(S)$.*

By applying Lemma 3.4 to the map $j_T \circ f$ and defining ι as the restriction of \bowtie_T to $\mathcal{C}(\text{im } f)$, we obtain item (1) of the following result. Item (2) follows easily.

Proposition 3.5.

- (1) For $(\vee, 0)$ -semilattices S and T , every $(\vee, 0)$ -homomorphism $f: S \rightarrow T$ extends to a unique $(\vee, 0)$ -homomorphism $\mathcal{R}(f): \mathcal{R}(S) \rightarrow \mathcal{R}(T)$ such that $\mathcal{R}(f)(\bowtie_S(\mathbf{u}, \mathbf{v}, \mathbf{w})) = \bowtie_T(f(\mathbf{u}), f(\mathbf{v}), f(\mathbf{w}))$, for all $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathcal{C}(S)$.
- (2) The assignment $S \mapsto \mathcal{R}(S)$, $f \mapsto \mathcal{R}(f)$ is a functor.

Putting $\mathcal{R}^0(S) = S$ and $\mathcal{R}^{n+1}(S) = \mathcal{R}(\mathcal{R}^n(S))$ for each n , the increasing union $\mathcal{D}(S) = \bigcup(\mathcal{R}^n(S) \mid n < \omega)$ is a distributive $(\vee, 0)$ -semilattice extending S . Furthermore, putting $\mathcal{D}(f) = \bigcup(\mathcal{R}^n(f) \mid n < \omega)$ for each $(\vee, 0)$ -homomorphism f , we obtain that \mathcal{D} is a functor. The proof of the following lemma is straightforward.

Lemma 3.6. *Let S be a $(\vee, 0)$ -semilattice and let $(S_i \mid i \in I)$ be a family of $(\vee, 0)$ -subsemilattices of S . The following statements hold:*

- (1) $\mathcal{R}(\bigcap_{i \in I} S_i) = \bigcap_{i \in I} \mathcal{R}(S_i)$ and $\mathcal{D}(\bigcap_{i \in I} S_i) = \bigcap_{i \in I} \mathcal{D}(S_i)$.
- (2) If I is a nonempty upward directed partially ordered set and $(S_i \mid i \in I)$ is isotone, then $\mathcal{R}(\bigcup_{i \in I} S_i) = \bigcup_{i \in I} \mathcal{R}(S_i)$ and $\mathcal{D}(\bigcup_{i \in I} S_i) = \bigcup_{i \in I} \mathcal{D}(S_i)$.

Definition 3.7. For a $(\vee, 0)$ -semilattice S and an element $\mathbf{x} \in \mathcal{D}(S)$, we define the *rank* of \mathbf{x} , denoted by $\text{rk } \mathbf{x}$, as the least natural number n such that $\mathbf{x} \in \mathcal{R}^n(S)$.

4. THE FUNCTORS \mathcal{L} AND \mathcal{G}

For a set Ω , we denote by $\mathcal{L}(\Omega)$ the $(\vee, 0)$ -semilattice defined by generators 1 and $\mathbf{a}_0^\xi, \mathbf{a}_1^\xi$ (for $\xi \in \Omega$), subjected to the relations

$$\mathbf{a}_0^\xi \vee \mathbf{a}_1^\xi = 1, \quad \text{for all } \xi \in \Omega. \quad (4.1)$$

Hence $\mathcal{L}(\Omega)$ is the same semilattice as the one presented in [23, Section 3]. It is a semilattice version of the dimension group $\mathbf{E}_K(\Omega)$ presented in [33, Section 2]. It can be ‘concretely’ represented as the (semi)lattice of all pairs $(X, Y) \in \mathfrak{P}(\Omega) \times \mathfrak{P}(\Omega)$ such that either X and Y are finite and disjoint or $X = Y = \Omega$, with

$$\mathbf{a}_0^\xi = (\{\xi\}, \emptyset) \text{ and } \mathbf{a}_1^\xi = (\emptyset, \{\xi\}), \quad \text{for all } \xi \in \Omega.$$

We shall identify $\mathcal{L}(X)$ with the $(\vee, 0, 1)$ -subsemilattice of $\mathcal{L}(\Omega)$ generated by the subset $\{\mathbf{a}_i^\xi \mid \xi \in X \text{ and } i < 2\}$, for all $X \subseteq \Omega$. For sets X and Y , any map $f: X \rightarrow Y$ gives rise to a unique $(\vee, 0, 1)$ -homomorphism $\mathcal{L}(f): \mathcal{L}(X) \rightarrow \mathcal{L}(Y)$ such that $\mathcal{L}(f)(\mathbf{a}_i^\xi) = \mathbf{a}_i^{f(\xi)}$, for all $(\xi, i) \in X \times \{0, 1\}$. Of course, the assignment $X \mapsto \mathcal{L}(X)$, $f \mapsto \mathcal{L}(f)$ is a functor from the category of sets with maps to the category of $(\vee, 0, 1)$ -semilattices and $(\vee, 0, 1)$ -homomorphisms.

Next, we put $\mathcal{G} = \mathcal{D} \circ \mathcal{L}$, the composition of the two functors \mathcal{D} and \mathcal{L} . Hence, for a set Ω , the semilattice $\mathcal{G}(\Omega)$ may be loosely described as a ‘free distributive $(\vee, 0)$ -semilattice defined by generators \mathbf{a}_i^ξ , for $\xi \in \Omega$ and $i < 2$, and relations (4.1)’. It is a distributive $(\vee, 0, 1)$ -semilattice, of the same cardinality as Ω in case Ω is infinite.

The proof of the following lemma is straightforward (see Lemma 3.6).

Lemma 4.1. *Let Ω be a set and let $(X_i \mid i \in I)$ be a family of subsets of Ω . The following statements hold:*

- (1) $\mathcal{L}(\bigcap_{i \in I} X_i) = \bigcap_{i \in I} \mathcal{L}(X_i)$ and $\mathcal{G}(\bigcap_{i \in I} X_i) = \bigcap_{i \in I} \mathcal{G}(X_i)$.

- (2) If I is a nonempty upward directed partially ordered set and the family $(X_i \mid i \in I)$ is isotone, then $\mathcal{L}(\bigcup_{i \in I} X_i) = \bigcup_{i \in I} \mathcal{L}(X_i)$ and $\mathcal{G}(\bigcup_{i \in I} X_i) = \bigcup_{i \in I} \mathcal{G}(X_i)$.

