



**HAL**  
open science

# Hitting Time of a Corner for a Reflected Diffusion in the Square

François Delarue

► **To cite this version:**

François Delarue. Hitting Time of a Corner for a Reflected Diffusion in the Square. 2006. hal-00016374v1

**HAL Id: hal-00016374**

**<https://hal.science/hal-00016374v1>**

Preprint submitted on 2 Jan 2006 (v1), last revised 15 May 2008 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

François Delarue

## Hitting Time of a Corner for a Reflected Diffusion in the Square

2nd January 2006

**Abstract.** We investigate the expectation of the hitting time of a neighborhood of the origin for a two dimensional reflected diffusion in the unit square. More specifically, we distinguish three different regimes depending on the sign of the correlation coefficient of the diffusion matrix at the point 0.

For a positive correlation coefficient, the expectation of the hitting time is uniformly bounded as the neighborhood shrinks. For a negative one, the expectation explodes in a polynomial way as the diameter of the neighborhood vanishes. In the null case, the expectation explodes in a logarithmic rate.

From a practical point of view, the considered hitting time appears as the deadlock time for various resource sharing problems.

---

### 1. Introduction

Several resource sharing problems arising in data processing or in mathematical finance may be modeled by a stochastic process moving inside a given bounded domain with mixed reflection and absorption conditions. For example, Knuth (Exercise 2.2.2–13) [8], Ellis [2], Yao [21], Flajolet [4] and Maier [13] investigate the “colliding stacks” problem, that is the allocation system of a finite quantity  $N$  of memory units between two different stacks. At a given discrete time  $n$ , the number of units used by each stack is represented by the value at time  $n$  of a two-dimensional process  $X$  that lives inside the triangle  $[(0, 0), (0, N), (N, 0)]$ . The process  $X$  is reflected on the sides parallel to the axes and killed on the segment  $[(0, N), (N, 0)]$ : when  $X$  hits the slope between  $(0, N)$  and  $(N, 0)$ , the system stops since the memory is exhausted. Another famous example of distributed algorithm is given by the “banker algorithm”: two customers  $C_1$  and  $C_2$  share a finite amount  $\rho N$ ,  $\rho > 1$ , of money lent by a banker, so that the two-dimensional process  $X$  now stands for the quantity of money lent to  $C_1$  and  $C_2$ . It is usually assumed that the maximum need for each customer is exactly given by  $N$ . In this new frame, the state space for the process  $X$  writes as a square of size  $N$  with a broken corner delimited by the slope between  $(0, \rho N)$  and  $(\rho N, 0)$  (at least for  $\rho < 2$ ). Again, the process is reflected on the sides parallel to

---

Laboratoire de Probabilités et Modèles Aléatoires,  
Université Paris VII, UFR de Mathématiques, Case 7012,  
2, Place Jussieu, 75251 Paris Cedex 05, France

*Correspondence to:* delarue@math.jussieu.fr

the axes and the hitting time of the slope appears as a deadlock time for the allocation system. In both situations, this quantity is of crucial interest for the manager of the system: for this reason, being able to provide *a priori* bounds for the deadlock time, or at least for its expectation, in terms of the dynamic for  $X$ , constitutes a relevant challenge for practical applications. In a series of papers due (among others) to Louchard *et al.* [10], [11] and [12] and to Maier *et al.* [13] and [14] and more recently to Guillotin and Schott [6] and Comets, Delarue and Schott [1], it is shown in various contexts how to reduce, through a normalization procedure, the analysis for large values of  $N$  to an absorption problem for a limit process living in the unit square.

In all of these papers, the dynamic for the process  $X$  is assumed to be Markovian (usually indexed by integers). In the continuous setting (*i.e.* in the limit setting after the normalization procedure),  $X$  can be written as the solution of a stochastic differential equation with reflection on the boundary of the unit square. Again, the hitting time of a given slope breaking the right upper corner of the square then plays a fundamental role for practical applications. Unfortunately, few estimates have been established in the literature for such a quantity.

From a general point of view, the existence and uniqueness of solutions to RSDE are established, at least for standard cases, in the paper of Lions and Sznitman [9] (see also the earlier article due to Tanaka [18]). When the dimension reduces to two and the underlying domain to the orthant (*i.e.* to  $\mathbb{R}_+^2$ ) or more generally to a wedge, the existence and long time behavior of such processes are investigated in a series of papers due to Varadhan and Williams [20], Reiman and Williams [15] and [16] and to Taylor and Williams [19]. The monograph due to Fayolle, Malyshev and Menshikov [3] focuses on the time-discrete counterpart (see Subsection 3.3 there in).

Our own situation differs from the former ones since the underlying domain is bounded. The objective now consists in estimating the expectation of the hitting time of a given slope for the solution to the RSDE. In addition to the standard Lipschitz setting, we then assume the diffusion matrix  $a$  of the equation to be uniformly elliptic. We then analyze the asymptotic trend for the expectation of the hitting time as the slope gets closer and closer to the right upper corner of the unit square. More precisely, we show that the resulting behavior depends on the sign of the off-diagonal elements of  $a$  computed in  $(1, 1)$ : when positive, the expectation of the hitting time is uniformly bounded; when negative it explodes in a polynomial way and when zero, the expectation explodes in a logarithmic rate.

The proof is organized as follows. In Section 2, we expose the basic background for our analysis as well as the main result of the paper. In Section 3, we give the main lines of the proof. The basic argument follows from the analysis for the recurrence and transience properties of the two-dimensional Brownian motion or more generally of a diffusion process. Generally speaking, it is based on a Lyapunov function argument. In Section 4, we exhibit suitable Lyapunov functions for our own setting. We finally complete the proof in Section 5.

## 2. Notation and Main Result

We now describe the basic background for our purpose and then state the main result of the paper.

For  $d \geq 1$ ,  $|\cdot|$  denotes the Euclidean norm on  $\mathbb{R}^d$  and for  $x \in \mathbb{R}^d$  and  $r > 0$ ,  $B(x, r)$  denotes the (open) Euclidean ball of center  $x$  and radius  $r$ .

### 2.1. Reflected SDE

Denote by  $\mathcal{S}_2(\mathbb{R})$  the set of  $2 \times 2$  symmetric real matrices and consider, for a given triple  $(K, \lambda, A) \in (\mathbb{R}_+^*)^3$ , a couple of  $K$ -Lipschitz continuous coefficients  $(b, \sigma) : [0, 1]^2 \rightarrow \mathbb{R}^2 \times \mathcal{S}_2(\mathbb{R})$  such that for all  $(\xi, x) \in (\mathbb{R}^2)^2$ :

$$\lambda|\xi|^2 \leq \langle \xi, a(x)\xi \rangle \leq A|\xi|^2,$$

where  $a(x)$  denotes the symmetric positive matrix  $\sigma\sigma^*(x)$ .

Denote also by  $(\Omega, \mathcal{A}, \mathbb{P})$  a probability space endowed with a two-dimensional Brownian motion  $(B_t)_{t \geq 0}$ , whose natural filtration, augmented with  $\mathbb{P}$ -null sets, is denoted by  $(\mathcal{F}_t)_{t \geq 0}$ , and consider the Reflected Stochastic Differential Equation (RSDE in short) driven by the pair  $(b, \sigma)$ , by the boundary of the unit square  $[0, 1]^2$  and by the initial condition  $(1, 1)$ . We are then seeking a triple  $(X, H, K)$  of continuous and  $(\mathcal{F}_t)_{t \geq 0}$ -adapted processes with values in  $[0, 1]^2 \times (\mathbb{R}_+)^2 \times (\mathbb{R}_+)^2$  such that:

**RSDE(1)** The coordinates of  $H$  and  $K$  are non-decreasing processes.

**RSDE(2)** For  $i \in \{1, 2\}$ , the  $i$ th coordinate process  $H^i$  is non-increasing on the set  $\{t \geq 0, X_t^i > 0\}$  and the  $i$ th coordinate process  $K^i$  is non-increasing on the set  $\{t \geq 0, X_t^i < 1\}$ , so that:

$$\forall t \geq 0, \int_0^t \mathbf{1}_{\{X_r^i > 0\}} dH_r^i = 0, \int_0^t \mathbf{1}_{\{X_r^i < 1\}} dK_r^i = 0.$$

**RSDE(3)**  $X$  is an Itô process whose differential form writes for  $t \geq 0$ :

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t + dH_t - dK_t, \text{ with } X_0 = (1, 1)^t.$$

Thanks to Theorem 3.1 in Lions and Sznitman [9], the equation **RSDE(1-2-3)** admits a unique solution (set, for  $i = 1, 2$ ,  $dH_t^i = -\xi_t^i \mathbf{1}_{\{X_t^i = 0\}} d|k|_t$  and  $dK_t^i = \xi_t^i \mathbf{1}_{\{X_t^i = 1\}} d|k|_t$  under the notations of [9]).

### 2.2. Regimes for Absorption

For  $X$  as above, we are interested in the first hitting time of a given neighborhood of the origin. In practical applications (up to a rotation of center  $(1/2, 1/2)$ , see the references mentioned in Introduction), the neighborhood is delimited by the line of equation  $x + y = \rho$ , for  $\rho \in ]0, 1[$  (see Figure 1 below). We then focus on the hitting time  $T_\rho \equiv \inf\{t \geq 0, X_t^1 + X_t^2 \leq \rho\}$  and more specifically on  $\mathbb{E}(T_\rho)$ .

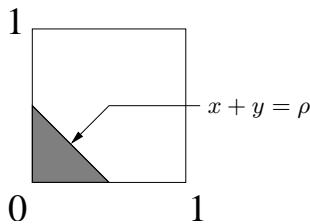


Fig. 1. Absorption Zone

When the matrix  $a$  is constant and diagonal, several explicit computations for the density of the process  $X$  before absorption are conceivable in terms of Bessel functions (see Louchard *et al.* [12]).

In the general framework, the story is rather different. In this paper, we manage to distinguish three different asymptotic regimes for the expectation  $\mathbb{E}(T_\rho)$  as the parameter  $\rho$  tends to zero, each of these regimes depending on the covariance matrix  $a(0)$ , and more precisely, on the sign of its off-diagonal components.

To detail these regimes, recall that the matrix  $a$  is assumed to be non-degenerate. It thus writes:

$$a(0) = \begin{pmatrix} \rho_1^2 & s\rho_1\rho_2 \\ s\rho_1\rho_2 & \rho_2^2 \end{pmatrix}, \quad (2.1)$$

with  $\rho_1, \rho_2 > 0$  and  $s \in ]-1, 1[$ . The matrix  $a(0)$  admits two eigenvalues:

$$\begin{cases} \lambda_1 = [\rho_1^2 + \rho_2^2 + \delta]/2, \\ \lambda_2 = [\rho_1^2 + \rho_2^2 - \delta]/2, \end{cases} \quad \text{with } \delta \equiv (\rho_1^4 + \rho_2^4 - 2(1 - 2s^2)\rho_1^2\rho_2^2)^{1/2}. \quad (2.2)$$

Denote by  $E_1$  and  $E_2$  the associated eigenvectors (up to a multiplicative constant). For  $s \neq 0$ ,

$$E_1 = \begin{pmatrix} 1 \\ (2s\rho_1\rho_2)^{-1}(\delta + \rho_2^2 - \rho_1^2) \end{pmatrix}, \quad E_2 = \begin{pmatrix} -(2s\rho_1\rho_2)^{-1}(\delta + \rho_2^2 - \rho_1^2) \\ 1 \end{pmatrix}.$$

Since  $\delta + \rho_2^2 - \rho_1^2 \geq 0$ , the signs of the non-trivial coordinates of  $E_1$  and  $E_2$  are given by the sign of  $s$ . The main eigenvector (*i.e.*  $E_1$ ) has two positive components for  $s > 0$ , and a positive one and a negative one for  $s < 0$ . Of course, if  $s$  vanishes,  $E_1$  and  $E_2$  reduce to the vectors of the canonical basis.

