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POSET REPRESENTATIONS OF DISTRIBUTIVE SEMILATTICES

FRIEDRICH WEHRUNG

ABSTRACT. We prove that for any distributive $\langle \vee, 0 \rangle$ -semilattice S , there are a meet-semilattice P with zero and a map $\mu: P \times P \rightarrow S$ such that $\mu(x, z) \leq \mu(x, y) \vee \mu(y, z)$ and $x \leq y$ implies that $\mu(x, y) = 0$, for all $x, y, z \in P$, together with the following conditions:

- (i) $\mu(v, u) = 0$ implies that $u = v$, for all $u \leq v$ in P .
- (ii) For all $u \leq v$ in P and all $\mathbf{a}, \mathbf{b} \in S$, if $\mu(v, u) = \mathbf{a} \vee \mathbf{b}$, then there are a positive integer n and a decomposition $u = x_0 \leq x_1 \leq \dots \leq x_n = v$ such that $\mu(x_{i+1}, x_i)$ lies either below \mathbf{a} or below \mathbf{b} , for all $i < n$.
- (iii) The subset $\{\mu(x, 0) \mid x \in P\}$ generates the semilattice S .

Furthermore, any finite, bounded subset of P has a join, and P is bounded in case S is bounded. Furthermore, the construction is functorial on lattice-indexed diagrams of finite distributive $\langle \vee, 0, 1 \rangle$ -semilattices.

1. INTRODUCTION

The classical congruence lattice representation problem, usually denoted by CLP, asks whether any distributive $\langle \vee, 0 \rangle$ -semilattice is isomorphic to the semilattice $\text{Con}_c L$ of all compact (i.e., finitely generated) congruences of some lattice L . (It is well-known, see [2] or [3, Theorem II.3.11], that $\text{Con}_c L$ is a distributive $\langle \vee, 0 \rangle$ -semilattice, for any lattice L .) This problem has finally found a negative solution [14]. This negative solution came out of a failed attempt to extend to *semilattices* a certain representation result of distributive semilattices by *posets* (i.e., partially ordered sets). The present paper gives the latter representation result. Unlike earlier representation results such as Grätzer and Schmidt's representation theorem of algebraic lattices as congruence lattices of abstract algebras [5], our result *characterizes* distributive algebraic lattices, see Corollary 9.3 and Proposition 9.4.

Most of the recent efforts at solving CLP were aimed at lifting not only individual semilattices, but also *diagrams* of semilattices, with respect to the congruence semilattice functor Con_c . It was observed by P. Pudlák [7] that every distributive $\langle \vee, 0 \rangle$ -semilattice S is the directed union of its finite distributive $\langle \vee, 0 \rangle$ -subsemilattices. Hence, in order to represent S as $\text{Con}_c L$ for some lattice L , it is sufficient to lift, with respect to the Con_c functor, the diagram of all finite $\langle \vee, 0 \rangle$ -subsemilattices of S , and then to define L as the direct limit of the corresponding lattices.

The possibility whether this could be done in full generality was raised by P. Pudlák in [7]. This question was answered in the negative by J. Tůma and F.

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Wehrung [10]: there exists a finite poset-indexed diagram of finite Boolean semilattices and $\langle \vee, 0, 1 \rangle$ -embeddings that cannot be lifted, with respect to the Con_c functor, by any diagram of lattices (or even by any diagram of algebras in any variety satisfying a nontrivial congruence lattice identity). Nevertheless this left the possibility open that any *lattice-indexed diagram* of finite distributive $\langle \vee, 0 \rangle$ -semilattices could be liftable by some diagram of lattices.

A caveat in the direction thus suggested appears in [9], where J. Tůma and the author provide a diagram of finite Boolean semilattices and $\langle \vee, 0, 1 \rangle$ -homomorphisms, indexed by the cube $\mathbf{2}^3$, that cannot be lifted by lattices with permutable (or even *almost permutable*) congruences (this is extended to arbitrary *algebras* with almost permutable congruences in [8]). Moreover, the contradiction that appears there makes a very limited use of the lattice structure: namely, only the *triangular inequality* for the map that with a pair of lattice elements $\langle x, y \rangle$ associates the least congruence $\Theta^+(x, y)$ forcing $x \leq y$. But this depends only on the *poset structure* of the lattices under question!

This suggests that before attempting to prove a representation result for lattices, we should tackle the problem *at poset level*. This is what we do in the present paper. More precisely, we lift any lattice-indexed diagram of finite distributive $\langle \vee, 0, 1 \rangle$ -semilattices and $\langle \vee, 0, 1 \rangle$ -homomorphisms by a diagram of finite lattices with so-called *p-measures* (where the letter ‘p’ stands for ‘poset’)—see Definition 6.1, the transition maps being what we shall call *interval extensions*—see Definition 2.1. Thus the objects of the resulting diagram are pairs $\langle P, \mu \rangle$, where P is a finite lattice and $\mu: P \times P \rightarrow D$ (where D is the corresponding finite distributive $\langle \vee, 0, 1 \rangle$ -semilattice) is a p-measure. Furthermore, we require that $\{\mu(x, 0) \mid x \in P\}$ generates D (as a join-semilattice) and that μ sends prime intervals of P to *join-irreducible* elements of D —the latter condition amounts to saying that μ satisfies the so-called *V-condition*, named this way after H. Dobbertin’s work [1] on measures on Boolean algebras (see also [8]). The V-condition is fundamental, as otherwise our construction would be meaningless (e.g., for a given distributive $\langle \vee, 0 \rangle$ -semilattice S , it is easy to construct a distributive lattice L and a surjective $\langle \vee, 0 \rangle$ -homomorphism from $\text{Con}_c L$ onto S).

Many proofs of positive representation results use transfinite iterations of ‘one-step constructions’, consisting of building a structure by adding one element at a time. For instance, such is the case for the main construction of [12] (that proves, among other things, that every lattice L such that $\text{Con}_c L$ is a lattice admits a relatively complemented congruence-preserving extension), or the construction used in [8] to prove that every distributive $\langle \vee, 0 \rangle$ -semilattice is the range of some ‘V-distance’ of type 2, or the construction used in [6] to establish that every bounded algebraic lattice is isomorphic to the congruence lattice of some groupoid. However, our result cannot be proved in such a way. The reason for this is presented in [13], where we construct, *at poset level*, an example of a p-measure that cannot be extended to a V-measure. This partly explains the complexity of our main construction: the posets and measures require a somehow ‘explicit’ construction, which in turn requires a relatively large technical background.

For elements a and b in a poset P , we shall use the abbreviations

$$\begin{aligned} a \prec_P b &\iff (a <_P b \text{ and there is no } x \text{ such that } a <_P x <_P b); \\ a \preceq_P b &\iff (\text{either } a \prec_P b \text{ or } a = b); \\ a \sim_P b &\iff (\text{either } a \leq_P b \text{ or } b \leq_P a); \\ a \parallel_P b &\iff (a \not\leq_P b \text{ and } b \not\leq_P a). \end{aligned}$$

We shall use \leq (instead of \leq_P), \prec , \preceq , \sim , or \parallel in case P is understood.

We shall sometimes view any poset P as a category, the usual way: namely, the objects of P are the elements of (the underlying poset of) P , while the set of morphisms from x to y is a singleton if $x \leq y$, empty otherwise. For a category \mathcal{C} , a P -indexed diagram of (members of) \mathcal{C} is a functor $\mathcal{D}: P \rightarrow \mathcal{C}$. This amounts to a map Φ from P to the objects of \mathcal{C} , together with a system of morphisms $\varphi_{x,y}: \Phi(x) \rightarrow \Phi(y)$, for $x \leq y$ in P , such that $\varphi_{x,x} = \text{id}_{\Phi(x)}$ and $\varphi_{x,z} = \varphi_{y,z} \circ \varphi_{x,y}$, for all $x \leq y \leq z$ in P . We shall also denote by $\mathcal{D}|_{\leq p}$ (resp., $\mathcal{D}|_{< p}$) the restriction of \mathcal{D} to $\{x \in P \mid x \leq p\}$ (resp., $\{x \in P \mid x < p\}$), for all $p \in P$.

A join-semilattice S is *distributive*, if for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in S$, if $\mathbf{c} \leq \mathbf{a} \vee \mathbf{b}$, then there are $\mathbf{x} \leq \mathbf{a}$ and $\mathbf{y} \leq \mathbf{b}$ in S such that $\mathbf{c} = \mathbf{x} \vee \mathbf{y}$. Equivalently, the ideal lattice of S is a distributive lattice, see [3, Section II.5].

We shall identify every natural number n with the set $\{0, 1, \dots, n-1\}$. We shall denote by $\mathfrak{P}(X)$ the powerset of a set X .

2. RELATIVELY COMPLETE AND INTERVAL EXTENSIONS OF POSETS

Definition 2.1. A poset Q is a *relatively complete extension* of a poset P , in notation $P \leq_{\text{rc}} Q$, if for all $x \in Q$, there exists a largest element of P below x (denoted by x_P) and a least element of P above x (denoted by x^P). Then we define binary relations \ll_P and \equiv_P on Q by

$$\begin{aligned} x \ll_P y &\iff x^P \leq y_P, \\ x \equiv_P y &\iff (x_P = y_P \text{ and } x^P = y^P), \end{aligned}$$

for all $x, y \in Q$. We say that Q is an *interval extension* of P , in notation $P \leq_{\text{int}} Q$, if $P \leq_{\text{rc}} Q$ and for all $x, y \in Q$, $x \leq y$ implies that either $x \ll_P y$ or $x \equiv_P y$.

The proofs of the following two lemmas are easy exercises.

Lemma 2.2. *Let P, Q , and R be posets. If $P \leq_{\text{rc}} Q$ and $Q \leq_{\text{rc}} R$, then $P \leq_{\text{rc}} R$.*

Lemma 2.3. *Let Q be a relatively complete extension of a poset P and let $X \subseteq P$. Then $\bigvee_P X$ exists iff $\bigvee_Q X$ exists, and then the two values are equal. The dual statement also holds.*

In particular, if $P \leq_{\text{rc}} Q$ and Q is a lattice, then P is a sublattice of Q . Hence, from now on, when dealing with relatively complete extensions, we shall often omit to mention in which subset the meets and joins are evaluated.

Lemma 2.4. *Let Q be an interval extension of a poset P , let $x \in P$ and $y, z \in Q$. If $x, y \leq z$ and $x \not\leq y$, then $y^P \leq z$; and dually.*

Proof. If $y \equiv_P z$, then, as $x \in P$ and $x \leq z$, we get $x \leq z_P = y_P \leq y$, a contradiction; hence $y \not\equiv_P z$. As $y \leq z$ and $P \leq_{\text{int}} Q$, we get $y \ll_P z$, and thus $y^P \leq z$. \square

Lemma 2.5. *Let Q be an interval extension of a poset P . Then Q is a lattice iff P is a lattice and the interval $[x_P, x^P]$ is a lattice for each $x \in Q$. Furthermore, if Q is a lattice, then for all incomparable $x, y \in Q$,*

$$x \vee y = \begin{cases} x^P \vee y^P, & \text{if } x \not\equiv_P y, \\ x \vee_{[u,v]} y, & \text{if } x_P = y_P = u \text{ and } x^P = y^P = v, \end{cases} \quad (2.1)$$

$$x \wedge y = \begin{cases} x^P \wedge y^P, & \text{if } x \not\equiv_P y, \\ x \wedge_{[u,v]} y, & \text{if } x_P = y_P = u \text{ and } x^P = y^P = v. \end{cases} \quad (2.2)$$

Proof. We prove the nontrivial direction. So suppose that P is a lattice and the interval $[x_P, x^P]$ is a lattice for each $x \in Q$. For incomparable $x, y \in P$, we prove that the join $x \vee y$ is defined in Q and given by (2.1). The proof for the meet is dual. So let $z \in Q$ such that $x, y \leq z$. If $x \ll_P z$, then $x^P \leq z$, hence, using Lemma 2.4 (with x^P instead of x), we obtain $y^P \leq z$, and hence, using Lemma 2.3, $x^P \vee y^P \leq z$. The conclusion is similar for $y \ll_P z$. As $P \leq_{\text{int}} Q$, the remaining case is where $x \equiv_P z \equiv_P y$. Putting $u = x_P = y_P$ and $v = x^P = y^P$, the interval $[u, v]$ is, by assumption, a lattice, so $x \vee_{[u,v]} y \leq z$, and hence $x \vee y = x \vee_{[u,v]} y \leq z$. \square

Lemma 2.6. *Let Q be an interval extension of a poset P . Then $x \sim y$ implies that $x_P \sim y$ and $x^P \sim y$, for all $x, y \in Q$.*

Proof. We prove the result for x_P . If $x \leq y$, then $x_P \leq y$ and we are done. Suppose that $y \leq x$. If $y \ll_P x$, then $y \leq x_P$. Suppose that $y \not\ll_P x$. As $y \leq x$ and $P \leq_{\text{int}} Q$, we get $x \equiv_P y$, and thus $x_P = y_P \leq y$. \square

Definition 2.7. A *standard interval scheme* is a family of the form $\langle P, \langle Q_{a,b} \mid \langle a, b \rangle \in \mathcal{I} \rangle \rangle$, where the following conditions are satisfied:

- (i) P is a poset, \mathcal{I} is a subset of $\{\langle a, b \rangle \in P \times P \mid a < b\}$, and $Q_{a,b}$ is a (possibly empty) poset, for all $\langle a, b \rangle \in \mathcal{I}$.
- (ii) $Q_{a,b} \cap P = \emptyset$, for all $\langle a, b \rangle \in \mathcal{I}$.
- (iii) $Q_{a,b} \cap Q_{c,d} = \emptyset$, for all distinct $\langle a, b \rangle, \langle c, d \rangle \in \mathcal{I}$.

We say that the standard interval scheme above is *based on P* .

The proofs of the following two lemmas are straightforward exercises.

Lemma 2.8. *Let $\langle P, \langle Q_{a,b} \mid \langle a, b \rangle \in \mathcal{I} \rangle \rangle$ be a standard interval scheme. Put $Q = P \cup \bigcup (Q_{a,b} \mid \langle a, b \rangle \in \mathcal{I})$. Furthermore, for all $x \in Q$, put $x_P = x^P = x$ if $x \in P$, while $x_P = a$ and $x^P = b$ if $x \in Q_{a,b}$, for $\langle a, b \rangle \in \mathcal{I}$. Let $x \leq y$ hold, if either $x^P \leq y_P$ or there exists $\langle a, b \rangle \in \mathcal{I}$ such that $x, y \in Q_{a,b}$ and $x \leq_{Q_{a,b}} y$, for all $x, y \in Q$. Then \leq is a partial ordering on Q and Q is an interval extension of P .*

In the context of Lemma 2.8, we shall use the notation

$$Q = P + \sum (Q_{a,b} \mid \langle a, b \rangle \in \mathcal{I}). \quad (2.3)$$

Conversely, the following lemma shows that any interval extension can be obtained by the $P + \sum (Q_{a,b} \mid \langle a, b \rangle \in \mathcal{I})$ construction. This construction is a special case of a construction presented in [4].

Lemma 2.9. *Let Q be an interval extension of a poset P . Put $\mathcal{I} = \{\langle a, b \rangle \in P \times P \mid a < b\}$, and $Q_{a,b} = \{x \in Q \mid x_P = a \text{ and } x^P = b\}$, for all $\langle a, b \rangle \in P$. Then $\langle P, \langle Q_{a,b} \mid \langle a, b \rangle \in \mathcal{I} \rangle \rangle$ is a standard interval scheme, and $Q = P + \sum (Q_{a,b} \mid \langle a, b \rangle \in \mathcal{I})$.*

It follows from Lemmas 2.8 and 2.9 that any standard interval scheme based on P defines an interval extension of P , and every interval extension of P is defined *via* some standard interval scheme on P .

3. COVERING EXTENSIONS OF POSETS

Definition 3.1. We say that a poset Q is a *covering extension* of a poset P , in notation $P \leq_{\text{cov}} Q$, if Q is an interval extension of P (see Definition 2.1) and the relation $x_P \preceq_P x^P$ holds for all $x \in Q$.