Corollary 4.2. For any set Ω and any $\mathbf{x} \in \mathcal{G}(\Omega)$, there exists a least (finite) subset X of Ω such that $\mathbf{x} \in \mathcal{G}(X)$.

We shall call the subset X of Corollary 4.2 the *support* of \mathbf{x} , and denote it by $\text{supp}(\mathbf{x})$.

Lemma 4.3. Let Ω be a set, let $\alpha \in \Omega$, and let $i < 2$. Then $\mathbf{x} \leq \mathbf{y} \vee \mathbf{a}_i^\alpha$ implies that $\mathbf{x} \leq \mathbf{y}$, for all $\mathbf{x}, \mathbf{y} \in \mathcal{G}(\Omega \setminus \{\alpha\})$.

Proof. There exists a unique retraction $r: \mathcal{L}(\Omega) \rightarrow \mathcal{L}(\Omega \setminus \{\alpha\})$ such that $r(\mathbf{a}_i^\alpha) = 0$. Put $s = \mathcal{D}(r)$, and observe that $s(\mathbf{x}) = \mathbf{x}$, $s(\mathbf{y}) = \mathbf{y}$, and $s(\mathbf{a}_i^\alpha) = 0$. By applying s to the inequality $\mathbf{x} \leq \mathbf{y} \vee \mathbf{a}_i^\alpha$, we get the conclusion. \square

The following crucial lemma describes an ‘evaporation process’ in $\mathcal{G}(\Omega)$.

Lemma 4.4. Let α, β, δ be distinct elements in a set Ω , let $i, j < 2$, $\mathbf{x} \in \mathcal{G}(\Omega \setminus \{\beta\})$, $\mathbf{y} \in \mathcal{G}(\Omega \setminus \{\alpha\})$, and $\mathbf{z} \in \mathcal{G}(\Omega \setminus \{\delta\})$. Then

$$\mathbf{z} \leq \mathbf{x} \vee \mathbf{y}, \quad \mathbf{x} \leq \mathbf{a}_0^\delta, \mathbf{a}_i^\alpha, \quad \text{and} \quad \mathbf{y} \leq \mathbf{a}_1^\delta, \mathbf{a}_j^\beta$$

implies that $\mathbf{z} = 0$.

Proof. For $s \in \omega$ and $\mathbf{u} \in \mathcal{R}^{s+1}\mathcal{L}(\Omega) \setminus \mathcal{R}^s\mathcal{L}(\Omega)$, we shall denote by $\pi(\mathbf{u})$ the image of \mathbf{u} under the canonical projection from $\mathcal{R}^{s+1}\mathcal{L}(\Omega)$ to $\mathcal{R}^s\mathcal{L}(\Omega)$. Put $m = \text{rk } \mathbf{x}$, $n = \text{rk } \mathbf{y}$, and $k = \text{rk } \mathbf{z}$. We argue by induction on $m + n + k$. If $\mathbf{z} \leq \mathbf{x}$, then $\mathbf{z} \leq \mathbf{a}_0^\delta$, thus, as $\mathbf{z} \in \mathcal{G}(\Omega \setminus \{\delta\})$, it follows from Lemma 4.3 that $\mathbf{z} = 0$ so we are done. The conclusion is similar in case $\mathbf{z} \leq \mathbf{y}$. So suppose that $\mathbf{z} \not\leq \mathbf{x}, \mathbf{y}$. If $m = 0$, then, as $\mathbf{x} \in \mathcal{L}(\Omega)$ and $\mathbf{x} \leq \mathbf{a}_0^\delta, \mathbf{a}_i^\alpha$ with $\alpha \neq \delta$, we get $\mathbf{x} = 0$, so $\mathbf{z} \leq \mathbf{y}$, a contradiction; hence $m > 0$. Similarly, $n > 0$. Put $l = \max\{m, n\}$, $\mathbf{x}^* = \mathbf{x} \setminus \{\pi(\mathbf{x})\}$, and $\mathbf{y}^* = \mathbf{y} \setminus \{\pi(\mathbf{y})\}$ (see Remark 3.3). Furthermore, we define (using again Remark 3.3) a finite subset \mathbf{w} of $\mathcal{C}\mathcal{R}^{l-1}\mathcal{L}(\Omega)$ as

$$\mathbf{w} = \begin{cases} \mathbf{x}^* \cup \mathbf{y}^* \cup \{\pi(\mathbf{x}) \vee \pi(\mathbf{y})\}, & \text{if } m = n, \\ \mathbf{y}^* \cup \{\mathbf{x} \vee \pi(\mathbf{y})\}, & \text{if } m < n, \\ \mathbf{x}^* \cup \{\pi(\mathbf{x}) \vee \mathbf{y}\}, & \text{if } m > n. \end{cases} \quad (4.2)$$

Claim. The set \mathbf{w} belongs to $\mathcal{R}^l\mathcal{L}(\Omega)$, and $\mathbf{x}, \mathbf{y} \leq \mathbf{w}$.

Proof of Claim. We need to verify that \mathbf{w} is a reduced subset of $\mathcal{C}\mathcal{R}^{l-1}\mathcal{L}(\Omega)$, modulo the identification of elements with diagonal triples (see Remark 3.3). It is obvious that there exists exactly one element in $\mathbf{w} \cap \mathcal{R}^{l-1}\mathcal{L}(\Omega)$, namely,

$$\pi(\mathbf{w}) = \begin{cases} \pi(\mathbf{x}) \vee \pi(\mathbf{y}), & \text{if } m = n, \\ \mathbf{x} \vee \pi(\mathbf{y}), & \text{if } m < n, \\ \pi(\mathbf{x}) \vee \mathbf{y}, & \text{if } m > n. \end{cases}$$

This settles item (1) of the definition of a reduced set.