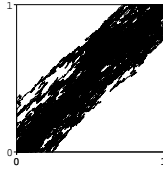
The three different regimes for  $\mathbb{E}(T_\rho)$  can be distinguished as follows:

**Positive Case.** If  $s > 0$ , the main eigenvector of  $a(0)$  (*i.e.*  $E_1$ ) is globally oriented from  $(0,0)$  to the neighborhood of the corner  $(1,1)$ , or up to a change of sign from  $(1,1)$  to the origin, and thus tends to push the reflected diffusion towards the absorption area. The reflection on the boundary cancels most of the effects of the second eigenvalue and keeps on bringing back the diffusion towards the main axis. As a consequence, the hitting time of

the border line is rather small and the following asymptotic holds for the diffusion starting from  $(1,1)$ :

$$\sup_{0 < \rho < 1} \mathbb{E}(T_\rho) < +\infty.$$

The following picture illustrates this phenomenon when  $b$  reduces to 0 and  $a$  is the constant matrix given by  $\rho_1 = \rho_2 = 1$  and  $s = 0,9$ . We plot below (see Figure 2) a simulated trajectory of the reflected diffusion process, starting from  $(1,1)$  at time 0, and running from time 0 to time 5 in the box  $[0, 1]^2$ . The algorithm used to simulate the reflected process is given in Słomiński [17]. The eigenvector  $E_1$  exactly matches  $(1, 1)^t$ .

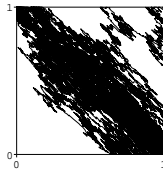


**Fig. 2.** Trajectory of the Reflected Process,  $s > 0$ .

**Negative Case.** If  $s < 0$ , the main eigenvector of  $a(0)$  is globally oriented from  $(1, 0)$  to the neighborhood of the corner  $(0, 1)$  and attracts the diffusion away from the border line. Again, the reflection on the boundary cancels most of the effects of the second eigenvalue, and thus, acts now as a trap: the diffusion stays for a long time along the main axis and hardly touches the boundary. The hitting time satisfies the following asymptotic behavior when the diffusion starts from  $(1, 1)$ :

$$\exists c_1, c_2 \geq 1, \forall \rho \in ]0, 1[, c_1^{-1} \rho^{-c_2} - c_1 \leq \mathbb{E}(T_\rho) \leq c_1 \rho^{-c_2^{-1}} + c_1.$$

This point is illustrated by Figure 3 below when  $b$  vanishes and  $a$  reduces to the constant matrix given by  $\rho_1 = \rho_2 = 1$  and  $s = -0,9$  (again, the initial condition is  $(1, 1)$ ). The eigenvector  $E_1$  is given, in this case, by  $(1, -1)^t$ .

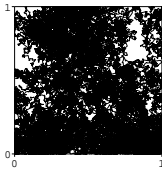


**Fig. 3.** Trajectory of the Reflected Process,  $s < 0$ .

**Null Case.** The case  $s = 0$  is intermediate. Eigenvectors are parallel to the axes and the behavior of the diffusion is close to the behavior of the two-dimensional Brownian motion. For the initial condition  $(1, 1)$ :

$$\exists c_1 \geq 1, \forall \rho \in ]0, 1[, -c_1^{-1} \ln(\rho) - c_1 \leq \mathbb{E}(T_\rho) \leq -c_1 \ln(\rho) + c_1.$$

This is illustrated by Figure 4 below when  $b$  vanishes and  $a$  reduces to the identity matrix (the initial condition of the process is  $(1, 1)$ ).



**Fig. 4.** Trajectory of the Reflected Process,  $s = 0$ .

### 2.3. Main Result

The following theorem summarizes the different cases detailed in the former subsection:

**Theorem 2.1** *There exists a constant  $C_{2.1} \geq 1$ , depending only on known parameters  $\lambda, \Lambda, K, \rho_1, \rho_2$  and  $s$ , such that:*

1. If  $s > 0$ ,  $\sup_{\rho \in ]0, 1[} \mathbb{E}(T_\rho) \leq C_{2.1}$
2. If  $s < 0$ , set  $\beta_- \equiv -s > 0$ ,  $\beta_+ \equiv s(s-3)(1+s)^{-1} > 0$ . Then,

$$\forall \rho \in ]0, 1[, C_{2.1}^{-1} \rho^{-\beta_-} - C_{2.1} \leq \mathbb{E}(T_\rho) \leq C_{2.1} \rho^{-\beta_+} + C_{2.1}.$$

3. If  $s = 0$ ,  $\forall \rho \in ]0, 1[, -C_{2.1}^{-1} \ln(\rho) - C_{2.1} \leq \mathbb{E}(T_\rho) \leq -C_{2.1} \ln(\rho) + C_{2.1}$ .

Note that Theorem 2.1 leaves open many questions. For example, we do not know how to compute, for  $s < 0$ , the exact value of the “true” exponent  $\beta \equiv \inf\{c > 0, \sup_{\rho \in ]0, 1[} [\rho^c \mathbb{E}(T_\rho)] < +\infty\}$ . We are even unable to specify the asymptotic behavior of  $\beta$  as  $s \rightarrow -1$  (note indeed that  $\lim_{s \rightarrow -1} \beta_- = 1$ ,  $\lim_{s \rightarrow -1} \beta_+ = +\infty$ ).

We have very few ideas about the extension of Theorem 2.1 to the upper dimensional cases. The only accessible case for us is  $a(0) = I_d$ ,  $I_d$  denoting the identity matrix of size  $d$ : in this case, the analysis derives from the transience properties of the Brownian motion in dimension  $d \geq 3$ . The arguments then mimic the ones used in the sequel for the two-dimensional setting  $s = 0$ .

*In the whole proof of Theorem 2.1, the constants appearing in various estimates just refer to known parameters  $\lambda, \Lambda, K, \rho_1, \rho_2$  and  $s$ . Even if denoted by the same letter, their values may vary from line to line.*

### 3. Description of the Method

Recall that the two-dimensional Brownian motion never hits zero at a positive time, but hits infinitely often any neighborhood of zero with probability one. The proof of this result (see e.g. Friedman [5], Chapter IX) relies on the differential form of the Bessel process of index 1, *i.e.* of the process  $|B|$ . In short, for  $B$  different from zero,  $d|B|$  writes  $d|B_t| = 1/(2|B_t|)dt + d\tilde{B}_t$ , where  $\tilde{B}$  denotes a one-dimensional Brownian motion.

The common strategy to investigate the recurrence and transience properties of the Brownian motion then consists in exhibiting a Lyapunov function for the process  $|B|$  (see again Friedman [5], Chapter IX, for a complete review on this topic). In dimension two, *i.e.* in our specific setting, the function  $\ln$  is harmonic for the process  $|B|$  (the Itô formula yields for  $B$  different from zero:  $d\ln(|B_t|) = |B_t|^{-1}d\tilde{B}_t$ ). The above asymptotic properties of  $B$  then follow from standard probabilistic arguments.

Roughly speaking, we aim to adapt this strategy to the reflected case. In this frame, the first question to answer is the following: what should be the auxiliary one-dimensional process for the reflected diffusion  $X$ ? Or, more precisely, what is the equivalent of the Bessel process in our setting?

#### 3.1. Natural Choice for the Underlying Functional

A good starting point seems to consider the following quadratic process:

$$\begin{aligned} \forall t \geq 0, R_t &\equiv \langle X_t, a^{-1}(0)X_t \rangle \\ &= (1-s^2)^{-1} [\rho_1^{-2}(X_t^1)^2 + \rho_2^{-2}(X_t^2)^2 - 2s\rho_1^{-1}\rho_2^{-1}X_t^1X_t^2], \end{aligned} \quad (3.1)$$

since  $a^{-1}(0) = (1-s^2)^{-1} \begin{pmatrix} \rho_1^{-2} & -s\rho_1^{-1}\rho_2^{-1} \\ -s\rho_1^{-1}\rho_2^{-1} & \rho_2^{-2} \end{pmatrix}$ .

The process  $R$  then aims to mimic the role played by  $|B|^2$  in the non-reflected Brownian case. Write indeed the differential form of  $R^{1/2}$ :

**Proposition 3.1** *There exist a constant  $C_{3.1}$  as well as a function  $\Gamma_{3.1} : \mathbb{R}^2 \rightarrow \mathbb{R}$ , bounded by  $C_{3.1}$ , such that for  $N \geq 1$  and  $t \in [0, \zeta_N]$ , with  $\zeta_N \equiv \inf\{t \geq 0, |X_t| \leq N^{-1}\}$ :*

$$\begin{aligned} dR_t^{1/2} &= \Gamma_{3.1}(X_t)dt + \frac{1}{2}R_t^{-1/2}dt \\ &\quad - \frac{s}{\sqrt{1-s^2}} [\rho_1^{-1}dH_t^1 + \rho_2^{-1}dH_t^2] - R_t^{-1/2} [\kappa_t^1 dK_t^1 + \kappa_t^2 dK_t^2] \\ &\quad + R_t^{-1/2} \langle \sigma(X_t) a^{-1}(0) X_t, dB_t \rangle, \end{aligned} \quad (3.2)$$

with:

$$\kappa_t^1 \equiv \frac{1}{1-s^2} [\rho_1^{-2} - s\rho_1^{-1}\rho_2^{-1}X_t^2], \quad \kappa_t^2 \equiv \frac{1}{1-s^2} [\rho_2^{-2} - s\rho_1^{-1}\rho_2^{-1}X_t^1]. \quad (3.3)$$

Proposition 3.1 is a particular case of the more general Proposition 3.3 given in the sequel. For this reason, the proof is put aside for the moment. Focus first on several consequences of Proposition 3.1:



- Csq(1)** For  $X$  close to 0, the scalar product  $R_t^{-1/2} \langle \sigma(X_t) a^{-1}(0) X_t, dB_t \rangle$  driving the martingale part of  $R^{1/2}$  looks like the process  $R_t^{-1/2} \langle \sigma^{-1}(0) X_t, dB_t \rangle$ , which is, thanks to Lévy's theorem, a Brownian motion.
- Csq(2)** For  $X$  close to 0, the term  $\Gamma_{3.1}(X_t)$  is negligible in front of  $R_t^{-1/2}$ .
- Csq(3)** For  $X$  close to 0, the  $dK$  terms vanish since  $dK^1$  is null for  $X^1 < 1$  and  $dK^2$  is null for  $X^2 < 1$ .
- Csq(4)** For  $s = 0$ , the  $dH$  terms reduce to zero.

Thus, from **Csq(1–4)**, the differential form of the process  $R_t^{1/2}$  looks like, for  $s = 0$  and for  $X$  close to 0, to the differential form of the Bessel process of index 1. We thus expect the function  $ln$  to be a kind of Lyapunov function (the exact meaning is to be specified) for  $R_t^{1/2}$  in the case  $s = 0$ , and thus to establish in this way the third point in Theorem 2.1.

