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**ON THE SPECTRAL DENSITY OF THE WAVELET
COEFFICIENTS OF LONG MEMORY TIME SERIES WITH
APPLICATION TO THE LOG-REGRESSION ESTIMATION
OF THE MEMORY PARAMETER**

E. MOULINES, F. ROUEFF, AND M.S. TAQQU

ABSTRACT. In the recent years, methods to estimate the memory parameter using wavelet analysis have gained popularity in many areas of science. Despite its widespread use, a rigorous semi-parametric asymptotic theory, comparable to the one developed for Fourier methods, is still missing. In this contribution, we adapt the classical semi-parametric framework introduced by Robinson and his co-authors for estimating the memory parameter of a (possibly) non-stationary process. As an application, we obtain min-max upper bounds for the log-scale regression estimator of the memory parameter for a Gaussian process and we derive an explicit expression of its variance.

AMS Keywords: 62M10, 60G18 Secondary: 62M15

1. INTRODUCTION

Let $X \stackrel{\text{def}}{=} \{X_k\}_{k \in \mathbb{Z}}$ be a real-valued process, not necessarily stationary and let $\Delta^K X$ denotes its K -th order difference. The first order difference is $[\Delta X]_k \stackrel{\text{def}}{=} X_k - X_{k-1}$ and Δ^K is defined recursively. The process X is said to have *memory parameter* d , $d \in \mathbb{R}$ (in short, is an $M(d)$ process) if for any integer $K > d - 1/2$, the K -th order difference process $\Delta^K X$ is weakly stationary with spectral density function

$$f_{\Delta^K X}(\lambda) \stackrel{\text{def}}{=} |1 - e^{-i\lambda}|^{2(K-d)} f^*(\lambda) \quad \lambda \in (-\pi, \pi), \quad (1)$$

where f^* is a non-negative symmetric function which is bounded on $(-\pi, \pi)$ and is bounded away from zero in a neighborhood of the origin. $M(d)$ processes encompass both stationary and non-stationary processes, depending on the value of the memory parameter d . When $d < 1/2$, the process X is covariance stationary and its spectral density is given by

$$f(\lambda) = |1 - e^{-i\lambda}|^{-2d} f^*(\lambda). \quad (2)$$

The process X is said to have long-memory if $0 < d < 1/2$, short-memory if $d = 0$ and negative memory if $d < 0$; the process is not invertible if $d < -1/2$. When $d > 1/2$, the process is non stationary but its (possibly higher-order) increments are covariance stationary. Stationarity of the increments is commonly assumed in time-series analysis, as in ARIMA models (in this case, $d = K$ is an integer and f^* is the spectral density of an autoregressive moving average short-memory process). Under this assumption, a finite number of integer differences produces a short-memory process, with the degree of differencing determined by diagnostics such as unit root tests. In this case, f is not integrable on $[-\pi, \pi]$ and is therefore not a spectral density. In the terminology introduced by Yaglom (1958), this referred to a generalized spectral density.

If $d > 0$ and $f^* \equiv \sigma^2$ in (2), one gets the so-called fractionally integrated white noise process. If

$$f_{\text{ARMA}}^*(\lambda) = \sigma^2 \frac{|1 - \sum_{k=1}^q \theta_k e^{-i\lambda k}|^2}{|1 - \sum_{k=1}^p \phi_k e^{-i\lambda k}|^2} \quad \text{with } 1 - \sum_{k=1}^p \phi_k z^k \neq 0 \text{ for } |z| = 1 \quad (3)$$

leads to the class of ARFIMA(p, d, q) processes. Another example are the discrete-time samples $\{B_H(k)\}_{k \in \mathbb{Z}}$ of the continuous-time fractional Brownian motion (FBM) $\{B_H(t), t \in \mathbb{R}\}$ with Hurst index $H \in (0, 1)$, which is a continuous centered Gaussian process with covariance

$$R_H(t, s) \stackrel{\text{def}}{=} \mathbb{E}[B_H(t)B_H(s)] = \frac{1}{2} \{|t|^{2H} + |s|^{2H} - |t - s|^{2H}\}.$$

(see for instance Mandelbrot and Van Ness (1968)). The process $\{B_H(k)\}_{k \in \mathbb{Z}}$ is increment stationary ($K = 1$) and its generalized spectral density is given

by $f_{\text{FBM}}(\lambda) \stackrel{\text{def}}{=} |1 - e^{-i\lambda}|^{-2d} f_{\text{FBM}}^*(\lambda)$, where $d = H + 1/2$ and

$$f_{\text{FBM}}^*(\lambda) = \left| \frac{2 \sin(\lambda/2)}{\lambda} \right|^{2d} + |2 \sin(\lambda/2)|^{2H+1} \sum_{k \neq 0} |\lambda + 2k\pi|^{-(2d)} \quad (4)$$

The memory parameter d plays a central role in the definition of $M(d)$ processes and is often the focus of empirical interest. Two classes of methods have emerged to estimate the memory parameter d : Fourier and Wavelet methods. Frequency-domain techniques are now well documented and understood (see for instance Hurvich and Ray (1995), Velasco (1999), Velasco and Robinson (2000) and Hurvich et al. (2002)).

In this paper, we focus on wavelet methods. The use of the DWT to estimate the memory parameter d was initiated by Flandrin (1992) and Wornell and Oppenheim (1992) for the continuous-time FBM. Flandrin (1992) showed that the wavelet coefficients of the FBM at a given scale is a covariance stationary sequence and provided explicit expression for the wavelet coefficient covariance sequence. Most importantly, the *log-scale spectrum*, defined as the logarithm of the variance of the wavelet coefficients as a function of the scale parameter, was shown to be a linear function of the scale index, with a slope proportional to the memory parameter d . The correlation of the wavelet coefficients of continuous time processes with stationary increments was studied by Masry (1993). Dijkerman and Mazumdar (1994) obtain bounds in the case of FBM.

In many applications, observations are in discrete-time and the DWT in discrete time should therefore be used (see below). One of the earliest reference in this context is Kaplan and Kuo (1993), who applied the DWT using the Haar basis to the discrete increments of the FBM, also known as the discrete fractional Gaussian noise (FGN). They have shown that the properties of the correlation structure of the Haar wavelet coefficients of the FGN are identical to that of the FBM. These results were later extended to the fractionally integrated white noise, defined as the $M(d)$ process with generalized spectral density density $f(\lambda) = \sigma^2 |1 - e^{i\lambda}|^{-2d}$ (see McCoy and Walden (1996) for the stationary case; Fan (2003) for nonstationary extensions). These authors emphasized the importance of the choice of the wavelet and in particular of the number of its vanishing moments; see for instance Percival and Walden, chapters 9,10 (2000) for an in-depth study. More recently, Craigmile and Percival (2005) study what happens when the number of zero moments tends to infinity. While these authors focus on bounds of the correlation between wavelet coefficients, our focus will be instead on bounds on their spectral densities. This allows us to develop a semiparametric approach to the estimation of the long memory parameter.

The particular structure of the scale spectrum suggested several estimators of the memory parameter d . In this paper, we focus on the regression estimator introduced in Abry and Veitch (1998), consisting in estimating d from the

slope in the regression of the logarithm of the scale spectrum on the scale index. This estimator is now widely used in many different fields (see *e.g.* Veitch and Abry (1999) for applications to network traffic; Percival and Walden (2000) and Papanicolaou and Sølna (2003) for applications in physical sciences; see *e.g.* Gençay et al. (2002) and Bayraktar et al. (2004) for applications in finance). This estimator is well-suited to process large data sets, since it has low computational complexity due to the pyramidal algorithm for computing the details coefficients. Also, it is robust with respect to additive polynomial trends (see for instance Veitch and Abry (1999) and Craigmile et al. (2005)).

Despite its widespread use, a rigorous semi-parametric asymptotic theory of this estimator, comparable to the one developed for corresponding estimators based on the periodogram, is still missing (the concluding remarks in Velasco (1999) are still valid). To our best knowledge, the only attempt in this direction was Bardet et al. (2000) (see also Bardet (2002)), where the log-scale regression estimator is studied in a semi-parametric setting. However their results cannot be compared with other estimators because the process is supposed to be observed in continuous-time processes (discretization issues were not discussed). Similar results were derived independently in Bayraktar et al. (2004). None of these results directly translate for discrete time observations. The main objective of this paper is to fill this gap.

The paper is organized as follows. In section 2, we introduce wavelets and wavelet transforms for time-series. We do not assume that the wavelets are orthonormal nor that they result from a multiresolution analysis. In section 3, we establish that the wavelet coefficients at a given scale of an $M(d)$ process are covariance stationary and derive an explicit expression for its spectral density. We then extend this result to two different scales by grouping, in appropriate way, the wavelet coefficients. These results apply to a general class of wavelets with bounded supports, which include but are not limited to Daubechies wavelets. We finally show that the spectral density of the wavelet coefficients of an $M(d)$ process can be approximated, at large scales, by the spectral density of the continuous-time wavelet coefficients of the FBM, and derive an explicit bound for the difference between these two quantities. In section 4, we apply the results obtained above to derive a minimax upper bound and an explicit expression of the limiting variance for the estimator of the memory parameter based on the regression of the log-scale spectrum for (possibly non-stationary) Gaussian processes.

2. DISCRETE WAVELET TRANSFORM

In this section, we introduce the main concepts required to define an (extended) discrete wavelet transform. Denote by $L^2(\mathbb{R})$ the set of square integrable functions with respect to the Lebesgue measure. Let $\phi \in L^2(\mathbb{R})$ and

$\psi \in L^2(\mathbb{R})$ be two functions and define their Fourier transforms as

$$\hat{\phi}(\xi) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \phi(t) e^{-i\xi t} dt \quad \text{and} \quad \hat{\psi}(\xi) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \psi(t) e^{-i\xi t} dt.$$

Consider the following assumptions:

- (W-1) ϕ and ψ are compactly-supported, integrable, and $\hat{\phi}(0) = \int_{-\infty}^{\infty} \phi(t) dt = 1$ and $\int_{-\infty}^{\infty} \psi^2(t) dt = 1$.
- (W-2) There exists $\alpha > 1$ such that $\sup_{\xi \in \mathbb{R}} |\hat{\psi}(\xi)| (1 + |\xi|)^\alpha < \infty$.
- (W-3) The function ψ has M vanishing moments, *i.e.* $\int_{-\infty}^{\infty} t^l \psi(t) dt = 0$ for all $l = 0, \dots, M-1$
- (W-4) The function $\sum_{k \in \mathbb{Z}} k^l \phi(\cdot - k)$ is a polynomial of degree l for all $l = 0, \dots, M-1$.

Assumption (W-1) implies that $\hat{\phi}$ and $\hat{\psi}$ are everywhere infinitely differentiable. Hence, under (W-1), (W-3) is equivalent to asserting that the first $M-1$ derivatives of $\hat{\psi}$ vanish at the origin which implies, using a Taylor expansion, that

$$|\hat{\psi}(\xi)| = O(|\xi|^M) \quad \text{as} \quad \xi \rightarrow 0. \quad (5)$$

By (Cohen, 2003, Theorem 2.8.1, Page 90), under (W-1), (W-4) is equivalent to

$$\sup_{k \neq 0} |\hat{\phi}(\xi + 2k\pi)| = O(|\xi|^M) \quad \text{as} \quad \xi \rightarrow 0. \quad (6)$$

Define the family $\{\psi_{j,k}, j > 0, k \in \mathbb{Z}\}$ of translated and dilated functions

$$\psi_{j,k}(t) = 2^{-j/2} \psi(2^{-j}t - k), \quad (7)$$

Many authors suppose that the $\psi_{j,k}$ are orthogonal and even that they are generated by a multiresolution analysis (MRA). We discuss in Appendix A, the relations between assumptions (W-1)-(W-4) and multiresolution analysis (in which case, ϕ is the scaling function and ψ is the associated wavelet). In this paper, we do not assume that wavelets are orthonormal nor that they are associated to a multiresolution analysis. We may therefore work with other convenient choices for ϕ and ψ as long as (W-1)-(W-4) are satisfied. A simple example is to set, for some positive integer N ,

$$\phi(x) \stackrel{\text{def}}{=} \mathbb{1}_{[0,1]}^{\otimes N}(x) \quad \text{and} \quad \psi(x) \stackrel{\text{def}}{=} \frac{d^N}{dx^N} \mathbb{1}_{[0,1]}^{\otimes 2N}(x),$$

where $\mathbb{1}_A$ is the indicator function of the set A and for a non negative function f , $f^{\otimes N}$ denotes the N -th self-convolution of f . It follows that

$$|\hat{\phi}(\xi)| = |2 \sin(\xi/2)/\xi|^N \quad \text{and} \quad \hat{\psi}(\xi) = |\xi|^N |2 \sin(\xi/2)/\xi|^{2N}.$$

Using (5) and (6), one easily checks that (W-1)-(W-4) are satisfied with M and α equal to N . Of course the family of functions $\{\psi_{j,k}\}$ are not orthonormal for this choice of the wavelet function ψ (and the function ϕ is not associated to a MRA). Nevertheless, to ease references to previously reported works, with

a slight abuse in the terminology, we still call ϕ and ψ the scaling and the wavelet functions.