Now suppose that there exists a non-diagonal triple $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ of elements of $\mathcal{R}^{l-1}\mathcal{L}(\Omega)$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{w}$ and $(\mathbf{b}, \mathbf{a}, \mathbf{c}) \in \mathbf{w}$. As both \mathbf{x} and \mathbf{y} are reduced sets, the only possibility is $m = n$ and, say, $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{x}$ and $(\mathbf{b}, \mathbf{a}, \mathbf{c}) \in \mathbf{y}$. As $\mathbf{x} \in \mathcal{G}(\Omega \setminus \{\beta\})$ and $\mathbf{y} \in \mathcal{G}(\Omega \setminus \{\alpha\})$, all elements $\mathbf{a}, \mathbf{b}, \mathbf{c}$ belong to $\mathcal{G}(\Omega \setminus \{\alpha, \beta\})$ (see

Lemma 4.1). As $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{x}$ and $\mathbf{x} \leq \mathbf{a}_i^\alpha$, it follows from (3.1) and the assumption that $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is non-diagonal that either $\mathbf{a} \leq \mathbf{a}_i^\alpha$ or $\mathbf{c} \leq \mathbf{a}_i^\alpha$. As $\mathbf{a}, \mathbf{c} \in \mathcal{G}(\Omega \setminus \{\alpha\})$, it follows from Lemma 4.3 that either $\mathbf{a} = 0$ or $\mathbf{c} = 0$, a contradiction. This settles item (2) of the definition of a reduced set.

Finally, let $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{w}$ be a non-diagonal triple of elements of $\mathcal{R}^{l-1}\mathcal{L}(\Omega)$, we must verify that $\mathbf{a}, \mathbf{b}, \mathbf{c} \not\leq \pi(\mathbf{w})$. Suppose, for example, that $\mathbf{a} \leq \pi(\mathbf{w})$. If $m = n$, then $\mathbf{a} \leq \pi(\mathbf{x}) \vee \pi(\mathbf{y})$ and, say, $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{x}^*$. From $\pi(\mathbf{y}) \leq \mathbf{y} \leq \mathbf{a}_j^\beta$ it follows that $\mathbf{a} \leq \pi(\mathbf{x}) \vee \mathbf{a}_j^\beta$. As $\mathbf{a}, \pi(\mathbf{x}) \in \mathcal{G}(\Omega \setminus \{\beta\})$ and by Lemma 4.3, it follows that $\mathbf{a} \leq \pi(\mathbf{x})$, which contradicts the assumption that $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a non-diagonal triple in \mathbf{x} . If $m < n$, then $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{y}^*$ and $\mathbf{a} \leq \mathbf{x} \vee \pi(\mathbf{y})$, so $\mathbf{a} \leq \mathbf{a}_i^\alpha \vee \pi(\mathbf{y})$, and so, as $\mathbf{a}, \pi(\mathbf{y}) \in \mathcal{G}(\Omega \setminus \{\alpha\})$ and by Lemma 4.3, it follows that $\mathbf{a} \leq \pi(\mathbf{y})$, which contradicts the assumption that $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a non-diagonal triple in \mathbf{y} . The proof for the case $m > n$ is similar. So we have proved that $\mathbf{a} \not\leq \pi(\mathbf{w})$. The proofs for \mathbf{b} and \mathbf{c} are similar. This settles item (3) of the definition of a reduced set.

The verification of the inequalities $\mathbf{x}, \mathbf{y} \leq \mathbf{w}$ (see (3.1)) is straightforward. In fact, it is not hard to verify, using Lemma 3.1, that $\mathbf{w} = \mathbf{x} \vee \mathbf{y}$. \square Claim.

Now we complete the proof of Lemma 4.4. From the claim above it follows that $\mathbf{z} \leq \mathbf{w}$. If $k < l$ then $\mathbf{z} \leq \pi(\mathbf{w})$, hence, as $\pi(\mathbf{w}) \in \{\pi(\mathbf{x}) \vee \pi(\mathbf{y}), \mathbf{x} \vee \pi(\mathbf{y}), \pi(\mathbf{x}) \vee \mathbf{y}\}$ and by the induction hypothesis, $\mathbf{z} = 0$. So suppose from now on that $k \geq l$; in particular, $k > 0$. As $\pi(\mathbf{z}) \leq \mathbf{z} \leq \mathbf{x} \vee \mathbf{y}$, it follows from the induction hypothesis that $\pi(\mathbf{z}) = 0$. Hence, if $\mathbf{z} \neq 0$, then there exists a non-diagonal triple $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{z} \cap \mathcal{R}^{l-1}\mathcal{L}(\Omega)$. As $\mathbf{z} \leq \mathbf{w}$, we obtain that either $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{w}$ or $\mathbf{a} \leq \mathbf{w}$ or $\mathbf{c} \leq \mathbf{w}$. In the first case, say, $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{x}$, we get $\bowtie(\mathbf{a}, \mathbf{b}, \mathbf{c}) \leq \mathbf{x} \leq \mathbf{a}_0^\delta$ with $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{G}(\Omega \setminus \{\delta\})$ (because $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{z}$, so $\bowtie(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0$ by Lemma 4.3, a contradiction). If either $\mathbf{a} \leq \mathbf{w}$ or $\mathbf{c} \leq \mathbf{w}$, then, by the induction hypothesis, either $\mathbf{a} = 0$ or $\mathbf{c} = 0$, a contradiction. Therefore, $\mathbf{z} = 0$. \square

5. THE SEMILATTICES $\text{Sem}(m, \Omega)$ AND THE STATEMENTS CLR , CLR_m

Many things can be said about weakly distributive homomorphisms (as defined in Section 1), but in the present paper we will only need the following one, proved in [34, Proposition 1.2].

Proposition 5.1. *Let K and L be lattices and let $e: K \rightarrow L$ be a lattice homomorphism with convex range. Then the induced map $\text{Con}_c e: \text{Con}_c K \rightarrow \text{Con}_c L$ is a weakly distributive $(\vee, 0)$ -homomorphism.*

Now we turn to the definition of the statements $\text{CLR}(\mathbf{e})$ and $\text{CLR}_m(\mathbf{e})$, see Definition 5.4.