Focus now on the cases  $s > 0$  and  $s < 0$ :

- Csq(5)** If  $s > 0$ , the  $dH$  terms are always negative. Each time the process  $X$  hits the border lines  $x^1 = 0$  or  $x^2 = 0$ , the process  $H$  attracts  $X$  towards 0.
- Csq(6)** If  $s < 0$ , the  $dH$  terms are always positive. At the opposite of the previous case, the process  $H$  repels  $X$  away from 0.

According to **Csq (5–6)** and to the previous discussion on the case  $s = 0$ , we then expect the case  $s < 0$  to be super-logarithmic and the case  $s > 0$  to be sub-logarithmic, and then hope to recover the three different regimes appearing in the statement of Theorem 2.1.

The very specific role of the function  $ln$  in the null case leads us to investigate the definition of a Lyapunov function for  $R^{1/2}$ ,  $s$  being possibly different from zero (refer to Chapter I in Hasminskii [7] for the definition of a Lyapunov function for an ordinary differential equation). In our setting, we seek for a function  $F$  satisfying at least:

- Lya(1)**  $F' \geq 0$ ,
- Lya(2)**  $L(F) \leq 0$ , where  $L$  denotes the second-order operator associated to the martingale and absolutely continuous parts of the differential form of  $R^{1/2}$ .

Due to the reflecting processes  $dH$  and  $dK$  and by analogy with the definition of a Lyapunov function for an ordinary differential equation (see (2.5), Chapter I in [7]), we also ask  $F$  to satisfy:

- Lya(3)**  $-sF'(R_t)[\rho_1^{-1}dH_t^1 + \rho_2^{-1}dH_t^2] \leq 0$ ,
- Lya(4)**  $-F'(R_t)[\kappa_t^1 dK_t^1 + \kappa_t^2 dK_t^2] \leq 0$ .

Seeking for a function  $F$  satisfying **Lya(1–2)** is rather conceivable, but seeking for a function  $F$  satisfying **Lya(1–4)** seems far from being trivial!

In fact, in view of **Lya(1)** and **Lya(4)**, a good solution for us would be  $\kappa^1, \kappa^2 \geq 0$ . Unfortunately, referring to (3.3),  $\kappa^1$  and  $\kappa^2$  may take, for  $s > 0$ , negative values for specific choices of  $\rho_1$  and  $\rho_2$ . Moreover, again in light of **Lya(1)**, the condition **Lya(3)** is hopeless for  $s < 0$ .

The plan is then the following: modify the choice of  $R$  to let  $\kappa^1$  and  $\kappa^2$  be positive and to get rid of the  $dH$  terms in the differential form of  $R^{1/2}$ .

### 3.2. A First Modification of the Auxiliary Process

We first modify  $R$  to take into account the conditions **Lya(1)** and **Lya(4)** in the above subsection. We set (note carefully that we keep the same notation  $R$ ):

$$\forall t \geq 0, R_t \equiv \langle X_t, a^{-1}(0)X_t \rangle + \gamma[X_t^1 X_t^2]^2, \quad (3.4)$$

where  $\gamma$  denotes a nonnegative real whose value is fixed in the sequel of the proof.

The choice of the correcting term  $\gamma[X^1 X^2]^2$  is rather natural. First, in light of **(RSDE2)**, the  $dK$  terms in the differential forms of  $(X^1)^2$  and  $(X^2)^2$  write respectively  $-2dK^1$  and  $-2dK^2$  and are thus nonpositive (as required in the condition **Lya(4)**). Second, the function  $(x_1, x_2) \mapsto x_1^2 x_2^2$  is of order four and is thus negligible in front of the scalar product  $\langle (x_1, x_2)^t, a^{-1}(0) (x_1, x_2)^t \rangle$  in the neighborhood of 0 (we thus expect the points **Csq(1–4)** to hold for this new form of  $R$  and for  $X$  close to 0).

Pay also attention to the fact that the connection between  $R$  and the usual Euclidean norm  $|X|^2$  is crucial. In short, both functionals measure in an equivalent way the distance from  $X$  to the corner  $(0, 0)$ . In this frame, the following lemma, whose proof is left to the reader, is very useful:

**Lemma 3.2** *For every  $x \in \mathbb{R}$ ,  $\lambda_1^{-1}|x|^2 \leq |\sigma^{-1}(0)x|^2 \leq \lambda_2^{-1}|x|^2$  (see (2.2) for the definition of  $\lambda_1$  and  $\lambda_2$ ). In particular, for every  $t \geq 0$ :*

$$\lambda_1^{-1}|X_t|^2 \leq \langle X_t, a^{-1}(0)X_t \rangle \leq \lambda_2^{-1}|X_t|^2 \leq 2\lambda_2^{-1}.$$

In particular, for all  $t \geq 0$ ,  $\lambda_1^{-1}|X_t|^2 \leq R_t \leq (\lambda_2^{-1} + \frac{\gamma}{2})|X_t|^2 \leq 2\lambda_2^{-1} + \gamma$ .

For a large fixed  $N$ , define the hitting time  $\zeta_N \equiv \inf\{t \geq 0, |X_t| \leq N^{-1}\}$ , and deduce that for every  $t \in [0, \zeta_N]$ ,  $R_t^{1/2} \geq \lambda_1^{-1/2}N^{-1} > 0$ .

**3.2.1. Differential Form of the Auxiliary Process  $R^{1/2}$**  We are now in position to write the differential form of the auxiliary process  $R^{1/2}$ :

**Proposition 3.3** *There exist a constant  $C_{3.3}$ , depending only on  $\gamma$  and on known parameters, as well as a function  $\Gamma_{3.3} : \mathbb{R}^2 \rightarrow \mathbb{R}$ , bounded by  $C_{3.3}$ , such that for  $t \in [0, \zeta_N]$ :*

$$\begin{aligned} dR_t^{1/2} &= \Gamma_{3.3}(X_t)dt + \frac{1}{2}R_t^{-1/2}dt \\ &\quad - \frac{s}{\sqrt{1-s^2}}[\rho_1^{-1}dH_t^1 + \rho_2^{-1}dH_t^2] - R_t^{-1/2}[\kappa_t^1 dK_t^1 + \kappa_t^2 dK_t^2] \\ &\quad + R_t^{-1/2}\langle \sigma(X_t)S(X_t)X_t, dB_t \rangle, \end{aligned} \quad (3.5)$$

with:

$$\begin{aligned} \kappa_t^1 &\equiv \frac{1}{1-s^2}[\rho_1^{-2} - s\rho_1^{-1}\rho_2^{-1}X_t^2 + (1-s^2)\gamma(X_t^2)^2], \\ \kappa_t^2 &\equiv \frac{1}{1-s^2}[\rho_2^{-2} - s\rho_1^{-1}\rho_2^{-1}X_t^1 + (1-s^2)\gamma(X_t^1)^2], \\ S(X_t) &\equiv a^{-1}(0) + \gamma(X_t^2)^2 \text{diag}(1, 0) + \gamma(X_t^1)^2 \text{diag}(0, 1), \end{aligned} \quad (3.6)$$

where the matrices  $\text{diag}(1, 0)$  and  $\text{diag}(0, 1)$  are diagonal matrices, whose diagonals are respectively equal to  $(1, 0)$  and to  $(0, 1)$ .

Focus a while on the consequences of Proposition 3.3:

1. First, the matrix  $S(x)$  remains close to  $a^{-1}(0)$  for  $x$  small. In particular, the martingale part of  $R^{1/2}$  still looks like a Brownian motion in the neighborhood of 0.
2. For  $s > 0$ , there is no difficulty to choose  $\gamma$  to make  $\kappa^1$  and  $\kappa^2$  positive.
3. For  $\gamma = 0$ , we recover Proposition 3.1.

*Proof of Proposition 3.3.* Write first the SDE satisfied by the process  $R$ :

**Lemma 3.4** *Keep the notation introduced in the statement of Proposition 3.3. Then, there exists a constant  $C_{3.4}$ , depending only on  $\gamma$  and on known parameters, as well as a function  $\Gamma_{3.4}$ , bounded by  $C_{3.4}$ , such that for  $t \geq 0$ :*

$$\begin{aligned} dR_t &= R_t^{1/2} \Gamma_{3.4}(X_t) dt + 2dt \\ &\quad - \frac{2s}{\sqrt{1-s^2}} R_t^{1/2} [\rho_1^{-1} dH_t^1 + \rho_2^{-1} dH_t^2] - 2[\kappa_t^1 dK_t^1 + \kappa_t^2 dK_t^2] \\ &\quad + 2\langle \sigma(X_t) S(X_t) X_t, dB_t \rangle. \end{aligned} \quad (3.7)$$

Assume for the moment that Lemma 3.4 holds and complete the proof of Proposition 3.3. Apply Itô's formula to  $R^{1/2}$  and derive from (3.7) (note that  $R$  does not vanish for  $t \in [0, \zeta_N]$ ) that for  $t \in [0, \zeta_N]$ ,

$$\begin{aligned} dR_t^{1/2} &= \left\{ \frac{1}{2} \Gamma_{3.4}(X_t) dt + R_t^{-1/2} dt \right\} \\ &\quad + \left\{ -\frac{s}{\sqrt{1-s^2}} [\rho_1^{-1} dH_t^1 + \rho_2^{-1} dH_t^2] - R_t^{-1/2} [\kappa_t^1 dK_t^1 + \kappa_t^2 dK_t^2] \right\} \\ &\quad + R_t^{-1/2} \langle \sigma(X_t) S(X_t) X_t, dB_t \rangle \\ &\quad - \frac{1}{2} R_t^{-3/2} \langle X_t, S(X_t) a(X_t) S(X_t) X_t \rangle dt \\ &\equiv [\Delta(1, X_t) + R_t^{-1/2}] dt + d\Delta_t(2) + d\Delta_t(3) - \frac{1}{2} R_t^{-3/2} \Delta(4, X_t) dt. \end{aligned} \quad (3.8)$$

Focus on  $\Delta(4, \cdot)$  in (3.8) and refer to (3.6) (definition of  $S$ ):

$$\begin{aligned} \forall t \in [0, \zeta_N], \quad \Delta(4, X_t) &= \langle X_t, S(X_t) X_t \rangle \\ &\quad + \langle X_t, [S(X_t) - a^{-1}(0)] a(0) S(X_t) X_t \rangle \\ &\quad + \langle X_t, S(X_t) [a(X_t) - a(0)] S(X_t) X_t \rangle \\ &= R_t + 2\gamma (X_t^1)^2 (X_t^2)^2 \\ &\quad + \langle X_t, [S(X_t) - a^{-1}(0)] a(0) S(X_t) X_t \rangle \\ &\quad + \langle X_t, S(X_t) [a(X_t) - a(0)] S(X_t) X_t \rangle \\ &\equiv R_t + \Delta(5, X_t). \end{aligned} \quad (3.9)$$

Thanks to the Lipschitz continuity and to the boundedness of  $a$ , to Lemma 3.2 (equivalence of  $R$  and  $|X|^2$ ) and to the definition of  $S$  (see the statement of Proposition 3.3), the function  $\Delta(5, \cdot)$  satisfies:

$$\forall t \in [0, \zeta_N], \quad |\Delta(5, X_t)| \leq CR_t^{3/2}. \quad (3.10)$$

where the constant  $C$  depends only on  $\gamma$  and on known parameters.