Lemma 3.2. *Let P , Q , and R be posets such that $P \leq_{\text{int}} Q$, $Q \leq_{\text{int}} R$, and there are no $x \in P$ and $y \in R$ such that $y_Q < x < y^Q$. Then $P \leq_{\text{int}} R$.*

Proof. First, $P \leq_{\text{rc}} R$ (see Lemma 2.2). Now let $x \leq y$ in R , and assume, towards a contradiction, that $x \not\ll_P y$ and $x \not\equiv_P y$.

If $x \equiv_Q y$, then $x \equiv_P y$, a contradiction. As $x \leq y$ and $Q \leq_{\text{int}} R$, we get $x \ll_Q y$, that is, $x^Q \leq y_Q$. If $x^Q \ll_P y_Q$, then $x \ll_P y$, a contradiction. As $x^Q \leq y_Q$ and $P \leq_{\text{int}} Q$, we get $x^Q \equiv_P y_Q$.

As $y_Q \leq y^Q$ and $P \leq_{\text{int}} Q$, either $y_Q \equiv_P y^Q$ or $y_Q \ll_P y^Q$. In the first case, we get, using the relation $x^Q \equiv_P y_Q$, the equalities $x^P = (x^Q)^P = (y_Q)^P = (y^Q)^P = y^P$. In the second case, we get, using again the relation $x^Q \equiv_P y_Q$, the inequalities $x^P = (x^Q)^P = (y_Q)^P \leq y^Q$. But $y_Q \leq (y_Q)^P = x^P$, and so $y_Q \leq x^P \leq y^Q$. Hence, by assumption, either $x^P = y_Q$ or $x^P = y^Q$. If $x^P = y_Q$, then $x^P = y_P$, so $x \ll_P y$, a contradiction; hence only the subcase where $x^P = y^Q$ remains, so $y^Q \in P$, and so $x^P = y^Q = y^P$.

So we have proved that in either case, the equality $x^P = y^P$ holds. Dually, the equality $x_P = y_P$ holds, and so $x \equiv_P y$, a contradiction. \square

Proposition 3.3. *For arbitrary posets P , Q , and R , the following statements hold:*

- (i) *If $P \leq_{\text{int}} Q$ and $Q \leq_{\text{cov}} R$, then $P \leq_{\text{int}} R$.*
- (ii) *If $P \leq_{\text{cov}} Q$ and $Q \leq_{\text{cov}} R$, then $P \leq_{\text{cov}} R$.*

Proof. (i) follows immediately from Lemma 3.2. Now we prove (ii). So assume that $P \leq_{\text{cov}} Q$ and $Q \leq_{\text{cov}} R$, let $x \in R$, we prove that $x_P \preceq_P x^P$. If $\{x_Q, x^Q\} \subseteq P$, then $x_P = x_Q$ and $x^P = x^Q$, but $Q \leq_{\text{cov}} R$, thus $x_Q \preceq_Q x^Q$, and thus, *a fortiori*, $x_P \preceq_P x^P$. So suppose, from now on, that $\{x_Q, x^Q\} \not\subseteq P$, say $x^Q \notin P$.

If $(x^Q)_P \not\leq x$, then, as $x, (x^Q)_P \leq x^Q$, as $P \leq_{\text{int}} R$ (proved in (i)), and by Lemma 2.4, we get $x^P \leq x^Q$, so $x^Q \in P$, a contradiction. Hence $(x^Q)_P \leq x$, but $(x^Q)_P$ lies above x_P and belongs to P , and so $(x^Q)_P = x_P$. As $P \leq_{\text{cov}} Q$, we get $x_P = (x^Q)_P \preceq_P (x^Q)^P = x^P$. \square

Example 3.4. The following example shows that interval extensions do not compose. We let K , L , and M the lattices diagrammed on Figure 3.1. Then $K \leq_{\text{int}} L$ and $L \leq_{\text{int}} M$, however $K \not\leq_{\text{int}} M$, as $x \leq y$ and $x \not\equiv_K y$ while $x^K \not\leq y^K$. In this example, $K \leq_{\text{cov}} L$ and $L \not\leq_{\text{cov}} M$.

4. STRONG AMALGAMS OF NORMAL DIAGRAMS OF POSETS

In this section we shall deal with families of posets indexed by meet-semilattices. We say that a meet-semilattice Λ is *lower finite*, if the principal ideal $\downarrow a = \{x \in \Lambda \mid x \leq a\}$ is finite, for all $a \in \Lambda$. Observe that this implies that Λ has a least element.

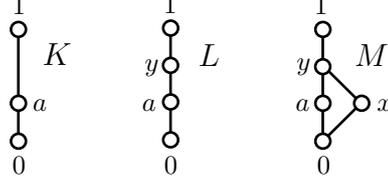


FIGURE 3.1. Interval extensions do not compose.

Definition 4.1. A *normal diagram of posets* consists of a family $\sigma = \langle Q_i \mid i \in \Lambda \rangle$ of posets, indexed by a meet-semilattice Λ , such that the following conditions hold (we denote by \leq_i the partial ordering of Q_i , for all $i \in \Lambda$):

- (1) Q_i is a sub-poset of Q_j , for all $i \leq j$ in Λ .
- (2) $Q_i \cap Q_j = Q_{i \wedge j}$ (set-theoretically!), for all $i, j \in \Lambda$.
- (3) For all $i, j, k \in \Lambda$ such that $i, j \leq k$ and all $\langle x, y \rangle \in Q_i \times Q_j$, if $x \leq_k y$, then there exists $z \in Q_{i \wedge j}$ such that $x \leq_i z$ and $z \leq_j y$.

Furthermore, we say that σ is a *normal interval diagram of posets*, if Q_j is an interval extension of Q_i , for all $i \leq j$ in Λ .

Let $\sigma = \langle Q_i \mid i \in \Lambda \rangle$ be a normal diagram of posets and set $P = \bigcup (Q_i \mid i \in \Lambda)$. For $x, y \in P$, let $x \leq y$ hold, if there are $i, j \in \Lambda$ and $z \in Q_{i \wedge j}$ such that $x \in Q_i$, $y \in Q_j$, and $x \leq_i z \leq_j y$. As the following lemma shows, this definition is independent of the chosen pair $\langle i, j \rangle$ such that $x \in Q_i$ and $y \in Q_j$.

Lemma 4.2. For all $x, y \in P$ and all $i, j \in \Lambda$ such that $x \in Q_i$ and $y \in Q_j$, $x \leq y$ iff there exists $z \in Q_{i \wedge j}$ such that $x \leq_i z \leq_j y$.

Proof. The given condition implies, by definition, that $x \leq y$. Conversely, suppose that $x \leq y$, and fix $i', j' \in \Lambda$ and $z' \in Q_{i' \wedge j'}$ such that $x \in Q_{i'}$, $y \in Q_{j'}$, and $x \leq_{i'} z' \leq_{j'} y$. As $x \in Q_{i \wedge i'}$, $z' \in Q_{i' \wedge j'}$, and $x \leq_{i'} z'$, there exists $t \in Q_{i \wedge i' \wedge j'}$ such that $x \leq_{i \wedge i'} t \leq_{i' \wedge j'} z'$. As $t \in Q_{i \wedge i' \wedge j'}$, $y \in Q_{j \wedge j'}$, and $t \leq_{j'} y$, there exists $z \in Q_{i \wedge i' \wedge j \wedge j'}$ such that $t \leq_{i \wedge i' \wedge j'} z \leq_{j \wedge j'} y$. In particular, $z \in Q_{i \wedge j}$ and $x \leq_i z \leq_j y$. \square

Lemma 4.3. The binary relation \leq defined above is a partial ordering of P . Furthermore, Q_i is a sub-poset of P , for all $i \in \Lambda$.

Proof. Reflexivity is obvious. Now let $x, y \in P$ such that $x \leq y$ and $y \leq x$. Fix $i, j \in \Lambda$ such that $x \in Q_i$ and $y \in Q_j$. By Lemma 4.2, there are $u, v \in Q_{i \wedge j}$ such that $x \leq_i u \leq_j y \leq_j v \leq_i x$. Hence $u \leq_{i \wedge j} v \leq_{i \wedge j} u$, and so $u = v$, and therefore $x = u = y$.

Now let $x \leq y \leq z$ in P , and fix $i, j, k \in \Lambda$ such that $x \in Q_i$, $y \in Q_j$, and $z \in Q_k$. By Lemma 4.2, there are $u \in Q_{i \wedge j}$ and $v \in Q_{j \wedge k}$ such that $x \leq_i u \leq_j y$ and $y \leq_j v \leq_k z$. As $u \leq_j v$, there exists $w \in Q_{i \wedge j \wedge k}$ such that $u \leq_{i \wedge j} w \leq_{j \wedge k} v$. Hence $x \leq_i w \leq_k z$, and so $x \leq z$. Therefore, \leq is a partial ordering on P .

Finally, let $i \in \Lambda$ and let $x, y \in Q_i$. If $x \leq_i y$, then $x \leq y$ trivially. Conversely, if $x \leq y$, then, by Lemma 4.2, there exists $z \in Q_i$ such that $x \leq_i z \leq_i y$, whence $x \leq_i y$. Therefore, $x \leq y$ iff $x \leq_i y$. \square

Hence, from now on, we shall drop the index i in \leq_i , for $i \in \Lambda$. We shall call the poset P the *strong amalgam* of $\langle Q_i \mid i \in \Lambda \rangle$.

Lemma 4.4. *Let $\sigma = \langle Q_i \mid i \in \Lambda \rangle$ be a normal diagram of posets, with strong amalgam $P = \bigcup(Q_i \mid i \in \Lambda)$. Then the following statements hold:*

- (i) *If $i \leq j$ implies that $Q_i \leq_{\text{rc}} Q_j$ for all $i, j \in \Lambda$, then $Q_i \leq_{\text{rc}} P$ for all $i \in \Lambda$.*
- (ii) *If $i \leq j$ implies that $Q_i \leq_{\text{int}} Q_j$ for all $i, j \in \Lambda$, then $Q_i \leq_{\text{int}} P$ for all $i \in \Lambda$.*

Proof. (i). Let $x \in P$, say $x \in Q_j$, for $j \in \Lambda$, and let $i \in \Lambda$. By Lemma 4.2, every element of Q_i below x lies below $x_{Q_{i \wedge j}}$; hence x_{Q_i} exists, and it is equal to $x_{Q_{i \wedge j}}$. Dually, x^{Q_i} exists, and it is equal to $x^{Q_{i \wedge j}}$. In particular, $Q_i \leq_{\text{rc}} P$.

To ease notation, we shall from now on use the abbreviations $x_{(i)} = x_{Q_i}$ and $x^{(i)} = x^{Q_i}$, for all $x \in P$ and all $i \in \Lambda$. Similarly, we shall abbreviate $x \equiv_{Q_i} y$ by $x \equiv_i y$ and $x \ll_{Q_i} y$ by $x \ll_i y$, for all $x, y \in P$ and all $i \in \Lambda$.

(ii). First, it follows from (i) above that $Q_i \leq_{\text{rc}} P$. Now let $x, y \in P$ such that $x \leq y$, we prove that either $x \equiv_i y$ or $x \ll_i y$. Fix $j, k \in \Lambda$ such that $x \in Q_j$ and $y \in Q_k$. Suppose first that $j = k$. As $Q_{i \wedge j} \leq_{\text{int}} Q_j$, either $x \equiv_{i \wedge j} y$ or $x \ll_{i \wedge j} y$. As $t_{(i)} = t_{(i \wedge j)}$ and $t^{(i)} = t^{(i \wedge j)}$ for all $t \in \{x, y\}$ (see proof of (i) above), this amounts to saying that either $x \equiv_i y$ or $x \ll_i y$, so we are done.

In the general case, there exists, by Lemma 4.2, $z \in Q_{j \wedge k}$ such that $x \leq_j z \leq_k y$. Applying the paragraph above to the pairs $\langle x, z \rangle$ and $\langle z, y \rangle$, we obtain that either $x \equiv_i z$ or $x \ll_i z$, and either $z \equiv_i y$ or $z \ll_i y$. If $x \equiv_i z$ and $z \equiv_i y$, then $x \equiv_i y$. In all other three cases, $x \ll_i y$. \square

Proposition 4.5. *Let $\sigma = \langle Q_i \mid i \in \Lambda \rangle$ be a normal interval diagram of lattices. Then the following statements hold:*

- (i) *The strong amalgam $P = \bigcup(Q_i \mid i \in \Lambda)$ is a lattice.*
- (ii) *Q_i is a sublattice of P , for all $i \in \Lambda$.*
- (iii) *For all $i, j \in \Lambda$ and all incomparable $a \in Q_i$ and $b \in Q_j$, both $a \vee b$ and $a \wedge b$ belong to $Q_{i \wedge j}$.*

Proof. We denote by \vee_k (resp., \wedge_k) the join (resp., meet) operation in Q_k , for all $k \in \Lambda$. We first establish a claim.

Claim. *Let $i, j, k \in \Lambda$ with $i, j \leq k$ and let $\langle x, y \rangle \in Q_i \times Q_j$. If $x \parallel y$, then both $x \vee_k y$ and $x \wedge_k y$ belong to $Q_{i \wedge j}$.*

Proof of Claim. If $x \equiv_i y$, then, as $x \in Q_i$, we obtain that $x = y$, which contradicts the assumption that $x \parallel y$; hence $x \not\equiv_i y$. As $Q_i \leq_{\text{int}} Q_k$, it follows from Lemma 2.5 that $x \vee_k y = x^{(i)} \vee_k y^{(i)}$, and thus, as Q_i is a sublattice of Q_k , $x \vee_k y \in Q_i$. Similarly, $x \wedge_k y \in Q_j$, and hence $x \vee_k y \in Q_{i \wedge j}$. The proof for the meet is dual. \square Claim.

Now we establish (iii). We give the proof for the meet; the proof for the join is dual. Suppose that $a \parallel b$, let $i, j \in \Lambda$ such that $a \in Q_i$ and $b \in Q_j$, and put $c = a_{(i \wedge j)} \wedge_{i \wedge j} b_{(i \wedge j)}$. Of course, $c \leq a, b$. Now let $x \in P$ such that $x \leq a, b$, we prove that $x \leq c$. Pick $k \in \Lambda$ such that $x \in Q_k$ and set $m = i \wedge j \wedge k$. By Lemma 4.2, there are $a' \in Q_{i \wedge k}$ and $b' \in Q_{j \wedge k}$ such that $x \leq a' \leq a$ and $x \leq b' \leq b$. Suppose first that $a' \parallel b'$. It follows from the Claim above that $a' \wedge_k b'$ belongs to Q_m , thus to $Q_{i \wedge j}$. As $x \leq a' \wedge_k b' \leq a, b$, we obtain that $x \leq a' \wedge_k b' \leq c$.

Suppose now that $a' \sim b'$, say $a' \leq b'$. By Lemma 4.2, there exists $a'' \in Q_m$ such that $a' \leq a'' \leq b'$. If $a'' \leq a$, then (as $a'' \leq b$ and $a'' \in Q_{i \wedge j}$) $a'' \leq c$, and so $x \leq c$. As $a \not\leq a''$ (for $a \not\leq b$), the only possibility left is $a \parallel a''$. By the Claim above, $a \wedge_i a''$

belongs to Q_m , but this element lies below both a and b , thus, again, below c . As $x \leq a' \leq a \wedge_i a''$, we thus obtain that $x \leq c$.

Hence c is the meet of $\{a, b\}$ in P , and so P is a meet-semilattice. Dually, P is a join-semilattice; this establishes (i). In case $a, b \in Q_i$, we take $i = j$, and thus $c = a \wedge_i b$, and so we obtain that Q_i is a meet-subsemilattice of P . Dually, Q_i is a join-subsemilattice of P ; this establishes (ii). \square

5. LOWER FINITE NORMAL INTERVAL DIAGRAMS; THE ELEMENTS x_\bullet AND x^\bullet

For a normal diagram $\sigma = \langle Q_i \mid i \in \Lambda \rangle$ of posets, it follows from Definition 4.1(2) that for every element x of $P = \bigcup(Q_i \mid i \in \Lambda)$, the set $\{i \in \Lambda \mid x \in Q_i\}$ is closed under finite meets. In particular, in case Λ is lower finite (we shall say that the diagram σ is *lower finite*), there exists a least $i \in \Lambda$ such that $x \in Q_i$. We shall denote this element by $\nu(x)$, and we shall call the map $\nu: P \rightarrow \Lambda$ the *valuation* associated with the normal diagram σ .