Notation 5.2. For a positive integer m and a nonempty set Ω , we denote by $\text{Sem}(m, \Omega)$ the set of all maps $p: \Omega \rightarrow m+2$ such that either $\{\xi \in \Omega \mid p(\xi) \neq 0\}$ is finite or $(\forall \xi \in \Omega)(p(\xi) = m+1)$, ordered componentwise. We denote by $\bar{0}$ (resp., $\bar{1}$) the bottom (resp., top) element of $\text{Sem}(m, \Omega)$. For all $\xi \in \Omega$, we put $(m+1) \cdot \xi = \bar{1}$; furthermore, for all $k \in \{0, 1, \dots, m\}$, we define an element $k \cdot \xi$ of $\text{Sem}(m, \Omega)$ by putting

$$(k \cdot \xi)(\eta) = \begin{cases} k, & \text{if } \xi = \eta, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for all } \eta \in \Omega.$$

In particular, $0 \cdot \xi = \bar{0}$.

It is not hard to verify that $\text{Sem}(m, \Omega)$ is the join-semilattice freely generated by chains $\bar{0} = 0 \cdot \dot{\xi} \leq 1 \cdot \dot{\xi} \leq \dots \leq m \cdot \dot{\xi} \leq (m+1) \cdot \dot{\xi} = \bar{1}$, for $\xi \in \Omega$.

Notation 5.3. For a positive integer m , a nonempty set Ω , a $(\vee, 0)$ -semilattice S , $e \in S$, and S -valued families

$$\vec{a} = (\mathbf{a}_i^\xi \mid (\xi, i) \in \Omega \times \{0, 1\}), \quad (5.1)$$

$$\vec{c} = (\mathbf{c}(p, q) \mid p, q \in \text{Sem}(m, \Omega)) \quad (5.2)$$

such that $e \leq \mathbf{a}_0^\xi \vee \mathbf{a}_1^\xi$ for all $\xi \in \Omega$ (we will abbreviate this by $e \leq \vec{a}_0 \vee \vec{a}_1$), let $\text{Meas}(\vec{a}, \vec{c})$ be the conjunction of the following statements:

- (1) $p \leq q$ implies that $\mathbf{c}(p, q) = 0$, for all $p, q \in \text{Sem}(m, \Omega)$;
- (2) $\mathbf{c}(p, r) \leq \mathbf{c}(p, q) \vee \mathbf{c}(q, r)$, for all $p, q, r \in \text{Sem}(m, \Omega)$;
- (3) $\mathbf{c}(p \vee q, r) = \mathbf{c}(p, r) \vee \mathbf{c}(q, r)$, for all $p, q, r \in \text{Sem}(m, \Omega)$;
- (4) $\mathbf{c}(\bar{1}, \bar{0}) = e$;
- (5) For all $\xi \in \Omega$ and all $k \leq m$, there exists $j < 2$ with $\mathbf{c}((k+1) \cdot \dot{\xi}, k \cdot \dot{\xi}) \leq \mathbf{a}_j^\xi$.

For a subset X of Ω , let $\vec{a}_{(X)}$ denote the restriction of \vec{a} to $X \times \{0, 1\}$.

Definition 5.4. For an element e in a $(\vee, 0)$ -semilattice S , we say that S satisfies $\text{CLR}(e)$, if for every nonempty set Ω and every S -valued family \vec{a} as in (5.1) with $e \leq \vec{a}_0 \vee \vec{a}_1$, there exists a decomposition $\Omega = \bigcup (\Omega_m \mid m \in \omega \setminus \{0\})$ together with families $\vec{c}_m: \text{Sem}(m, \Omega_m) \times \text{Sem}(m, \Omega_m) \rightarrow S$, for $m \in \omega \setminus \{0\}$, such that S satisfies $\text{Meas}(\vec{a}_{(\Omega_m)}, \vec{c}_m)$ for all $m \in \omega \setminus \{0\}$. If, for a fixed $m \in \omega \setminus \{0\}$, we can always take $\Omega_m = \Omega$ while $\Omega_n = \emptyset$ for all $n \neq m$, we say that S satisfies $\text{CLR}_m(e)$.

The statement $\text{CLR}(e)$ is an analogue, for arbitrary lattices, of the ‘uniform refinement property’ introduced in [34], denoted by ‘URP⁻ at e ’ in [31].

Lemma 5.5. *Let S and T be $(\vee, 0)$ -semilattices, let $e \in S$, and let $\mu: S \rightarrow T$ be a $(\vee, 0)$ -homomorphism. Suppose that μ is weakly distributive at e . Then the following statements hold.*

- (1) *If S satisfies $\text{CLR}(e)$, then T satisfies $\text{CLR}(\mu(e))$.*
- (2) *If S satisfies $\text{CLR}_m(e)$, then T satisfies $\text{CLR}_m(\mu(e))$, for any positive integer m .*

Proof. We present the proof for $\text{CLR}(e)$. The proof for $\text{CLR}_m(e)$ is similar. Let Ω be a nonempty set and let $\vec{b}: \Omega \times \{0, 1\} \rightarrow T$ be a family with $\mu(e) \leq \vec{b}_0 \vee \vec{b}_1$. As μ is weakly distributive at e , there exists a family $\vec{a}: \Omega \times \{0, 1\} \rightarrow S$ with $e \leq \vec{a}_0 \vee \vec{a}_1$ such that $\mu(\mathbf{a}_i^\xi) \leq \mathbf{b}_i^\xi$ for all $(\xi, i) \in \Omega \times \{0, 1\}$. As S satisfies $\text{CLR}(e)$, there are a decomposition $\Omega = \bigcup (\Omega_m \mid m \in \omega \setminus \{0\})$ and families $\vec{c}_m: \text{Sem}(m, \Omega_m) \times \text{Sem}(m, \Omega_m) \rightarrow S$, for $m \in \omega \setminus \{0\}$, such that S satisfies $\text{Meas}(\vec{a}_{(\Omega_m)}, \vec{c}_m)$ for all $m \in \omega \setminus \{0\}$. Then T satisfies $\text{Meas}(\vec{b}_{(\Omega_m)}, \mu \vec{c}_m)$, for all $m \in \omega \setminus \{0\}$. \square

Theorem 5.6. *If an algebra L has a polynomially definable structure of $(\vee, 0, 1)$ -semilattice, then $\text{Con}_c L$ satisfies $\text{CLR}(e)$ at the largest congruence e (necessarily compact) of L . If, in addition, L has m -permutable congruences (where m is a positive integer), then $\text{Con}_c L$ satisfies $\text{CLR}_m(e)$.*