Having defined the functions ϕ and ψ , we now define what we call the Discrete Wavelet Transform in discrete time. Start with a real valued sequence $\{x_k, k \in \mathbb{Z}\}$. Using the scaling function ϕ , we first associate to the sequence $\{x_k, k \in \mathbb{Z}\}$ the functions

$$\mathbf{x}_n(t) \stackrel{\text{def}}{=} \sum_{k=1}^n x_k \phi(t-k) \quad \text{and} \quad \mathbf{x}(t) \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} x_k \phi(t-k) \quad (8)$$

The (details) wavelet coefficients are then defined as

$$W_{j,k}^{\mathbf{x}} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \mathbf{x}(t) \psi_{j,k}(t) dt \quad j \geq 0, k \in \mathbb{Z}. \quad (9)$$

If the support of the scaling function ϕ is included in $(-T, 0)$ for some integer $T \geq 1$, then $\mathbf{x}_n(t) = \mathbf{x}(t)$ for all $t = 0, \dots, n - T + 1$. If the support of the wavelet function ψ is included in $(0, T)$, then, the support of $\psi_{j,k}$ is included in the interval $(2^j k, 2^j(k + T))$. Hence

$$W_{j,k}^{\mathbf{x}} = W_{j,k}^{\mathbf{x}_n} = \int_{-\infty}^{\infty} \mathbf{x}_n(t) \psi_{j,k}(t) dt, \quad (10)$$

for all $(j, k) \in \mathcal{I}_n$, where

$$\mathcal{I}_n \stackrel{\text{def}}{=} \{(j, k) : j \geq 0, 0 \leq k \leq 2^{-j}(n - T + 1) - T\}. \quad (11)$$

For any j , the wavelet coefficients $\{W_{j,k}^{\mathbf{x}}\}_{k \in \mathbb{Z}}$ are obtained by discrete convolution and downsampling. More precisely, under (W-1), for all $j \geq 0, k \in \mathbb{Z}$,

$$W_{j,k}^{\mathbf{x}} = \sum_{l \in \mathbb{Z}} x_l h_{j, 2^j k - l} = (h_{j, \cdot} \star \mathbf{x})_{2^j k} = (\downarrow^j [h_{j, \cdot} \star \mathbf{x}])_k, \quad (12)$$

where $h_{j,l} \stackrel{\text{def}}{=} 2^{-j/2} \int_{-\infty}^{\infty} \phi(t+l) \psi(2^{-j}t) dt$, \star denotes the convolution of discrete sequences and, for any sequence $\{c_k\}_{k \in \mathbb{Z}}$ and any integer l , $(\downarrow^j c)_k = c_{k2^j}$. For all $j \geq 0$, $H_j(\lambda) \stackrel{\text{def}}{=} \sum_{l \in \mathbb{Z}} h_{j,l} e^{-i\lambda l}$ denotes the discrete Fourier transform of $\{h_{j,l}\}_{l \in \mathbb{Z}}$,

$$H_j(\lambda) \stackrel{\text{def}}{=} 2^{-j/2} \int_{-\infty}^{\infty} \sum_{l \in \mathbb{Z}} \phi(t+l) e^{-i\lambda l} \psi(2^{-j}t) dt. \quad (13)$$

For all $j \geq 0$ and all $m = 0, \dots, M - 1$,

$$\sum_{l \in \mathbb{Z}} h_{j,l} l^m = 2^{-j/2} \int_{-\infty}^{\infty} \psi(2^{-j}t) \sum_{l \in \mathbb{Z}} \phi(t+l) l^m dt.$$

Under assumption (W-4), $t \mapsto \sum_{l \in \mathbb{Z}} \phi(t+l) l^m$ is a polynomial of degree m and (W-3) therefore implies that $\sum_{l \in \mathbb{Z}} h_{j,l} l^m = 0$; equivalently, the trigonometric polynomial H_j satisfies $\left. \frac{d^m H_j(\lambda)}{d\lambda^m} \right|_{\lambda=0} = 0$, $m = 0, \dots, M - 1$ and thus admits a

zero at 0 of degree at least equal to M . Therefore, $H_j(\lambda)$ can be factorized as $H_j(\lambda) = (1 - e^{i\lambda})^M \tilde{H}_j(\lambda)$, where $\tilde{H}_j(\lambda)$ is a trigonometric polynomial. Hence, the wavelet coefficient (12) may be computed as

$$W_{j,k}^{\mathbf{x}} = (\downarrow^j [\tilde{h}_{j,\cdot} \star \Delta^M \mathbf{x}])_k \quad (14)$$

where $\{\tilde{h}_{j,l}\}_{l \in \mathbb{Z}}$ are the coefficients of the trigonometric polynomial \tilde{H}_j and $\Delta^M \mathbf{x}$ is the M -th order difference of the sequence \mathbf{x} . In words, the use of a wavelet and a scaling function satisfying (W-4) and (W-3) implicitly perform a M -th order differentiation of the time-series. Therefore, we may process a K -th order integrated processes X without specific preprocessing, provided that $M \geq K$. This is in sharp contrast with Fourier methods: in this case, the time series must be explicitly differentiated at least K times and a data taper must be applied on the differenced series to avoid frequency-domain leakage (see for instance Hurvich et al. (2002)).

3. SPECTRAL DENSITY OF THE WAVELET COEFFICIENTS

Because the wavelet coefficients at a given scale are obtained by applying time-invariant linear filters, computing the covariance of the wavelet coefficients of K -th order stationary processes is an easy exercise. The following proposition provides an integral expression for calculating the covariance between two wavelet coefficients on possibly different scales, expressed in terms of the transfer function H_j of the wavelet filters and the generalized spectral density of the process X . This proposition extends Theorem 2 in Masry (1993) on the spectral measure of the DWT coefficients of increment stationary continuous time processes to the discrete-time setting and Lemma 1 in Craigmile and Percival (2005) to functions ψ and ϕ that do not necessarily define a MRA. For a K -th order integrated process $X \stackrel{\text{def}}{=} \{X_k\}_{k \in \mathbb{Z}}$ with generalized spectral density f , we denote by Var_f and Cov_f the variance and covariance of random variables which can be expressed as linear transformations of the K -th order increments of the process (note that these quantities only depend on f , and not on the precise distribution of the process). In view of (14), the wavelet coefficients are examples of such transformations, provided that M is larger than K .

Proposition 1. *Let X be a K -th order integrated process with generalized spectral density f . Assume (W-1)-(W-4) with $M \geq K$. Then, for all $j, j' \geq 0$ and $k, k' \in \mathbb{Z}$,*

$$\text{Cov}_f(W_{j,k}^X, W_{j',k'}^X) = \int_{-\pi}^{\pi} e^{i\lambda(k2^j - k'2^{j'})} f(\lambda) H_j(\lambda) \overline{H_{j'}(\lambda)} d\lambda, \quad (15)$$

where the wavelet coefficient $W_{j,k}^X$ is defined in (9).

The proof follows from elementary results on time-invariant linear filtering of covariance stationary processes, using the representation (12) of the wavelet coefficients.

By (14), for a given scale j , the process $\{W_{j,k}^X\}_{k \in \mathbb{Z}}$ is covariance stationary. The situation is more complicated when considering two different scales $j \neq j'$, because the two-dimensional sequence $\{[W_{j,k}^X \ W_{j',k}^X]^T\}_{k \in \mathbb{Z}}$ is not stationary for $j \neq j'$. This is a consequence of the pyramidal wavelet scheme, where at scale j , the wavelet coefficients are downsampled by a factor 2^j which depends on j . Thus, to obtain a stationary sequence, one should consider the process $\{[W_{j,k}^X \ W_{j',2^{j-j'}k}^X]^T\}_{k \in \mathbb{Z}}$ for $j > j'$, which involves a downsampled subsequence of the coefficients at the finer scale j' . One can also consider the process $\{[W_{j,k}^X \ W_{j',2^{j-j'}k+l}^X]^T\}_{k \in \mathbb{Z}}$ for $j > j'$ any integer l . It turns out that the most convenient is to merge the processes corresponding to $l = 0, \dots, 2^{j-j'} - 1$ and hence to consider the *between-scale* process $\{[W_{j,k}^X \ \mathbf{W}_{j,k}^X(j-j')^T]^T\}_{k \in \mathbb{Z}}$, where for any $j \geq u \geq 0$,

$$\mathbf{W}_{j,k}^X(u) \stackrel{\text{def}}{=} [W_{j-u,2^uk}^X, \dots, W_{j-u,2^uk+2^u-1}^X]^T. \quad (16)$$

Proposition 1 allows to determine the spectral density of the within scale process $\{W_{j,k}^X\}_{k \in \mathbb{Z}}$ and the between scale process $\{[W_{j,k}^X \ \mathbf{W}_{j,k}^X(j-j')^T]^T\}_{k \in \mathbb{Z}}$ in terms of the generalized spectral density of X and the transfer function of the wavelet filters folded on the interval $[-\pi, \pi]$.

Corollary 2. *Define for all $0 \leq u \leq j$ and $\lambda \in [-\pi, \pi]$,*

$$\mathbf{D}_{j,u}(\lambda; f, \phi, \psi) \stackrel{\text{def}}{=} 2^{-j} \times \sum_{l=0}^{2^j-1} \mathbf{e}_u(|\lambda| + 2l\pi) f(2^{-j}(|\lambda| + 2l\pi)) H_j(2^{-j}(|\lambda| + 2l\pi)) \overline{H_{j-u}(2^{-j}(|\lambda| + 2l\pi))}, \quad (17)$$

where for all $\xi \in \mathbb{R}$, $\mathbf{e}_u(\xi) \stackrel{\text{def}}{=} [1, e^{-i2^{-u}\xi}, \dots, e^{-i(2^u-1)2^{-u}\xi}]^T$. Then,

- for all $j \geq 0$, the within-scale process $\{W_{j,k}^X\}_{k \in \mathbb{Z}}$ is covariance stationary with spectral density $\mathbf{D}_{j,0}(\cdot; f, \phi, \psi)$,
- for all $j \geq u > 0$, the between-scale process $\{[W_{j,k}^X \ \mathbf{W}_{j,k}^X(u)^T]^T\}_{k \in \mathbb{Z}}$ is covariance stationary with cross spectral density $\mathbf{D}_{j,u}(\cdot; f, \phi, \psi)$.

We now specialize the results above to the class of processes with memory parameter $d \in \mathbb{R}$ (see (1)). To obtain bounds on the deviation of the correlation and the spectral density of the wavelet coefficients from those of FBM, some additional assumptions are required on the smoothness of f^* at zero frequency. For $0 < \beta \leq 2$ and $L > 0$, define the function class $\mathcal{H}(\beta, L)$ as the set of positive even functions g on $[-\pi, \pi]$ such that, for all $\lambda \in [-\pi, \pi]$,

$$|g(\lambda) - g(0)| \leq L g(0) |\lambda|^\beta. \quad (18)$$

This type of assumption is typical in the *semiparametric* estimation setting (see for instance Robinson (1995) and Moulines and Soulier (2002)). From (3) and (4), f_{ARMA}^* always belongs to $\mathcal{H}(L, 2)$ (for some L) and f_{FBM}^* belongs to $\mathcal{H}(L, 2d \wedge 2)$ (where $a \wedge b = \min(a, b)$).