In this section, we shall fix a lower finite normal interval diagram $\sigma = \langle Q_i \mid i \in \Lambda \rangle$ of lattices, with strong amalgam $P = \bigcup(Q_i \mid i \in \Lambda)$. It follows from Proposition 4.5 that P is a lattice.

Lemma 5.1. *For all $x \in P \setminus Q_0$, there exist a largest $x_\bullet < x$ such that $\nu(x_\bullet) < \nu(x)$ and a least $x^\bullet > x$ such that $\nu(x^\bullet) < \nu(x)$. Furthermore, the following hold:*

- (i) $\nu(x_\bullet)$ and $\nu(x^\bullet)$ are comparable.
- (ii) Putting $i = \max\{\nu(x_\bullet), \nu(x^\bullet)\}$, both equalities $x_\bullet = x_{(i)}$ and $x^\bullet = x^{(i)}$ hold.
- (iii) For all $y \in P$ such that $\nu(x) \not\leq \nu(y)$, $x \leq y$ implies that $x^\bullet \leq y$, and $y \leq x$ implies that $y \leq x_\bullet$.

Proof. Put $\Lambda' = \{i \in \Lambda \mid i < \nu(x)\}$ and $X = \{x_{(i)} \mid i \in \Lambda'\}$. As $x \notin Q_0$, the set X is nonempty. As X is finite (because Λ' is finite), it has a join in P , say x_\bullet . It follows easily from Proposition 4.5 that $\nu(x_\bullet) < \nu(x)$, whence x_\bullet is the largest element of X . The proof of the existence of x^\bullet is similar.

As $x_\bullet \leq x^\bullet$, there exists $y \in Q_{\nu(x_\bullet) \wedge \nu(x^\bullet)}$ such that $x_\bullet \leq y \leq x^\bullet$. If $x \leq y$, then, as $\nu(y) < \nu(x)$, we get $y = x^\bullet$, and thus $\nu(x^\bullet) \leq \nu(x_\bullet)$. Similarly, if $y \leq x$, then $y = x_\bullet$, and thus $\nu(x_\bullet) \leq \nu(x^\bullet)$. Now suppose that $x \parallel y$. It follows from Proposition 4.5 that $\nu(x \wedge y), \nu(x \vee y) \leq \nu(y) < \nu(x)$, thus, as $x \vee y \leq x^\bullet$ and $x_\bullet \leq x \wedge y$, we get $x \vee y = x^\bullet$ and $x \wedge y = x_\bullet$. By using the first equality, we get $\nu(x^\bullet) = \nu(x \vee y) \leq \nu(y) \leq \nu(x_\bullet)$, while by using the second one, we get $\nu(x_\bullet) \leq \nu(x^\bullet)$, and hence $\nu(x_\bullet) = \nu(x^\bullet)$. This takes care of (i).

Now we deal with (ii). From $\nu(x_{(i)}), \nu(x^{(i)}) \leq i < \nu(x)$ it follows that $x_{(i)} \leq x_\bullet$ and $x^\bullet \leq x^{(i)}$. As both x_\bullet and x^\bullet belong to Q_i , we get $x_{(i)} = x_\bullet$ and $x^{(i)} = x^\bullet$.

Let $y \in P$ with $\nu(x) \not\leq \nu(y)$. If $x \leq y$, then there exists $z \in Q_{\nu(x) \wedge \nu(y)}$ such that $x \leq z \leq y$, but $\nu(z) < \nu(x)$, thus $x^\bullet \leq z$, and so $x^\bullet \leq y$. The proof for x_\bullet is dual. This takes care of (iii). \square

6. EXTENDING A P-MEASURE TO AN INTERVAL EXTENSION

Definition 6.1. Let S be a $\langle \vee, 0 \rangle$ -semilattice and let P be a poset. A map $\mu: P \times P \rightarrow S$ is a *S-valued p-measure on P*, if $\mu(x, z) \leq \mu(x, y) \vee \mu(y, z)$ and $x \leq y$ implies $\mu(x, y) = 0$, for all $x, y, z \in P$. The pair $\langle P, \mu \rangle$ is a *S-valued p-measured poset*.

The inequality $\mu(x, z) \leq \mu(x, y) \vee \mu(y, z)$ will be referred to as the *triangular inequality*.

Notation 6.2. We shall always denote by P, Q, \dots , the underlying posets of p-measured posets $\mathbf{P}, \mathbf{Q}, \dots$. For a p-measured poset $\mathbf{P} = \langle P, \mu \rangle$, we shall use the notation $\|x \leq y\|_{\mathbf{P}} = \mu(x, y)$, for $x, y \in P$. Elements of the form $\|x \leq y\|_{\mathbf{P}}$ will be called *Boolean values*.

We shall define the *distance function* on a p-measured poset \mathbf{P} by

$$\|x = y\|_{\mathbf{P}} = \|\max\{x, y\} \leq \min\{x, y\}\|_{\mathbf{P}}, \quad \text{for comparable } x, y \in P.$$

Obviously, the distance function on P satisfies the *triangular inequality* $\|x = z\|_{\mathbf{P}} \leq \|x = y\|_{\mathbf{P}} \vee \|y = z\|_{\mathbf{P}}$, for all pairwise comparable $x, y, z \in P$. Furthermore, the equality holds for $x \geq y \geq z$.

For $\langle \vee, 0 \rangle$ -semilattices S and T and a $\langle \vee, 0 \rangle$ -homomorphism $\varphi: S \rightarrow T$, a S -valued p-measured poset \mathbf{P} , and a T -valued p-measured poset \mathbf{Q} , we shall say that \mathbf{Q} *extends* \mathbf{P} with respect to φ , if P is a sub-poset of Q and

$$\|x \leq y\|_{\mathbf{Q}} = \varphi(\|x \leq y\|_{\mathbf{P}}), \quad \text{for all } x, y \in P.$$

We shall then say that the inclusion map from P into Q , together with φ , form a *morphism* from \mathbf{P} to \mathbf{Q} , and define diagrams of p-measured posets accordingly. (Obviously, we could have defined morphisms more generally by involving an order-embedding from P into Q , but the present definition is sufficient, and more convenient, for our purposes.)

Until Lemma 6.10, we fix a distributive lattice D with zero and a D -valued p-measured poset \mathbf{P} . We are given an interval extension Q of P in which each interval of Q of the form $[x_P, x^P]$, for $x \in Q$, is endowed with a p-measure $\|_ - \leq - \|_{[x_P, x^P]}$. We assume *compatibility* between those p-measures, in the sense that $\|x^P = x_P\|_{\mathbf{P}} = \|x^P = x_P\|_{[x_P, x^P]}$, for all $x \in Q$. We define a map $\|_ - \leq - \|_{\mathbf{Q}}$ from $Q \times Q$ to D , by setting $\|x \leq y\|_{\mathbf{Q}} = \|x \leq y\|_{[x_P, x^P]}$ in case $x \equiv_P y$, and

$$\begin{aligned} \|x \leq y\|_{\mathbf{Q}} = & \|x^P \leq y_P\|_{\mathbf{P}} \wedge (\|x_P \leq y_P\|_{\mathbf{P}} \vee \|x = x_P\|_{[x_P, x^P]}) \\ & \wedge (\|x^P \leq y^P\|_{\mathbf{P}} \vee \|y^P = y\|_{[y_P, y^P]}) \\ & \wedge (\|x_P \leq y^P\|_{\mathbf{P}} \vee \|x = x_P\|_{[x_P, x^P]} \vee \|y^P = y\|_{[y_P, y^P]}), \end{aligned} \tag{6.1}$$

if $x \not\equiv_P y$.

The proof of the following lemma is straightforward.

Lemma 6.3. *The new map $\|_ - \leq - \|$ extends the original one $\|_ - \leq - \|_{\mathbf{P}}$, and also all maps of the form $\|_ - \leq - \|_{[x_P, x^P]}$, for $x \in Q$. Furthermore, for all $x, y \in Q$, the following statements hold:*

- (i) $x \in P$ implies that $\|x \leq y\|_{\mathbf{Q}} = \|x \leq y_P\|_{\mathbf{P}} \wedge (\|x \leq y^P\|_{\mathbf{P}} \vee \|y^P = y\|_{[y_P, y^P]})$;
- (ii) $y \in P$ implies that $\|x \leq y\|_{\mathbf{Q}} = \|x^P \leq y\|_{\mathbf{P}} \wedge (\|x_P \leq y\|_{\mathbf{P}} \vee \|x = x_P\|_{[x_P, x^P]})$.

Lemma 6.4. $x \leq y$ implies that $\|x \leq y\|_{\mathbf{Q}} = 0$, for all $x, y \in Q$.

Proof. If $x \equiv_P y$, then $\|x \leq y\|_{\mathbf{Q}} = \|x \leq y\|_{[x_P, x^P]} = 0$. If $x \not\equiv_P y$, then, as $P \leq_{\text{int}} Q$, we get $x^P \leq y_P$, thus $\|x^P \leq y_P\|_{\mathbf{P}} = 0$, and so $\|x \leq y\|_{\mathbf{Q}} = 0$. \square

Lemma 6.5. *The inequality $\|x \leq z\|_{\mathbf{Q}} \leq \|x \leq y\|_{\mathbf{Q}} \vee \|y \leq z\|_{\mathbf{Q}}$ holds, for all $x, y, z \in Q$ two of which belong to P .*

Proof. Suppose first that $x, y \in P$. By applying Lemma 6.3 to $\|x \leq z\|_{\mathbf{Q}}$ and $\|y \leq z\|_{\mathbf{Q}}$, we reduce the problem to the two inequalities

$$\begin{aligned} \|x \leq z_P\|_{\mathbf{P}} &\leq \|x \leq y\|_{\mathbf{P}} \wedge \|y \leq z_P\|_{\mathbf{P}}, \\ \|x \leq z^P\|_{\mathbf{P}} &\leq \|x \leq y\|_{\mathbf{P}} \wedge \|y \leq z^P\|_{\mathbf{P}}, \end{aligned}$$

which hold by assumption. The proof is dual for the case $y, z \in P$.

Suppose now that $x, z \in P$. By applying Lemma 6.3 to $\|x \leq y\|_{\mathbf{Q}}$ and $\|y \leq z\|_{\mathbf{Q}}$, we reduce the problem to four inequalities, which we proceed to verify:

$$\begin{aligned} \|x \leq z\|_{\mathbf{P}} &\leq \|x \leq y_P\|_{\mathbf{P}} \vee \|y_P \leq z\|_{\mathbf{P}} \\ &\leq \|x \leq y_P\|_{\mathbf{P}} \vee \|y^P \leq z\|_{\mathbf{P}}. \end{aligned}$$

$$\begin{aligned} \|x \leq z\|_{\mathbf{P}} &\leq \|x \leq y_P\|_{\mathbf{P}} \vee \|y_P \leq z\|_{\mathbf{P}} \\ &\leq \|x \leq y_P\|_{\mathbf{P}} \vee \|y_P \leq z\|_{\mathbf{P}} \vee \|y = y_P\|_{[y_P, y^P]}. \end{aligned}$$

$$\begin{aligned} \|x \leq z\|_{\mathbf{P}} &\leq \|x \leq y^P\|_{\mathbf{P}} \vee \|y^P \leq z\|_{\mathbf{P}} \\ &\leq \|x \leq y^P\|_{\mathbf{P}} \vee \|y^P \leq z\|_{\mathbf{P}} \vee \|y^P = y\|_{[y_P, y^P]}. \end{aligned}$$

$$\begin{aligned} \|x \leq z\|_{\mathbf{P}} &\leq \|x \leq y^P\|_{\mathbf{P}} \vee \|y^P = y_P\|_{\mathbf{P}} \vee \|y_P \leq z\|_{\mathbf{P}} \\ &= \|x \leq y^P\|_{\mathbf{P}} \vee \|y^P = y\|_{[y_P, y^P]} \vee \|y = y_P\|_{[y_P, y^P]} \vee \|y_P \leq z\|_{\mathbf{P}}. \end{aligned}$$

Hence $\|x \leq z\|_{\mathbf{P}} \leq \|x \leq y\|_{\mathbf{Q}} \vee \|y \leq z\|_{\mathbf{Q}}$. \square

Lemma 6.6. *The Boolean value $\|x \leq y\|_{\mathbf{Q}}$ lies below each of the semilattice elements $\|x^P \leq y_P\|_{\mathbf{P}}$, $\|x_P \leq y_P\|_{\mathbf{P}} \vee \|x = x_P\|_{[x_P, x^P]}$, $\|x^P \leq y^P\|_{\mathbf{P}} \vee \|y^P = y\|_{[y_P, y^P]}$, and $\|x_P \leq y^P\|_{\mathbf{P}} \vee \|x = x_P\|_{[x_P, x^P]} \vee \|y^P = y\|_{[y_P, y^P]}$, for all $x, y \in Q$.*

Proof. This is obvious by the definition of $\|x \leq y\|_{\mathbf{Q}}$ in case $x \not\equiv_P y$. If $x \equiv_P y$, then, putting $u = x_P = y_P$ and $v = x^P = y^P$, the four semilattice elements in the statement above are respectively equal to $\|v = u\|_{\mathbf{P}}$, $\|x = u\|_{[u, v]}$, $\|v = y\|_{[u, v]}$, and $\|x = u\|_{[u, v]} \vee \|v = y\|_{[u, v]}$. As $\|v = u\|_{\mathbf{P}} = \|v = u\|_{[u, v]}$, we need to prove that $\|x \leq y\|_{[u, v]}$ lies below both $\|x = u\|_{[u, v]}$ and $\|v = y\|_{[u, v]}$, which is obvious (for example, $\|x \leq y\|_{[u, v]} \leq \|x \leq u\|_{[u, v]} \vee \|u \leq y\|_{[u, v]} = \|x = u\|_{[u, v]}$). \square

Lemma 6.7. *The inequalities $\|x_P \leq y\|_{\mathbf{Q}} \leq \|x \leq y\|_{\mathbf{Q}} \leq \|x^P \leq y\|_{\mathbf{Q}}$ and $\|x \leq y^P\|_{\mathbf{Q}} \leq \|x \leq y\|_{\mathbf{Q}} \leq \|x \leq y_P\|_{\mathbf{Q}}$ hold, for all $x, y \in Q$.*

Proof. As the two sets of inequalities are dual, it suffices to prove that $\|x_P \leq y\|_{\mathbf{Q}} \leq \|x \leq y\|_{\mathbf{Q}} \leq \|x^P \leq y\|_{\mathbf{Q}}$. As the conclusion is obvious in case $x \equiv_P y$, it suffices to consider the case where $x \not\equiv_P y$. As, by Lemma 6.3, the equality

$$\|x^P \leq y\|_{\mathbf{Q}} = \|x^P \leq y_P\|_{\mathbf{P}} \wedge (\|x^P \leq y^P\|_{\mathbf{P}} \vee \|y^P = y\|_{[y_P, y^P]})$$

holds, it follows from Lemma 6.6 that $\|x \leq y\|_{\mathbf{Q}} \leq \|x^P \leq y\|_{\mathbf{Q}}$. Moreover, again by Lemma 6.3, the equality

$$\|x_P \leq y\|_{\mathbf{Q}} = \|x_P \leq y_P\|_{\mathbf{P}} \wedge (\|x_P \leq y^P\|_{\mathbf{P}} \vee \|y^P = y\|_{[y_P, y^P]})$$

holds, and so $\|x_P \leq y\|_{\mathbf{Q}}$ lies below each of the four meetands defining $\|x \leq y\|_{\mathbf{Q}}$ on the right hand side of (6.1), and hence $\|x_P \leq y\|_{\mathbf{Q}} \leq \|x \leq y\|_{\mathbf{Q}}$. \square

Lemma 6.8. *The inequalities $\|x \leq y\|_{\mathbf{Q}} \leq \|x_P \leq y\|_{\mathbf{Q}} \vee \|x = x_P\|_{[x_P, x^P]}$ and $\|x \leq y\|_{\mathbf{Q}} \leq \|x \leq y^P\|_{\mathbf{Q}} \vee \|y^P = y\|_{[y_P, y^P]}$ hold, for all $x, y \in Q$.*

Proof. By symmetry, it suffices to prove the first inequality. Using the expression of $\|x_P \leq y\|_{\mathbf{Q}}$ given by Lemma 6.3(i), we reduce the problem to the following two inequalities,

$$\begin{aligned} \|x \leq y\|_{\mathbf{Q}} &\leq \|x_P \leq y_P\|_{\mathbf{P}} \vee \|x = x_P\|_{[x_P, x^P]}, \\ \|x \leq y\|_{\mathbf{Q}} &\leq \|x_P \leq y^P\|_{\mathbf{P}} \vee \|x = x_P\|_{[x_P, x^P]} \vee \|y^P = y\|_{[y_P, y^P]}, \end{aligned}$$

that follow immediately from Lemma 6.6. \square

Lemma 6.9. *The inequalities $\|x^P \leq y\|_{\mathbf{Q}} \leq \|x^P = x\|_{[x_P, x^P]} \vee \|x \leq y\|_{\mathbf{Q}}$ and $\|x \leq y_P\|_{\mathbf{Q}} \leq \|y = y_P\|_{[y_P, y^P]} \vee \|x \leq y\|_{\mathbf{Q}}$ hold, for all $x, y \in Q$.*

Proof. By symmetry, it suffices to prove the first inequality. Suppose first that $x \equiv_P y$, put $u = x_P = y_P$ and $v = x^P = y^P$. We need to prove that $\|v = y\|_{\mathbf{Q}} \leq \|v = x\|_{\mathbf{Q}} \vee \|x \leq y\|_{[u, v]}$, which is obvious since $\|v = y\|_{\mathbf{Q}} = \|v = y\|_{[u, v]}$ and $\|v = x\|_{\mathbf{Q}} = \|v = x\|_{[u, v]}$.