Plug now (3.9) into (3.8). For all  $t \in [0, \zeta_N]$ :

$$dR_t^{1/2} = \left[ \Delta(1, X_t) - \frac{1}{2}R_t^{-3/2}\Delta(5, X_t) + \frac{1}{2}R_t^{-1/2} \right] dt + d\Delta_t(2) + d\Delta_t(3). \quad (3.11)$$

Set finally  $\Gamma_{3.3}(X_t) \equiv \Delta(1, X_t) - (1/2)R_t^{-3/2}\Delta(5, X_t)$ . Thanks to (3.8), (3.10) and to (3.11), this completes the proof.  $\square$

*Proof of Lemma 3.4.* Thanks to Itô's formula,  $d\langle X, a^{-1}(0)X \rangle$  writes for  $t \geq 0$ :

$$\begin{aligned} d\langle X_t, a^{-1}(0)X_t \rangle &= 2\langle X_t, a^{-1}(0)b(X_t) \rangle dt + \text{trace}[\sigma^{-1}(0)a(X_t)\sigma^{-1}(0)] dt \\ &\quad + 2\langle X_t, a^{-1}(0)(dH_t - dK_t) \rangle + 2\langle a^{-1}(0)X_t, \sigma(X_t)dB_t \rangle \\ &= 2dt \\ &\quad + 2\langle X_t, a^{-1}(0)b(X_t) \rangle dt + \text{trace}[\sigma^{-1}(0)(a(X_t) - a(0))\sigma^{-1}(0)] dt \\ &\quad + 2\langle X_t, a^{-1}(0)(dH_t - dK_t) \rangle + 2\langle a^{-1}(0)X_t, \sigma(X_t)dB_t \rangle \end{aligned} \quad (3.12)$$

Focus for the moment on the  $dH$  and  $dK$  terms in the above r.h.s. Due to **(RSDE2)**, note first that  $X_t^1 dH_t^1 = 0$  and  $X_t^2 dH_t^2 = 0$  and that  $\rho_2^{-1} X_t^2 dH_t^1 = (1 - s^2)^{1/2} R_t^{1/2} dH_t^1$  and  $\rho_1^{-1} X_t^1 dH_t^2 = (1 - s^2)^{1/2} R_t^{1/2} dH_t^2$ . Derive that:

$$\begin{aligned} \langle X_t, a^{-1}(0)dH_t \rangle &= X_t^1 (a^{-1}(0))_{1,1} dH_t^1 + X_t^2 (a^{-1}(0))_{2,2} dH_t^2 \\ &\quad + (a^{-1}(0))_{1,2} [X_t^2 dH_t^1 + X_t^1 dH_t^2] \\ &= -\frac{s}{1-s^2} \rho_1^{-1} \rho_2^{-1} [X_t^2 dH_t^1 + X_t^1 dH_t^2] \\ &= -\frac{s}{\sqrt{1-s^2}} R_t^{1/2} [\rho_1^{-1} dH_t^1 + \rho_2^{-1} dH_t^2]. \end{aligned} \quad (3.13)$$

Note in the same way that  $X_t^1 dK_t^1 = dK_t^1$  and  $X_t^2 dK_t^2 = dK_t^2$ . Thus:

$$\begin{aligned} \langle X_t, a^{-1}(0)dK_t \rangle &= X_t^1 (a^{-1}(0))_{1,1} dK_t^1 + X_t^2 (a^{-1}(0))_{2,2} dK_t^2 \\ &\quad + (a^{-1}(0))_{1,2} [X_t^1 dK_t^2 + X_t^2 dK_t^1] \\ &= \frac{1}{1-s^2} [\rho_1^{-2} - s\rho_1^{-1}\rho_2^{-1}X_t^2] dK_t^1 \\ &\quad + \frac{1}{1-s^2} [\rho_2^{-2} - s\rho_1^{-1}\rho_2^{-1}X_t^1] dK_t^2. \end{aligned} \quad (3.14)$$

Plug now (3.13) and (3.14) into (3.12) and deduce from the boundedness of  $b$  and the Lipschitz continuity of  $a$  and from Lemma 3.2 (equivalence between  $R$  and  $|X|^2$ ) that there exists a function  $\Delta(6, \cdot)$  from  $\mathbb{R}^2$  into  $\mathbb{R}$ , bounded by a constant  $C$  (depending on  $\gamma$  and on known parameters) such that for all  $t \geq 0$ :

$$\begin{aligned}
& d\langle X_t, a^{-1}(0)X_t \rangle \\
&= 2dt + R_t^{1/2} \Delta(6, X_t) dt \\
&\quad - \frac{2s}{\sqrt{1-s^2}} R_t^{1/2} [\rho_1^{-1} dH_t^1 + \rho_2^{-1} dH_t^2] \\
&\quad - \frac{2}{1-s^2} [\rho_1^{-2} - s\rho_1^{-1}\rho_2^{-1}X_t^2] dK_t^1 - \frac{2}{1-s^2} [\rho_2^{-2} - s\rho_1^{-1}\rho_2^{-1}X_t^1] dK_t^2 \\
&\quad + 2\langle a^{-1}(0)X_t, \sigma(X_t)dB_t \rangle.
\end{aligned} \tag{3.15}$$

Note again from **(RSDE2)** that:

$$\begin{aligned}
d(X_t^1)^2 &= 2X_t^1 b_1(X_t) dt + a_{1,1}(X_t) dt - 2dK_t^1 + 2\langle X_t, \text{diag}(1, 0) \sigma(X_t) dB_t \rangle, \\
d(X_t^2)^2 &= 2X_t^2 b_2(X_t) dt + a_{2,2}(X_t) dt - 2dK_t^2 + 2\langle X_t, \text{diag}(0, 1) \sigma(X_t) dB_t \rangle.
\end{aligned} \tag{3.16}$$

Thus,  $d[X_t^1 X_t^2]^2$  writes:

$$\begin{aligned}
& d[X_t^1 X_t^2]^2 \\
&= \{ (X_t^2)^2 [2X_t^1 b_1(X_t) + a_{1,1}(X_t)] dt + (X_t^1)^2 [2X_t^2 b_2(X_t) + a_{2,2}(X_t)] dt \\
&\quad + 4\langle X_t, \text{diag}(1, 0) a(X_t) \text{diag}(0, 1) X_t \rangle \} dt \\
&\quad - 2(X_t^2)^2 dK_t^1 - 2(X_t^1)^2 dK_t^2 \\
&\quad + 2(X_t^2)^2 \langle X_t, \text{diag}(1, 0) \sigma(X_t) dB_t \rangle + 2(X_t^1)^2 \langle X_t, \text{diag}(0, 1) \sigma(X_t) dB_t \rangle \\
&\equiv \Delta(7, X_t) dt \\
&\quad - 2(X_t^2)^2 dK_t^1 - 2(X_t^1)^2 dK_t^2 \\
&\quad + 2(X_t^2)^2 \langle X_t, \text{diag}(1, 0) \sigma(X_t) dB_t \rangle + 2(X_t^1)^2 \langle X_t, \text{diag}(0, 1) \sigma(X_t) dB_t \rangle.
\end{aligned} \tag{3.17}$$

Due to the regularity of the coefficients and to Lemma 3.2, the term  $\Delta(7, X_t)$  is bounded by  $CR_t$ . Deduce then (3.7) from (3.15) and (3.17).  $\square$

*3.2.2. Choice of the Extra Quartic Perturbation* We now choose the parameter  $\gamma$  in (3.4). Focus to this end on (3.6) in Proposition 3.3.

**Lemma 3.5** *There exists a constant  $C_{3.5} > 0$  such that  $\forall t \geq 0$ ,  $S(X_t) \geq C_{3.5}^{-1} I_2$ , where  $I_2$  denotes the  $2 \times 2$ -identity matrix. Moreover,*



From now on, we always assume that  $\gamma$  is given by Lemma 3.5, *i.e.*  $\gamma = 0$  for  $s \leq 0$  and  $\gamma = (1 - s^2)^{-1} \max(\rho_1^{-2}, \rho_2^{-2})$  for  $s > 0$ . These choices ensure that the  $dK$  terms in the differential form of the auxiliary process (*cf.* (3.5)) are always uniformly negative. This point is crucial in the sequel of the paper.

### 3.3. Definitive Auxiliary Process

We now aim to correct the auxiliary process  $R^{1/2}$  to remove the component  $dH$  in the differential form (3.5) (*cf.* the discussion in the end of Subsection 3.1). We consider to this end the so-called definitive auxiliary process  $A_t \equiv R_t^{1/2} + Z_t$ ,  $t \geq 0$ , with:

$$\forall t \geq 0, Z_t \equiv \frac{s}{\sqrt{1-s^2}} \langle (\rho_1^{-1}, \rho_2^{-1})^t, X_t \rangle = \frac{s}{\sqrt{1-s^2}} [\rho_1^{-1} X_t^1 + \rho_2^{-1} X_t^2].$$

Again, it is crucial to verify first that the auxiliary process  $A$  and the process  $|X|$  are equivalent (in the sense of Lemma 3.2). According to Lemma 3.2, it is sufficient to prove that  $A$  and  $R^{1/2}$  are equivalent:

**Lemma 3.6** *The following bounds hold for  $A$ :*

1. For  $s > 0$  and  $t \in [0, \zeta_N]$ ,  $R_t^{1/2} \leq A_t \leq [1 + (2s^2(1-s)^{-1})^{1/2}] R_t^{1/2}$ .
2. For  $s < 0$  and  $t \in [0, \zeta_N]$ ,  $(1+s)(1-\sqrt{2s})^{-1} R_t^{1/2} \leq A_t \leq (1+s) R_t^{1/2}$ .

*Proof.* Note first that the following bound is obvious:  $R_t^{1/2} \leq A_t$  for  $s > 0$ . Focus now on the upper bound for  $s > 0$ . For all  $t \in [0, \zeta_N]$ ,

$$\begin{aligned} R_t &\geq (1-s^2)^{-1} [\rho_1^{-2} (X_t^1)^2 + \rho_2^{-2} (X_t^2)^2 - 2s\rho_1^{-1}\rho_2^{-1} X_t^1 X_t^2] \\ &= (1-s^2)^{-1} [(1-s)\rho_1^{-2} (X_t^1)^2 + (1-s)\rho_2^{-2} (X_t^2)^2 \\ &\quad + s(\rho_1^{-1} X_t^1 - \rho_2^{-1} X_t^2)^2] \\ &\geq (1-s^2)^{-1} (1-s) [\rho_1^{-2} (X_t^1)^2 + \rho_2^{-2} (X_t^2)^2] \\ &\geq (1-s) [2(1-s^2)]^{-1} [\rho_1^{-1} X_t^1 + \rho_2^{-1} X_t^2]^2 \\ &= (1-s) [2s^2]^{-1} Z_t^2. \end{aligned} \tag{3.18}$$

This completes the proof for  $s > 0$ .