The expressions of the within- and between-scale wavelet coefficient spectral densities $\mathbf{D}_{j,u}(\cdot; d, f^*, \phi, \psi)$ given in corollary 2 depends both on d and the function f^* . However, we are going to show how these quantities may be approximated by quantities which depend only on the memory parameter d and $f^*(0)$. When X has a generalized spectral density $f(\lambda) = |1 - e^{i\lambda}|^{-2d} f^*(\lambda)$, we use the notations Var_{d,f^*} , Cov_{d,f^*} and $\mathbf{D}_{j,u}(\lambda; d, f^*)$ for Var_f , Cov_f and $\mathbf{D}_{j,u}(\lambda; f)$, respectively. Define

$$\sigma_j^2(d, f^*) \stackrel{\text{def}}{=} \text{Var}_{d,f^*}[W_{j,0}^X] = \int_{-\pi}^{\pi} |1 - e^{-i\lambda}|^{-2d} f^*(\lambda) |H_j(\lambda)|^2 d\lambda, \quad (19)$$

the variance of the wavelet coefficient of such process at scale j .

Theorem 3. *Let $M \geq 1$ be an integer and α, L, β be constants such that $\alpha > 1, 0 < L < \infty$ and $\beta \in (0, 2]$. Assume that (W-1)-(W-4) hold with M and α .*

(a) *Let d_{\min} and d_{\max} be two constants such that*

$$[d_{\min}, d_{\max}] \subset ((1 + \beta)/2 - \alpha, M + 1/2). \quad (20)$$

Then, there exists a constant $C > 0$ (only depending on the constants $\beta, d_{\min}, d_{\max}$ and the functions ϕ and ψ) such that, for all $j \geq 0, d \in [d_{\min}, d_{\max}]$ and $f^ \in \mathcal{H}(L, \beta)$,*

$$|\sigma_j^2(d, f^*) - f^*(0) \mathbf{K}(d, \psi) 2^{2jd}| \leq C f^*(0) L 2^{(2d-\beta)j} \quad (21)$$

where $\mathbf{K}(d, \psi)$ is given by

$$\mathbf{K}(d, \psi) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} |\xi|^{-2d} |\hat{\psi}(\xi)|^2 d\xi. \quad (22)$$

(b) *Let d_{\min} and d_{\max} be two constants such that*

$$[d_{\min}, d_{\max}] \subset ((1 + \beta)/2 - \alpha, M]. \quad (23)$$

Then, for all $u \geq 0$, there exists $C > 0$ (only depending on the constants $\beta, d_{\min}, d_{\max}$ and the functions ϕ and ψ) such that, for all $\lambda \in (-\pi, \pi), j \geq 0, f^ \in \mathcal{H}(L, \beta)$ and $d \in [d_{\min}, d_{\max}]$,*

$$|\mathbf{D}_{j,u}(\lambda; d, f^*, \phi, \psi) - f^*(0) \mathbf{D}_{\infty,u}(\lambda; d, \psi) 2^{2jd}| \leq C f^*(0) L 2^{(2d-\beta)j} \quad (24)$$

where $|\cdot|$ denotes the Euclidean norm in any dimension and, for all $u \geq 0$,

$$\mathbf{D}_{\infty,u}(\lambda; d, \psi) \stackrel{\text{def}}{=} \sum_{l \in \mathbb{Z}} |\lambda + 2l\pi|^{-2d} \mathbf{e}_u(\lambda + 2l\pi) \overline{\hat{\psi}(\lambda + 2l\pi)} \hat{\psi}(2^{-u}(\lambda + 2l\pi)). \quad (25)$$

The proof is based on approximating the wavelet filter transfer function and is postponed to the Section 5.

Proposition 3 states that for any $u \geq 0$, $f^*(0)\mathbf{D}_{\infty,u}(\lambda; d, \psi)2^{2jd}$ is a good approximation of the spectral density $\mathbf{D}_{j,u}(\lambda; d, f^*, \phi, \psi)$. When the memory parameter d belongs to $(-1/2, 1/2)$, the limiting spectral density can be identified as the spectral density of the process $\{[W_{0,k}^{B_H}, \mathbf{W}_{0,k}^{B_H}(u)]\}_{k \in \mathbb{Z}}$, where, with a slight abuse of notations, $\{W_{j,k}^{B_H}\}_{k \in \mathbb{Z}}$ are the wavelet coefficients of the *continuous time* FBM with Hurst index $H = d + 1/2$ defined as the integrals

$$\tilde{d}_{j,k}^{B_H} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} B_H(s) \psi_{j,k}(s) ds, \quad k \in \mathbb{Z}.$$

The asymptotic properties of the within- and between-scale spectral density $\mathbf{D}_{j,u}(\lambda; d, f^*, \phi, \psi)$ (and of the associated correlation function) may thus be deduced from the corresponding properties of the DWT of the FBM.

Remark 1. If $d = 0$ and $\{2^{j/2}\psi(2^j(\cdot - k)), k \in \mathbb{Z}, j \in \mathbb{Z}\}$ is an orthonormal system, then $\mathbf{D}_{\infty,0}(\lambda; 0, \psi) = \sum_{l \in \mathbb{Z}} |\hat{\psi}(\lambda + 2l\pi)|^2 = 1$ (see e.g. Cohen (2003)) and, by a straightforward computation based on the Parseval Formula,

$$\int_{-\pi}^{\pi} |\mathbf{D}_{\infty,u}(\lambda; 0, \psi)|^2 d\lambda = 2^{u+1} \pi \sum_{k \in \mathbb{Z}} \left| \int_{\mathbb{R}} \psi(t) \psi(2^u t - k) dt \right|^2$$

which then vanishes for all integer $u > 0$. Hence, when the memory parameter $d = 0$ and the wavelets are orthonormal, the wavelet coefficients $\{W_{j,k}^X, k \in \mathbb{Z}\}$ are asymptotically uncorrelated as $j \rightarrow \infty$.

Remark 2. Let us examine how Theorem 3 applies when X is FBM with Hurst index H . From the discussion above, we have $d = H + 1/2$ and $f^* \in \mathcal{H}(2d, L)$ for some constant L . The condition on M is then $M > H$ for (a) and $M \geq H + 1/2$ for (b). The condition on α is $\alpha > -H$ in (b) and $\alpha > 1/2$ in (a), which are both satisfied because $\alpha > 1$ and $H \in (0, 1)$. Theorem 3 can therefore be applied when ψ is a Daubechies wavelet with at least 2 vanishing moments.

4. ANALYSIS OF THE MEMORY PARAMETER ESTIMATOR BASED ON THE REGRESSION OF THE WAVELET VARIANCE

We now apply Theorem 3 to study the wavelet estimator of the memory parameter d , based on the regression of the scale spectrum $\sigma_j^2(d, f^*)$ with respect to the scale index j . This is reasonable because, for large scale j , $\log \sigma_j^2(d, f^*)$ is approximately an affine function of j with slope $(2 \log 2) d$ (see (21) in Theorem 3). Given n observations X_1, \dots, X_n , $\sigma_j^2(d, f^*)$ can be estimated by the empirical variance

$$\hat{\sigma}_j^2 \stackrel{\text{def}}{=} n_j^{-1} \sum_{k=0}^{n_j-1} (W_{j,k}^X)^2,$$

where for any j , n_j denotes the number of available wavelet coefficients at scale index j , namely, from (11),

$$n_j = [2^{-j}(n - T + 1) - T + 1] , \quad (26)$$

where T is the size of the time series and $[x]$ denotes the integer part of x . An estimator of the memory parameter d is then obtained by regressing the logarithm of the empirical variance $\log(\hat{\sigma}_i^2)$ for a finite number of scale indices $j \in \{J_0, \dots, J_0 + \ell\}$ where J_0 is the lower scale and $1 + \ell \geq 2$ is the number of scales in the regression. For a sample size equal to n , this estimator is well defined for J_0 and ℓ such that $\ell \geq 1$ and $J_0 + \ell \leq J(n)$ where

$$J(n) \stackrel{\text{def}}{=} [\log_2(n - T + 1) - \log_2(T)] \quad (27)$$

is the maximum index j such that $n_j \geq 1$. The regression estimator can be expressed formally as

$$\hat{d}_n(J_0, \mathbf{w}) \stackrel{\text{def}}{=} \sum_{j=J_0}^{J_0+\ell} w_{j-J_0} \log(\hat{\sigma}_j^2) , \quad (28)$$

where the vector $\mathbf{w} \stackrel{\text{def}}{=} [w_0, \dots, w_\ell]^T$ satisfies

$$\sum_{i=0}^{\ell} w_i = 0 \quad \text{and} \quad 2 \log(2) \sum_{i=0}^{\ell} i w_i = 1 . \quad (29)$$

One may choose, for example, \mathbf{w} corresponding to the weighted least-squares regression vector, defined by

$$\mathbf{w} = DB(B^T DB)^{-1} \mathbf{b} ,$$

where $B \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 2 & \dots & \ell \end{bmatrix}^T$ is the so-called design matrix, D is a definite positive matrix and

$$\mathbf{b} \stackrel{\text{def}}{=} [0 \ (2 \log(2))^{-1}]^T . \quad (30)$$

Ordinary least square regression corresponds to the case where D is the identity matrix.

Consider now a *Gaussian* process X with memory parameter d and generalized spectral density $f(\lambda) = |1 - e^{-i\lambda}|^{-2d} f^*(\lambda)$, and let $K \geq (d - 1/2) \vee 1$. Then the distribution of the K -th order increment process $\Delta^K X$ only depends on d and f^* . In this section we denote by $\mathbb{E}_{d,f^*}^{\mathcal{G}}$ and $\text{Var}_{d,f^*}^{\mathcal{G}}$ the expectation and the variance computed on random variables which can be expressed as $\Phi(\Delta^K X)$, where Φ is a possibly non-linear function. Note that any function of the wavelet coefficients of X can be written this way, provided that $M \geq d - 1/2$ (see (14)). The superscript \mathcal{G} indicates that X is Gaussian.

We now compute a bound of the mean square error and an asymptotic equivalent of the variance of $\hat{d}_n(J_0, \mathbf{w})$ in the usual semiparametric framework

adopted by Robinson and his co-authors for studying Fourier estimators. For the wavelet estimator defined above, these quantities primarily depend on n and on the scale index J_0 , while in the Fourier case, the bounds are generally expressed as functions of n and a *bandwidth* parameter m , equal to the number of discrete Fourier frequencies used. To ease comparison, we will express our results with respect to n and m , where m is the number of wavelet coefficients appearing in $\hat{d}_n(J_0, \mathbf{w})$, namely,

$$m \stackrel{\text{def}}{=} \sum_{j=J_0}^{J_0+\ell} n_j .$$

From (26), one gets immediately that $|m - n2^{-J_0}(2 - 2^{-\ell})| \leq 2(\ell + 1)(T - 1)$. Thus $m \rightarrow \infty$ is equivalent to having $n2^{-J_0} \rightarrow \infty$, and, when these conditions holds, we have

$$m \sim n2^{-J_0}(2 - 2^{-\ell}) . \quad (31)$$

The next result provides a bound to the bias $\mathbb{E}_{d,f^*}^{\mathcal{G}} [\hat{d}_n(J_0, \mathbf{w})] - d$ and to the variance $\text{Var}_{d,f^*}^{\mathcal{G}} [\hat{d}_n(J_0, \mathbf{w})]$.