Proof. Of course, $e = \Theta_L(0, 1)$ is compact. Furthermore, as the join operation \vee is a polynomial of L , every congruence of L is a congruence for \vee . Let Ω be a nonempty set and let $\vec{a}: \Omega \times \{0, 1\} \rightarrow \text{Con}_c L$ be a family with $e \leq \vec{a}_0 \vee \vec{a}_1$. For all

$\xi \in \Omega$, there are an integer $n_\xi \geq 2$ and elements $z_i^\xi \in L$, for $0 \leq i \leq n_\xi$, such that $z_0^\xi = 0$, $z_{n_\xi}^\xi = 1$, and

$$\text{either } \Theta_L(z_i^\xi, z_{i+1}^\xi) \leq \mathbf{a}_0^\xi \text{ or } \Theta_L(z_i^\xi, z_{i+1}^\xi) \leq \mathbf{a}_1^\xi, \quad \text{for all } i < n_\xi. \quad (5.3)$$

After replacing z_k^ξ by $\bigvee_{i \leq k} z_i^\xi$, for each $k \leq n_\xi$, we may also assume that

$$0 = z_0^\xi \leq z_1^\xi \leq \cdots \leq z_{n_\xi-1}^\xi \leq z_{n_\xi}^\xi = 1. \quad (5.4)$$

We put $\Omega_m = \{\xi \in \Omega \mid n_\xi = m+1\}$, for all $m \in \omega \setminus \{0\}$. In case L has m -permutable congruences, we may take $n_\xi = m+1$ for all $\xi \in \Omega$, so $\Omega_m = \Omega$ while $\Omega_n = \emptyset$ for all $n \neq m$. Then we put

$$\mathbf{c}_m(p, q) = \Theta_L^+ \left(\bigvee_{\xi \in \Omega_m} z_{p(\xi)}^\xi, \bigvee_{\xi \in \Omega_m} z_{q(\xi)}^\xi \right), \quad \text{for all } p, q \in \text{Sem}(m, \Omega_m) \quad (5.5)$$

(observe that the two joins inside the Θ^+ in (5.5) are, in fact, *finite* joins). It is straightforward that $\text{Con}_c L$ satisfies items (1)–(3) of the definition of $\text{Meas}(\vec{\mathbf{a}}_{(\Omega_m)}, \mathbf{c}_m)$. Furthermore, $\mathbf{c}_m(\bar{1}, \bar{0}) = \Theta_L^+(1, 0) = \mathbf{e}$, while for all $\xi \in \Omega$ and all $k \leq m$, $\mathbf{c}_m((k+1) \cdot \xi, k \cdot \xi) = \Theta_L^+(z_{k+1}^\xi, z_k^\xi)$ lies below \mathbf{a}_j^ξ for some $j < 2$. \square

6. FAILURE OF CLR IN LARGE ENOUGH $\mathcal{G}(\Omega)$

The main result of the present section is the following.

Theorem 6.1. *Let Ω be a set and let m be a positive integer.*

- (1) *If $|\Omega| \geq \aleph_{\omega+1}$, then $\mathcal{G}(\Omega)$ does not satisfy CLR(1).*
- (2) *If $|\Omega| \geq \aleph_{2^m}$, then $\mathcal{G}(\Omega)$ does not satisfy CLR $_m$ (1).*

The remainder of this section will be devoted to a proof of Theorem 6.1. We prove, for example, item (1) above. Assume that $\mathcal{G}(\Omega)$ satisfies CLR(1). Since $\mathbf{a}_0^\xi \vee \mathbf{a}_1^\xi = 1$ for all $\xi \in \Omega$, there exists a decomposition $\Omega = \bigcup (\Omega_m \mid m \in \omega \setminus \{0\})$ of Ω together with families $\vec{\mathbf{c}}_m : \text{Sem}(m, \Omega_m) \times \text{Sem}(m, \Omega_m) \rightarrow \mathcal{G}(\Omega)$, for $m \in \omega \setminus \{0\}$, such that $\text{Meas}(\vec{\mathbf{a}}_{(\Omega_m)}, \vec{\mathbf{c}}_m)$ holds for all $m \in \omega \setminus \{0\}$. As $\aleph_{\omega+1}$ is regular (*this is the reason why \aleph_ω would not work* a priori), there exists $m \in \omega \setminus \{0\}$ such that $|\Omega_m| \geq \aleph_{\omega+1}$. By composing $\vec{\mathbf{c}}$ with $\mathcal{G}(r)$, for some retraction $r : \Omega \twoheadrightarrow \Omega_m$, we reduce the problem to the case where $\Omega = \Omega_m$. In case we are dealing with CLR $_m$, it is sufficient to take $\Omega_m = \Omega$. In any case, the only cardinality assumption that we shall use is $|\Omega| \geq \aleph_{2^m}$. We pick $\vec{\mathbf{c}} : \text{Sem}(m, \Omega) \times \text{Sem}(m, \Omega) \rightarrow \mathcal{G}(\Omega)$ such that $\text{Meas}(\vec{\mathbf{a}}, \vec{\mathbf{c}})$ holds. We put

$$\Phi(X) = \bigcup (\text{supp } \mathbf{c}(p, q) \mid p, q \in \text{Sem}(m, X)), \quad \text{for all finite nonempty } X \subseteq \Omega. \quad (6.1)$$

Of course, we identify in (6.1) the $(\vee, 0, 1)$ -semilattice $\text{Sem}(m, X)$ with its canonical image in $\text{Sem}(m, \Omega)$, for all nonempty $X \subseteq \Omega$: send any $p \in \text{Sem}(m, X)$ to its extension by 0 in case $p \neq \bar{1}$, and to $\bar{1}$ if $p = \bar{1}$. So, $\Phi : [\Omega]^{<\omega} \rightarrow [\Omega]^{<\omega}$.

The following crucial lemma carries a flavor of what could be called ‘semilattice-theoretical commutator theory’.