Now suppose that $x \not\equiv_P y$. As in (6.1), $\|x \leq y\|_{\mathbf{Q}}$ is the meet of four meetands, so the first inequality reduces to four inequalities, which we proceed to prove:

$$\begin{aligned} \|x^P \leq y\|_{\mathbf{Q}} &\leq \|x^P \leq y_P\|_{\mathbf{P}} && \text{(by Lemma 6.6)} \\ &\leq \|x^P = x\|_{\mathbf{Q}} \vee \|x \leq y_P\|_{\mathbf{Q}} && \text{(by Lemma 6.5)} \\ &\leq \|x^P = x\|_{[x_P, x^P]} \vee \|x^P \leq y_P\|_{\mathbf{P}} && \text{(by Lemma 6.7)}. \end{aligned}$$

(We have used the easy observation that $\|x^P = x\|_{\mathbf{Q}} = \|x^P = x\|_{[x_P, x^P]}$.)

$$\begin{aligned} \|x^P \leq y\|_{\mathbf{Q}} &\leq \|x^P \leq y_P\|_{\mathbf{P}} && \text{(by Lemma 6.7)} \\ &\leq \|x^P = x_P\|_{\mathbf{P}} \vee \|x_P \leq y_P\|_{\mathbf{P}} \\ &= \|x^P = x\|_{[x_P, x^P]} \vee \|x_P \leq y_P\|_{\mathbf{P}} \vee \|x = x_P\|_{[x_P, x^P]}. \end{aligned}$$

$$\begin{aligned} \|x^P \leq y\|_{\mathbf{Q}} &\leq \|x^P \leq y^P\|_{\mathbf{P}} \vee \|y^P = y\|_{\mathbf{Q}} && \text{(by Lemma 6.5)} \\ &\leq \|x^P = x\|_{[x_P, x^P]} \vee \|x^P \leq y^P\|_{\mathbf{P}} \vee \|y^P = y\|_{[y_P, y^P]}. \end{aligned}$$

$$\begin{aligned} \|x^P \leq y\|_{\mathbf{Q}} &\leq \|x^P \leq y^P\|_{\mathbf{P}} \vee \|y^P = y\|_{\mathbf{Q}} && \text{(by Lemma 6.5)} \\ &\leq \|x^P = x_P\|_{\mathbf{P}} \vee \|x_P \leq y^P\|_{\mathbf{P}} \vee \|y^P = y\|_{[y_P, y^P]} \\ &= \|x^P = x\|_{[x_P, x^P]} \vee \|x_P \leq y^P\|_{\mathbf{P}} \vee \|x = x_P\|_{[x_P, x^P]} \vee \|y^P = y\|_{[y_P, y^P]}, \end{aligned}$$

which completes the proof of the inequality

$$\|x^P \leq y\|_{\mathbf{Q}} \leq \|x^P = x\|_{[x_P, x^P]} \vee \|x \leq y\|_{\mathbf{Q}}.$$

The proof of the inequality $\|x \leq y_P\|_{\mathbf{Q}} \leq \|y = y_P\|_{[y_P, y^P]} \vee \|x \leq y\|_{\mathbf{Q}}$ is dual. \square

Lemma 6.10. *The inequality $\|x \leq z\|_{\mathbf{Q}} \leq \|x \leq y\|_{\mathbf{Q}} \vee \|y \leq z\|_{\mathbf{Q}}$ holds, for all $x, y, z \in Q$.*

Proof. This is obvious in case $x \equiv_P y \equiv_P z$, as $\|_ - \leq -\|_{[x_P, x^P]}$ is a p-measure. So suppose that either $x \not\equiv_P y$ or $y \not\equiv_P z$, say $x \not\equiv_P y$. Expressing the Boolean value

$\|x \leq y\|_{\mathbf{Q}}$ as in (6.1), we reduce the problem to four inequalities, that we proceed to prove:

$$\begin{aligned} \|x \leq z\|_{\mathbf{Q}} &\leq \|x^P \leq z\|_{\mathbf{Q}} && \text{(by Lemma 6.7)} \\ &\leq \|x^P \leq y_P\|_{\mathbf{P}} \vee \|y_P \leq z\|_{\mathbf{Q}} && \text{(by Lemma 6.5)} \\ &\leq \|x^P \leq y_P\|_{\mathbf{P}} \vee \|y \leq z\|_{\mathbf{Q}} && \text{(by Lemma 6.7)}. \end{aligned}$$

$$\begin{aligned} \|x \leq z\|_{\mathbf{Q}} &\leq \|x_P \leq z\|_{\mathbf{Q}} \vee \|x = x_P\|_{[x_P, x^P]} && \text{(by Lemma 6.8)} \\ &\leq \|x_P \leq y_P\|_{\mathbf{P}} \vee \|y_P \leq z\|_{\mathbf{Q}} \vee \|x = x_P\|_{[x_P, x^P]} && \text{(by Lemma 6.5)} \\ &\leq \|x_P \leq y_P\|_{\mathbf{P}} \vee \|x = x_P\|_{[x_P, x^P]} \vee \|y \leq z\|_{\mathbf{Q}} && \text{(by Lemma 6.7)}. \end{aligned}$$

$$\begin{aligned} \|x \leq z\|_{\mathbf{Q}} &\leq \|x^P \leq z\|_{\mathbf{Q}} && \text{(by Lemma 6.7)} \\ &\leq \|x^P \leq y^P\|_{\mathbf{P}} \vee \|y^P \leq z\|_{\mathbf{Q}} && \text{(by Lemma 6.5)} \\ &\leq \|x^P \leq y^P\|_{\mathbf{P}} \vee \|y^P = y\|_{[y_P, y^P]} \vee \|y \leq z\|_{\mathbf{Q}} && \text{(by Lemma 6.9)}. \end{aligned}$$

$$\begin{aligned} \|x \leq z\|_{\mathbf{Q}} &\leq \|x_P \leq z\|_{\mathbf{Q}} \vee \|x = x_P\|_{[x_P, x^P]} && \text{(by Lemma 6.8)} \\ &\leq \|x_P \leq y^P\|_{\mathbf{P}} \vee \|y^P \leq z\|_{\mathbf{P}} \vee \|x = x_P\|_{[x_P, x^P]} && \text{(by Lemma 6.5)} \\ &\leq \|x_P \leq y^P\|_{\mathbf{P}} \vee \|y^P = y\|_{[y_P, y^P]} \vee \|x = x_P\|_{[x_P, x^P]} \vee \|y \leq z\|_{\mathbf{Q}} && \text{(by Lemma 6.9)}. \end{aligned}$$

This completes the proof. \square

So we have reached the following result.

Proposition 6.11. *Let D be a distributive lattice with zero, let \mathbf{P} be a D -valued p -measured poset, and let Q be an interval extension of P in which each interval of Q of the form $[x_P, x^P]$, for $x \in Q$, is endowed with a p -measure $\|_ - \leq -\|_{[x_P, x^P]}$ such that $\|x^P = x_P\|_{\mathbf{P}} = \|x^P = x_P\|_{[x_P, x^P]}$, for all $x \in Q$. Then there exists a common extension of all p -measures $\|_ - \leq -\|_{\mathbf{P}}$ and $\|_ - \leq -\|_{[x_P, x^P]}$, for $x \in Q$, to a p -measure on Q , given by (6.1) on pairs $\langle x, y \rangle$ such that $x \not\equiv_P y$.*

7. DOUBLING EXTENSIONS; THE CONDITIONS (DB1) AND (DB2)

For a poset Λ and a Λ -indexed diagram $\mathcal{S} = \langle S_i, \varphi_{i,j} \mid i \leq j \text{ in } \Lambda \rangle$ of $\langle \vee, 0 \rangle$ -semilattices and $\langle \vee, 0 \rangle$ -homomorphisms, we shall say that a Λ -indexed diagram $\langle \mathbf{Q}_i \mid i \in \Lambda \rangle$ of p -measured posets is \mathcal{S} -valued, if \mathbf{Q}_i is S_i -valued and \mathbf{Q}_j extends \mathbf{Q}_i with respect to $\varphi_{i,j}$, for all $i \leq j$ in Λ .

We shall also use the convention of notation and terminology that consists of extending to p -measured posets the notions defined for posets, by restricting them to the underlying posets and stating that the poset extensions involved preserve the corresponding p -measures. For example, we say that a p -measured poset \mathbf{Q} is an *interval extension* of a p -measured poset \mathbf{P} , in notation $\mathbf{P} \leq_{\text{int}} \mathbf{Q}$, if \mathbf{Q} extends \mathbf{P} and the underlying posets (see Notation 6.2) satisfy $P \leq_{\text{int}} Q$. In particular, a normal interval diagram of p -measured lattices is a diagram of p -measured lattices whose underlying posets form a normal interval diagram.

Definition 7.1. Let \mathbf{P} and \mathbf{Q} be p -measured posets such that $\mathbf{P} \leq_{\text{rc}} \mathbf{Q}$. We say that \mathbf{Q} is a *doubling extension* of \mathbf{P} , in notation $\mathbf{P} \leq_{\text{db}} \mathbf{Q}$, if $\|x = x_P\| \sim \|x^P = x\|$,

for all $x \in Q$. Equivalently, either $\|x^P = x_P\| = \|x = x_P\|$ or $\|x^P = x_P\| = \|x^P = x\|$, for all $x \in Q$.

The following lemma shows that under mild assumptions, doubling extensions are transitive.

Lemma 7.2. *Let \mathbf{P} , \mathbf{Q} , and \mathbf{R} be p -measured posets. If $P \leq_{\text{rc}} Q \leq_{\text{rc}} R$, $P \leq_{\text{int}} R$, and $\mathbf{P} \leq_{\text{db}} \mathbf{Q} \leq_{\text{db}} \mathbf{R}$, then $\mathbf{P} \leq_{\text{db}} \mathbf{R}$.*

Proof. Let $x \in R$, we prove $\|x = x_P\| \sim \|x^P = x\|$. As $\mathbf{Q} \leq_{\text{db}} \mathbf{R}$, we get $\|x = x_Q\| \sim \|x^Q = x\|$. Hence, if $\{x_Q, x^Q\} \subseteq P$, then $x_P = x_Q$ and $x^P = x^Q$, thus we are done. Suppose that $\{x_Q, x^Q\} \not\subseteq P$, say $x^Q \notin P$. As $P \leq_{\text{int}} R$ and $x \sim x^Q$, it follows from Lemma 2.6 that $x \sim (x^Q)_P$. If $x \leq (x^Q)_P$, then, as $(x^Q)_P \leq x^Q$ and $(x^Q)_P$ belongs to P , we get $x^Q = (x^Q)_P \in P$, a contradiction; hence $(x^Q)_P \leq x$. As $x_P \leq (x^Q)_P$ and $(x^Q)_P \in P$, we get $(x^Q)_P = x_P$. As $\mathbf{P} \leq_{\text{db}} \mathbf{Q}$, we get $\|x^P = x^Q\| \sim \|x^Q = x_P\|$. But this also holds trivially in case $x^Q \in P$, so it holds in every case. So we have proved the following:

$$\|x^P = x^Q\| \sim \|x^Q = x_P\|. \quad (7.1)$$

The dual argument gives

$$\|x_Q = x_P\| \sim \|x^P = x_Q\|. \quad (7.2)$$

If $\|x^P = x_Q\| \leq \|x_Q = x_P\|$, then we get $\|x_Q = x_P\| = \|x^P = x_P\|$, and thus $\|x = x_P\| = \|x^P = x_P\|$, and we are done. Dually, the same conclusion follows from $\|x^Q = x_P\| \leq \|x^P = x^Q\|$.

By (7.1) and (7.2), it remains to consider the case where both inequalities $\|x_Q = x_P\| \leq \|x^P = x_Q\|$ and $\|x^P = x^Q\| \leq \|x^Q = x_P\|$ hold, in which case

$$\|x^Q = x_P\| = \|x^P = x_Q\| = \|x^P = x_P\|. \quad (7.3)$$

From $\mathbf{Q} \leq_{\text{db}} \mathbf{R}$ it follows that $\|x^Q = x\| \sim \|x = x_Q\|$. Suppose, for example, that $\|x = x_Q\| \leq \|x^Q = x\|$. Hence $\|x^Q = x\| = \|x^Q = x_Q\|$, and we get

$$\begin{aligned} \|x^P = x\| &= \|x^P = x^Q\| \vee \|x^Q = x\| \\ &= \|x^P = x^Q\| \vee \|x^Q = x_Q\| \\ &= \|x^P = x_Q\| \\ &= \|x^P = x_P\| \quad (\text{see (7.3)}) \\ &\geq \|x = x_P\|. \end{aligned} \quad \square$$

From now on until the end of this section, we shall fix a finite lattice Λ with largest element ℓ , a Λ -indexed diagram $\mathcal{D} = \langle D_i, \varphi_{i,j} \mid i \leq j \text{ in } \Lambda \rangle$ of distributive lattices with zero and $\langle \vee, 0 \rangle$ -homomorphisms, a $\mathcal{D}|_{<\ell}$ -valued normal interval diagram $\langle \mathbf{Q}_i \mid i < \ell \rangle$ of p -measured lattices. In addition, we assume that the following statements hold:

- (DB1) \mathbf{Q}_j is a doubling extension of \mathbf{Q}_i , for all $i \leq j < \ell$.
- (DB2) For all $i < \ell$ and all $x, y \in \mathbf{Q}_i$ with $\nu(x) \not\leq \nu(y)$, $\|x = x_\bullet\|_{\mathbf{Q}_i} = \|x^\bullet = x_\bullet\|_{\mathbf{Q}_i}$ implies that $\|x \leq y\|_{\mathbf{Q}_i} = \|x^\bullet \leq y\|_{\mathbf{Q}_i}$ and $\|x^\bullet = x\|_{\mathbf{Q}_i} = \|x^\bullet = x_\bullet\|_{\mathbf{Q}_i}$ implies that $\|y \leq x\|_{\mathbf{Q}_i} = \|y \leq x_\bullet\|_{\mathbf{Q}_i}$.

As usual, we denote by $P = \bigcup(Q_i \mid i < \ell)$ the strong amalgam of $\langle \mathbf{Q}_i \mid i < \ell \rangle$.