Theorem 4. *Assume that (W-1)-(W-4) hold with $M \geq 1$ and $\alpha > 1$. Let \mathbf{w} be a vector satisfying (29) for some $\ell \geq 1$. Let d_{\min}, d_{\max} be two scalars such that $d_{\min} < d_{\max}$ and $[d_{\min}, d_{\max}] \subset ((1 + \beta)/2 - \alpha, M]$, where $\beta \in (0, 2]$. Then, there exist a finite constant C and an integer J_{\min} (depending only on $\mathbf{w}, \beta, L, d_{\min}, d_{\max}, \phi$ and ψ) such that for all $J_0 \in \{J_{\min}, \dots, J(n) - \ell\}$, $d \in [d_{\min}, d_{\max}]$, and $f^* \in \mathcal{H}(\beta, L)$ with $f^*(0) > 0$*

$$\left| \mathbb{E}_{d,f^*}^{\mathcal{G}} [\hat{d}_n(J_0, \mathbf{w})] - d \right| \leq C \left\{ \left(\frac{m}{n} \right)^{\beta} + m^{-1} \right\} , \quad (32)$$

$$\text{Var}_{d,f^*}^{\mathcal{G}} [\hat{d}_n(J_0, \mathbf{w})] \leq Cm^{-1} . \quad (33)$$

By combining (32) and (33) it is possible to obtain a bound on the mean square error of $\hat{d}_n(J_0, \mathbf{w})$. More precisely, there exist constants C and J_{\min} (depending only on $M, \alpha, \beta, L, d_{\min}$ and d_{\max}) such that, for any $f^* \in \mathcal{H}(\beta, L)$, $d \in [d_{\min}, d_{\max}]$ and $J_0 \in \{J_{\min}, \dots, J(n) - \ell + 1\}$,

$$\mathbb{E}_{d,f^*}^{\mathcal{G}} \left[\left\{ \hat{d}_n(J_0, \mathbf{w}) - d \right\}^2 \right] \leq C \left\{ \left(\frac{m}{n} \right)^{2\beta} + m^{-1} \right\} . \quad (34)$$

This shows in particular that, for any non-decreasing sequence $\{J_0(n), n \geq 0\}$ such that $m^{-1} + m/n \rightarrow 0$, $\hat{d}_n(\mathbf{w}) \stackrel{\text{def}}{=} \hat{d}_n(J_0(n), \mathbf{w})$ is a consistent estimator of d . If the regularity exponent β is known, it is possible to choose $J_0(n)$ to balance these two terms, that is, set $(m/n)^{2\beta} \asymp m^{-1}$ or equivalently $2^{J_0(n)} \asymp n^{1/(1+2\beta)}$

as $n \rightarrow \infty$. If we choose $J_0(n)$ in such a way, (31) and (34) imply

$$\limsup_{n \rightarrow \infty} \sup_{d \in [d_{\min}, d_{\max}]} \sup_{f^* \in \mathcal{H}(\beta, L)} n^{2\beta/(1+2\beta)} \mathbb{E}_{d, f^*}^{\mathcal{G}} \left[\left\{ \hat{d}_n(\mathbf{w}) - d \right\}^2 \right] < \infty .$$

As shown in Giraitis et al. (1997), $n^{-2\beta/(1+2\beta)}$ is the minimax rate of convergence for the memory parameter d in this semi-parametric setting. Therefore,

Corollary 5. *The wavelet estimator is rate optimal in the minimax sense.*

We shall now obtain the asymptotic behavior of $\text{Var}_{d, f^*}^{\mathcal{G}} \left[\hat{d}_n(\mathbf{w}) \right]$ as $n \rightarrow \infty$.

Theorem 6. *Assume that (W-1)-(W-4) hold with $M \geq 1$ and $\alpha > 1$. Let \mathbf{w} be a vector satisfying (29) for some $\ell \geq 1$. Let $\{J_0(n), n \in \mathbb{N}\}$ be a sequence such that $m \rightarrow \infty$ as $n \rightarrow \infty$. For any $f^* \in \mathcal{H}(\beta, L)$, where $\beta \in (0, 2]$, and $d \in ((1 + \beta)/2 - \alpha, M]$,*

$$\lim_{n \rightarrow \infty} m \text{Var}_{d, f^*}^{\mathcal{G}} \left[\hat{d}_n(\mathbf{w}) \right] = (2 - 2^{-\ell}) \mathbf{w}^T \mathbf{V}(d, \psi) \mathbf{w} , \quad (35)$$

where $\mathbf{V}(d, \psi)$ is the $(1 + \ell) \times (1 + \ell)$ matrix defined as

$$\mathbf{V}_{i,j}(d, \psi) \stackrel{\text{def}}{=} \frac{4\pi 2^{2d|j-i|} 2^{i \wedge j}}{\mathbf{K}(d, \psi)^2} \int_{-\pi}^{\pi} |\mathbf{D}_{\infty, |j-i|}(\lambda; d, \psi)|^2 d\lambda \quad 0 \leq i, j \leq \ell . \quad (36)$$

Remark 3. The asymptotic expression of the variance (35) is a quadratic form of \mathbf{w} defined by the matrix $\mathbf{V}(d, \psi)$, which depends only on d and ψ (see (36)). The standard theory of linear regression shows that, for any $\ell \geq 1$, the optimal regression vector of length $\ell + 1$ is

$$\mathbf{w}^{\text{opt}}(d, \psi) \stackrel{\text{def}}{=} \mathbf{V}^{-1}(d, \psi) B (B^T \mathbf{V}^{-1}(d, \psi) B)^{-1} \mathbf{b}$$

and the associated limiting variance is $(2 - 2^{-\ell}) \mathbf{b}^T (B^T \mathbf{V}^{-1}(d, \psi) B)^{-1} \mathbf{b}$. This optimal regression vector cannot be used directly since it depends on d which is unknown, but one may apply a two-step procedure using a preliminary estimate of d as in Bardet (2002) in a similar context.

If we choose m (or $J_0(n)$) such that the bias in (32) is asymptotically negligible, then we can obtain the asymptotic behavior of the mean square error $\mathbb{E}_{d, f^*}^{\mathcal{G}} \left(\hat{d}_n(\mathbf{w}) - d \right)^2$. In view of (32) and (35), we need $m \rightarrow \infty$ and $\{(m/n)^\beta + m^{-1}\}^2 \ll m^{-1}$, or equivalently

$$n 2^{-J_0(n)(1+2\beta)} + n^{-1} 2^{J_0(n)} \rightarrow 0, \quad n \rightarrow \infty . \quad (37)$$

Corollary 7. *If (37) holds, then for $f^* \in \mathcal{H}(\beta, L)$ and $d \in ((1 + \beta)/2 - \alpha, M]$,*

$$\lim_{n \rightarrow \infty} n 2^{-J_0(n)} \mathbb{E}_{d, f^*}^{\mathcal{G}} \left(\hat{d}_n(\mathbf{w}) - d \right)^2 = \mathbf{w}^T \mathbf{V}(d, \psi) \mathbf{w} .$$

This result of course hints at the existence of a central limit theorem for the estimator $\hat{d}_n(\mathbf{w})$. Such result can be obtained using a central limit of quadratic form of Gaussian variables which is established in a companion paper Moulines et al. (2005).

5. PROOF OF THEOREM 3

From now on, we denote by C constants possibly depending on d , β , ϕ and ψ , which may change from line to line and we omit the dependence in ϕ and ψ in the notations. We assume, without loss of generality that $f^*(0) = 1$.

Proof of (a). Let $j \geq 0$ and define

$$A_j \stackrel{\text{def}}{=} 2^j \int_{-\pi}^{\pi} |1 - e^{-i\lambda}|^{-2d} f^*(\lambda) |\hat{\phi}(\lambda) \hat{\psi}(2^j \lambda)|^2 d\lambda \quad \text{and} \quad R_j \stackrel{\text{def}}{=} \sigma_j^2(d, f^*) - A_j .$$

Using (68), we have

$$|R_j| \leq C 2^{j(1+M-\alpha)} \int_{-\pi}^{\pi} |1 - e^{-i\lambda}|^{-2d} f^*(\lambda) |\lambda|^{2M} (1 + 2^j |\lambda|)^{-\alpha-M} d\lambda . \quad (38)$$

We consider A_j and R_j separately starting with A_j . Define the function g by

$$\lambda \mapsto g(\lambda) \stackrel{\text{def}}{=} f^*(\lambda) |\hat{\phi}(\lambda)|^2 |\lambda/(1 - e^{i\lambda})|^{2d}, \quad \lambda \in (-\pi, \pi) . \quad (39)$$

Since $\hat{\phi}$ is infinitely differentiable by (W-1), $\lambda \mapsto |\hat{\phi}(\lambda)|^2 |\lambda/(1 - e^{i\lambda})|^{2d}$ is infinitely differentiable on $[-\pi, \pi]$. Because $f^* \in \mathcal{H}(\beta, L)$ and $f^*(0) = 1$, there exists a constant C (depending only on d_{\min} , d_{\max} and ϕ) such that for all $\lambda \in [-\pi, \pi]$,

$$|g(\lambda) - g(0)| \leq C L |\lambda|^\beta . \quad (40)$$

We can now express A_j as

$$A_j = 2^j \int_{-\pi}^{\pi} g(\lambda) |\lambda|^{-2d} |\hat{\psi}(2^j \lambda)|^2 d\lambda . \quad (41)$$

Observe that A_j is finite since g is bounded and (5) applies since $M > d - 1/2$. We now replace the function $\lambda \mapsto g(\lambda)$ by the constant $g(0) = 1$ and extends the interval of integration from $[-\pi, \pi]$ to the whole real line in (41). Eqs. (40) and (41) imply

$$\left| A_j - 2^j \int_{-\pi}^{\pi} g(0) |\lambda|^{-2d} |\hat{\psi}(2^j \lambda)|^2 d\lambda \right| \leq C L 2^j \int_{-\pi}^{\pi} |\lambda|^{\beta-2d} |\hat{\psi}(2^j \lambda)|^2 d\lambda .$$

First observe that, after a change of variable,

$$2^j \int_{-\pi}^{\pi} |\lambda|^{\beta-2d} |\hat{\psi}(2^j \lambda)|^2 d\lambda \leq 2^{j(2d-\beta)} \int_{-\infty}^{\infty} \{ |\lambda|^{\beta-2d_{\min}} \vee |\lambda|^{\beta-2d_{\max}} \} |\hat{\psi}(\lambda)|^2 d\lambda$$

In the RHS of this inequality, using the behavior of $|\hat{\psi}(\lambda)|$ at infinity and at the origin implied by (W-2) and (W-3) respectively, and because $d_{\max} < M + 1/2$

and $d_{\min} > (1 + \beta)/2 - \alpha$, the integral is a finite constant depending only on ψ and β . We further observe that, by (W-2), since $d_{\min} > 1/2 - \alpha$, we may write

$$2^j \int_{|\lambda| > \pi} |\lambda|^{-2d} |\hat{\psi}(2^j \lambda)|^2 d\lambda \leq C 2^{j(1-2\alpha)} \int_{|\lambda| > \pi} |\lambda|^{-2(\alpha+d_{\min})} d\lambda.$$

Therefore, there exists a constant C , depending only on $\beta, \phi, \psi, d_{\min}$ and d_{\max} such that

$$|A_j - K(d) 2^{2jd}| \leq C L 2^{(2d-\beta)j}. \quad (42)$$

We now compute a bound for R_j from (38). Note that, there exists a constant C depending only on β, d_{\min} and d_{\max} such that, for all $\lambda \in [-\pi, \pi]$,

$$f(\lambda) = f^*(\lambda) \left| \frac{\lambda}{1 - e^{i\lambda}} \right|^{2d} |\lambda|^{-2d} \leq C L |\lambda|^{-2d}. \quad (43)$$

Plugging into (38) and then separating $\lambda < 1$ and $\lambda \geq 1$, we obtain

$$\begin{aligned} R_j &\leq C L 2^{2jd} 2^{-j(M+\alpha)} \int_0^{2^j \pi} \{ \lambda^{2(M-d_{\min})} \vee \lambda^{2(M-d_{\max})} \} (1 + \lambda)^{-\alpha-M} d\lambda \\ &\leq C L 2^{j(2d-\beta)} 2^{-j(M+\alpha-\beta)} \left\{ \int_0^1 \lambda^{2(M-d_{\max})} d\lambda + \int_1^{2^j \pi} \lambda^{M-2d_{\min}-\alpha} d\lambda \right\}. \end{aligned}$$

Since $2(M - d_{\max}) > -1$, the first integral is a finite constant. Depending on whether $M - 2d_{\min} - \alpha$ is less than, equal to or larger than -1 the second integral is bounded by a finite constant, $\log \pi + j \log 2$ or $C 2^{j(1+M-2d_{\min}-\alpha)}$, where C only depends on M, α and d_{\min} . In the two first cases, we simply observe that we always have $M + \alpha - \beta > 0$, and in the last case that $-(M + \alpha - \beta) + 1 + M - 2d_{\min} - \alpha = 1 - 2d_{\min} - 2\alpha - \beta \leq 0$ by (20) so that, in all cases, $R_j \leq C L 2^{(2d-\beta)j}$. This, with (42), shows (21).