Turn now to the case  $s < 0$ . Since for every  $t \in [0, \zeta_N]$ ,  $R_t^{1/2} - Z_t \geq R_t^{1/2} > 0$ , write for  $t \in [0, \zeta_N]$ :

$$\begin{aligned} A_t &= (R_t - Z_t^2) (R_t^{1/2} - Z_t)^{-1} \\ &= (R_t^{1/2} - Z_t)^{-1} (1-s^2)^{-1} [(1-s^2)\rho_1^{-2} (X_t^1)^2 + (1-s^2)\rho_2^{-2} (X_t^2)^2 \\ &\quad - 2s(1+s)\rho_1^{-1}\rho_2^{-1} X_t^1 X_t^2] \\ &= (R_t^{1/2} - Z_t)^{-1} (1-s)^{-1} [(1-s)\rho_1^{-2} (X_t^1)^2 + (1-s)\rho_2^{-2} (X_t^2)^2 \\ &\quad - 2s\rho_1^{-1}\rho_2^{-1} X_t^1 X_t^2]. \end{aligned} \tag{3.19}$$

Derive first the lower bound. Note that, for every  $t \geq 0$ ,  $-Z_t \leq -s(2R_t)^{1/2}$ . Hence, from (3.19), for  $t \in [0, \zeta_N]$ :

$$\begin{aligned} A_t &\geq (R_t^{1/2} - Z_t)^{-1}(1-s)^{-1} \\ &\quad \times [\rho_1^{-2}(X_t^1)^2 + \rho_2^{-2}(X_t^2)^2 - 2s\rho_1^{-1}\rho_2^{-1}X_t^1X_t^2] \\ &= (1+s)R_t(R_t^{1/2} - Z_t)^{-1} \\ &\geq (1+s)(1-\sqrt{2}s)^{-1}R_t^{1/2}. \end{aligned} \tag{3.20}$$

Turn finally to the upper bound. Note now that, for every  $t \geq 0$ ,  $-Z_t \geq -s(R_t)^{1/2}$ . Hence, from (3.19), for all  $t \in [0, \zeta_N]$ :

$$\begin{aligned} A_t &\leq (R_t^{1/2} - Z_t)^{-1}[\rho_1^{-2}(X_t^1)^2 + \rho_2^{-2}(X_t^2)^2 - 2s\rho_1^{-1}\rho_2^{-1}X_t^1X_t^2]. \\ &= (R_t^{1/2} - Z_t)^{-1}(1-s^2)R_t \\ &\leq (1-s)^{-1}(1-s^2)R_t^{1/2} \\ &= (1+s)R_t^{1/2}. \end{aligned} \tag{3.21}$$

Thanks to (3.20) and (3.21), we complete the proof for  $s < 0$ .  $\square$

Write now the differential form of the auxiliary process  $A$  (the proof derives from Proposition 3.3 and is thus left to the reader):

**Proposition 3.7** *There exist a constant  $C_{3.7}$  and a function  $\Gamma_{3.7}$ , bounded by  $C_{3.7}$ , such that for all  $t \in [0, \zeta_N]$ :*

$$\begin{aligned} dA_t &= \Gamma_{3.7}(X_t)dt + \frac{1}{2}R_t^{-1/2}dt - [\bar{\kappa}_t^1 dK_t^1 + \bar{\kappa}_t^2 dK_t^2] \\ &\quad + \langle R_t^{-1/2}\sigma(X_t)S(X_t)X_t + \frac{s}{\sqrt{1-s^2}}\sigma(X_t)(\rho_1^{-1}, \rho_2^{-1})^t, dB_t \rangle, \end{aligned}$$

$$\text{with } \bar{\kappa}_t^1 = R_t^{-1/2}\kappa_t^1 + \frac{s}{\sqrt{1-s^2}}\rho_1^{-1}, \quad \bar{\kappa}_t^2 = R_t^{-1/2}\kappa_t^2 + \frac{s}{\sqrt{1-s^2}}\rho_2^{-1}.$$

Note that the processes  $\bar{\kappa}^1$  and  $\bar{\kappa}^2$  are uniformly positive for  $X^1$  or  $X^2$  matching one. Indeed, if  $s < 0$ , the first case in Lemma 3.5 applies, and if  $s \geq 0$ ,  $\bar{\kappa}^1$  (resp.  $\bar{\kappa}^2$ ) is greater than  $\kappa^1$  (resp.  $\kappa^2$ ).

### 3.4. Definition of the New Boundary

At this stage of the paper, it is rather judicious to introduce a new boundary that is more appropriate to the auxiliary process  $A$ . The following choice is the most natural:

**Definition 3.8** *For a real  $N > 0$ , define:*

$$\begin{aligned} \mathcal{B}_N \text{ (resp. } \partial\mathcal{B}_N) &\equiv \{(x, y) \in [0, 1]^2, \\ &\quad (|\sigma^{-1}(0)(x, y)^t|^2 + \gamma x^2 y^2)^{1/2} + \frac{s}{\sqrt{1-s^2}}(\rho_1^{-1}x + \rho_2^{-1}y) \leq (\text{resp. } =)N^{-1}\}, \end{aligned}$$

so that the associated hitting time writes  $\xi_N \equiv \inf\{t \geq 0, A_t \leq N^{-1}\}$ .



Thanks to Lemmas 3.2 and 3.6, there exists a constant  $c_{3.8} \geq 1$  such that, for  $N > c_{3.8}$ ,  $\Delta_{c_{3.8}N} \subset \mathcal{B}_N \subset \Delta_{c_{3.8}^{-1}N}$  where  $\Delta_t$ , for  $t \geq 0$ , denotes the quarter ball  $B(0, t^{-1}) \cap \mathbb{R}_+^2$ . It then comes  $\zeta_{c_{3.8}^{-1}N} \leq \xi_N \leq \zeta_{c_{3.8}N}$  for  $N > c_{3.8}$ .

In the sequel,  $N$  is chosen greater than  $c_{3.8}$ , so that the condition  $t \in [0, \zeta_N]$  can be replaced by  $t \in [0, \xi_N]$  in all the former statements.

#### 4. Lyapunov Functions for the Definitive Auxiliary Process

We now look for a Lyapunov function for the process  $A$ .

##### 4.1. Itô's Formula for the Auxiliary Functional

We first investigate the differential form of  $F(A)$ , for  $F$  smooth:

**Proposition 4.1** *The following bounds hold for all  $t \in [0, \xi_N]$ :*

$$\left(1 + \frac{2s^2}{1-s}\right) + 2Z_t R_t^{-1/2} \geq \begin{cases} 1 & \text{for } s \geq 0, \\ (1+s)^2(1-s)^{-2} > 0 & \text{for } s < 0. \end{cases} \quad (4.1)$$

Moreover, there exist a constant  $C_{4.1} \geq 1$  and two functions  $\Gamma_{4.1}$  and  $\Psi_{4.1}$  from  $\mathbb{R}^2$  into  $\mathbb{R}$  such that  $|\Gamma_{4.1}|, \Psi_{4.1} \leq C_{4.1}$  and  $\Psi_{4.1} \geq C_{4.1}^{-1}$  and such that for every  $F \in \mathcal{C}^2(\mathbb{R}_+^*, \mathbb{R})$  and every  $t \in [0, \xi_N]$ :

$$\begin{aligned} dF(A_t) &= \Psi_{4.1}(X_t) \left\{ F''(A_t) + \left[ \left(1 + \frac{2s^2}{1-s}\right) R_t^{1/2} + 2Z_t \right]^{-1} + \Gamma_{4.1}(X_t) \right\} F'(A_t) dt \\ &\quad - F'(A_t) [\bar{\kappa}_t^1 dK_t^1 + \bar{\kappa}_t^2 dK_t^2] \\ &\quad + F'(A_t) \langle R_t^{-1/2} \sigma(X_t) S(X_t) X_t + \frac{s}{\sqrt{1-s^2}} \sigma(X_t) (\rho_1^{-1}, \rho_2^{-1})^t, dB_t \rangle. \end{aligned}$$

*Proof.* Assume for the moment the following statement (whose proof is postponed to the end of the subsection):

**Lemma 4.2** *Bounds given in (4.1) hold and there exists a constant  $C_{4.2} > 0$  such that for all  $t \in [0, \xi_N]$ :*

$$\Psi(X_t) \equiv \left| R_t^{-1/2} \sigma(X_t) S(X_t) X_t + \frac{s}{\sqrt{1-s^2}} \sigma(X_t) (\rho_1^{-1}, \rho_2^{-1})^t \right|^2 \geq C_{4.2}. \quad (4.2)$$

Turn now to the differential form of  $F(A)$ . Deduce first from Proposition 3.7 and Itô's formula that for all  $t \in [0, \xi_N]$ :

$$\begin{aligned} dF(A_t) &= F'(A_t) \left[ \Gamma_{3.7}(X_t) + \frac{1}{2} R_t^{-1/2} \right] dt - F'(A_t) [\bar{\kappa}_t^1 dK_t^1 + \bar{\kappa}_t^2 dK_t^2] \\ &\quad + F'(A_t) \langle R_t^{-1/2} \sigma(X_t) S(X_t) X_t + \frac{s}{\sqrt{1-s^2}} \sigma(X_t) (\rho_1^{-1}, \rho_2^{-1})^t, dB_t \rangle, \\ &\quad + \frac{1}{2} F''(A_t) \left| R_t^{-1/2} \sigma(X_t) S(X_t) X_t + \frac{s}{\sqrt{1-s^2}} \sigma(X_t) (\rho_1^{-1}, \rho_2^{-1})^t \right|^2 dt, \end{aligned}$$

so that the last term writes  $(1/2)F''(A_t)\Psi(X_t)$ . From Lemma 4.2,  $\Psi(X)$  is uniformly positive on  $[0, \xi_N]$ . Moreover, due to the boundedness of  $\sigma$  and to Lemmas 3.2 and 3.5, it is also bounded. Hence, for all  $t \in [0, \xi_N]$ :

$$\begin{aligned} dF(A_t) &= -F'(A_t)[\bar{\kappa}_t^1 dK_t^1 + \bar{\kappa}_t^2 dK_t^2] \\ &+ \frac{\Psi(X_t)}{2} [F''(A_t) + (\Psi(X_t))^{-1}F'(A_t)(R_t^{-1/2} + 2\Gamma_{3.7}(X_t))] dt \\ &+ F'(A_t)\langle R_t^{-1/2}\sigma(X_t)S(X_t)X_t + \frac{s}{\sqrt{1-s^2}}\sigma(X_t)(\rho_1^{-1}, \rho_2^{-1})^t, dB_t \rangle. \end{aligned} \quad (4.3)$$

Develop now  $\Psi(X)$ . For all  $t \in [0, \xi_N]$ :

$$\begin{aligned} \Psi(X_t) &= R_t^{-1}|\sigma(X_t)S(X_t)X_t|^2 + \frac{s^2}{1-s^2}|\sigma(X_t)(\rho_1^{-1}, \rho_2^{-1})^t|^2 \\ &+ 2R_t^{-1/2}\frac{s}{\sqrt{1-s^2}}\langle a(X_t)S(X_t)X_t, (\rho_1^{-1}, \rho_2^{-1})^t \rangle \\ &= R_t^{-1}|\sigma(0)S(0)X_t + [\sigma(X_t)S(X_t) - \sigma(0)S(0)]X_t|^2 \\ &+ \frac{s^2}{1-s^2}|\sigma(0)(\rho_1^{-1}, \rho_2^{-1})^t + [\sigma(X_t) - \sigma(0)](\rho_1^{-1}, \rho_2^{-1})^t|^2 \\ &+ 2R_t^{-1/2}\frac{s}{\sqrt{1-s^2}}\langle a(0)S(0)X_t \\ &+ [a(X_t)S(X_t) - a(0)S(0)]X_t, (\rho_1^{-1}, \rho_2^{-1})^t \rangle. \end{aligned}$$