Lemma 6.2. *Let $X_0, X_1 \subseteq \Omega$ finite, let $\delta \in \Omega$, and let $x_j \in \text{Sem}(m, X_j)$, for $j < 2$. Then there are $\mathbf{u}_j \in \mathcal{G}\Phi(X_j \cup \{\delta\})$, for $j < 2$, such that*

$$\mathbf{c}(\bar{1}, x_0 \vee x_1) \leq \mathbf{u}_0 \vee \mathbf{u}_1 \quad \text{and} \quad \mathbf{u}_j \leq \mathbf{c}(\bar{1}, x_j), \mathbf{a}_j^\delta \quad \text{for all } j < 2.$$

Proof. Put $\mathbf{b}_j = \mathbf{c}(\bar{1}, x_j)$, for all $j < 2$. For $\mathbf{a} \in \mathcal{G}(\Omega)$ and $x, y \in \text{Sem}(m, \Omega)$, write $x \leq_{\mathbf{a}} y$ for $\mathbf{c}(x, y) \leq \mathbf{a}$ and $x \equiv_{\mathbf{a}} y$ for $x \leq_{\mathbf{a}} y \leq_{\mathbf{a}} x$. Clearly, $\equiv_{\mathbf{a}}$ is a \vee -congruence of $\text{Sem}(m, \Omega)$. For all $i \leq m$, pick $\varepsilon(i) \in \{0, 1\}$ such that $\mathbf{c}((i+1) \cdot \dot{\delta}, i \cdot \dot{\delta}) \leq \mathbf{a}_{\varepsilon(i)}^{\delta}$.

Claim. For all $i \leq m$, there exists $\mathbf{v}_i \in \mathcal{G}\Phi(X_{\varepsilon(i)} \cup \{\delta\})$ such that $\mathbf{v}_i \leq \mathbf{b}_{\varepsilon(i)}, \mathbf{a}_{\varepsilon(i)}^{\delta}$ and $i \cdot \dot{\delta} \vee x_{\varepsilon(i)} \equiv_{\mathbf{v}_i} (i+1) \cdot \dot{\delta} \vee x_{\varepsilon(i)}$.

Proof of Claim. Put $j = \varepsilon(i)$. On the one hand,

$$(i+1) \cdot \dot{\delta} \leq (i+1) \cdot \dot{\delta} \vee x_j \equiv_{\mathbf{a}_j^{\delta}} i \cdot \dot{\delta} \vee x_j \quad (\text{because } (i+1) \cdot \dot{\delta} \equiv_{\mathbf{a}_j^{\delta}} i \cdot \dot{\delta}),$$

on the other hand

$$(i+1) \cdot \dot{\delta} \leq \bar{1} = i \cdot \dot{\delta} \vee \bar{1} \equiv_{\mathbf{b}_j} i \cdot \dot{\delta} \vee x_j \quad (\text{because } x_j \equiv_{\mathbf{b}_j} \bar{1}).$$

Thus $\mathbf{c}((i+1) \cdot \dot{\delta} \vee x_j, i \cdot \dot{\delta} \vee x_j) \leq \mathbf{a}_j^{\delta}, \mathbf{b}_j$, hence, as $i \cdot \dot{\delta} \leq (i+1) \cdot \dot{\delta}$, the element $\mathbf{v}_i = \mathbf{c}((i+1) \cdot \dot{\delta} \vee x_j, i \cdot \dot{\delta} \vee x_j)$ is as required. \square Claim.

Now we put

$$\mathbf{u}_j = \bigvee(\mathbf{v}_i \mid i \leq m, \varepsilon(i) = j), \quad \text{for all } j < 2.$$

Hence $\mathbf{u}_j \in \mathcal{G}\Phi(X_j \cup \{\delta\})$ for all $j < 2$, and $\mathbf{u}_0 \vee \mathbf{u}_1 = \bigvee(\mathbf{v}_i \mid i \leq m)$. In particular, it follows from the Claim above that $i \cdot \dot{\delta} \vee x_0 \vee x_1 \equiv_{\mathbf{u}_0 \vee \mathbf{u}_1} (i+1) \cdot \dot{\delta} \vee x_0 \vee x_1$, for all $i \leq m$. Therefore, $0 \cdot \dot{\delta} \vee x_0 \vee x_1 \equiv_{\mathbf{u}_0 \vee \mathbf{u}_1} (m+1) \cdot \dot{\delta} \vee x_0 \vee x_1$, that is, $x_0 \vee x_1 \equiv_{\mathbf{u}_0 \vee \mathbf{u}_1} \bar{1}$. \square

As $|\Omega| \geq \aleph_{2m}$, it follows from Kuratowski's Theorem that there exists a $(2^m + 1)$ -element subset U of Ω which is free with respect to the restriction of Φ to 2^m -elements subsets of Ω .

For all natural numbers k, l with $k \leq m-1$ and $l \leq 2^k$, let $P(k, l)$ hold, if for all disjoint $X, Y \subseteq U$ with $|X| = 2^k - l$ and $|Y| = 2l$, the following relation $E(X, Y)$ holds:

$$\bigvee((m+1-k) \cdot \dot{\xi} \mid \xi \in X) \vee \bigvee((m-k) \cdot \dot{\eta} \mid \eta \in Y) \equiv_0 \bar{1}. \quad (E(X, Y))$$

The following decisive lemma describes an 'erosion process' on the chains $(i \cdot \dot{\xi} \mid 0 \leq i \leq m+1)$.

Lemma 6.3. *The statement $P(k, l)$ holds, for all possible values of k and l .*

Proof. We argue by induction on $2^k + l$. Obviously, $P(0, 0)$ holds. Assuming that $P(k, l)$ holds, we shall establish $P(k', l')$ for the next value (k', l') . As $P(k, 2^k)$ is equivalent to $P(k+1, 0)$, we may assume that $l < 2^k$, so $k' = k$ and $l' = l+1$. So let $X, Y \subseteq U$ disjoint with $|X| = 2^k - l - 1$ and $|Y| = 2l + 2$. As $|X| + |Y| = 2^k + l + 1 \leq 2^m$ and $|U| = 2^m + 1$, there exists an element $\delta \in U \setminus (X \cup Y)$. Pick two distinct elements $\eta_0, \eta_1 \in Y$, set $Y' = Y \setminus \{\eta_0, \eta_1\}$ and

$$x_j = \bigvee((m+1-k) \cdot \dot{\xi} \mid \xi \in X) \vee \bigvee((m-k) \cdot \dot{\eta} \mid \eta \in Y' \cup \{\eta_j\}), \quad (6.2)$$

for all $j < 2$. It follows from the induction hypothesis that

$$\bigvee((m+1-k) \cdot \dot{\xi} \mid \xi \in X \cup \{\eta_j\}) \vee \bigvee((m-k) \cdot \dot{\eta} \mid \eta \in Y') \equiv_0 \bar{1}, \quad (6.3)$$

for all $j < 2$. Now there are $p_0, p_1 \in \{0, 1\}$ such that

$$\mathbf{c}((m+1-k) \cdot \dot{\eta}_j, (m-k) \cdot \dot{\eta}_j) \leq \mathbf{a}_{p_j}^{\eta_j}, \quad \text{for all } j < 2.$$