Proof of (b). For ease of notation, we only consider the case $u = 0$. Pick $j \geq 1$. In (17), the summands are $2^j(2\pi)$ -periodic; hence, omitting the summands, $\sum_{l=0}^{2^j-1} = \sum_{l=0}^{2^{j-1}-1} + \sum_{j=2^{j-1}}^{2^j-1} = \sum_{l=0}^{2^{j-1}-1} + \sum_{l=-2^{j-1}}^{-1} = \sum_{l=-2^{j-1}}^{2^{j-1}-1}$. Note that, for $l \in \{-2^{j-1}, \dots, 2^{j-1}-1\}$ and $\lambda \in (0, \pi)$, the $2^{-j}(\lambda + 2l\pi) \in (-\pi, \pi)$ so that (68) applies. Hence, $\mathbf{D}_{j,0}(\lambda; d, f^*)$ is expressed as the sum of two functions $A_j(\lambda) + R_j(\lambda)$, defined for all $\lambda \in (0, \pi)$ by

$$A_j(\lambda) \stackrel{\text{def}}{=} \sum_{l=-2^{j-1}}^{2^{j-1}-1} |2^{-j}(\lambda + 2l\pi)|^{-2d} g(2^{-j}(\lambda + 2l\pi)) |\hat{\psi}(\lambda + 2l\pi)|^2 \quad (44)$$

where g is defined in (39) and

$$R_j(\lambda) \leq C L 2^{j(2d-M-\alpha)} \sum_{l=-2^{j-1}}^{2^{j-1}} |\lambda + 2l\pi|^{2(M-d)} (1 + |\lambda + 2l\pi|)^{-\alpha-M}. \quad (45)$$

From (40), we get, for all $\lambda \in (0, \pi)$,

$$\left| A_j(\lambda) - 2^{2dj} g(0) \sum_{l=-2^{j-1}}^{2^j-1} |\lambda + 2l\pi|^{-2d} |\hat{\psi}(\lambda + 2l\pi)|^2 \right| \leq C L 2^{(2d-\beta)j} B_j(\lambda), \quad (46)$$

where, by (5) and (6), for all $\lambda \in (0, \pi)$,

$$\begin{aligned} B_j(\lambda) &\stackrel{\text{def}}{=} \sum_{l=-2^{j-1}}^{2^j-1} |\lambda + 2l\pi|^{\beta-2d} |\hat{\psi}(\lambda + 2l\pi)|^2 \\ &\leq C \left(|\lambda|^{\beta+2(M-d)} + 2 \sum_{l \geq 1} |\lambda + 2l\pi|^{\beta-2d-2\alpha} \right) \\ &\leq C \left(1 + 2 \sum_{l \geq 1} (2l-1)^{\beta-2d_{\min}-2\alpha} \right) < \infty \end{aligned}$$

since $|\lambda + 2l\pi| \geq \pi(2l-1)$, $M \geq d$ and $\beta - 2d_{\min} - 2\alpha < -1$. Using the same arguments, for all $\lambda \in (0, \pi)$,

$$\sum_{|l| \geq 2^{j-1}-1} |\lambda + 2l\pi|^{-2d} |\hat{\psi}(\lambda + 2l\pi)|^2 \leq C 2^{j(1-2(d_{\min}+\alpha))}.$$

Eqs. (25) with $u = 0$ and (46), and the above inequalities yield that, for all $\lambda \in (0, \pi)$,

$$|A_j(\lambda) - \mathbf{D}_{\infty,0}(\lambda; d) 2^{2dj}| \leq C L 2^{(2d-\beta)j}.$$

We now turn to bounding $R_j(\lambda)$. For or all $\lambda \in (0, \pi)$,

$$R_j(\lambda) \leq C L 2^{j(2d-\beta)} 2^{-j(M+\alpha-\beta)} \left(1 + \sum_{l=1}^{2^j} l^{-2d_{\min}+M-\alpha} \right)$$

and the proof follows as for bounding R_j in the proof of (a), by considering the cases $M - 2d_{\min} - \alpha <, =$ or > -1 .

6. PROOF OF THEOREM 4

From now on, we take $d \in [d_{\min}, d_{\max}]$, $f^* \in \mathcal{H}(\beta, L)$ and denote by J_{\min} , C , C_1 , C_2 , ... some constants that depend only on \mathbf{w} , β , d_{\min} , d_{\max} , ϕ , and ψ but that may take different values upon each appearance. For any measurable function vector-valued function φ on $[-\pi, +\pi]$ and any $p > 0$,

$$\|\varphi\|_p = \left(\int_{-\pi}^{\pi} |\varphi(\lambda)|^p d\lambda \right)^{1/p}.$$

Lemma 8. *There exist a constant C and an integer J_{\min} such that, for all $j \geq J_{\min}$, and $1 \leq q < p \leq \infty$,*

$$\frac{\|\mathbf{D}_{j,0}(\cdot; d, f^*)\|_p}{\|\mathbf{D}_{j,0}(\cdot; d, f^*)\|_q} \leq C(2\pi)^{1/p-1/q+1}. \quad (47)$$

Proof. Using $L^p([-\pi, \pi])$ spaces embeddings, it suffices to prove the result for $p = \infty$ and $q = 1$. Note that, for all $d \in [d_{\min}, d_{\max}]$, $0 < K_{\min} \leq K(d) \leq K_{\max} < \infty$, where

$$K_{\min} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} (|\xi|^{-2d_{\min}} \wedge |\xi|^{-2d_{\max}}) |\hat{\psi}(\xi)|^2 d\xi, \quad (48)$$

$$K_{\max} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} (|\xi|^{-2d_{\min}} \vee |\xi|^{-2d_{\max}}) |\hat{\psi}(\xi)|^2 d\xi. \quad (49)$$

By Theorem 3, (24), there exists a constant C_0 such that

$$|\mathbf{D}_{j,0}(\lambda; d, f^*)| \leq 2^{2jd} (K_{\max} \mathbf{D}_{\infty,0}(\lambda; d) + C_0 L 2^{-2\beta j}),$$

where $\mathbf{D}_{\infty,0}(\lambda; d)$ is defined in (25). Assumptions (W-2) and (W-3) imply that $|\hat{\psi}(\xi)| \leq C_1 |\xi|^M (1 + |\xi|)^{-\alpha-M}$ for all $\xi \in \mathbb{R}$. Therefore, for all $\lambda \in (-\pi, \pi)$, there exists a constant C_2 ,

$$\begin{aligned} \mathbf{D}_{\infty,0}(\lambda; d) &\leq C_2 \left\{ |\lambda|^{2(M-d)} + 2 \sum_{l>0} |\lambda + 2l\pi|^{-2(\alpha+d)} \right\} \\ &\leq C_2 \left\{ \pi^{2(M-d_{\min})} + 2 \sum_{l>0} [(2l-1)\pi]^{-2(\alpha+d_{\min})} \right\}. \end{aligned}$$

The two previous displays show that there exists a constant C_3 such that, for all $j \geq 0$, $d \in [d_{\min}, d_{\max}]$ and $f^* \in \mathcal{H}(\beta, L)$

$$\|\mathbf{D}_{j,0}(\cdot; d, f^*)\|_{\infty} \leq C_3 2^{2jd}. \quad (50)$$

On the other hand, we have

$$\begin{aligned} &\|\mathbf{D}_{j,0}(\cdot; d, f^*)\|_1 \\ &\geq 2^{2jd} \|\mathbf{D}_{\infty,0}(\cdot; d, f^*)\|_1 - \|\mathbf{D}_{j,0}(\cdot; d, f^*) - 2^{2jd} \mathbf{D}_{\infty,0}(\cdot; d, f^*)\|_1. \end{aligned}$$

By Theorem 3, (24),

$$\begin{aligned} &\|\mathbf{D}_{j,0}(\cdot; d, f^*) - 2^{2jd} \mathbf{D}_{\infty,0}(\cdot; d, f^*)\|_2 \\ &\leq (2\pi)^{1/2} \|\mathbf{D}_{j,0}(\cdot; d, f^*) - 2^{2jd} \mathbf{D}_{\infty,0}(\cdot; d, f^*)\|_{\infty} \leq (2\pi)^{1/2} C_0 L 2^{2j(d-\beta)}. \end{aligned}$$

By definition, $\|\mathbf{D}_{\infty,0}(d; \cdot)\|_1 = K(d) \geq K_{\min}$. The two previous display therefore imply that if $2^{-\beta j} \leq 4\pi C_0 L / K_{\min}$, $\|\mathbf{D}_{j,0}(\cdot; d, f^*)\|_1 \geq C 2^{2jd}$, which, with (50), shows (47). \square

Proof of Theorem 4. The bias $\mathbb{E}_{d,f^*}^{\mathcal{G}}[\hat{d}_n(J_0, \mathbf{w})] - d$ can be decomposed into two terms as follows

$$\begin{aligned} \sum_{j=J_0}^{J_0+\ell} w_{j-J_0} \mathbb{E}_f [\log(\hat{\sigma}_j^2)] - d &= \sum_{j=J_0}^{J_0+\ell} w_{j-J_0} \log [\sigma_j^2(d, f^*)] - d + \\ &\quad \sum_{j=J_0}^{J_0+\ell} w_{j-J_0} \{ \mathbb{E}_f [\log(\hat{\sigma}_j^2)] - \log [\sigma_j^2(d, f^*)] \} . \end{aligned} \quad (51)$$

Using (29), the first term on the RHS of the previous display may be rewritten as

$$\begin{aligned} \sum_{j=J_0}^{J_0+\ell} w_{j-J_0} \log [\sigma_j^2(d, f^*)] - d &= \sum_{j=J_0}^{J_0+\ell} w_{j-J_0} \log \left[1 + \frac{\sigma_j^2(d, f^*) - K(d) 2^{2jd}}{K(d) 2^{2jd}} \right] \end{aligned}$$

Using Theorem 3-(21), there exists a constant C such that $\sigma_j^2(d, f^*) - K(d) 2^{2jd} \leq CL2^{2j(d-\beta)}$. Using that $\log(1+x) \leq 2x$ for $x \in (-1/2, 1/2)$, for any J_0 such that $CL2^{-\beta J_0} \leq K_{\min}/2$,

$$\left| \sum_{j=J_0}^{J_0+\ell} w_{j-J_0} \log[\sigma_j^2(d, f^*)] - d \right| \leq \frac{2C}{K_{\min}} 2^{-jJ_0\beta} L \sum_{j=0}^{\ell} |w_j| 2^{-j\beta} . \quad (52)$$

We now consider the second term in the RHS of the display (51). We apply Lemma 10: the empirical wavelet coefficient variance at scale j is a quadratic form in the wavelet coefficients at $[W_{j,0}, \dots, W_{j,n_j-1}]$ which are, by Proposition 1, a part of a stationary process with spectral density $\mathbf{D}_{j,0}(\cdot; d, f^*)$, defined in Corollary 2, (17) The spectral radius of the covariance matrix $\Gamma_j(d, f^*)$ of the random vector $[W_{j,0}, \dots, W_{j,n_j-1}]$ is bounded by the supremum of the spectral density,

$$\rho[\Gamma_j(d, f^*)] \leq 2\pi \|\mathbf{D}_{j,0}(\cdot; d, f^*)\|_{\infty} . \quad (53)$$