Since  $S(0) = a^{-1}(0)$ , we can write for  $t \in [0, \xi_N]$ :

$$\begin{aligned} \Psi(X_t) &\equiv R_t^{-1}|\sigma^{-1}(0)X_t + e(1, X_t)X_t|^2 \\ &+ \frac{s^2}{1-s^2}|\sigma(0)(\rho_1^{-1}, \rho_2^{-1})^t + e(2, X_t)|^2 \\ &+ 2R_t^{-1/2}\frac{s}{\sqrt{1-s^2}}\langle X_t + e(3, X_t)X_t, (\rho_1^{-1}, \rho_2^{-1})^t \rangle. \end{aligned} \quad (4.4)$$

From the Lipschitz continuity of the coefficients and from Lemma 3.2, there exists a constant  $C$  such that  $|e(1, X_t)|, |e(2, X_t)|, |e(3, X_t)| \leq CR_t^{1/2}$ . Develop now the squares related to  $e(1, \cdot)$  and  $e(2, \cdot)$  in (4.4). There exists a function  $E(1, \cdot) : \mathbb{R}^2 \rightarrow \mathbb{R}$ , bounded by  $C$ , such that:

$$\begin{aligned} \Psi(X_t) &= R_t^{-1}|\sigma^{-1}(0)X_t|^2 + \frac{s^2}{1-s^2}|\sigma(0)(\rho_1^{-1}, \rho_2^{-1})^t|^2 \\ &+ 2R_t^{-1/2}\frac{s}{\sqrt{1-s^2}}\langle X_t, (\rho_1^{-1}, \rho_2^{-1})^t \rangle + R_t^{1/2}E(1, X_t). \end{aligned} \quad (4.5)$$

Write, in (4.5),  $|\sigma^{-1}(0)X_t|^2 = \langle X_t, a^{-1}(0)X_t \rangle = R_t - \gamma(X_t^1)^2(X_t^2)^2$  and  $|\sigma(0)(\rho_1^{-1}, \rho_2^{-1})^t|^2 = 2(1+s)$ . Up to a modification of  $E(1, \cdot)$ , deduce that:

$$\begin{aligned} \Psi(X_t) &= 1 + \frac{2s^2}{1-s} + 2R_t^{-1/2}\frac{s}{\sqrt{1-s^2}}\langle X_t, (\rho_1^{-1}, \rho_2^{-1})^t \rangle + R_t^{1/2}E(1, X_t) \\ &= 1 + \frac{2s^2}{1-s} + 2R_t^{-1/2}Z_t + R_t^{1/2}E(1, X_t). \end{aligned} \quad (4.6)$$

Thanks to (4.1) and (4.2), we can consider the inverses in (4.6):

$$\begin{aligned}
(\Psi(X_t))^{-1} &= \left(1 + \frac{2s^2}{1-s} + 2R_t^{-1/2}Z_t\right)^{-1} - \left(1 + \frac{2s^2}{1-s} + 2R_t^{-1/2}Z_t\right)^{-1} \\
&\quad + \left(1 + \frac{2s^2}{1-s} + 2R_t^{-1/2}Z_t + R_t^{1/2}E(1, X_t)\right)^{-1} \\
&= \left(1 + \frac{2s^2}{1-s} + 2R_t^{-1/2}Z_t\right)^{-1} \\
&\quad - R_t^{1/2}E(1, X_t) \left[\left(1 + \frac{2s^2}{1-s} + 2R_t^{-1/2}Z_t\right)\right. \\
&\quad \left. \times \left(1 + \frac{2s^2}{1-s} + 2R_t^{-1/2}Z_t + R_t^{1/2}E(1, X_t)\right)\right]^{-1} \\
&\equiv \left(1 + \frac{2s^2}{1-s} + 2R_t^{-1/2}Z_t\right)^{-1} - R_t^{1/2}E(2, X_t).
\end{aligned} \tag{4.7}$$

Thanks to the boundedness of  $E(1, X_t)$ , to (4.1), (4.6) and (4.2),  $E(2, \cdot)$  is bounded. Plug (4.7) into (4.3) and complete the proof.  $\square$

*Proof (Lemma 4.2).* Prove first (4.1). Since the result is obvious for  $s \geq 0$ , we focus on the case  $s < 0$ . For  $s < 0$ ,  $\gamma$  reduces to 0 and  $R$  writes as a quadratic functional. In particular, for  $t \in [0, \xi_N]$ :

$$\begin{aligned}
&\left(1 + \frac{2s^2}{1-s}\right) + 2Z_t R_t^{-1/2} \\
&= 1 + \frac{2s^2}{1-s} + 2\langle R_t^{-1/2}\sigma^{-1}(0)X_t, \frac{s}{\sqrt{1-s^2}}\sigma(0)(\rho_1^{-1}, \rho_2^{-1})^t \rangle \\
&\geq 1 + \frac{2s^2}{1-s} - 2|R_t^{-1/2}\sigma^{-1}(0)X_t| \left| \frac{s}{\sqrt{1-s^2}}\sigma(0)(\rho_1^{-1}, \rho_2^{-1})^t \right|.
\end{aligned} \tag{4.8}$$

Since  $|R_t^{-1/2}\sigma^{-1}(0)X_t| = 1$  and  $|\sigma(0)(\rho_1^{-1}, \rho_2^{-1})^t|^2 = 2(1+s)$ , deduce:

$$\begin{aligned}
\left(1 + \frac{2s^2}{1-s}\right) + 2Z_t R_t^{-1/2} &\geq 1 + \frac{2s^2}{1-s} - 2\frac{\sqrt{2}|s|}{\sqrt{1-s}} \\
&= \left[1 - \frac{\sqrt{2}|s|}{\sqrt{1-s}}\right]^2 \\
&= \left[1 - \frac{2s^2}{1-s}\right]^2 \left[1 + \frac{\sqrt{2}|s|}{\sqrt{1-s}}\right]^{-2} \\
&= \frac{(1+s)^2(1-2s)^2}{(1-s)^2} \left[1 + \frac{\sqrt{2}|s|}{\sqrt{1-s}}\right]^{-2} \\
&= (1+s)^2\varphi(s),
\end{aligned} \tag{4.9}$$

with  $\forall s \in [-1, 0]$ ,  $\varphi(s) \equiv \frac{(1-2s)^2}{(1-s)^2} \left[1 + \frac{\sqrt{2}|s|}{\sqrt{1-s}}\right]^{-2}$ .

Note in this frame that:

$$\begin{aligned} \forall s \in [-1, 0], \varphi(s) &= \frac{(1-2s)^2}{(1-s)^2} \left[ 1 + \frac{\sqrt{2}|s|}{\sqrt{1-s}} \right]^{-2} \\ &\geq \frac{(1-2s)^2}{(1-s)^2} [1 + 2|s|]^{-2} = (1-s)^{-2}. \end{aligned} \quad (4.10)$$

Deduce (4.1) for  $s < 0$  from (4.9) and (4.10).

Complete now the proof of Lemma 4.2. Since the matrices  $(\sigma(X_t))_{t \geq 0}$  are uniformly elliptic, derive for a suitable constant  $C > 0$  and for all  $t \in [0, \xi_N]$ :

$$\begin{aligned} \Psi(X_t) &= |R_t^{-1/2} \sigma(X_t) S(X_t) X_t + \frac{s}{\sqrt{1-s^2}} \sigma(X_t) (\rho_1^{-1}, \rho_2^{-1})^t|^2 \\ &\geq C |R_t^{-1/2} S(X_t) X_t + \frac{s}{\sqrt{1-s^2}} (\rho_1^{-1}, \rho_2^{-1})^t|^2. \end{aligned} \quad (4.11)$$

Note now that the matrices  $(S(X_t))_{t \geq 0}$  are also uniformly elliptic (see Lemma 3.5). Hence, deduce from (4.11) that for a new constant  $C > 0$ :

$$\begin{aligned} \Psi(X_t) &\geq C |R_t^{-1/2} S^{1/2}(X_t) X_t + \frac{s}{\sqrt{1-s^2}} S^{-1/2}(X_t) (\rho_1^{-1}, \rho_2^{-1})^t|^2 \\ &= C [R_t^{-1} |S^{1/2}(X_t) X_t|^2 + 2 \frac{s}{\sqrt{1-s^2}} R_t^{-1/2} \langle X_t, (\rho_1^{-1}, \rho_2^{-1})^t \rangle \\ &\quad + \frac{s^2}{1-s^2} |S^{-1/2}(X_t) (\rho_1^{-1}, \rho_2^{-1})^t|^2]. \end{aligned} \quad (4.12)$$

For  $s \geq 0$ , the result then derives from the inequality  $S \geq a^{-1}(0)$ . For  $s < 0$ , the function  $S^{1/2}$  is constant and matches  $\sigma^{-1}(0)$ . In this frame, up to the constant  $C$ , the last term in (4.12) coincides with  $(1 + 2s^2(1-s)^{-1}) + 2Z_t R_t^{-1/2}$ .  $\square$

#### 4.2. Typical Example for $F$

We now exhibit the typical choice for  $F$ . For  $\alpha \in \mathbb{R}$  and  $C \in \mathbb{R}$ :

$$\forall r > 0, F'(r) = r^{\alpha-1} \exp(-Cr). \quad (4.13)$$

The parameters  $\alpha$  and  $C$  are to be fixed in the sequel of the proof.  $F$  satisfies the ODE  $F''(r) + (1-\alpha)r^{-1}F'(r) = -CF'(r)$ ,  $r > 0$ . Derive then from Proposition 4.1 that for all  $t \in [0, \xi_N]$ :

$$\begin{aligned} dF(A_t) &= -F'(A_t) [\bar{\kappa}_t^1 dK_t^1 + \bar{\kappa}_t^2 dK_t^2] \\ &\quad + \Psi_{4.1}(X_t) \left[ (\alpha-1)A_t^{-1} + \left(1 + \frac{2s^2}{1-s}\right) R_t^{1/2} + 2Z_t \right]^{-1} \\ &\quad + \Gamma_{4.1}(X_t) - C] F'(A_t) dt \\ &\quad + F'(A_t) \langle R_t^{-1/2} \sigma(X_t) S(X_t) X_t + \frac{s}{\sqrt{1-s^2}} \sigma(X_t) (\rho_1^{-1}, \rho_2^{-1})^t, dB_t \rangle. \end{aligned} \quad (4.14)$$

Of course, it is crucial to estimate the primitive of  $F'$ , or more precisely, to choose a suitable version of  $F$ . Thanks to Lemmas 3.2 and 3.6, deduce the following proposition:

**Proposition 4.3** Set  $\delta \equiv (1 + (2s^2(1-s)^{-1})^{1/2} \mathbf{1}_{s>0})(2\lambda_2^{-1} + \gamma)^{1/2}$  (so that  $\forall t \geq 0, A_t \leq \delta$ ). Then, for every  $r \in ]0, \delta]$ :

1. If  $\alpha > 0$  and  $F(0) = 0$ , then:
  - (a) If  $C \geq 0$ ,  $0 \leq \alpha^{-1} \exp(-\delta C) r^\alpha \leq F(r) \leq \alpha^{-1} r^\alpha$ ,
  - (b) If  $C \leq 0$ ,  $0 \leq \alpha^{-1} r^\alpha \leq F(r) \leq \alpha^{-1} \exp(-\delta C) r^\alpha$ ,
2. If  $\alpha = 0$  and  $F(\delta) = 0$ , then:
  - (a) If  $C \geq 0$ ,  $\ln(r/\delta) \leq F(r) \leq \exp(-\delta C) \ln(r/\delta) \leq 0$ ,
  - (b) If  $C \leq 0$ ,  $\exp(-\delta C) \ln(r/\delta) \leq F(r) \leq \ln(r/\delta) \leq 0$ ,
3. If  $\alpha < 0$  and  $F(\delta) = 0$ , then:
  - (a) If  $C \geq 0$ ,  $\alpha^{-1} [r^\alpha - (\delta)^\alpha] \leq F(r) \leq \alpha^{-1} \exp(-\delta C) [r^\alpha - (\delta)^\alpha] \leq 0$ ,
  - (b) If  $C \leq 0$ ,  $\alpha^{-1} \exp(-\delta C) [r^\alpha - (\delta)^\alpha] \leq F(r) \leq \alpha^{-1} [r^\alpha - (\delta)^\alpha] \leq 0$ .