Using (6.2) and (6.3), it follows that $\mathbf{c}(\bar{1}, x_j) \leq \mathbf{a}_{p_j}^{\eta_j}$, for all $j < 2$. Therefore, by Lemma 6.2, there are $\mathbf{u}_j \in \mathcal{G}\Phi(X \cup Y' \cup \{\eta_j, \delta\})$, for $j < 2$, such that

$$\mathbf{c}(\bar{1}, x_0 \vee x_1) \leq \mathbf{u}_0 \vee \mathbf{u}_1 \quad \text{and} \quad \mathbf{u}_j \leq \mathbf{a}_{p_j}^{\eta_j}, \mathbf{a}_j^\delta, \quad \text{for all } j < 2. \quad (6.4)$$

Using the monotonicity of Φ and the freeness of U with respect to the restriction of Φ to 2^m -element subsets, we obtain

$$\begin{aligned} \Phi(X \cup Y) &\subseteq \Omega \setminus \{\delta\}, \\ \Phi(X \cup Y' \cup \{\eta_j, \delta\}) &\subseteq \Omega \setminus \{\eta_{1-j}\}, \quad \text{for all } j < 2. \end{aligned}$$

As $\mathbf{c}(\bar{1}, x_0 \vee x_1)$ belongs to $\mathcal{G}\Phi(X \cup Y)$ and by using (6.4) together with Lemma 4.4, we obtain that $\mathbf{c}(\bar{1}, x_0 \vee x_1) = 0$, that is, $x_0 \vee x_1 \equiv_0 \bar{1}$, which completes the proof of $E(X, Y)$. \square

Now pick $\delta \in U$ and put $Y = U \setminus \{\delta\}$, so $|Y| = 2^m$. By applying Lemma 6.3 to $k = m - 1$ and $l = 2^k$, we obtain the relation

$$\bigvee (1 \cdot \dot{\eta} \mid \eta \in Y) \equiv_0 \bar{1}. \quad (6.5)$$

For each $\eta \in Y$, pick $j_\eta \in \{0, 1\}$ such that $\mathbf{c}(1 \cdot \dot{\eta}, \bar{0}) \leq \mathbf{a}_{j_\eta}^\eta$. By using (6.5), we obtain $1 = \mathbf{c}(\bar{1}, \bar{0}) \leq \bigvee (\mathbf{a}_{j_\eta}^\eta \mid \eta \in Y)$, and hence $\bigvee (\mathbf{a}_{j_\eta}^\eta \mid \eta \in Y) = 1$, a contradiction. This completes the proof of Theorem 6.1.

7. CONSEQUENCES ON CONGRUENCE LATTICES OF ALGEBRAS

An immediate application of Theorems 5.6 and 6.1 together with Lemma 5.5 yields the following result.

Theorem 7.1. *Let Ω be a set and let L be an algebra. Suppose that L has a polynomially definable structure of $(\vee, 0, 1)$ -semilattice. Then the following statements hold:*

- (1) *If $|\Omega| \geq \aleph_{\omega+1}$, then there exists no $(\vee, 0, 1)$ -homomorphism from $\text{Con}_c L$ to $\mathcal{G}(\Omega)$ which is weakly distributive at the largest congruence of L .*
- (2) *For any positive integer m , if $|\Omega| \geq \aleph_{2^m}$ and L has m -permutable congruences, then there exists no $(\vee, 0, 1)$ -homomorphism from $\text{Con}_c L$ to $\mathcal{G}(\Omega)$ which is weakly distributive at the largest congruence of L .*

Restricting the statement CLR to sets of cardinality at most $\aleph_{\omega+1}$ yields the $\mathcal{L}_{(2^\kappa)^+, \kappa^+}$ (where $\kappa = \aleph_{\omega+1}$) semilattice-theoretical statement announced in Section 1, which holds in all weakly distributive images of $\text{Con}_c L$ for a bounded semilattice L but fails in $\mathcal{G}(\Omega)$ for $|\Omega| \geq \aleph_{\omega+1}$.

Observe that Theorem 7.1 applies to L a *bounded lattice*. We now extend this result to lattices that are not necessarily bounded. In order to obtain Corollary 7.3(2), we need the following variant of Theorem 5.6.

Lemma 7.2. *Let L be a lattice and let m be a positive integer. If L has m -permutable congruences, then $\text{Con}_c L$ satisfies $\text{CLR}_m(\mathbf{e})$ for any principal congruence \mathbf{e} of L .*

Proof. We proceed as in the proof of Theorem 5.6. Put $\mathbf{e} = \Theta_L(u, v)$, where $u \leq v$ in L . Let Ω be a nonempty set and let $\bar{\mathbf{a}}: \Omega \times \{0, 1\} \rightarrow \text{Con}_c L$ be a family with

$e \leq \vec{a}_0 \vee \vec{a}_1$. For all $\xi \in \Omega$, there are elements $z_i^\xi \in L$, for $0 \leq i \leq m+1$, such that $z_0^\xi = u$, $z_{m+1}^\xi = v$, and

$$\text{either } \Theta_L(z_i^\xi, z_{i+1}^\xi) \leq \mathbf{a}_0^\xi \text{ or } \Theta_L(z_i^\xi, z_{i+1}^\xi) \leq \mathbf{a}_1^\xi, \quad \text{for all } i \leq m. \quad (7.1)$$

After replacing z_k^ξ by $(z_k^\xi \vee u) \wedge v$, we may assume that $u \leq z_k^\xi \leq v$ for all $k \leq m+1$. (Here we use the full lattice structure of L , not only the join-semilattice structure.) Furthermore, after replacing z_k^ξ by $\bigvee_{i \leq k} z_i^\xi$, for each $k \leq m+1$, we may also assume that

$$u = z_0^\xi \leq z_1^\xi \leq \cdots \leq z_m^\xi \leq z_{m+1}^\xi = v. \quad (7.2)$$