On the other hand, the stationarity of the wavelet coefficients at scale j implies

$$\begin{aligned} \text{Var}_{d,f^*}^{\mathcal{G}}(\hat{\sigma}_j^2) &= \frac{1}{n_j^2} \sum_{k,l=1}^{n_j} \text{Cov}_{d,f^*}^{\mathcal{G}}(W_{j,k}^2, W_{j,l}^2) \\ &= \frac{1}{n_j} \sum_{k=1-n_j}^{n_j-1} \left(1 - \frac{|k|}{n_j} \right) \text{Cov}_{d,f^*}^{\mathcal{G}}(W_{j,0}^2, W_{j,k}^2) \end{aligned}$$

Because $W_{j,0}$ and $W_{j,k}$ are jointly Gaussian,

$$\text{Cov}_{d,f^*}^{\mathcal{G}}(W_{j,0}^2, W_{j,k}^2) = 2 (\text{Cov}_{d,f^*}^{\mathcal{G}}[W_{j,0}, W_{j,k}])^2 .$$

The Parseval identity gives that

$$\sum_{k=1-n_j}^{n_j-1} \left(1 - \frac{|k|}{n_j}\right) \text{Cov}_{d,f^*}^{\mathcal{G}}(W_{j,0}^2, W_{j,k}^2) \leq 2 \sum_{k=-\infty}^{\infty} \left\{ \text{Cov}_{d,f^*}^{\mathcal{G}}(W_{j,0}, W_{j,k}) \right\}^2 = \frac{1}{\pi} \|\mathbf{D}_{j,0}(\cdot; d, f^*)\|_2^2,$$

showing that the variance $\text{Var}_{d,f^*}^{\mathcal{G}}[\hat{\sigma}_j^2]$ of the empirical wavelet variance at scale j satisfies

$$\text{Var}_{d,f^*}^{\mathcal{G}}[\hat{\sigma}_j^2] \leq \frac{1}{\pi n_j} \|\mathbf{D}_{j,0}(\cdot; d, f^*)\|_2^2. \quad (54)$$

By Lemma 10-(70), using (53) and (54),

$$\left| \mathbb{E}_{d,f^*}^{\mathcal{G}}[\log(\hat{\sigma}_j^2)] - \log[\mathbb{E}_{d,f^*}^{\mathcal{G}}(\hat{\sigma}_j^2)] \right| \leq C n_j^{-1} \frac{\|\mathbf{D}_{j,0}(\cdot; d, f^*)\|_{\infty}^2}{\|\mathbf{D}_{j,0}(\cdot; d, f^*)\|_2^2} \quad (55)$$

where C is a universal constant. The bound (32) on the bias follows from (52), (55), Lemma 8 and (31).

We now compute the variance of the estimator $\hat{d} \stackrel{\text{def}}{=} \hat{d}_n(J_0, \mathbf{w})$. By Lemma 8 with $p = \infty$ and $q = 2$, Lemma 10, and using again (53) and (54), there are constants C and J_{\min} such that, for all $J_0 \geq J_{\min}$,

$$\begin{aligned} & \left| \text{Var}_{d,f^*}^{\mathcal{G}}(\hat{d}) - \sum_{i,j=J_0}^{J_0+\ell} w_{i-J_0} w_{j-J_0} \frac{\text{Cov}_{d,f^*}^{\mathcal{G}}[\hat{\sigma}_i^2, \hat{\sigma}_j^2]}{\sigma_i^2(d, f^*) \sigma_j^2(d, f^*)} \right| \\ & \leq \sum_{i,j=J_0}^{J_0+\ell} |w_{i-J_0} w_{j-J_0}| \left| \text{Cov}_{d,f^*}^{\mathcal{G}}[\log(\hat{\sigma}_i^2), \log(\hat{\sigma}_j^2)] - \frac{\text{Cov}_{d,f^*}^{\mathcal{G}}[\hat{\sigma}_i^2, \hat{\sigma}_j^2]}{\sigma_i^2(d, f^*) \sigma_j^2(d, f^*)} \right| \\ & \leq C \sum_{i,j=J_0}^{J_0+\ell} |w_{i-J_0} w_{j-J_0}| \left\{ \frac{\|\mathbf{D}_{i,0}(\cdot; d, f^*)\|_{\infty}^3}{n_i^{3/2} \|\mathbf{D}_{i,0}(\cdot; d, f^*)\|_2^3} \vee \frac{\|\mathbf{D}_{j,0}(\cdot; d, f^*)\|_{\infty}^3}{n_j^{3/2} \|\mathbf{D}_{j,0}(\cdot; d, f^*)\|_2^3} \right\} \\ & \leq C n_{J_0}^{-3/2} \leq C (n 2^{-J_0})^{-3/2}. \end{aligned} \quad (56)$$

The Cauchy-Schwarz inequality and (54) yield

$$\frac{|\text{Cov}_{d,f^*}^{\mathcal{G}}[\hat{\sigma}_i^2, \hat{\sigma}_j^2]|}{\sigma_i^2(d, f^*) \sigma_j^2(d, f^*)} \leq \frac{1}{\pi \sqrt{n_i n_j}} \frac{\|\mathbf{D}_{i,0}(\cdot; d, f^*)\|_2 \|\mathbf{D}_{j,0}(\cdot; d, f^*)\|_2}{\|\mathbf{D}_{i,0}(\cdot; d, f^*)\|_1 \|\mathbf{D}_{j,0}(\cdot; d, f^*)\|_1}.$$

Using Lemma 8 with $p = 2$ and $q = 1$ thus gives for all $J_0 \geq J_{\min}$,

$$\left| \sum_{i,j=J_0}^{J_0+\ell} w_{i-J_0} w_{j-J_0} \frac{\text{Cov}_{d,f^*}^{\mathcal{G}}[\hat{\sigma}_i^2, \hat{\sigma}_j^2]}{\sigma_i^2(d, f^*) \sigma_j^2(d, f^*)} \right| \leq C n_{J_0}^{-1}.$$

This bound with (56) and (31) shows (33). \square

7. PROOF OF THEOREM 6

For any $0 \leq a \leq b \leq \ell$, set $a_n \stackrel{\text{def}}{=} J_0(n) + a$ and $b_n \stackrel{\text{def}}{=} J_0(n) + b$.

$$\begin{aligned} \text{Cov}_{d,f^*}^{\mathcal{G}} [\hat{\sigma}_{b_n}^2, \hat{\sigma}_{a_n}^2] &= \frac{1}{n_{a_n} n_{b_n}} \sum_{k,l=0}^{n_{b_n}-1} \sum_{v=0}^{2^{b-a}-1} \text{Cov}_{d,f^*}^{\mathcal{G}} [W_{b_n,k}^2, W_{a_n,l2^{b-a}+v}^2] \\ &= \frac{2}{n_{a_n} n_{b_n}} \sum_{k,l=0}^{n_{b_n}-1} |\text{Cov}_{d,f^*}^{\mathcal{G}} [W_{b_n,k}, \mathbf{W}_{b_n,l}(b-a)]|^2, \end{aligned} \quad (57)$$

where we have used that if X and $\mathbf{Y} = [Y_1 \dots Y_K]^T$ are Gaussian vectors,

$$\text{Cov}(X^2, |\mathbf{Y}|^2) = \text{Cov}\left(X^2, \sum_{k=1}^K Y_k^2\right) = 2 \sum_{k=1}^K \text{Cov}^2(X, Y_k) = 2 |\text{Cov}(X, \mathbf{Y})|^2.$$

By Corollary 2, using the fact that the processes $\{W_{b_n,k}\}_{k \in \mathbb{Z}}$ and $\{\mathbf{W}_{b_n,k}(b-a)\}_{k \in \mathbb{Z}}$ (see (16)) are jointly stationary and Gaussian, we may write

$$\begin{aligned} \text{Cov}_{d,f^*}^{\mathcal{G}} [\hat{\sigma}_{b_n}^2, \hat{\sigma}_{a_n}^2] &= \frac{2}{n_{a_n}} \sum_{k=1-n_{b_n}}^{n_{b_n}-1} \left(1 - \frac{|k|}{n_{b_n}}\right) |\text{Cov}_{d,f^*}^{\mathcal{G}} [W_{b_n,0}, \mathbf{W}_{b_n,k}(b-a)]|^2 \\ &\leq \frac{2}{n_{a_n}} \sum_{l=-\infty}^{\infty} |\text{Cov}_{d,f^*}^{\mathcal{G}} (W_{b_n,0}, \mathbf{W}_{b_n,l}(b-a))|^2 \leq \frac{4\pi}{n_{a_n}} \|\mathbf{D}_{b_n,b-a}(\cdot; d, f^*)\|_2^2 \end{aligned} \quad (58)$$

where the last inequality follows from the Parseval inequality. Hence, by Theorem 3, (21)-(24), since $n_{a_n} \sim n2^{-J_0(n)-a}$,

$$\limsup_{n \rightarrow \infty} n2^{-J_0(n)} \frac{\text{Cov}_{d,f^*}^{\mathcal{G}} [\hat{\sigma}_{b_n}^2, \hat{\sigma}_{a_n}^2]}{\sigma_{a_n}^2(d, f^*) \sigma_{b_n}^2(d, f^*)} \leq 4\pi \frac{2^{2d(b-a)} 2^a}{K(d)^2} \|\mathbf{D}_{\infty,b-a}(\cdot; d)\|_2^2. \quad (59)$$

For any given $l \in \mathbb{Z}$, Theorem 3, (21)-(24) also shows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left\{ \left(1 - \frac{|l|}{n_{b_n}}\right)_+^{1/2} \frac{\text{Cov}_{d,f^*}^{\mathcal{G}} (W_{b_n,0}, \mathbf{W}_{b_n,l}(b-a))}{\sqrt{\sigma_{a_n}^2(d, f^*) \sigma_{b_n}^2(d, f^*)}} \right\} \\ = \frac{2^{d(b-a)}}{K(d)} \int_{-\pi}^{+\pi} \mathbf{D}_{\infty,b-a}(\lambda; d) e^{i\lambda l} d\lambda, \end{aligned}$$

where $x_+ = \max(x, 0)$. The Parseval inequality then gives

$$\begin{aligned} \sum_{l=-\infty}^{\infty} \lim_{n \rightarrow \infty} \left\{ \left(1 - \frac{|l|}{n_{b_n}} \right)_+ \frac{|\text{Cov}_{d, f^*}^{\mathcal{G}}(W_{b_n, 0}, \mathbf{W}_{b_n, l}(b-a))|^2}{\sigma_{a_n}^2(d, f^*) \sigma_{b_n}^2(d, f^*)} \right\} \\ = \frac{2\pi 2^{2d(b-a)}}{\mathbf{K}(d)^2} \|\mathbf{D}_{\infty, b-a}(\cdot; d)\|_2^2. \end{aligned}$$

Hence, by the Fatou Lemma and (58),

$$4\pi \frac{2^{2d(b-a)} 2^a}{\mathbf{K}(d)^2} \|\mathbf{D}_{\infty, b-a}(\cdot; d)\|_2^2 \leq \liminf_{n \rightarrow \infty} n 2^{-J_0(n)} \frac{\text{Cov}_{d, f^*}^{\mathcal{G}}[\hat{\sigma}_{b_n}^2, \hat{\sigma}_{a_n}^2]}{\sigma_{a_n}^2(d, f^*) \sigma_{b_n}^2(d, f^*)} \quad (60)$$

Observe that the RHS of (59) and the LHS of (60) both are equal to $\mathbf{V}_{a,b}(d, \psi)$ defined in (36). The proof then follows from (56), (59), (60) and (31).