#### 4.3. Standard Quadratic Process and Related Lyapunov Functions

The objective now consists in choosing suitable parameters  $\alpha$  and  $C$  in (4.14) to get rid of the  $dt$  term in (4.14) and then in adding another differential form to get rid of the  $dK$  terms. In short, this extra differential form is given by a functional of the square of the Euclidean norm of  $X$ . Define to this end:

$$\forall t \geq 0, \Sigma_t \equiv |X_t|^2. \quad (4.15)$$

The differential of  $\Sigma$  is easily computed (see (3.16)):

**Proposition 4.4** There exist a constant  $C_{4.4}$  and a function  $\phi_{4.4}$  from  $\mathbb{R}^2$  into  $\mathbb{R}$ , bounded by  $C_{4.4}$ , such that the differential  $d\Sigma$  writes for  $t \geq 0$ :

$$d\Sigma_t = \text{trace}(a(0))dt + \phi_{4.4}(X_t) \Sigma_t^{1/2} dt - 2[dK_t^1 + dK_t^2] + 2\langle X_t, \sigma(X_t) dB_t \rangle. \quad (4.16)$$

As done for  $A$  and  $F$ , we need to associate to  $\Sigma$  a suitable family of functionals. Define for a given  $D \in \mathbb{R}$ :

$$\forall r > 0, G(r) \equiv \int_0^r \exp(Dv^{1/2}) dv. \quad (4.17)$$

It is well seen that  $\forall r > 0$ ,  $G'(r) = \exp(Dr^{1/2})$ ,  $G''(r) = (D/2)r^{-1/2}G'(r)$ . Again, the following proposition mimics the previous results obtained for  $F$  and is thus left to the reader:

**Lemma 4.5** Let  $G$  be given by (4.17). Then for every  $r \in [0, 2]$  (recall  $\forall t \geq 0, \Sigma_t \leq 2$ ):

1. If  $D \geq 0$ , then  $0 \leq r \leq G(r) \leq r \exp(2^{1/2}D)$ .
2. If  $D \leq 0$ , then  $0 \leq r \exp(2^{1/2}D) \leq G(r) \leq r$ .

We are finally in position to express the differential form of  $G(\Sigma)$  (again, the proof is left to the reader):

**Proposition 4.6** *Let  $G$  be given by (4.17). Then, for  $t \in [0, \xi_N]$ :*

$$\begin{aligned} dG(\Sigma_t) &= G'(\Sigma_t)\text{trace}(a(0))dt \\ &+ [D\Sigma_t^{-1/2}|\sigma(X_t)X_t|^2 + \phi_{4.4}(X_t)\Sigma_t^{1/2}]G'(\Sigma_t)dt \\ &- 2G'(\Sigma_t)[dK_t^1 + dK_t^2] + 2G'(\Sigma_t)\langle X_t, \sigma(X_t)dB_t \rangle. \end{aligned}$$

The strategy now consists in balancing  $\phi_{4.4}$  with a suitable choice for  $D$  (depending on the ellipticity constant of  $a$ ). From Lemma 4.5 and Proposition 4.6, we claim:

**Corollary 4.7** *Under the notations of Proposition 4.4, it comes for  $t \in [0, \xi_N]$ :*

1. *If  $D = -\lambda^{-1}C_{4.4}$ ,  $dG(\Sigma_t) \leq (\rho_1 + \rho_2)dt - 2\exp(-2^{1/2}\lambda^{-1}C_{4.4})[dK_t^1 + dK_t^2] + 2\exp(D\Sigma_t^{1/2})\langle X_t, \sigma(X_t)dB_t \rangle$ ,*
2. *If  $D = \lambda^{-1}C_{4.4}$ ,  $dG(\Sigma_t) \geq (\rho_1 + \rho_2)dt - 2\exp(2^{1/2}\lambda^{-1}C_{4.4})[dK_t^1 + dK_t^2] + 2\exp(D\Sigma_t^{1/2})\langle X_t, \sigma(X_t)dB_t \rangle$ .*

## 5. Proof of the Main Bounds

We now investigate the asymptotic behavior of  $\mathbb{E}[\xi_N]$ .

### 5.1. Intermediate Case: $s = 0$

We first focus on the simple case  $s = 0$ . Choose to this end  $F$  as given in (4.13) with  $\alpha = 0$  and  $F(\delta) = 0$ . Since  $Z$  vanishes and  $A$  reduces to  $R^{1/2}$  (see Subsection 3.3), it comes from (4.14) for  $t \in [0, \xi_N]$ :

$$\begin{aligned} dF(R_t^{1/2}) &= \Psi_{4.1}(X_t)[\Gamma_{4.1}(X_t) - C]F'(R_t^{1/2})dt \\ &- F'(R_t^{1/2})[\bar{\kappa}_t^1 dK_t^1 + \bar{\kappa}_t^2 dK_t^2] + F'(R_t^{1/2})\langle R_t^{-1/2}\sigma(X_t)S(X_t)X_t, dB_t \rangle. \end{aligned} \quad (5.1)$$

#### 5.1.1. Upper Bound for the Expectation of $\xi_N$

**Theorem 5.1** *Assume that  $s = 0$ . Then, there exists a constant  $C_{5.1}$  such that  $\mathbb{E}[\xi_N] \leq C_{5.1}(1 + \ln(N))$ .*

*Proof.* Choose  $F$  as in (5.1) with  $C = C_{4.1}$  (see Proposition 4.1). Then, due to the positivity of  $\Psi_{4.1}$  (and the one of  $F'$ ):

$$\begin{aligned} \forall t \in [0, \xi_N], \quad dF(R_t^{1/2}) &\leq -F'(R_t^{1/2})[\bar{\kappa}_t^1 dK_t^1 + \bar{\kappa}_t^2 dK_t^2] \\ &+ F'(R_t^{1/2})\langle R_t^{-1/2}\sigma(X_t)S(X_t)X_t, dB_t \rangle. \end{aligned} \quad (5.2)$$

Focus on the  $dK$  terms. Again, the functional  $F'(R^{1/2})$  is uniformly bounded from below by a positive constant. Moreover, thanks to Proposition 3.7, the processes  $\bar{\kappa}^1$  and  $\bar{\kappa}^2$  are also uniformly bounded from below by a positive

constant when one of the coordinate of  $X$  matches one. Hence, there exists a constant  $m > 0$ , such that for all  $t \in [0, \xi_N]$ :

$$dF(R_t^{1/2}) \leq -m^{-1} [dK_t^1 + dK_t^2] + F'(R_t^{1/2}) \langle R_t^{-1/2} \sigma(X_t) S(X_t) X_t, dB_t \rangle. \quad (5.3)$$

Apply now the point 2 in Corollary 4.7 (with the same  $D$ ) and deduce that for every  $t \in [0, \xi_N]$ :

$$\begin{aligned} -dG(\Sigma_t) &\leq -(\rho_1 + \rho_2)dt + 2 \exp(2^{1/2} \lambda^{-1} C_{4.4}) [dK_t^1 + dK_t^2] \\ &\quad - 2 \exp(D \Sigma_t^{1/2}) \langle X_t, \sigma(X_t) dB_t \rangle. \end{aligned} \quad (5.4)$$

Hence, from (5.3) and (5.4), there exist a constant  $\theta > 0$  and a square integrable martingale  $M$  such that:

$$\forall t \in [0, \xi_N], \quad d[F(R_t^{1/2}) - \theta G(\Sigma_t)] \leq -\theta(\rho_1 + \rho_2)dt + dM_t. \quad (5.5)$$

For  $t \geq 0$ , take the expectation in (5.5) between 0 and  $t \wedge \xi_N$ :

$$\mathbb{E}[F(R_{t \wedge \xi_N}^{1/2}) - \theta G(\Sigma_{t \wedge \xi_N})] - F(R_0) + \theta G(\Sigma_0) \leq -\theta(\rho_1 + \rho_2) \mathbb{E}[t \wedge \xi_N].$$

Letting  $t \rightarrow +\infty$ , deduce that  $\mathbb{E}(\xi_N) < +\infty$  and that:

$$\theta(\rho_1 + \rho_2) \mathbb{E}[\xi_N] \leq \theta [\mathbb{E}[G(\Sigma_{\xi_N})] - G(\Sigma_0)] - [F(N^{-1}) - F(R_0)].$$

Recall from 2-(a) in Proposition 4.3 that the growth of  $F$  is logarithmic and from Lemma 4.5 that  $G$  is bounded. This completes the proof.  $\square$

### 5.1.2. Lower Bound for the Expectation of $\xi_N$

**Theorem 5.2** *Assume that  $s = 0$ . Then, there exists a constant  $C_{5.2} > 0$  such that  $\mathbb{E}[\xi_N] \geq C_{5.2}^{-1} \ln(N) - C_{5.2}$ .*

*Proof.* Choose  $F$  as in (5.1) and  $C = -C_{4.1}$ . As in the proof of Theorem 5.1:

$$\begin{aligned} \forall t \in [0, \xi_N], \quad dF(R_t^{1/2}) &\geq -F'(R_t^{1/2}) [\bar{\kappa}_t^1 dK_t^1 + \bar{\kappa}_t^2 dK_t^2] \\ &\quad + F'(R_t^{1/2}) \langle R_t^{-1/2} \sigma(X_t) S(X_t) X_t, dB_t \rangle. \end{aligned} \quad (5.6)$$

The functionals  $F'(R^{1/2})$ ,  $\bar{\kappa}^1$  and  $\bar{\kappa}^2$  are uniformly bounded when  $X$  belongs to one of the two border lines  $x^1 = 1$  or  $x^2 = 1$ . Thus, there exists a constant  $m' > 0$  such that:

$$\begin{aligned} \forall t \in [0, \xi_N], \quad dF(R_t^{1/2}) &\geq -(m')^{-1} [dK_t^1 + dK_t^2] \\ &\quad + F'(R_t^{1/2}) \langle R_t^{-1/2} \sigma(X_t) S(X_t) X_t, dB_t \rangle. \end{aligned} \quad (5.7)$$

Apply now the point 1 in Corollary 4.7 (with the same  $D$ ). For every  $t \in [0, \xi_N]$ :

$$\begin{aligned} -dG(\Sigma_t) &\geq -(\rho_1 + \rho_2)dt + 2 \exp(-2^{1/2} \lambda^{-1} C_{4.4}) [dK_t^1 + dK_t^2] \\ &\quad - 2 \exp(D \Sigma_t^{1/2}) \langle X_t, \sigma(X_t) dB_t \rangle. \end{aligned} \quad (5.8)$$

Hence, from (5.7) and (5.8), there exist a constant  $\theta' > 0$  and a square integrable martingale  $M'$  such that:

$$\forall t \in [0, \xi_N], d[F(R_t^{1/2}) - \theta'G(\Sigma_t)] \geq -\theta'(\rho_1 + \rho_2)dt + dM'_t. \quad (5.9)$$

Take the expectation between 0 and  $t \wedge \xi_N$  and let  $t$  tend to  $+\infty$ :

$$\mathbb{E}[F(R_{\xi_N}^{1/2}) - \theta'G(\Sigma_{\xi_N})] - F(R_0) + \theta'G(\Sigma_0) \geq -\theta'(\rho_1 + \rho_2)\mathbb{E}[\xi_N].$$

Hence:

$$\theta'(\rho_1 + \rho_2)\mathbb{E}[\xi_N] \geq \theta'[\mathbb{E}[G(\Sigma_{\xi_N})] - G(\Sigma_0)] - [F(N^{-1}) - F(R_0)].$$

Recall again from 2-(b) in Proposition 4.3 that the growth of  $F$  is logarithmic and from Lemma 4.5 that  $G$  is bounded.  $\square$

### 5.2. Positive Case

Assume now that  $s > 0$  and return back to (4.13). To establish the uniform boundedness in  $N$  of the expectation of  $\xi_N$ , we aim at choosing  $\alpha$  positive. To this end, we need to bound efficiently the coefficient  $[(1 + 2s^2(1 - s)^{-1})R^{1/2} + 2Z]^{-1}$  in (4.14).