Remark 7.3. It suffices to verify (DB2) in case $x \parallel y$. Indeed, let $x, y \in Q_i$ such that $\nu(x) \not\leq \nu(y)$ and $\|x = x_\bullet\| = \|x^\bullet = x_\bullet\|$ (to ease the notation, we drop the indices Q_i). If $x \leq y$, then, by Lemma 5.1, $x^\bullet \leq y$, thus $\|x \leq y\| = \|x^\bullet \leq y\| = 0$. If $y \leq x$, then, again by Lemma 5.1, $y \leq x_\bullet$, and so

$$\begin{aligned} \|x \leq y\| &= \|x = x_\bullet\| \vee \|x_\bullet = y\| && \text{(because } y \leq x_\bullet \leq x\text{)} \\ &= \|x^\bullet = x_\bullet\| \vee \|x_\bullet = y\| && \text{(because } \|x^\bullet = x_\bullet\| = \|x = x_\bullet\|\text{)} \\ &= \|x^\bullet \leq y\|. \end{aligned}$$

The proof that $x \sim y$ and $\nu(x) \not\leq \nu(y)$ and $\|x^\bullet = x\| = \|x^\bullet = x_\bullet\|$ implies that $\|y \leq x\| = \|y \leq x_\bullet\|$ is dual.

Notation 7.4. We add a largest element, denoted by 1, to D_ℓ , and for all $x, y \in P$, we define an element $\llbracket x \leq y \rrbracket$ of D_ℓ as follows:

$$\llbracket x \leq y \rrbracket = \begin{cases} \varphi_{i,\ell}(\|x \leq y\|_{Q_i}), & \text{if } \nu(x) \vee \nu(y) \leq i < \ell, \\ 1, & \text{otherwise.} \end{cases} \quad (7.4)$$

It is obvious that the value of $\llbracket x \leq y \rrbracket$ defined in the first case is independent of the chosen i such that $\nu(x) \vee \nu(y) \leq i < \ell$. We also put

$$\llbracket x = y \rrbracket = \llbracket \max\{x, y\} \leq \min\{x, y\} \rrbracket, \quad \text{for all comparable } x, y \in P.$$

Lemma 7.5. *The elements $\llbracket x = x_{(i)} \rrbracket$ and $\llbracket x^{(i)} = x \rrbracket$ are comparable, for all $x \in P$ and all $i < \ell$. Furthermore, $\llbracket x^\bullet = x \rrbracket \sim \llbracket x = x_\bullet \rrbracket$, for all $x \in P \setminus Q_0$.*

Proof. Let $x \in P$. As $x \in Q_j$ for some $j < \ell$, we get $x_{(i)} = x_{(i \wedge j)}$ and $x^{(i)} = x^{(i \wedge j)}$ (see Lemma 4.4(i)). As $Q_{i \wedge j} \leq_{\text{db}} Q_j$, we get $\|x = x_{(i \wedge j)}\|_{Q_j} \sim \|x^{(i \wedge j)} = x\|_{Q_j}$, that is, $\|x = x_{(i)}\|_{Q_j} \sim \|x^{(i)} = x\|_{Q_j}$, and thus, applying $\varphi_{j,\ell}$, we obtain the relation $\llbracket x = x_{(i)} \rrbracket \sim \llbracket x^{(i)} = x \rrbracket$.

It follows from Lemma 5.1 that $\nu(x_\bullet)$ and $\nu(x^\bullet)$ are comparable and that, if i denotes their maximum, then $x_\bullet = x_{(i)}$ and $x^\bullet = x^{(i)}$. By applying the result of the previous paragraph, we obtain $\llbracket x^\bullet = x \rrbracket \sim \llbracket x = x_\bullet \rrbracket$. \square

Now we put

$$\begin{aligned} P^\oplus &= \{x \in P \setminus Q_0 \mid \llbracket x = x_\bullet \rrbracket = \llbracket x^\bullet = x_\bullet \rrbracket\}, \\ P^\ominus &= \{x \in P \setminus Q_0 \mid \llbracket x^\bullet = x \rrbracket = \llbracket x^\bullet = x_\bullet \rrbracket\}. \end{aligned}$$

If x belongs to $Q_i \setminus Q_0$, then $x^\bullet, x_\bullet \in Q_i$. Hence, both $\llbracket x = x_\bullet \rrbracket$ and $\llbracket x^\bullet = x \rrbracket$ are evaluated by the formula giving the case $\nu(x) \vee \nu(y) < \ell$ of (7.4). Therefore, it follows from Lemma 7.5 that $P \setminus Q_0 = P^\oplus \cup P^\ominus$.

8. STRONG AMALGAMS OF P-MEASURED POSETS; FROM $\llbracket x \leq y \rrbracket$ TO $\|x \leq y\|$

From now on until Lemma 8.12, we shall fix a finite lattice Λ with largest element ℓ , a Λ -indexed diagram $\mathcal{D} = \langle D_i, \varphi_{i,j} \mid i \leq j \text{ in } \Lambda \rangle$ of *finite* distributive lattices and $\langle \vee, 0 \rangle$ -homomorphisms, a $\mathcal{D}|_{<\ell}$ -valued normal interval diagram $\langle Q_i \mid i < \ell \rangle$ of p-measured lattices. Furthermore, we assume that the conditions (DB1) and (DB2) introduced in Section 7 are satisfied.

We denote by P the strong amalgam of $\langle Q_i \mid i < \ell \rangle$ and by $\rho(x)$ the height of $\nu(x)$ in Λ , for all $x \in P$.

Lemma 8.1. *For every positive integer n and all elements $x_0, x_1, \dots, x_n \in P$, $\nu(x_0) \vee \nu(x_n) < \ell$ implies that $\llbracket x_0 \leq x_n \rrbracket \leq \bigvee_{i < n} \llbracket x_i \leq x_{i+1} \rrbracket$.*

Proof. We argue by induction on the pair $\langle n, \sum_{k=0}^n \rho(x_k) \rangle$, ordered lexicographically. The conclusion is trivial for $n = 1$.

Now suppose that $n = 2$. If either $\nu(x_0) \vee \nu(x_1) = \ell$ or $\nu(x_1) \vee \nu(x_2) = \ell$, then the right hand side of the desired inequality is equal to 1 and we are done; so suppose that $\nu(x_0) \vee \nu(x_1), \nu(x_1) \vee \nu(x_2) < \ell$. If $\nu(x_1) \leq \nu(x_0) \vee \nu(x_2)$, then, putting $k = \nu(x_0) \vee \nu(x_2)$ (which is smaller than ℓ), all the Boolean values under consideration are images under $\varphi_{k,\ell}$ of the corresponding Boolean values in \mathbf{Q}_k , so the conclusion follows from the inequality $\llbracket x_0 \leq x_2 \rrbracket_{\mathbf{Q}_k} \leq \llbracket x_0 \leq x_1 \rrbracket_{\mathbf{Q}_k} \vee \llbracket x_1 \leq x_2 \rrbracket_{\mathbf{Q}_k}$ (we will often encounter this kind of reduction, and we will summarize it by “everything happens below level k ”). Now suppose that $\nu(x_1) \not\leq \nu(x_0) \vee \nu(x_2)$. In particular, $x_1 \notin Q_0$ and $\nu(x_1) \not\leq \nu(x_0), \nu(x_2)$. By Lemma 7.5, x_1 belongs to $P^\oplus \cup P^\ominus$. If $x_1 \in P^\oplus$, then, as $\nu(x_1) \vee \nu(x_2) < \ell$ and by (DB2), $\llbracket x_1 \leq x_2 \rrbracket = \llbracket (x_1)^\bullet \leq x_2 \rrbracket$, hence

$$\begin{aligned} \llbracket x_0 \leq x_2 \rrbracket &\leq \llbracket x_0 \leq (x_1)^\bullet \rrbracket \vee \llbracket (x_1)^\bullet \leq x_2 \rrbracket && \text{(by the induction hypothesis)} \\ &\leq \llbracket x_0 \leq x_1 \rrbracket \vee \llbracket x_1 \leq x_2 \rrbracket && \text{(because } \llbracket x_0 \leq (x_1)^\bullet \rrbracket \leq \llbracket x_0 \leq x_1 \rrbracket) \end{aligned}$$

so we are done. The proof is symmetric in case $x_1 \in P^\ominus$. This concludes the case where $n = 2$.

Now assume that $n \geq 3$. It $\nu(x_i) \vee \nu(x_{i+1}) = \ell$ for some $i < n$, then the right hand side of the desired inequality is equal to 1 and we are done, so suppose that $\nu(x_i) \vee \nu(x_{i+1}) < \ell$ for all $i < n$. Suppose that there are i, j such that $0 \leq i \leq j \leq n$ and $2 \leq j - i < n$ such that $\nu(x_i) \vee \nu(x_j) < \ell$. It follows from the induction hypothesis that $\llbracket x_i \leq x_j \rrbracket \leq \bigvee_{i \leq k < j} \llbracket x_k \leq x_{k+1} \rrbracket$. Hence, using again the induction hypothesis, we get

$$\begin{aligned} \llbracket x_0 \leq x_n \rrbracket &\leq \bigvee_{k < i} \llbracket x_k \leq x_{k+1} \rrbracket \vee \llbracket x_i \leq x_j \rrbracket \vee \bigvee_{j \leq k < n} \llbracket x_k \leq x_{k+1} \rrbracket \\ &\leq \bigvee_{k < n} \llbracket x_k \leq x_{k+1} \rrbracket, \end{aligned}$$

so we are done again. Hence suppose that $0 \leq i \leq j \leq n$ and $2 \leq j - i < n$ implies that $\nu(x_i) \vee \nu(x_j) = \ell$, for all i, j . As $\nu(x_1) \vee \nu(x_n) = \ell$ while $\nu(x_0) \vee \nu(x_n) < \ell$ (we use here the assumption that $n \geq 3$), we get $\nu(x_1) \not\leq \nu(x_0)$. As $\nu(x_2) \vee \nu(x_3) < \ell$ and $\nu(x_1) \vee \nu(x_3) = \ell$, we get $\nu(x_1) \not\leq \nu(x_2)$. Hence, if $x_1 \in P^\oplus$, then, as $\nu(x_1) \not\leq \nu(x_2)$ and by (DB2), $\llbracket x_1 \leq x_2 \rrbracket = \llbracket (x_1)^\bullet \leq x_2 \rrbracket$, and hence, by using the induction hypothesis and the obvious inequality $\llbracket x_0 \leq (x_1)^\bullet \rrbracket \leq \llbracket x_0 \leq x_1 \rrbracket$ (“everything there happens below level $\nu(x_0) \vee \nu(x_1)$ ”), we get

$$\begin{aligned} \llbracket x_0 \leq x_n \rrbracket &\leq \llbracket x_0 \leq (x_1)^\bullet \rrbracket \vee \llbracket (x_1)^\bullet \leq x_2 \rrbracket \vee \bigvee_{2 \leq i < n} \llbracket x_i \leq x_{i+1} \rrbracket \\ &\leq \bigvee_{k < n} \llbracket x_k \leq x_{k+1} \rrbracket, \end{aligned}$$

so we are done. If $x_1 \in P^\ominus$, then, as $\nu(x_1) \not\leq \nu(x_0)$ and by (DB2), $\llbracket x_0 \leq x_1 \rrbracket = \llbracket x_0 \leq (x_1)^\bullet \rrbracket$, hence, by using the induction hypothesis and the obvious inequality

$\llbracket (x_1)_\bullet \leq x_2 \rrbracket \leq \llbracket x_1 \leq x_2 \rrbracket$, we get

$$\begin{aligned} \llbracket x_0 \leq x_n \rrbracket &\leq \llbracket x_0 \leq (x_1)_\bullet \rrbracket \vee \llbracket (x_1)_\bullet \leq x_2 \rrbracket \vee \bigvee_{2 \leq i < n} \llbracket x_i \leq x_{i+1} \rrbracket \\ &\leq \bigvee_{k < n} \llbracket x_k \leq x_{k+1} \rrbracket, \end{aligned}$$

so we are done. As $x_1 \in P^\oplus \cup P^\ominus$ (see Lemma 7.5), this completes the induction step. \square

Notation 8.2. We put

$$\begin{aligned} P(z) &= \{t \in P \mid \nu(t) < \nu(z)\}, \\ P^\oplus(z) &= P(z) \cap P^\oplus, \quad P^\ominus(z) = P(z) \cap P^\ominus, \end{aligned}$$

for all $z \in P$. Furthermore, for all $x, y \in P$, we define

$$\|x \leq y\|^+ = \bigwedge (\llbracket x \leq t \rrbracket \vee \llbracket t \leq y \rrbracket \mid t \in P(y)), \quad (8.1)$$

$$\|x \leq y\|^- = \bigwedge (\llbracket x \leq t \rrbracket \vee \llbracket t \leq y \rrbracket \mid t \in P(x)), \quad (8.2)$$

$$\|x \leq y\|^\pm = \bigwedge (\llbracket x \leq u \rrbracket \vee \llbracket u \leq v \rrbracket \vee \llbracket v \leq y \rrbracket \mid \langle u, v \rangle \in P^\ominus(x) \times P^\oplus(y)), \quad (8.3)$$

$$\|x \leq y\| = \llbracket x \leq y \rrbracket \wedge \|x \leq y\|^+ \wedge \|x \leq y\|^- \wedge \|x \leq y\|^\pm. \quad (8.4)$$

(All meets are evaluated in D_ℓ , the empty meet being defined as equal to 1.) We observe that the meet on the right hand side of (8.1) may be taken over all $t \in P(y)$ such that $\nu(x) \vee \nu(t) < \ell$: indeed, for all other $t \in P(y)$, we get $\llbracket x \leq t \rrbracket = 1$. Similarly, the meet on the right hand side of (8.2) may be taken over all $t \in P(x)$ such that $\nu(t) \vee \nu(y) < \ell$, and the meet on the right hand side of (8.3) may be taken over all $\langle u, v \rangle \in P^\ominus(x) \times P^\oplus(y)$ such that $\nu(u) \vee \nu(v) < \ell$.

Lemma 8.3. $\|x \leq y\| \leq \llbracket x \leq t \rrbracket \vee \llbracket t \leq y \rrbracket$, for all $x, y, t \in P$.

Proof. We argue by induction on $\rho(x) + \rho(y) + \rho(t)$. If $\nu(x) \vee \nu(t) = \ell$ or $\nu(t) \vee \nu(y) = \ell$ then the right hand side of the desired inequality is equal to 1 so we are done. Suppose, from now on, that $\nu(x) \vee \nu(t), \nu(t) \vee \nu(y) < \ell$. If $\nu(x) \vee \nu(y) < \ell$, then it follows from Lemma 8.1 (for $n = 2$) that $\llbracket x \leq y \rrbracket \leq \llbracket x \leq t \rrbracket \vee \llbracket t \leq y \rrbracket$, so we are done as $\|x \leq y\| \leq \llbracket x \leq y \rrbracket$. Now suppose that $\nu(x) \vee \nu(y) = \ell$. In particular, $x, y \notin Q_0$. If $t \in Q_0$, then $t \in P(y)$, thus

$$\|x \leq y\| \leq \|x \leq y\|^+ \leq \llbracket x \leq t \rrbracket \vee \llbracket t \leq y \rrbracket.$$

So suppose that $t \notin Q_0$. If $\nu(x) \leq \nu(t)$, then “everything happens below level $\nu(t) \vee \nu(y)$ ” (which is smaller than ℓ), so we are done. The conclusion is similar in case $\nu(y) \leq \nu(t)$.

So suppose that $\nu(x), \nu(y) \not\leq \nu(t)$. If $\nu(t) \leq \nu(y)$, then $\nu(t) < \nu(y)$ (because $\nu(y) \not\leq \nu(t)$), thus $t \in P(y)$, and thus

$$\|x \leq y\| \leq \|x \leq y\|^+ \leq \llbracket x \leq t \rrbracket \vee \llbracket t \leq y \rrbracket,$$

so we are done. If $t \in P^\oplus$ and $\nu(t) \not\leq \nu(y)$, then, by (DB2), $\llbracket t \leq y \rrbracket = \llbracket t^\bullet \leq y \rrbracket$, and thus

$$\begin{aligned} \|x \leq y\| &\leq \llbracket x \leq t^\bullet \rrbracket \vee \llbracket t^\bullet \leq y \rrbracket && \text{(by the induction hypothesis)} \\ &\leq \llbracket x \leq t \rrbracket \vee \llbracket t \leq y \rrbracket, \end{aligned}$$

so we are done again. This covers the case where $t \in P^\oplus$. The proof is symmetric for $t \in P^\ominus$. \square

Lemma 8.4. $\|x \leq y\| \leq \llbracket x \leq u \rrbracket \vee \llbracket u \leq v \rrbracket \vee \llbracket v \leq y \rrbracket$, for all $x, y, u, v \in P$.