APPENDIX A. MULTIREOLUTION ANALYSIS

In a multiresolution analysis, the scaling function ϕ satisfies the dilation equation $\phi(x) = 2 \sum_{k \in \mathbb{Z}} c_k \phi(2x - k)$ where $\sum_k c_k = 1$. In the Fourier domain, the dilation equation reads

$$\hat{\phi}(\xi) = m(\xi/2) \hat{\phi}(\xi/2), \quad (61)$$

where m is the “symbol” associated to the function ϕ defined as

$$m(\xi) = \frac{1}{2} \sum_{k \in \mathbb{Z}} c_k e^{-ik\xi}. \quad (62)$$

The scaling function ϕ has a finite support (assumption (W-1)) if the symbol m in (62) is a trigonometric polynomial. The wavelet function ψ associated to the scaling function ϕ is defined as $\psi(x) = \sum_{k \in \mathbb{Z}} (-1)^{1-k} c_{1-k} \phi(2(x - \ell) - k)$ or equivalently,

$$\hat{\psi}(\xi) = e^{-i\xi(\ell+1/2)} \overline{m(\xi/2 + \pi)} \hat{\phi}(\xi/2) \quad (63)$$

where $\ell \in \mathbb{Z}$ is an arbitrarily chosen translation parameter. The key property of MRA is that the family of dilated and translated wavelets $\{\psi_{j,k}, j \in \mathbb{Z}, k \in \mathbb{Z}\}$ (7) forms an orthonormal basis of $L^2(\mathbb{R})$.

Many properties of ϕ and ψ can be deduced from (61) and (63) and the specific form of the symbol m . In particular, (W-3) and (W-4), or, equivalently, (5) and (6) are equivalent to having that m can be factorized as

$$m(\xi) = \left(\frac{1 + e^{-i\xi}}{2} \right)^M p(\xi), \quad (64)$$

where p is a trigonometric polynomial (see (Cohen, 2003, Eq. (2.7.12) and Section 2.8)).

The maximal M for which (64) (or, equivalently, (W-3) and (W-4)) is satisfied is called *the number of vanishing moments*. For instance, the so-called

Daubechies DB- M wavelets have M vanishing moments and are associated to a trigonometric polynomial symbol m of degree $2M$ and admitting a zero at π of multiplicity M (see Daubechies (1992), (see (Mallat, 1998, Proposition 7.3, Page 242)). The maximal value of the α appearing in (W-2) (or equivalently, using Eq. (63) $\sup_{\xi}(1 + |\xi|^{\alpha})|\hat{\phi}(\xi)| < \infty$) has been studied for some families of compactly supported wavelets (see for instance (Daubechies, 1992, Chapter 7) and (Cohen, 2003, Section 2.7)). For the DB- M wavelets, a lower bound $\alpha \geq (1 - \log_2(3)/2)M > 0.207M$ can be established, see (Cohen, 2003, Eq 2.10.27). Therefore, α can be made arbitrarily large by increasing M . This implies that, for a given value of M and α , it is always possible to find a Daubechies wavelet DB- M' for some $M' \geq M$ satisfying (W-1)-(W-4).

APPENDIX B. APPROXIMATION OF WAVELET FILTER TRANSFER FUNCTIONS

Proposition 9. *Under (W-1)-(W-4), there exist positive constants C_i , $i = 1, \dots, 4$ only depending on ϕ and ψ , such that, for all $j \geq 0$ and $\lambda \in (-\pi, \pi)$,*

$$|H_j(\lambda) - 2^{j/2}\hat{\phi}(\lambda)\overline{\hat{\psi}(2^j\lambda)}| \leq C_1 2^{j(1/2-\alpha)} |\lambda|^M, \quad (65)$$

$$|\hat{\phi}(\lambda)\hat{\psi}(2^j\lambda)| \leq C_2 |2^j\lambda|^M (1 + 2^j|\lambda|)^{-\alpha-M}, \quad (66)$$

$$|H_j(\lambda)| \leq C_3 2^{j/2} |2^j\lambda|^M (1 + 2^j|\lambda|)^{-\alpha-M}, \quad (67)$$

$$\left| |H_j(\lambda)|^2 - 2^j |\hat{\phi}(\lambda)\hat{\psi}(2^j\lambda)|^2 \right| \leq C_4 2^{j(1+M-\alpha)} |\lambda|^{2M} (1 + 2^j|\lambda|)^{-\alpha-M}. \quad (68)$$

Proof. Under (W-1) and (W-2), we have that, for all $t \in \mathbb{R}$, $\sum_{k \in \mathbb{Z}} \hat{\phi}(\lambda + 2k\pi) e^{it(\lambda+2k\pi)}$ is a 2π -periodic function, integrable on $(-\pi, \pi)$ and whose l -th Fourier coefficients is

$$\int_{-\pi}^{\pi} \sum_{k \in \mathbb{Z}} \hat{\phi}(\lambda + 2k\pi) e^{it(\lambda+2k\pi)} e^{-i\lambda l} d\lambda = \int_{-\infty}^{\infty} \hat{\phi}(\lambda) e^{it\lambda} e^{-i\lambda l} d\lambda = 2\pi \phi(t - l).$$

It follows that, for all λ and t in \mathbb{R} ,

$$\sum_{l \in \mathbb{Z}} \phi(t - l) e^{i\lambda l} = \sum_{k \in \mathbb{Z}} \hat{\phi}(\lambda + 2k\pi) e^{it(\lambda+2k\pi)},$$

which is a form of the Poisson summation formula. Inserting this in (13) gives

$$\begin{aligned} H_j(\lambda) &= 2^{-j/2} \int_{-\infty}^{\infty} \left(\sum_{k \in \mathbb{Z}} \hat{\phi}(\lambda + 2k\pi) e^{it(\lambda+2k\pi)} \right) \psi(2^{-j}t) dt \\ &= 2^{-j/2} \sum_{k \in \mathbb{Z}} \hat{\phi}(\lambda + 2k\pi) \int_{-\infty}^{\infty} e^{it(\lambda+2k\pi)} \psi(2^{-j}t) dt \\ &= 2^{j/2} \sum_{k \in \mathbb{Z}} \hat{\phi}(\lambda + 2k\pi) \overline{\hat{\psi}(2^j(\lambda + 2k\pi))}. \end{aligned}$$

From this expression of H_j , we get, for all $j \geq 0$ and $\lambda \in (-\pi, \pi)$,

$$|H_j(\lambda) - 2^{j/2} \hat{\phi}(\lambda) \overline{\hat{\psi}(2^j \lambda)}| = 2^{j/2} \left| \sum_{|k| \geq 1} \hat{\phi}(\lambda + 2k\pi) \overline{\hat{\psi}(2^j(\lambda + 2k\pi))} \right|. \quad (69)$$

Now using successively (6) and (W-2), there is a constant C such that, for all non-zero integer k and all $\lambda \in (-\pi, \pi)$, $|\hat{\phi}(\lambda + 2k\pi)| \leq C|\lambda|^M$ and

$$|\overline{\hat{\psi}(2^j(\lambda + 2k\pi))}| \leq C(2^j|\lambda + 2k\pi|)^{-\alpha} \leq C2^{-\alpha j}(2|k|\pi - |\lambda|)^{-\alpha} \leq \frac{C2^{-\alpha j}}{\pi^\alpha(2|k| - 1)^\alpha}.$$

Inserting these bounds into (69) gives (65).

The bound (66) follows from (W-1) ($|\hat{\phi}(\xi)| \leq \int_{-\infty}^{\infty} |\phi(t)| dt$), (W-2) ($|\hat{\psi}(\xi)| \leq C(1 + |\xi|)^\alpha$) and (W-3) ($\lim_{\xi \rightarrow 0^+} \xi^{-M} |\hat{\psi}(\xi)| = 0$).

The two last bounds (67) and (68) follow from the two first (65) and (66). Indeed, let $\tilde{H}_j(\lambda) \stackrel{\text{def}}{=} 2^{j/2} \hat{\phi}(\lambda) \hat{\psi}(2^j \lambda)$. For (67) we write

$$|H_j(\lambda)| \leq |H_j(\lambda) - \tilde{H}_j(\lambda)| + |\tilde{H}_j(\lambda)|.$$

Applying (65) and (66), the RHS of this equation is bounded by

$$C_1 2^{j(1/2-\alpha)} |\lambda|^M + C_2 2^{j/2} |2^j \lambda|^M (1 + 2^j |\lambda|)^{-\alpha-M} \leq 2^{j/2} |2^j \lambda|^M (1 + 2^j |\lambda|)^{-\alpha-M} (C_1 2^{-j(\alpha+M)} (1 + 2^j |\lambda|)^{\alpha+M} + C_2).$$

By observing that, for all $j \geq 0$ and $\lambda \in (-\pi, \pi)$, the last term between parentheses is bounded by $C_1 2^{-j(\alpha+M)} (2^{1+j}\pi)^{\alpha+M} + 2C_2 \leq C_1 (2\pi)^{\alpha+M} + C_2$, we get (67). For (68), we write

$$\left| |H_j(\lambda)|^2 - |\tilde{H}_j(\lambda)|^2 \right| \leq \left| H_j(\lambda) - \tilde{H}_j(\lambda) \right| \left(|\tilde{H}_j(\lambda)| + |H_j(\lambda)| \right)$$

and apply (65), (66) and (67). \square

APPENDIX C. A USEFUL INEQUALITY

Denote by $\text{Tr}(A)$ and $\rho(A)$ the trace and the spectral radius of a matrix A .

Lemma 10. *Let n be a positive integer. There exists a constant C (depending only on n) such that for any $n \times n$ non-negative symmetric matrices A and Γ satisfying $\text{Tr}(A\Gamma) > 0$,*

$$|\mathbb{E}(\log[\xi^T A \xi]) - \log[\text{Tr}(A\Gamma)]| \leq C \left(1 \wedge \frac{\rho^2(A)\rho^2(\Gamma)}{\text{Var}(\xi^T A \xi)} \right); \quad (70)$$

$$\text{Var}(\log[\xi^T A \xi]) \leq C, \quad (71)$$

where ξ is a zero-mean $d \times 1$ Gaussian vector with covariance Γ .

There exists a constant C (depending only on n) such that for any $n \times n$ non-negative symmetric matrix A , \tilde{A} and Γ , $\tilde{\Gamma}$ satisfying $\text{Tr}(A\Gamma) > 0$ and $\text{Tr}(\tilde{A}\tilde{\Gamma}) > 0$,

$$\left| \text{Cov} \left(\log[\xi^T A \xi], \log[\tilde{\xi}^T \tilde{A} \tilde{\xi}] \right) - \frac{\text{Cov}(\xi^T A \xi, \tilde{\xi}^T \tilde{A} \tilde{\xi})}{\text{Tr}(AQ)\text{Tr}(\tilde{A}\tilde{Q})} \right| \leq C \left\{ \frac{\rho^3(\tilde{A})\rho^3(\tilde{\Gamma})}{\text{Var}^{3/2}(\tilde{\xi}^T \tilde{A} \tilde{\xi})} \vee \frac{\rho^3(\tilde{A})\rho^3(\tilde{\Gamma})}{\text{Var}^{3/2}(\xi^T A \xi)} \right\}. \quad (72)$$

where $[\xi^T, \tilde{\xi}^T]^T$ is a zero-mean Gaussian vector such that $\text{Cov}(\xi) = \Gamma$ and $\text{Cov}(\tilde{\xi}) = \tilde{\Gamma}$.