The rest of the proof proceeds as in the proof of Theorem 5.6, with $\Omega_m = \Omega$ and $\mathbf{c} = \mathbf{c}_m$. \square

Corollary 7.3. *The following statements hold, for any set Ω and any lattice L :*

- (1) *If $|\Omega| \geq \aleph_{\omega+1}$, then there exists no weakly distributive $(\vee, 0)$ -homomorphism $\mu: \text{Con}_c L \rightarrow \mathcal{G}(\Omega)$ with 1 in its range.*
- (2) *For any positive integer m , if $|\Omega| \geq \aleph_{2^m}$ and L has m -permutable congruences, then there exists no weakly distributive $(\vee, 0)$ -homomorphism $\mu: \text{Con}_c L \rightarrow \mathcal{G}(\Omega)$ with 1 in its range.*

Proof. Suppose that (1) does not hold. There are $u \leq v$ in L such that $\mu \Theta_L(u, v) = 1$. Put $K = \{x \in L \mid u \leq x \leq v\}$. It follows from Proposition 5.1 that the canonical homomorphism $j: \text{Con}_c K \rightarrow \text{Con}_c L$ is weakly distributive. Hence $\mu \circ j$ is a weakly distributive $(\vee, 0, 1)$ -homomorphism from $\text{Con}_c K$ to $\mathcal{G}(\Omega)$, which contradicts the result of Theorem 7.1.

Item (2) requires a slightly different proof, because K may not have m -permutable congruences. By Lemma 7.2, $\text{Con}_c L$ satisfies $\text{CLR}_m(\mathbf{e})$, where $\mathbf{e} = \Theta_L(u, v)$. As $\mu(\mathbf{e}) = 1$, it follows from Lemma 5.5 that $\mathcal{G}(\Omega)$ satisfies $\text{CLR}_m(1)$, a contradiction by Theorem 6.1(2). \square

In particular, we obtain a negative solution to CLP.

Corollary 7.4. *Let Ω be a set.*

- (1) *If $|\Omega| \geq \aleph_{\omega+1}$, then there exists no lattice L with $\text{Con}_c L \cong \mathcal{G}(\Omega)$.*
- (2) *For any positive integer m , if $|\Omega| \geq \aleph_{2^m}$, then there exists no lattice L with m -permutable congruences such that $\text{Con}_c L \cong \mathcal{G}(\Omega)$.*

For $m = 1$, the bound \aleph_2 is obtained in the series of works [24, 30, 33, 34]. By contrast, Lampe proved in [22] that *every $(\vee, 0, 1)$ -semilattice is isomorphic to $\text{Con}_c G$ for some groupoid G with 3-permutable congruences*. In particular, if $|\Omega| \geq \aleph_8$, then $\mathcal{G}(\Omega) \cong \text{Con } G$ for some groupoid G with 3-permutable congruences, while there is no lattice L with 3-permutable congruences such that $\mathcal{G}(\Omega) \cong \text{Con } L$. This shows a critical discrepancy between general algebras and lattices.

8. OPEN PROBLEMS

The most obvious problem suggested by the present paper is to fill the cardinality gap between \aleph_2 and \aleph_ω .

Problem 1. Let S be a distributive $(\vee, 0)$ -semilattice with $\aleph_2 \leq |S| \leq \aleph_\omega$. Does there exist a lattice L such that $\text{Con}_c L \cong S$?

Another natural problem asks whether the bound \aleph_{2^m} can be improved in such results as Theorem 6.1 or Corollary 7.4 (we only know that it is optimal for $m = 1$).

Problem 2. For a positive integer m , what is the least positive integer $\rho(m)$ such that there is no lattice L with m -permutable congruences with $\text{Con}_c L \cong \mathcal{G}(\aleph_{\rho(m)})$?

It follows from Theorem 6.1 that $\rho(m) \leq 2^m$. Furthermore, $\rho(1) = 2$ (this is the only known value of ρ so far). Better estimates of $\rho(m)$ could probably be obtained *via* appropriate strengthenings of Lemmas 4.4 and 6.3.

Now that we know that the answer to CLP is negative, a natural question is the corresponding one in congruence-distributive varieties.

Problem 3. Is every distributive algebraic lattice isomorphic to the congruence lattice of some algebra generating a congruence-distributive variety?

Recall the classical open problem asking whether every algebraic distributive lattice isomorphic to the congruence lattice of some algebra with finitely many operations. In view of Theorem 7.1, we may try to find the algebra with a polynomially definable structure of $(\vee, 0)$ -semilattice (but not $(\vee, 0, 1)$ -semilattice).

Kearnes proves in [19] that there exists an algebraic lattice that is not isomorphic to the congruence lattice of any locally finite algebra. In light of this result, one may ask the following question.

Problem 4. Does there exist a lattice L such that $\text{Con } L$ is not isomorphic to the congruence lattice of any locally finite lattice (resp., algebra)?

In [30], infinite semilattices considered earlier in [34, 24, 33] are approximated by finite semilattices, yielding, in particular, a $\{0, 1\}^3$ -indexed diagram of finite Boolean semilattices that cannot be lifted, with respect to the Con_c functor, by congruence-permutable lattices. The methods used in the present paper suggest that those works could be extended to solve the following problem.

Problem 5. For any positive integer m , prove that there exists a $\{0, 1\}^{2^m+1}$ -indexed diagram of finite Boolean semilattices that cannot be lifted, with respect to the Con_c functor, by lattices with m -permutable congruences.

The approach of lifting diagrams of finite semilattices originated in Pudlák [25]. Tůma and Wehrung prove in [32] that there exists a diagram of finite Boolean semilattices, indexed by a finite *partially ordered set*, that cannot be lifted, with respect to the Con_c functor, by any diagram of lattices (or even algebras in any variety satisfying a nontrivial congruence lattice identity). This leaves open the following probably very difficult problem.

Problem 6. Prove that any diagram of finite distributive $(\vee, 0)$ -semilattices and $(\vee, 0)$ -homomorphisms, indexed by a finite lattice, can be lifted, with respect to the Con_c functor, by a diagram of (finite?) lattices and lattice homomorphisms.

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