Proof. We argue by induction on $\rho(x) + \rho(y) + \rho(u) + \rho(v)$. If either $\nu(x) \vee \nu(u) = \ell$ or $\nu(u) \vee \nu(v) = \ell$ or $\nu(v) \vee \nu(y) = \ell$, then the right hand side of the desired inequality is equal to 1 and we are done. So suppose that $\nu(x) \vee \nu(u), \nu(u) \vee \nu(v), \nu(v) \vee \nu(y) < \ell$. If $\nu(x) \vee \nu(v) < \ell$, then, by Lemma 8.1, we get $\llbracket x \leq v \rrbracket \leq \llbracket x \leq u \rrbracket \vee \llbracket u \leq v \rrbracket$, and so

$$\begin{aligned} \|x \leq y\| &\leq \llbracket x \leq v \rrbracket \vee \llbracket v \leq y \rrbracket && \text{(by Lemma 8.3)} \\ &\leq \llbracket x \leq u \rrbracket \vee \llbracket u \leq v \rrbracket \vee \llbracket v \leq y \rrbracket. \end{aligned}$$

The conclusion is similar for $\nu(u) \vee \nu(y) < \ell$. So suppose that $\nu(x) \vee \nu(v) = \nu(u) \vee \nu(y) = \ell$. In particular, $\nu(x) \not\leq \nu(u)$, $\nu(y) \not\leq \nu(v)$, and $\nu(u) \parallel \nu(v)$, so $x, y, u, v \notin Q_0$.

Suppose that $u \in P^\oplus$. As $\nu(u) \not\leq \nu(v)$ and by (DB2), we get $\llbracket u \leq v \rrbracket = \llbracket u^\bullet \leq v \rrbracket$, hence

$$\begin{aligned} \|x \leq y\| &\leq \llbracket x \leq u^\bullet \rrbracket \vee \llbracket u^\bullet \leq v \rrbracket \vee \llbracket v \leq y \rrbracket && \text{(by the induction hypothesis)} \\ &\leq \llbracket x \leq u \rrbracket \vee \llbracket u \leq v \rrbracket \vee \llbracket v \leq y \rrbracket && \text{(because } \llbracket x \leq u^\bullet \rrbracket \leq \llbracket x \leq u \rrbracket \text{)}. \end{aligned}$$

Suppose that $u \in P^\ominus$ and $\nu(u) \not\leq \nu(x)$. As $\nu(x) \not\leq \nu(u)$, we get $\nu(u) \not\leq \nu(x)$, thus $\llbracket x \leq u \rrbracket = \llbracket x \leq u_\bullet \rrbracket$, and so

$$\begin{aligned} \|x \leq y\| &\leq \llbracket x \leq u_\bullet \rrbracket \vee \llbracket u_\bullet \leq v \rrbracket \vee \llbracket v \leq y \rrbracket && \text{(by the induction hypothesis)} \\ &\leq \llbracket x \leq u \rrbracket \vee \llbracket u \leq v \rrbracket \vee \llbracket v \leq y \rrbracket && \text{(because } \llbracket u_\bullet \leq v \rrbracket \leq \llbracket u \leq v \rrbracket \text{)}. \end{aligned}$$

The case where either $v \in P^\ominus$ or ($v \in P^\oplus$ and $\nu(v) \not\leq \nu(y)$) is symmetric. The only remaining case is where $u \in P^\ominus(x)$ and $v \in P^\oplus(y)$, in which case

$$\|x \leq y\| \leq \|x \leq y\|^\pm \leq \llbracket x \leq u \rrbracket \vee \llbracket u \leq v \rrbracket \vee \llbracket v \leq y \rrbracket. \quad \square$$

Consequently, we get the formula

$$\|x \leq y\| = \bigwedge (\llbracket x \leq u \rrbracket \vee \llbracket u \leq v \rrbracket \vee \llbracket v \leq y \rrbracket \mid u, v \in P), \quad \text{for all } x, y \in P. \quad (8.5)$$

Lemma 8.5. $\|x \leq z\| \leq \|x \leq y\| \vee \|y \leq z\|$, for all $x, y, z \in P$.

Proof. If $\nu(y) \vee \nu(z) = \ell$ then $\llbracket y \leq z \rrbracket = 1$ and the conclusion is trivial. Suppose that $\nu(y) \vee \nu(z) < \ell$. A direct use of Lemma 8.3 yields the inequality $\|x \leq z\| \leq \llbracket x \leq y \rrbracket \vee \llbracket y \leq z \rrbracket$, while a direct use of Lemma 8.4 together with the distributivity of D_ℓ yields that $\|x \leq z\| \leq \|x \leq y\|^+ \wedge \|x \leq y\|^- \vee \llbracket y \leq z \rrbracket$. It remains to establish the inequality $\|x \leq z\| \leq \|x \leq y\|^\pm \vee \llbracket y \leq z \rrbracket$, which reduces, by the distributivity of D_ℓ , to proving the inequality

$$\|x \leq z\| \leq \llbracket x \leq u \rrbracket \vee \llbracket u \leq v \rrbracket \vee \llbracket v \leq y \rrbracket \vee \llbracket y \leq z \rrbracket, \quad (8.6)$$

for all $\langle u, v \rangle \in P^\ominus(x) \times P^\oplus(y)$. From $\nu(v) < \nu(y)$ it follows that $\llbracket v \leq z \rrbracket \leq \llbracket v \leq y \rrbracket \vee \llbracket y \leq z \rrbracket$ (“everything there happens below level $\nu(y) \vee \nu(z)$ ”), and hence

$$\begin{aligned} \|x \leq z\| &\leq \llbracket x \leq u \rrbracket \vee \llbracket u \leq v \rrbracket \vee \llbracket v \leq z \rrbracket && \text{(by Lemma 8.4)} \\ &\leq \llbracket x \leq u \rrbracket \vee \llbracket u \leq v \rrbracket \vee \llbracket v \leq y \rrbracket \vee \llbracket y \leq z \rrbracket, \end{aligned}$$

which completes the proof of (8.6). \square

Lemma 8.6. $\|x \leq z\| \leq \|x \leq y\| \vee \|y \leq z\|$, for all $x, y, z \in P$.

Proof. For elements $u, v \in P$, we get, by three successive applications of Lemma 8.5, the inequalities

$$\begin{aligned} \|x \leq u\| &\leq \|x \leq y\| \vee \|y \leq u\|; \\ \|x \leq v\| &\leq \|x \leq u\| \vee \|u \leq v\|; \\ \|x \leq z\| &\leq \|x \leq v\| \vee \|v \leq z\|. \end{aligned}$$

Hence, combining these inequalities, we obtain

$$\|x \leq z\| \leq \|x \leq y\| \vee \|y \leq u\| \vee \|u \leq v\| \vee \|v \leq z\|.$$

Evaluating the meets of both sides over $u, v \in P$ and using (the easy direction of) (8.5) yields the desired conclusion. \square

As a consequence, we obtain the following simple expression of $\|x \leq y\|$.

Corollary 8.7. *The Boolean value $\|x \leq y\|$ is equal to the meet in D_ℓ of all elements of D_ℓ of the form*

$$\|x \leq z_1\| \vee \|z_1 \leq z_2\| \vee \cdots \vee \|z_{n-1} \leq y\|, \quad (8.7)$$

where n is a natural number and $z_0, z_1, \dots, z_n \in P$ such that $z_0 = x$, $z_n = y$, and $\nu(z_i) \vee \nu(z_{i+1}) < \ell$ for all $i < n$. Furthermore, it is sufficient to restrict the meet to finite sequences $\langle z_0, z_1, z_2, z_3 \rangle$ (so $n = 3$).

Proof. Denote temporarily by $\|x \leq y\|^*$ the meet in D_ℓ of all elements of D_ℓ of the form (8.7). An immediate application of the easy direction of (8.5) yields the inequality $\|x \leq y\|^* \leq \|x \leq y\|$. Conversely, for every natural number n and all $z_0, z_1, \dots, z_n \in P$ such that $z_0 = x$, $z_n = y$, and $\nu(z_i) \vee \nu(z_{i+1}) < \ell$ for all $i < n$,

$$\begin{aligned} \|x \leq y\| &\leq \bigvee_{i < n} \|z_i \leq z_{i+1}\| && \text{(by Lemma 8.6)} \\ &\leq \bigvee_{i < n} \|z_i \leq z_{i+1}\| && \text{(because } \|z_i \leq z_{i+1}\| \leq \|z_i \leq z_{i+1}\|), \end{aligned}$$

which concludes the proof of the first part. The bound $n = 3$ follows from the easy direction of (8.5). \square

As an immediate consequence of Lemma 8.1, we obtain that the equality $\|x \leq y\| = \|x \leq y\|$ holds for all $x, y \in P$ such that $\nu(x) \vee \nu(y) < \ell$. Hence we obtain the following lemma.

Lemma 8.8. *The p -measure $\|_p \leq _p$ extends the p -measure $\|_p \leq _p$ with respect to $\varphi_{i,\ell}$, for all $i < \ell$.*

Definition 8.9. The strong amalgam $P = \bigcup(Q_i \mid i < \ell)$, endowed with the p -measure $\|_p \leq _p$ constructed above, will be called the *strong amalgam of the family $\langle Q_i \mid i < \ell \rangle$ with respect to \mathcal{D}* .

So we have reached the main goal of the present section.

Proposition 8.10. *Let Λ be a finite lattice with largest element ℓ , let $\mathcal{D} = \langle D_i, \varphi_{i,j} \mid i \leq j \text{ in } \Lambda \rangle$ be a Λ -indexed diagram of finite distributive lattices and $\langle \nu, 0 \rangle$ -homomorphisms, and let $\langle Q_i \mid i < \ell \rangle$ be a $\mathcal{D}|_{<\ell}$ -valued normal interval diagram of p -measured lattices satisfying (DB1) and (DB2). Then the strong amalgam P of $\langle Q_i \mid i < \ell \rangle$ (see Definition 8.9) is a D_ℓ -valued p -measured lattice, which extends Q_i with respect to $\varphi_{i,\ell}$, for all $i < \ell$.*

Lemma 8.11. *Under the assumptions of Proposition 8.10, the p -measured poset \mathbf{P} is a doubling extension of \mathbf{Q}_i , for all $i < \ell$.*

Proof. An immediate consequence of Lemmas 7.5 and 8.8. \square

The goal of the following lemma is to propagate the assumption (DB2) through the induction process that will appear in the constructions of Theorems 9.1 and 9.2.

Lemma 8.12. *For all $x, y \in P$, the following statements hold:*

- (i) $(\nu(x) \not\leq \nu(y) \text{ and } x \in P^\oplus)$ implies that $\|x \leq y\| = \|x^\bullet \leq y\|$.
- (ii) $(\nu(y) \not\leq \nu(x) \text{ and } y \in P^\ominus)$ implies that $\|x \leq y\| = \|x \leq y_\bullet\|$.

Proof. As (i) and (ii) are dual, it suffices to establish (i). We first claim that for all $x \in P^\oplus$ and all $y \in P$, $\nu(x) \not\leq \nu(y)$ implies that $\|x^\bullet \leq y\| \leq \|x \leq y\|$. Indeed, the equality holds by assumption (DB2) in case $\nu(x) \vee \nu(y) < \ell$. If $\nu(x) \vee \nu(y) = \ell$, then $\|x \leq y\| = 1$ and we are done again.

Now let $x \in P^\oplus$ and $y \in P$ such that $\nu(x) \not\leq \nu(y)$, we must prove that $\|x^\bullet \leq y\|$ lies below $\|x \leq y\|$, $\|x \leq y\|^+$, $\|x \leq y\|^-$, and $\|x \leq y\|^\pm$.

$$\begin{aligned} \|x^\bullet \leq y\| &\leq \|x^\bullet \leq y\| && \text{(see (8.4))} \\ &\leq \|x \leq y\| && \text{(as } \nu(x) \not\leq \nu(y) \text{ and by the claim above).} \end{aligned}$$

Now let $t \in P(y)$.

$$\begin{aligned} \|x^\bullet \leq y\| &\leq \|x^\bullet \leq t\| \vee \|t \leq y\| && \text{(by Lemma 8.3)} \\ &\leq \|x \leq t\| \vee \|t \leq y\| && \text{(as } \nu(x) \not\leq \nu(t) \text{ and by the claim above).} \end{aligned}$$

Evaluating the meets of both sides over $t \in P^\oplus(y)$ yields $\|x^\bullet \leq y\| \leq \|x \leq y\|^+$. A similar (but not symmetric!) proof yields $\|x^\bullet \leq y\| \leq \|x \leq y\|^-$. Finally, let $u \in P^\ominus(x)$ and $v \in P^\oplus(y)$. Then

$$\begin{aligned} \|x^\bullet \leq y\| &\leq \|x^\bullet \leq u\| \vee \|u \leq v\| \vee \|v \leq y\| && \text{(by Lemma 8.4)} \\ &\leq \|x \leq u\| \vee \|u \leq v\| \vee \|v \leq y\| && \text{(as } \nu(x) \not\leq \nu(u) \text{ and by the claim above),} \end{aligned}$$

hence, evaluating the meets of both sides over $\langle u, v \rangle \in P^\ominus(x) \times P^\oplus(y)$, we get $\|x^\bullet \leq y\| \leq \|x \leq y\|^\pm$, which completes the proof. \square

9. CONSTRUCTING A P-MEASURE ON A COVERING, DOUBLING EXTENSION OF A STRONG AMALGAM OF LATTICES

Let Λ , \mathcal{D} , and $\langle \mathbf{Q}_i \mid i < \ell \rangle$ satisfy the assumptions of Proposition 8.10, with strong amalgam \mathbf{P} (see Definition 8.9). By Proposition 8.10, $\|_-\leq_-\|_{\mathbf{P}}$ is a D_ℓ -valued p -measure on \mathbf{P} , which extends each p -measured lattice \mathbf{Q}_i with respect to the corresponding $\langle \vee, 0 \rangle$ -homomorphism $\varphi_{i,\ell}$.

Now we let Q be a covering extension of P . Furthermore, we assume that each closed interval $[x_P, x^P]$ of Q , for $x \in Q$, is endowed with a p -measure $\|_-\leq_-\|_{[x_P, x^P]}$ such that

$$\|x = x_P\|_{[x_P, x^P]} \sim \|x^P = x\|_{[x_P, x^P]}, \quad \text{for all } x \in Q, \quad (9.1)$$

$$\|x^P = x_P\|_{[x_P, x^P]} = \|x^P = x_P\|_{\mathbf{P}}, \quad \text{for all } x \in Q. \quad (9.2)$$

(Observe that the notation $\|x^P = x_P\|_{\mathbf{P}}$ in (9.2) above does not involve the full definition of the strong amalgam given in Definition 8.9: indeed, from $P \leq_{\text{cov}} Q$ it follows that $x_P \leq_P x^P$; as P is the strong amalgam of $\langle \mathbf{Q}_i \mid i < \ell \rangle$, x_P and x^P

belong to some Q_i , and so we can just put $\|x^P = x_P\|_{\mathbf{P}} = \varphi_{i,\ell}(\|x^P = x_P\|_{Q_i})$, which is independent of the chosen i .)

The goal of the present section is to extend $\|_-\leq_-\|_{\mathbf{P}}$ to a p-measure on \mathbf{Q} such that, setting $\mathbf{Q}_\ell = \mathbf{Q}$, the extended diagram $\langle \mathbf{Q}_i \mid i \leq \ell \rangle$ satisfies the assumptions of Proposition 8.10.