Proof. Let k be the rank of Γ and Q be $n \times k$ full rank matrix such that $QQ^T = \Gamma$. Let $\zeta \sim \mathcal{N}(0, I_k)$, where I_k is the identity matrix of size $k \times k$. For any unitary matrix U , $U\zeta \sim \mathcal{N}(0, I_k)$ and hence $QU\zeta$ has same distribution as ξ . Moreover, since A is symmetric, so is $Q^T A Q$. We may choose an unitary matrix U such that $\Lambda \stackrel{\text{def}}{=} U^T(Q^T A Q)U$ is a diagonal matrix with non-negative entries. Furthermore,

$$\zeta^T \Lambda \zeta = (QU\zeta)^T A (QU\zeta) \stackrel{\text{d}}{=} \xi^T A \xi, \quad (73)$$

where $\stackrel{\text{d}}{=}$ denotes the equality of distributions. Since Λ is diagonal, $\zeta^T \Lambda \zeta$ is a sum of independent r.v.'s of the form $\sum_{i=1}^k \lambda_i \zeta_i^2$ where λ_i are the (non-negative) diagonal entries of Λ . Since $\mathbb{E}\zeta_i^2 = 1$ and $\text{Var}(\zeta_i^2) = 2$, we get from (73) that $\sum_{i=1}^k \lambda_i = \mathbb{E}[\xi^T A \xi] = \text{Tr}(A\Gamma)$ and $2 \sum_{i=1}^k \lambda_i^2 = \text{Var}[\xi^T A \xi]$. Finally we may write

$$\mathbb{E}(\log[\xi^T A \xi]) - \log[\mathbb{E}(\xi^T A \xi)] = \mathbb{E}(\log[S]) \quad (74)$$

where

$$S \stackrel{\text{def}}{=} \frac{\xi^T A \xi}{\mathbb{E}[\xi^T A \xi]} = \sum_{i=1}^k d_i \zeta_i^2 \quad \text{with} \quad d_i \stackrel{\text{def}}{=} \frac{\lambda_i}{\sum_{j=1}^k \lambda_j}. \quad (75)$$

Set $\|d\|^2 \stackrel{\text{def}}{=} \sum_{i=1}^k d_i^2$ and observe that $\rho(\Lambda) = \max_{1 \leq i \leq k} \lambda_i$. Since $\sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \lambda_i^2 / \max_{1 \leq i \leq k} \lambda_i$, we get

$$\|d\|^2 = \frac{\sum_{i=1}^k \lambda_i^2}{\left(\sum_{i=1}^k \lambda_i\right)^2} \leq \frac{\rho^2(\Lambda)}{\sum_{i=1}^k \lambda_i^2} = \frac{2\rho^2(\Lambda)}{\text{Var}(\xi^T A \xi)} \leq \frac{2\rho^2(A)\rho^2(\Gamma)}{\text{Var}(\xi^T A \xi)}, \quad (76)$$

For $t > -(2 \max_{1 \leq i \leq k} d_i)^{-1}$, we may write, using standard computations on the central chi-square, that

$$\mathbb{E}[e^{-tS}] = \prod_{i=1}^k \mathbb{E}[e^{-td_i \zeta_i^2}] = \prod_{i=1}^k (1 + 2d_i t)^{-1/2} \quad (77)$$

Denote by F the distribution function of S , that is $F(x) = \mathbb{P}(S \leq x)$. Observe that $F(0) = 0$ since S is a non-negative weighted sum of independent central chi-squares and that all the weights do not vanish. For any $t > 0$ and $x > 0$,

$$\log [F(x)] \leq \log [e^{xt} \mathbb{E}(e^{-tS})] = xt - (1/2) \sum_{i=1}^k \log(1 + 2d_i t), \quad (78)$$

where the equality follows from (77). Because, for $t \geq 0$, $\prod_{i=1}^k (1 + 2d_i t) \geq 1 + 2t \sum_{i=1}^k d_i = 1 + 2t$, $\sum_{i=1}^k \log(1 + 2d_i t) \geq \log(1 + 2t)$. Plugging this inequality in (78) and setting $t = x/2$ yields to

$$F(x) \leq e^{1/2} \left(\frac{x}{1+x} \right)^{1/2}, \quad x > 0. \quad (79)$$

By (79), $\lim_{x \rightarrow 0^+} \log(x)F(x) = 0 = F(0)$ and $\int_0^1 x^{-1} F(x) dx < \infty$. Integration by parts together with (79) gives that

$$\int_0^1 |\log(x)| dF(x) = \int_0^1 x^{-1} F(x) dx \leq e^{1/2} \int_0^1 x^{-1/2} dx = \frac{2}{\sqrt{e}}.$$

Since $\log(x) \leq x - 1 \leq x$ for all $x \geq 1$ and $\mathbb{E}S = 1$, we get $\int_1^\infty \log(x) dF(x) \leq 1$ and thus

$$\mathbb{E}|\log S| = \int_0^1 |\log(x)| dF(x) + \int_1^\infty \log(x) dF(x) \leq \frac{2}{\sqrt{e}} + 1. \quad (80)$$

This bound proves the left part of the \wedge sign in (70). We now provide a new bound for $F(x)$ which will yield the right part. Since the second derivative of the $\log(x)$ has absolute value at most 1 for all $x \geq 1$, we have that, for any $t \geq 0$,

$$\log(1 + 2d_i t) \geq 2d_i t - 2d_i^2 t^2$$

which gives that

$$\log[F(x)] \leq (x - 1)t + t^2 \|d\|^2.$$

Setting $t = \|d\|^{-1}$ yields to the following exponential bound

$$F(x) \leq \exp [(x - 1)\|d\|^{-1} + 1], \quad x > 0. \quad (81)$$

Having this new bound of F at hand, we can improve the bound established in (80) as follows. Integration by parts together with (81) and (79) gives that, for any $0 < b < a < 1$,

$$\begin{aligned} \int_0^a |\log(x)| dF(x) &= |\log(a)|F(a) + \int_0^a x^{-1} F(x) dx \\ &\leq |\log(a)|F(a) + C \int_0^b x^{-1/2} dx + Cb^{-1} \int_b^a \exp [-(1-x)\|d\|^{-1}] dx \\ &\leq 2Cb^{1/2} + (|\log(a)| + Cb^{-1}\|d\|) \exp [-(1-a)\|d\|^{-1}]. \end{aligned}$$

Set $b \stackrel{\text{def}}{=} \exp \left[-\frac{2}{3}(1-a)\|d\|^{-1} \right] \wedge a/2$, so that $b^{1/2} \leq \exp \left[-\frac{1}{3}(1-a)\|d\|^{-1} \right]$ and $b^{-1} \leq 2/a \exp \left[\frac{2}{3}(1-a)\|d\|^{-1} \right]$ (since both $2/a$ and the latter exponential are larger than 1). The latter displayed inequality then implies that there exists a constant C only depending on a such that

$$\mathbb{E} (|\log(S)| \mathbb{1}\{S \leq a\}) \leq C(1 + \|d\|) \exp \left[-(1-a)\|d\|^{-1}/3 \right] \leq C\|d\|^\alpha, \quad (82)$$

where the last inequality is valid for all $\alpha \geq 1$ with C only depending on a and α . Pick $a \in (0, 1)$. Since the second derivative of the $\log(x)$ has absolute value at most a^{-2} for all $x \geq a$ and that $|\log(x) - (x-1)| \leq |\log(x)|$ for all $x \in (0, 1)$, we have that, for any $x > 0$,

$$|\log(x) - (x-1)| \leq \frac{1}{2a^2} (1-x)^2 \mathbb{1}_{[a, \infty)}(x) + |\log(x)| \mathbb{1}_{[0, a]}(x). \quad (83)$$

Since $\mathbb{E}[(1-S)^2] = \text{Var}(S) = \|d\|^2 \text{Var}(\zeta_1^2) = 2\|d\|^2$ and using (82) and (83), we get, for some positive constant C only depending on a ,

$$\begin{aligned} |\mathbb{E}[\log(S)]| &\leq \frac{1}{2a^2} \mathbb{E}[(1-S)^2] + \mathbb{E}[|\log(S)| \mathbb{1}\{|S| \leq a\}] \\ &\leq a^{-2}\|d\|^2 + C\|d\|^2, \end{aligned}$$

which shows (70) by applying (74) and (76).

We now prove (71). We have, by definition of S ,

$$\text{Var}(\log[\xi^T A \xi]) = \text{Var}(\log(S)) \leq \mathbb{E} \log^2 S. \quad (84)$$

By arguing as for (80), that is, by using the bound (79) for $F(x)$, $x \in [0, 1]$, that the derivative of $\log^2(x)$ is $2 \log x/x$ and that $\log^2(x) \leq 2x$ for all $x \geq 1$, we obtain $\mathbb{E} \log^2 S \leq \frac{8}{\sqrt{e}} + 2$, which yields (71).

We now prove (72). Define \tilde{k} , \tilde{d}_i and \tilde{S} as we did k , d_i and S . The LHS in (72) then reads

$$\begin{aligned} &\mathbb{E} \left[\log(S) \log(\tilde{S}) \right] - \mathbb{E} \left[(S-1)(\tilde{S}-1) \right] = \mathbb{E} \left[(S-1)(\log(\tilde{S}) - (\tilde{S}-1)) \right] \\ &+ \mathbb{E} \left[(\tilde{S}-1)(\log(S) - (S-1)) \right] + \mathbb{E} \left[(\log(S) - (S-1))(\log(\tilde{S}) - (\tilde{S}-1)) \right]. \end{aligned}$$

We will provide a bound for the first term of the RHS of this display, the other terms being treated similarly. By using (83) for some $a \in (0, 1)$ and the Cauchy-Schwarz inequality,

$$\begin{aligned} &\left| \mathbb{E} \left[(S-1) \left(\log(\tilde{S}) - (\tilde{S}-1) \right) \right] \right| \\ &\leq \frac{1}{2a^2} \mathbb{E} \left[(S-1)(\tilde{S}-1)^2 \right] + \mathbb{E} \left[(S-1) |\log(\tilde{S})| \mathbb{1}_{[0, a]}(\tilde{S}) \right] \\ &\leq \frac{1}{2a^2} \left(\mathbb{E}|S-1|^2 \mathbb{E}|\tilde{S}-1|^4 \right)^{1/2} + \left(\mathbb{E}|S-1|^2 \mathbb{E} \left[|\log(\tilde{S})|^2 \mathbb{1}_{[0, a]}(\tilde{S}) \right] \right)^{1/2}. \end{aligned}$$

By (76), it remains to show that the two last terms are $O(\|d\|^3 \vee \|\tilde{d}\|^3)$. By definition, $\tilde{S} - 1 = \sum_{i=1}^{\tilde{k}} \tilde{d}_i(\zeta_i^2 - 1)$, where $\{\zeta_i\}_{1 \leq i \leq \tilde{k}}$ are i.i.d. standard normal. Therefore,

$$\mathbb{E}|\tilde{S} - 1|^4 = \sum_{i=1}^{\tilde{k}} \tilde{d}_i^4 \text{cum}_4(\zeta_1^2) + 3 \left(\sum_{i=1}^{\tilde{k}} \tilde{d}_i^2 \right)^2 \text{Var}(\zeta_1^2),$$

where $\text{cum}_4(Z)$ is the fourth-order cumulant of the random variable Z . Since $\sum_{i=1}^{\tilde{k}} \tilde{d}_i^4 \leq \left(\sum_{i=1}^{\tilde{k}} \tilde{d}_i^2 \right)^2$, we obtain that $\mathbb{E}|\tilde{S} - 1|^4 \leq C\|\tilde{d}\|^4$ for some constant C . Therefore,

$$\left(\mathbb{E}|S - 1|^2 \right)^{1/2} \left(\mathbb{E}|\tilde{S} - 1|^4 \right)^{1/2} \leq C\|d\|\|\tilde{d}\|^2 \leq C(\|d\|^3 \vee \|\tilde{d}\|^3).$$

Proceeding as in (82), it can be shown that

$$\mathbb{E} \left[|\log(\tilde{S})|^2 \mathbb{1}_{[0,a]}(\tilde{S}) \right] \leq C\|d\|^\alpha,$$

for all $\alpha \geq 0$ with C only depending on a and α . In particular

$$\left(\mathbb{E}|S - 1|^2 \right)^{1/2} \left(\mathbb{E} \left[|\log(\tilde{S})|^2 \mathbb{1}_{[0,a]}(\tilde{S}) \right] \right)^{1/2} \leq C(\|d\|^3 \vee \|\tilde{d}\|^3),$$

for some constant C , which concludes the proof. \square

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