**Lemma 5.3** *Assume that  $s > 0$  and define  $\bar{s} \equiv \min(\frac{2s^2}{1-s}, 1)$ . Then,*

$$\forall t \in [0, \xi_N], [(1 + \frac{2s^2}{1-s})R_t^{1/2} + 2Z_t]^{-1} \leq [1 + \bar{s}]^{-1}A_t^{-1}.$$

**Proof.** Recall that  $Z$  (see Subsection 3.3) is positive since  $s > 0$ . Hence,

$$\begin{aligned} \forall t \in [0, \xi_N], (1 + \frac{2s^2}{1-s})R_t^{1/2} + 2Z_t &= A_t + 2s^2(1-s)^{-1}R_t^{1/2} + Z_t \\ &\geq A_t + \bar{s}(R_t^{1/2} + Z_t) = (1 + \bar{s})A_t. \end{aligned}$$

We are now in position to prove the following theorem:

**Theorem 5.4** *Assume that  $s > 0$ . Then, there exists a constant  $C_{5.4}$  such that  $\mathbb{E}[\xi_N] \leq C_{5.4}$ .*

*Proof.* Choose  $\alpha = 1 - (1 + \bar{s})^{-1}$  (so that  $\alpha > 0$ ),  $C = C_{4.1}$  and  $F(0) = 0$  in (4.13). Then, due to the positivity of  $\Psi_{4.1}$  (see Proposition 4.1) and the one of  $F'$ , deduce from (4.14) and Lemma 5.3 that for  $t \in [0, \xi_N]$ ,

$$\begin{aligned} dF(A_t) &\leq -F'(A_t)[\bar{\kappa}_t^1 dK_t^1 + \bar{\kappa}_t^2 dK_t^2] \\ &\quad + F'(A_t)\langle R_t^{-1/2}\sigma(X_t)S(X_t)X_t + \frac{s}{\sqrt{1-s^2}}\sigma(X_t)(\rho_1^{-1}, \rho_2^{-1})^t, dB_t \rangle. \end{aligned}$$

Follow now the proof of Theorem 5.1 (due to the equivalence between  $A$  and  $|X|$ , see Lemma 3.6,  $F'(A_t)$  is uniformly bounded from above and from below when one of the coordinate of  $X$  matches one) and deduce for a suitable constant  $\theta > 0$ :

$$\theta(\rho_1 + \rho_2)\mathbb{E}[\xi_N] \leq \theta[\mathbb{E}[G(\Sigma_{\xi_N})] - G(\Sigma_0)] - [F(N^{-1}) - F(R_0)].$$

Recall from 1-(a) in Proposition 4.3 and from Lemma 4.5 that  $F$  and  $G$  are bounded. This completes the proof.  $\square$



### 5.3. Negative Case

Consider finally the case  $s < 0$ . As done in the latter subsection, we need to bound efficiently the coefficient  $(1 + 2s^2(1 - s)^{-1})R_t^{1/2} + 2Z$  in (4.14).

**Lemma 5.5** *Assume that  $s < 0$ . Then,*

$$\forall t \in [0, \xi_N], (1 - s) A_t^{-1} \leq \left[ \left(1 + \frac{2s^2}{1 - s}\right) R_t^{1/2} + 2Z_t \right]^{-1} \leq A_t^{-1} \frac{(1 - s)^2}{(1 + s)}.$$

*Proof.* The upper bound follows from (4.1) and Lemma 3.6.

Turn to the lower bound. For  $t \in [0, \xi_N]$ ,

$$\begin{aligned} \left(1 + \frac{2s^2}{1 - s}\right) R_t^{1/2} + 2Z_t &= \left(1 + \frac{2s^2}{1 - s}\right) R_t^{1/2} + 2(A_t - R_t^{1/2}) \\ &= 2A_t + \left(-1 + \frac{2s^2}{1 - s}\right) R_t^{1/2} \\ &= 2A_t + \frac{(2s - 1)(s + 1)}{1 - s} R_t^{1/2}. \end{aligned} \quad (5.10)$$

Note that  $(2s - 1)(1 - s)^{-1} < 0$  and apply Lemma 3.6 to (5.10). For  $t \in [0, \xi_N]$ :

$$\left(1 + \frac{2s^2}{1 - s}\right) R_t^{1/2} + 2Z_t \leq 2A_t + \frac{(2s - 1)}{1 - s} A_t = (1 - s)^{-1} A_t.$$

This completes the proof.  $\square$

**5.3.1. Upper Bound** We are now in position to give an upper bound for the expectation of  $\xi_N$ .

**Theorem 5.6** *Assume  $s < 0$  and set  $\beta_+ \equiv (1 - s)^2(1 + s)^{-1} - 1 = s(s - 3)(1 + s)^{-1} > 0$ . Then, there exists a constant  $C_{5.6}$  such that  $\mathbb{E}(\xi_N) \leq C_{5.6}(N^{\beta_+} + 1)$ .*

*Proof.* Choose  $\alpha = -\beta_+$ ,  $C = C_{4.1}$  and  $F(\delta) = 0$  in (4.13), apply (4.14) and follow the proof of Theorem 5.1.  $\square$

**5.3.2. Lower Bound** Turn now to the lower bound:

**Theorem 5.7** *Assume  $s < 0$  and set  $\beta_- \equiv 1 - s - 1 = -s > 0$ . Then, there exists a constant  $C_{5.7}$  such that  $\mathbb{E}(\xi_N) \geq C_{5.7}^{-1} N^{\beta_-} - C_{5.7}$ .*

*Proof.* Choose  $\alpha = -\beta_-$ ,  $C = C_{4.1}$  and  $F(\delta) = 0$  in (4.13), apply (4.14) and follow the proof of Theorem 5.2.  $\square$

### 5.4. Conclusion

Derive now Theorem 2.1 from Theorems 5.1, 5.2, 5.4, 5.6 and 5.7. Note indeed from the equivalence between  $A$  and the Euclidean norm (see Lemmas 3.2 and 3.6) that there exists a parameter  $c > 1$ , such that, for  $\ell \in ]0, c^{-1}[$ ,  $\xi_{c\ell^{-1}} \leq T_\ell \leq \xi_{c^{-1}\ell^{-1}}$ , where  $T_\ell = \inf\{t \geq 0, X_t^1 + X_t^2 \leq \ell\}$ .

## References

1. Comets, F., Delarue, F., Schott, R.: Distributed algorithms in an ergodic Markovian environment. Technical report of the Laboratoire PMA, Universities Paris 6 and 7, **996**, available at <http://hal.ccsd.cnrs.fr/ccsd-00005841> (2005)
2. Ellis, C.A.: Probabilistic models of computer deadlock. *Inform. Sci.* **12**, 43–60 (1977)
3. Fayolle, G., Malyshev, V.A., Menshikov, M.V.: *Topics in the Constructive Theory of Countable Markov Chains*. Cambridge University Press, Cambridge, 1995
4. Flajolet, P.: The evolution of two stacks in bounded space and random walks in a triangle. In: *Proceedings of FCT'86, LNCS No 233*, Springer Verlag, 1986, pp. 425–340
5. Friedman, A.: *Stochastic Differential Equations and Applications*. Vol. 1. Probability and Mathematical Statistics, Academic Press New York-London, 1975
6. Guillotin-Plantard, N., Schott, R.: Distributed algorithms with dynamic random transitions. *Random Structures and Algorithms* **21**, 371–396 (2002)
7. Hasminskii, R.Z.: *Stochastic Stability of Differential Equations*. Translated from the Russian by D. Louvish. Monographs and Textbooks on Mechanics of Solids and Fluids: Mechanics and Analysis, Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md, 1980
8. Knuth, D.E.: *The Art of Computer Programming*. Volume 1: Fundamental algorithms. Second printing of the Second edition. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1975
9. Lions, P.-L., Sznitman, A.-S.: Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.* **37**, 511–537 (1984)
10. Louchard, G.: Some distributed algorithms revisited. *Commun. Statist. Stochastic models* **11**, 563–586 (1995)
11. Louchard, G., Schott, R.: Probabilistic analysis of some distributed algorithms. *Random Structures and Algorithms* **2**, 151–186 (1991)
12. Louchard, G., Schott, R., Tolley, M., Zimmermann, P.: Random walks, heat equations and distributed algorithms. *Journal of Computational and Applied Mathematics* **53**, 243–274 (1994)
13. Maier, R.S.: Colliding Stacks: A large deviations analysis. *Random Structures and Algorithms* **2**, 379–420 (1991)
14. Maier, R.S., Schott, R.: Exhaustion of shared memory: stochastic results. In: *Proceedings of WADS'93, LNCS No 709*, Springer Verlag, 1993, pp. 494–505
15. Reiman, M.I., Williams, R.J.: A boundary property of semimartingale reflecting Brownian motions. *Probab. Theory Related Fields* **77**, 87–97 (1988)
16. Reiman, M.I., Williams, R.J.: Correction to: “A boundary property of semimartingale reflecting Brownian motions”, [*Probab. Theory Related Fields* **77**, 87–97]. *Probab. Theory Related Fields* **80**, 633 (1989)
17. Słomiński, L.: Euler’s approximations of solutions of SDEs with reflecting boundary. *Stochastic Process. Appl.* **94**, 317–337 (2001)
18. Tanaka, H.: Stochastic differential equations with reflecting boundary condition in convex regions. *Hiroshima Math. J.* **9**, 163–177 (1979)
19. Taylor, L.M., Williams, R.J.: Existence and uniqueness of semimartingale reflecting Brownian motions in an orthant. *Probab. Theory Related Fields* **96**, 283–317 (1993)
20. Varadhan, S.R.S., Williams, R.J.: Brownian motion in a wedge with oblique reflection. *Comm. Pure Appl. Math.* **38**, 405–443 (1985)
21. Yao, A.C.: An analysis of a memory allocation scheme for implementing stacks. *SIAM J. Comput.* **10**, 398–403 (1981).