We need to verify several points. First, for all $i < \ell$, as $Q_i \leq_{\text{int}} P$ and $P \leq_{\text{cov}} Q$, we obtain from Lemma 4.4 that $Q_i \leq_{\text{int}} Q$. Item (3) of Definition 4.1 for the extended diagram $\langle \mathbf{Q}_i \mid i \leq \ell \rangle$ follows from the definition of the ordering of P_ℓ (see Section 4). Further, the new valuation on the extended diagram $\langle \mathbf{Q}_i \mid i \leq \ell \rangle$ extends the original one (so we shall still denote it by ν), and $\nu(x) = \ell$ for all $x \in Q \setminus P$. In addition, the elements x_\bullet and x^\bullet (see Lemma 5.1) remain the same for $x \in P \setminus Q_0$, while $x_\bullet = x_P$ and $x^\bullet = x^P$ for all $x \in Q \setminus P$.

Now we denote by $\|_-\leq_-\|_{\mathbf{Q}}$ the p-measure that we constructed in Section 6 (see Proposition 6.11), extending $\|_-\leq_-\|_{\mathbf{P}}$ and all p-measures $\|_-\leq_-\|_{[x_P, x^P]}$, for $x \in Q$ —this is made possible by (9.2). It follows from the assumption (9.1) that $\|x^P = x\|_{\mathbf{Q}} \sim \|x = x_P\|_{\mathbf{Q}}$, for all $x \in Q$; that is, \mathbf{Q} is a doubling extension of \mathbf{P} . As $Q_i \leq_{\text{int}} Q$ and by Lemmas 8.11 and 7.2 (applied to the extensions $\mathbf{Q}_i \leq \mathbf{P} \leq \mathbf{Q}$), we obtain that \mathbf{Q} is a doubling extension of \mathbf{Q}_i . This takes care of extending (DB1) to the larger diagram.

It remains to verify that $\langle \mathbf{Q}_i \mid i \leq \ell \rangle$ satisfies (DB2). So let $x, y \in Q$ such that $\nu(x) \not\leq \nu(y)$, we need to verify that $\|x = x_\bullet\|_{\mathbf{Q}} = \|x^\bullet = x_\bullet\|_{\mathbf{Q}}$ implies that $\|x \leq y\|_{\mathbf{Q}} = \|x^\bullet \leq y\|_{\mathbf{Q}}$ and $\|x^\bullet = x\|_{\mathbf{Q}} = \|x^\bullet = x_\bullet\|_{\mathbf{Q}}$ implies that $\|y \leq x\|_{\mathbf{Q}} = \|y \leq x_\bullet\|_{\mathbf{Q}}$. We prove for example the first statement. From $\nu(x) \not\leq \nu(y)$ it follows that $y \in P$. If $x \in P$ then we are done by Lemma 8.12, so the remaining case is where $x \in Q \setminus P$. Observe that $x_\bullet = x_P$ and $x^\bullet = x^P$. As $y \in P$, the Boolean value $\|x \leq y\|_{\mathbf{Q}}$ is given by Lemma 6.3(ii). Hence proving the inequality $\|x^\bullet \leq y\|_{\mathbf{Q}} \leq \|x \leq y\|_{\mathbf{Q}}$ reduces to proving that $\|x^P \leq y\|_{\mathbf{Q}}$ (of course equal to $\|x^P \leq y\|_{\mathbf{P}}$) lies below both $\|x^P \leq y\|_{\mathbf{P}}$ and $\|x_P \leq y\|_{\mathbf{P}} \vee \|x = x_P\|_{[x_P, x^P]}$. The first inequality is a tautology, and the second one is proved as follows:

$$\begin{aligned} \|x^P \leq y\|_{\mathbf{P}} &\leq \|x^P = x_P\|_{\mathbf{P}} \vee \|x_P \leq y\|_{\mathbf{P}} && \text{(because } \|_-\leq_-\|_{\mathbf{P}} \text{ is a p-measure)} \\ &= \|x = x_P\|_{\mathbf{Q}} \vee \|x_P \leq y\|_{\mathbf{P}} && \text{(because } \|x = x_\bullet\|_{\mathbf{Q}} = \|x^\bullet = x_\bullet\|_{\mathbf{Q}}) \\ &= \|x = x_P\|_{[x_P, x^P]} \vee \|x_P \leq y\|_{\mathbf{P}}. \end{aligned}$$

As the inequality $\|x \leq y\|_{\mathbf{Q}} \leq \|x^\bullet \leq y\|_{\mathbf{Q}}$ always holds, we have proved the equality, and hence the extended diagram $\langle \mathbf{Q}_i \mid i \leq \ell \rangle$ satisfies (DB2). So we have reached the following theorem, which is the main technical result of the present paper. It refers to the conditions (DB1) and (DB2) introduced in Section 7.

Theorem 9.1. *Let Λ be a finite lattice with largest element ℓ , let $\mathcal{D} = \langle D_i, \varphi_{i,j} \mid i \leq j \text{ in } \Lambda \rangle$ be a Λ -indexed diagram of finite distributive lattices and $\langle \vee, 0 \rangle$ -homomorphisms, and let $\langle \mathbf{Q}_i \mid i < \ell \rangle$ be a $\mathcal{D}|_{< \ell}$ -valued normal interval diagram of p-measured lattices satisfying (DB1) and (DB2). Let Q be a covering extension of the strong amalgam P of $\langle \mathbf{Q}_i \mid i < \ell \rangle$. Furthermore, we assume that for all $x \in Q$, the closed interval $[x_P, x^P]$ of Q is endowed with a p-measure $\|_-\leq_-\|_{[x_P, x^P]}$ (depending only on the interval $[x_P, x^P]$) such that*

$$\|x = x_P\|_{[x_P, x^P]} \sim \|x^P = x\|_{[x_P, x^P]} \quad \text{and} \quad \|x^P = x_P\|_{[x_P, x^P]} = \|x^P = x_P\|_{\mathbf{P}}.$$

Then there exists a D_ℓ -valued p-measure on Q extending all p-measures $\|_-\leq_-\|_{[x_P, x^P]}$ such that, defining \mathbf{Q}_ℓ as the corresponding p-measured poset, the extended diagram

$\langle Q_i \mid i \leq \ell \rangle$ is a \mathcal{D} -valued normal interval diagram of p -measured posets satisfying (DB1) and (DB2).

This result makes it possible to state and prove our main theorem.

Theorem 9.2. *Let Λ be a lower finite lattice and let $\mathcal{D} = \langle D_i, \varphi_{i,j} \mid i \leq j \text{ in } \Lambda \rangle$ be a Λ -indexed diagram of finite distributive lattices and zero-separating $\langle \vee, 0, 1 \rangle$ -homomorphisms. Then there exists a \mathcal{D} -valued normal interval diagram $\langle Q_i \mid i \in \Lambda \rangle$ of finite p -measured lattices satisfying (DB1) and (DB2) together with the following additional conditions:*

- (i) *For all $i < j$ in Λ and all $x < y$ in Q_i , there exists $z \in Q_j$ such that $x < z < y$.*
- (ii) *$\|y = x\|_{Q_i}$ is join-irreducible in D_i , for all $i \in \Lambda$ and all $x \prec y$ in Q_i .*
- (iii) *For all $i \in \Lambda$ and all $p \in J(D_i)$, there exists $x \in Q_i$ such that $0 \prec_{Q_i} x$ and $\|x = 0\|_{Q_i} = p$.*

Proof. We construct Q_i by induction on the height of i in Λ . After possibly adding a new zero element to Λ , we may assume that $D_0 = \{0, 1\}$, so we take $Q_0 = \{0, 1\}$, with the p -measure defined by $\|1 = 0\|_{Q_0} = 1$. Put $\Lambda_n = \{i \in \Lambda \mid \text{height}(i) \leq n\}$ and denote by $\mathcal{D}_{(n)}$ the restriction of \mathcal{D} to Λ_n , for every natural number n . Suppose having constructed a $\mathcal{D}_{(n)}$ -valued normal interval diagram $\langle Q_i \mid i \in \Lambda_n \rangle$ of finite p -measured lattices satisfying (DB1), (DB2), and Conditions (i)–(iii) of the statement of the theorem, we show how to extend it to a $\mathcal{D}_{(n+1)}$ -valued normal interval diagram of finite p -measured lattices satisfying (DB1) and (DB2). In order to propagate Item (2) of Definition 4.1, we shall add the induction hypothesis that every $x \in \bigcup(Q_i \mid i \in \Lambda_n)$ can be written in the form $\langle \bar{x}, \nu(x) \rangle$, where ν denotes the valuation associated with the diagram $\langle Q_i \mid i \in \Lambda_n \rangle$.

Let $\ell \in \Lambda_{n+1} \setminus \Lambda_n$ and denote by P_ℓ the strong amalgam of $\langle Q_i \mid i < \ell \rangle$ with respect to \mathcal{D} given in Definition 8.9. It follows from Proposition 4.5 that P_ℓ is a lattice and every Q_i , for $i < \ell$, is a sublattice of P_ℓ . For $x \prec y$ in P_ℓ , we put

$$\begin{aligned} 1_{x,y,\ell} &= \{p \in J(L) \mid p \leq \|y = x\|_{P_\ell}\} \text{ (that we shall sometimes simply denote by 1),} \\ B_{x,y,\ell} &= \mathfrak{P}(1_{x,y,\ell}) \text{ (the powerset lattice of } 1_{x,y,\ell}\text{),} \\ \overline{Q}_{x,y,\ell} &= \{\langle X, 0 \rangle \mid X \subseteq 1_{x,y,\ell}\} \cup \{\langle 1, Y \rangle \mid Y \subseteq 1_{x,y,\ell}\} \text{ (a sublattice of } B_{x,y,\ell} \times B_{x,y,\ell}\text{).} \end{aligned}$$

Hence $\overline{Q}_{x,y,\ell}$ can be viewed as the ordinal sum of two copies of the Boolean lattice $B_{x,y,\ell}$, with the top of the lower copy of $B_{x,y,\ell}$ (namely, $\langle X, 0 \rangle$ where $X = 1_{x,y,\ell}$) identified with the bottom of the upper copy of $B_{x,y,\ell}$ (namely, $\langle 1, Y \rangle$ where $Y = \emptyset$). As $x \prec_{P_\ell} y$, there exists $i < \ell$ such that $x, y \in Q_i$. By induction hypothesis (Condition (ii)), $\|y = x\|_{Q_i} > 0$, thus, as $\varphi_{i,\ell}$ separates zero, $\|y = x\|_{P_\ell} = \varphi_{i,\ell}(\|y = x\|_{Q_i}) > 0$, and thus $1_{x,y,\ell}$ is nonempty.

We endow $\overline{Q}_{x,y,\ell}$ with the p -measure $\|_-\leq_-\|_{x,y,\ell}$ defined by

$$\begin{aligned} \|\langle X_0, 0 \rangle \leq \langle X_1, 0 \rangle\|_{x,y,\ell} &= \bigvee (X_0 \setminus X_1), \\ \|\langle 1, Y_0 \rangle \leq \langle 1, Y_1 \rangle\|_{x,y,\ell} &= \bigvee (Y_0 \setminus Y_1), \\ \|\langle X, 0 \rangle \leq \langle 1, Y \rangle\|_{x,y,\ell} &= 0, \\ \|\langle 1, Y \rangle \leq \langle X, 0 \rangle\|_{x,y,\ell} &= \bigvee (\mathbb{C}X \cup Y), \end{aligned}$$

(where we put $\mathbb{C}X = \Omega_{x,y,\ell} \setminus X$), for all $X, X_0, X_1, Y, Y_0, Y_1 \subseteq 1$ (it is easy to verify that this way we get, indeed, a p-measure on $\overline{Q}_{x,y,\ell}$). Further, we put

$$\begin{aligned} Q'_{x,y,\ell} &= \overline{Q}_{x,y,\ell} \setminus \{\langle \emptyset, 0 \rangle, \langle 1, 1 \rangle\} && \text{("truncated } \overline{Q}_{x,y,\ell} \text{"}, \\ Q_{x,y,\ell} &= Q'_{x,y,\ell} \times \{\ell\}, \end{aligned}$$

where $Q'_{x,y,\ell}$ is endowed with the restrictions of both the ordering and the p-measure of $\overline{Q}_{x,y,\ell}$ and $Q_{x,y,\ell}$ is endowed with the ordering and p-measure for which the map $t \mapsto \langle t, \ell \rangle$ is a measure-preserving isomorphism. So $Q_{x,y,\ell}$ is the result of applying to $\overline{Q}_{x,y,\ell}$ the following two transformations:

- Remove the top and bottom elements of $\overline{Q}_{x,y,\ell}$; get $Q'_{x,y,\ell}$.
- Replace t by $\langle t, \ell \rangle$, for all $t \in Q'_{x,y,\ell}$; get $Q_{x,y,\ell}$.

Put $Q_\ell = P_\ell + \sum(Q_{x,y,\ell} \mid x \prec y \text{ in } P_\ell)$ (see (2.3)). Then Q_ℓ is an interval extension of P_ℓ (see Lemma 2.8). In fact, as $Q_{x,y,\ell}$ is defined only for $x \prec y$ in P_ℓ , the poset Q_ℓ is a covering extension of P_ℓ (see Definition 3.1). We shall still denote by $\|_-\leq -\|_{x,y,\ell}$ the p-measure on $Q_{x,y,\ell} \cup \{x, y\}$ inherited from the p-measure on $\overline{Q}_{x,y,\ell}$. As P_ℓ is a lattice and $[x, y]_{Q_\ell} = Q_{x,y,\ell} \cup \{x, y\} \cong \overline{Q}_{x,y,\ell}$ is a lattice, for all $x \prec y$ in P_ℓ , it follows from Lemma 2.5 that Q_ℓ is a lattice.

Now we verify Conditions (9.1) and (9.2) with respect to $\|_-\leq -\|_{P_\ell}$ and all p-measures $\|_-\leq -\|_{x,y,\ell}$. Fix $x \prec y$ in P_ℓ . Condition (9.1) follows immediately from the inequalities

$$\begin{aligned} \|\langle X, 0 \rangle = \langle \emptyset, 0 \rangle\|_{x,y,\ell} &= \bigvee X \leq \|y = x\|_{P_\ell} = \|\langle 1, 1 \rangle = \langle X, 0 \rangle\|_{x,y,\ell}, \\ \|\langle 1, 1 \rangle = \langle 1, X \rangle\|_{x,y,\ell} &= \bigvee (\mathbb{C}X) \leq \|y = x\|_{P_\ell} = \|\langle 1, X \rangle = \langle \emptyset, 0 \rangle\|_{x,y,\ell}, \end{aligned}$$

for all $X \subseteq 1_{x,y,\ell}$. Condition (9.2) follows from the equalities

$$\|\langle 1, 1 \rangle = \langle \emptyset, 0 \rangle\|_{x,y,\ell} = \bigvee 1_{x,y,\ell} = \|y = x\|_{P_\ell}.$$

Hence, by Theorem 9.1, there is a p-measure on Q_ℓ , extending all p-measures $\|_-\leq -\|_{x,y,\ell}$, such that $\langle \mathbf{Q}_i \mid i \leq \ell \rangle$ is a $\mathcal{D}_{\leq \ell}$ -valued normal interval diagram of p-measured lattices satisfying (DB1) and (DB2).

Now we verify Conditions (i)–(iii) of the statement of Theorem 9.2. Let $i < \ell$ and let $x < y$ in Q_i , we prove that $x \not\prec_{Q_\ell} y$. If $x \not\prec_{P_\ell} y$ this is trivial, so suppose that $x \prec_{P_\ell} y$. Pick any element $z \in Q_{x,y,\ell}$ (e.g., $z = \langle \langle 1, 0 \rangle, \ell \rangle$, the ‘middle element’—here we use the nonemptiness of $1_{x,y,\ell}$); then $x < z < y$ in Q_ℓ . Condition (i) follows.

In order to verify Condition (ii) at level Q_ℓ , it suffices to prove that $\|v = u\|_{x,y,\ell}$ belongs to $J(D_\ell)$, for all $x \prec y$ in P_ℓ and all $u \prec v$ in $\overline{Q}_{x,y,\ell}$. There are a proper subset X of $1_{x,y,\ell}$ and an element $p \in \mathbb{C}X$ such that either $(u = \langle X, 0 \rangle$ and $v = \langle X \cup \{p\}, 0 \rangle)$ or $(u = \langle 1, X \rangle$ and $v = \langle 1, X \cup \{p\} \rangle)$. In both cases, $\|v = u\|_{x,y,\ell} = p$ belongs to $J(D_\ell)$.

Now we verify Condition (iii). Let $p \in J(D_\ell)$ and pick $k \prec \ell$ in Λ . As $p \leq \varphi_{k,\ell}(1) = \bigvee(\varphi_{k,\ell}(q) \mid q \in J(D_k))$ and p is join-irreducible, there exists $q \in J(D_k)$ such that $p \leq \varphi_{k,\ell}(q)$. By the induction hypothesis (Condition (iii)), there exists $x \in Q_k$ such that $0 \prec_{Q_k} x$ and $\|x = 0\|_{Q_k} = q$. Suppose that there exists $y \in P_\ell$ such that $0 < y < x$, and let $i < \ell$ such that $y \in Q_i$. As $y \leq x$, there exists $z \in Q_{i \wedge k}$ such that $y \leq z \leq x$. As $0 < z \leq x$ with $z \in Q_k$ and $0 \prec_{Q_k} x$, we get $z = x$, and so $x \in Q_{i \wedge k}$. If $i \wedge k < k$, then, by Condition (i) on $\mathcal{D}_{(n)}$, we get

$0 \not\prec_{Q_k} x$, a contradiction. Therefore, $k = i \wedge k \leq i$, but $k \prec \ell$, and thus $i = k$. As $0 < y < x$, $y \in Q_k$, and $0 \prec_{Q_k} x$, we get again a contradiction. So we have proved that $0 \prec_{P_\ell} x$. As $p \leq \varphi_{k,\ell}(q) = \|x = 0\|_{P_\ell}$, we get $p \in 1_{0,x,\ell}$. We consider the element $t = \langle \{p\}, 0, \ell \rangle$ of $Q_{0,x,\ell}$ (so $0 \prec t < x$ in Q_ℓ). We compute

$$\|t = 0\|_{Q_\ell} = \|\langle \{p\}, 0 \rangle = \langle \emptyset, 0 \rangle\|_{0,x,\ell} = p,$$

which completes the verification of Condition (iii) at level ℓ .

In order to verify that $\langle Q_i \mid i \in \Lambda_{n+1} \rangle$ is as required, it remains to verify that $\langle Q_i \mid i \in \Lambda_{n+1} \rangle$ satisfies Item (2) of Definition 4.1. So let $i, j \in \Lambda_{n+1}$, we need to verify that $Q_i \cap Q_j = Q_{i \wedge j}$. This holds by induction hypothesis for $i, j \in \Lambda_n$. As it trivially holds for $i = j$, we assume that $i \neq j$. If $\text{height}(i) = \text{height}(j) = n$, then

$$Q_i = P_i \cup \bigcup (Q_{x,y,i} \mid x \prec y \text{ in } P_i), \quad (9.3)$$

$$Q_j = P_j \cup \bigcup (Q_{x,y,j} \mid x \prec y \text{ in } P_j), \quad (9.4)$$

and thus, as $i \parallel j$,

$$Q_i \cap Q_j = P_i \cap P_j = \bigcup (Q_{i'} \cap Q_{j'} \mid i' < i, j' < j) = Q_{i \wedge j}.$$

If $\text{height}(i) = n$ while $\text{height}(j) < n$, then Q_i is still given by (9.3), and so

$$Q_i \cap Q_j = P_i \cap Q_j = \bigcup (Q_{i'} \cap Q_j \mid i' < i) = Q_{i \wedge j},$$

which completes the verification of Item (2) of Definition 4.1. This completes the proof of the induction step. \square

Corollary 9.3. *For every distributive $\langle \vee, 0 \rangle$ -semilattice S , there are a $\langle \wedge, 0 \rangle$ -semilattice P and a S -valued p -measure $\|_ - \leq - \|$ on P satisfying the following additional conditions:*

- (i) $\|y = x\| > 0$, for all $x < y$ in P .
- (ii) For all $x \leq y$ in P and all $\mathbf{a}, \mathbf{b} \in S$, if $\|y = x\| \leq \mathbf{a} \vee \mathbf{b}$, there are a positive integer n and a decomposition $x = z_0 \leq z_1 \leq \dots \leq z_n = y$ such that either $\|z_{i+1} = z_i\| \leq \mathbf{a}$ or $\|z_{i+1} = z_i\| \leq \mathbf{b}$, for all $i < n$.
- (iii) The subset $\{\|x = 0\| \mid x \in P\}$ generates the semilattice S .

Furthermore, if S is bounded, then P can be taken a bounded lattice.

Proof. Suppose first that S is bounded. By a well-known result of P. Pudlák, see [7, Fact 4, p. 100], S is the directed union of its finite distributive $\langle \vee, 0, 1 \rangle$ -subsemilattices. Hence one can write $S = \varinjlim_{i \in \Lambda} D_i$, where Λ is the (lower finite) lattice of all finite subsets of $S \times \omega$, all the D_i are finite distributive $0, 1$ -subsemilattices of S , and the transition map from D_i to D_j is the inclusion map for all $i \leq j$ in Λ . Let $\langle Q_i \mid i \in \Lambda \rangle$ be as in Theorem 9.2. Then the union of all the p -measures $\|_ - \leq - \|_{Q_i}$ on $Q = \bigcup (Q_i \mid i \in \Lambda)$ is as required. Conditions (i) and (ii) above follow from Condition (ii) in Theorem 9.2, while Condition (iii) above follows from Condition (iii) in Theorem 9.2.

In the general case, we apply the result above to $S \cup \{1\}$ (for some new unit element 1), and then, denoting by \mathbf{P} the corresponding p -measured lattice, we set $Q = \{x \in P \mid \|x = 0\| \in S\}$, which is a lower subset of P . The restriction of the p -measure of \mathbf{P} to $Q \times Q$ is as required. \square

The following easy result shows that distributivity cannot be removed from the assumptions of Corollary 9.3.

Proposition 9.4. *Let S be a $\langle \vee, 0 \rangle$ -semilattice, let P be a poset, and let $\|_ - \leq \|_ -$ be a S -valued p -measure on P satisfying condition (ii) of Corollary 9.3 such that the subset $\Sigma = \{\|y = x\| \mid x \leq y \text{ in } P\}$ join-generates S . Then S is distributive.*

Proof. Let $\mathbf{a}_0, \mathbf{a}_1, \mathbf{b} \in S$ such that $\mathbf{b} \leq \mathbf{a}_0 \vee \mathbf{a}_1$, we find $\mathbf{b}_i \leq \mathbf{a}_i$, for $i < 2$, such that $\mathbf{b} = \mathbf{b}_0 \vee \mathbf{b}_1$. Suppose first that $\mathbf{b} \in \Sigma$, so $\mathbf{b} = \|y = x\|$, for some $x \leq y$ in P . By assumption, there are a positive integer m and a decomposition $x = z_0 \leq z_1 \leq \dots \leq z_m = y$ such that for all $i < m$, there exists $\varepsilon(i) \in \{0, 1\}$ with $\|z_{i+1} = z_i\| \leq \mathbf{a}_{\varepsilon(i)}$. Put $\mathbf{b}_j = \bigvee (\|z_{i+1} = z_i\| \mid i \in \varepsilon^{-1}\{j\})$, for all $j < 2$. Then $\mathbf{b}_j \leq \mathbf{a}_j$ and $\mathbf{b} = \|y = x\| = \mathbf{b}_0 \vee \mathbf{b}_1$.

In the general case, $\mathbf{b} = \bigvee (\mathbf{c}_j \mid j < n)$ for a positive integer n and elements $\mathbf{c}_0, \dots, \mathbf{c}_{n-1} \in \Sigma$. By the above paragraph, there are decompositions $\mathbf{c}_j = \mathbf{c}_{j,0} \vee \mathbf{c}_{j,1}$ with $\mathbf{c}_{j,k} \leq \mathbf{a}_k$ for all $j < n$ and $k < 2$. The elements $\mathbf{b}_k = \bigvee (\mathbf{c}_{j,k} \mid j < n)$, for $k < 2$, are as required. \square

The following example shows that the conditions (DB1) and (DB2) cannot be removed from the assumptions of Theorem 9.1. The construction is inspired by the one of the cube \mathcal{D}_c presented in [9, Section 3].

Example 9.5. *Put $\Lambda = \mathfrak{P}(3)$ (the three-dimensional cube) and $\Lambda^* = \Lambda \setminus \{3\}$. There are a Λ -indexed diagram $\mathcal{B} = \langle B_p \mid p \in \Lambda \rangle$ of finite Boolean lattices and $\langle \vee, 0, 1 \rangle$ -embeddings, whose restriction to Λ^* we denote by \mathcal{B}^* , and a \mathcal{B}^* -valued normal interval diagram $\langle \mathbf{Q}_p \mid p \in \Lambda^* \rangle$ of finite p -measured lattices that cannot be extended to any \mathcal{B} -valued normal diagram of p -measured posets.*

Proof. We first put $B_{\{0,1,2\}} = \mathfrak{P}(5)$ (where, as usual, $5 = \{0, 1, 2, 3, 4\}$). Further, we define elements $\gamma_{i,j}$ of $\mathfrak{P}(5)$, for $i < 3$ and $j < 4$, by

$$\begin{array}{llll} \gamma_{0,0} = \{0, 4\}, & \gamma_{0,1} = \{3\}, & \gamma_{0,2} = \{2\}, & \gamma_{0,3} = \{1, 4\}; \\ \gamma_{1,0} = \{0, 4\}, & \gamma_{1,1} = \{1, 4\}, & \gamma_{1,2} = \{2\}, & \gamma_{1,3} = \{3, 4\}; \\ \gamma_{2,0} = \{0, 4\}, & \gamma_{2,1} = \{1\}, & \gamma_{2,2} = \{3\}, & \gamma_{2,3} = \{2, 4\}. \end{array}$$

Observe that the equality $5 = \bigcup (\gamma_{i,j} \mid j < 4)$ holds, for all $i < 3$.

We shall now define certain subsemilattices of $\langle \mathfrak{P}(5), \cup, \emptyset \rangle$. For $\{i, j, k\} = 3$, we define $B_{\{i,j\}}$ as the $\langle \vee, 0 \rangle$ -subsemilattice of $\langle \mathfrak{P}(5), \cup, \emptyset \rangle$ generated by the subset $\{\gamma_{k,0}, \gamma_{k,1}, \gamma_{k,2}, \gamma_{k,3}\}$.

Further, for all $i < 3$, let $B_{\{i\}}$ be the $\langle \vee, 0 \rangle$ -subsemilattice of $\mathfrak{P}(5)$ generated by $\{\alpha_i, \beta_i\}$, where we put

$$\begin{array}{ll} \alpha_0 = \{0, 1, 4\}, & \beta_0 = \{2, 3, 4\}; \\ \alpha_1 = \{0, 3, 4\}, & \beta_1 = \{1, 2, 4\}; \\ \alpha_2 = \{0, 2, 4\}, & \beta_2 = \{1, 3, 4\}. \end{array}$$

At the bottom of the diagram, we put the two-element semilattice $B_\emptyset = \{\emptyset, 5\}$. Observe, in particular, that 5 is the largest element of B_p , for all $p \subseteq 3$.

It is a matter of routine to verify that B_p is a $\langle \vee, 0, 1 \rangle$ -subsemilattice of B_q if $p \subseteq q$, for all $p, q \subseteq 3$. In that case, we denote by $\varphi_{p,q}$ the inclusion map from B_p into B_q . Set

$$\begin{aligned} \mathcal{B} &= \langle \langle B_p, \varphi_{p,q} \rangle \mid p \subseteq q \text{ in } \mathfrak{P}(3) \rangle, \\ \mathcal{B}^* &= \langle \langle B_p, \varphi_{p,q} \rangle \mid p \subseteq q \text{ in } \mathfrak{P}(3) \setminus \{3\} \rangle. \end{aligned}$$

Let Q_p , for $p \in \Lambda^*$, and P be the lattices diagrammed on Figure 9.1. We observe that $\langle Q_p \mid p \in \Lambda^* \rangle$ is a normal interval diagram of finite lattices. We endow Q_\emptyset

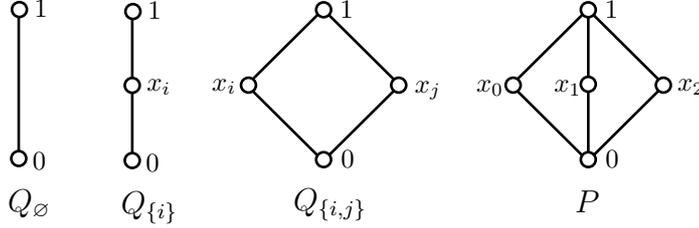


FIGURE 9.1. The posets Q_\emptyset , $Q_{\{i\}}$, $Q_{\{i,j\}}$, and P .

with the unique p-measure $\|_-\leq -\|_\emptyset$ that satisfies $\|1 = 0\|_\emptyset = 5$, the largest element of B_\emptyset . For $i < 3$, we endow $Q_{\{i\}}$ with the unique p-measure $\|_-\leq -\|_{\{i\}}$ that satisfies $\|x_i = 0\|_{\{i\}} = \alpha_i$ and $\|1 = x_i\|_{\{i\}} = \beta_i$. Finally, for $\{i, j, k\} = 3$, it is not hard to verify that there exists a unique p-measure $\|_-\leq -\|_{\{i,j\}}$ on $Q_{\{i,j\}}$ such that $\|x_i \leq x_j\|_{\{i,j\}} = \gamma_{k,1}$ and $\|x_j \leq x_i\|_{\{i,j\}} = \gamma_{k,2}$.

Suppose that the \mathcal{B}^* -valued diagram $\langle Q_p \mid p \in \Lambda^* \rangle$ extends to some \mathcal{B} -valued diagram $\langle Q_p \mid p \in \Lambda \rangle$. Evaluating the Boolean values in $Q_{\{0,1,2\}}$, we obtain

$$\begin{aligned} \|x_0 \leq x_1\| &= \|x_0 \leq x_1\|_{\{0,1\}} = \gamma_{2,1}, \\ \|x_1 \leq x_2\| &= \|x_1 \leq x_2\|_{\{1,2\}} = \gamma_{0,1}, \\ \|x_0 \leq x_2\| &= \|x_0 \leq x_2\|_{\{0,2\}} = \gamma_{1,1}, \end{aligned}$$

hence, by the triangular inequality, $\gamma_{1,1} \subseteq \gamma_{0,1} \cup \gamma_{2,1}$, a contradiction. \square

10. COMPARISON WITH SEMILATTICE-VALUED DISTANCES

The main result of the present paper, Theorem 9.2, is formally similar to one of the results of [8], that states that *every distributive $\langle \vee, 0 \rangle$ -semilattice is, functorially, the range of a V-distance of type 2 on some set*. By definition, for a $\langle \vee, 0 \rangle$ -semilattice S , a S -valued distance on a set X is a map $\delta: X \times X \rightarrow S$ such that $\delta(x, x) = 0$, $\delta(x, y) = \delta(y, x)$, and $\delta(x, z) \leq \delta(x, y) \vee \delta(y, z)$, for all $x, y, z \in X$. Furthermore, δ satisfies the V-condition of type 2, if for all $\mathbf{a}, \mathbf{b} \in S$ and all $x, y \in X$, if $\delta(x, y) = \mathbf{a} \vee \mathbf{b}$, then there are $u, v \in X$ such that $\delta(x, u) \vee \delta(v, y) \leq \mathbf{a}$ and $\delta(u, v) \leq \mathbf{b}$. As every distance on a set X is obviously a p-measure on X viewed as a discrete poset, the problem of functorially lifting distributive $\langle \vee, 0 \rangle$ -semilattices by p-measures does not appear as difficult. The main problems encountered in the present work were (1) to get our posets *connected* (which is the case here as they are meet-semilattices), and (2) to get the subset $\{\|y = x\| \mid x \leq y \text{ in } P\}$ join-generating the semilattice S under question.

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