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The action of the Frobenius map on rank 2 vector bundles over genus 2 curves in small characteristics

Laurent Ducrohet

27th December 2005

Abstract

Let X be genus 2 curve defined over an algebraically closed field of characteristic p and let X_1 be its p-twist. Let M_X (resp. M_{X_1}) be the (coarse) moduli space of semi-stable rank 2 vector bundles with trivial determinant over X (resp. X_1). The moduli space M_X is isomorphic to the 3 dimensional projective space and is endowed with an action of the group $J[2]$ of order 2 line bundles over X. When $3 \leq p \leq 7$, we show that the Verschiebung (i.e., the separable part of the action of Frobenius by pull-back) $V : M_{X_1} \dashrightarrow M_X$ is completely determined by its restrictions to the lines that are invariant under the action of a non zero element of $J[2]$. As those lines correspond to elliptic curves that appear as Prym varieties, the Verschiebung restricts to the morphism induced by multiplication by p and we are able to compute the explicit equations for the Verschiebung.

1 Introduction

Let k be a algebraically closed field of positive characteristic p and let X be a proper and smooth (connected) curve of genus 2 over k. Let X_s ($s \in \mathbb{Z}$ since k is perfect) be the p^s-twist of X and let J (resp. J_s) denote its Jacobian variety (resp. the p^s-twist of its Jacobian variety). Denote by $M_X(r)$ the (coarse) moduli space of semi-stable rank r vector bundles with trivial determinant over X. The map $E \mapsto F^*_{\text{abs}} E$ defines a rational map the separable part of which, the generalized Verschiebung, will be denoted by $V_r : M_{X_1}(r) \dashrightarrow M_X(r)$.

For $r = 2$ and $k = \mathbb{C}$, we let M_X (resp. V) be M_X (resp. V_2). [NR] have constructed an isomorphism $D : M_X \to |2\Theta| \cong \mathbb{P}^3$ that remains valid for an algebraically closed field of positive characteristic (it is straightforward for $p \neq 2$ and [LP1] (section 5) give a sketch of proof for $p = 2$). Furthermore, the semistable boundary of M_X identifies with the Kummer surface Kum_X, which is canonically contained in the linear system $|2\Theta|$. We have the commutative diagram

(1.1)

$$\begin{tikzcd} M_{X_1} \arrow[r, dashed, "V"] \arrow[d, "D"'] & M_X \arrow[d, "D"] \\ |2\Theta_1| \arrow[r, dashed, "\widetilde{V}"] & |2\Theta| \end{tikzcd}$$

and the induced rational map $\widetilde{V}$ is given by degree p polynomials ([LP2]). Note that all the map in that diagram are equivariant under the action of $J[2]$.

Our interest in the situation described by the diagram (1.1) comes from the fact ([LS]) that, given a proper and smooth curve of genus g over a field k, a semistable rank r vector bundle E over X corresponds to an (irreducible) continuous representation of the algebraic fundamental group $\pi_1(X)$ in $\mathrm{GL}_r(\bar{k})$ (endowed with the discrete topology) if and only if one can find an integer $n > 0$ such that ${F_{\mathrm{abs}}^{(n)}}^* E \cong E$. Thus, natural questions about the generalized Verschiebung $V_r : M_{X_1}(r) \dashrightarrow M_X(r)$ arise like, e.g., its surjectivity, its degree, the density of Frobenius-stable bundles, and the loci of Frobenius-destabilized bundles.

For general (g, r, p), not much seems to be known (see the introductions of [LP1] and [LP2] for an overview of this subject).

When $g = 2$, $r = 2$, and $p = 2$ and X is an ordinary curve, [LP1] determined the quadric equations of $\widetilde{V}$ in terms of the generalized theta constants of the curve X. In [LP2] (resp. in [Du]), one could give the equations of $\widetilde{V}$ in case of a nonordinary curve X with Hasse-Witt invariant equal to 1 (resp. a supersingular curve X) by specializing a family $\mathcal{X}$ of genus 2 curves parameterized by a discrete valuation ring with ordinary generic fiber and special fiber isomorphic to X.

When $g = 2$, $r = 2$, and $p = 3$, [LP2] determined the cubic equations of $\widetilde{V}$ in showing that this rational map coincided with the polar map of a Kummer surface, isomorphic to Kum_X, in $\mathbb{P}^3$. To reach this result, they used a striking relationship (see [vG]) between cubics and quartics on $|2\Theta_1|$.

In this paper, we shall suppose that $p \geq 3$. Given τ a line bundle of order 2 over X and a τ-invariant semi-stable vector bundle E, i.e., satisfying $E \otimes \tau \xrightarrow{\sim} E$, of degree 0, one can give E a structure of invertible $\mathcal{O}_X \oplus \tau$-module. In other words, if $\pi : \widetilde{X} \to X$ is the degree 2 étale cover corresponding to τ, there is a degree 0 line bundle q over $\widetilde{X}$ such that $E \cong \pi_*(q)$. On the one hand, the τ-invariant locus of M_X is the union of two projective lines. On the other hand, π being étale, one has $F_{\mathrm{abs}}^*(\pi_*(q)) \cong \pi_*(F_{\mathrm{abs}}^*(q))$. Requiring that E has trivial determinant forces q to be in some translate of the Prym variety P associated to π (which is an elliptic curve) and, as multiplication by p over an elliptic curve commutes with the inversion, it induces a map $P/\{\pm\} \cong \mathbb{P}^1 \to P/\{\pm\} \cong \mathbb{P}^1$. Let $V_{\mathbb{P}^1}$ be the separable part of the latter map and choose a τ-invariant line $\Delta(\tau)$ in M_X. There is a natural isomorphism $P/\{\pm\} \xrightarrow{\sim} \Delta(\tau)$ and $V_{\mathbb{P}^1}$ coincides with the restriction of $V : M_{X_1} \dashrightarrow M_X$ to $\Delta(\tau)$. The main result of this paper is the following theorem (4.16) :

Theorem *Let X be a smooth and proper curve of genus 2, sufficiently general, over an algebraically closed field of characteristic $p = 3, 5$ or 7. The generalized Verschiebung $V : M_{X_1} \dashrightarrow M_X$ is completely determined by its restriction to the projective lines that are*

invariant under the action of a non zero element of $J[2]$.

In particular, V can be computed explicitly in these cases and we recover the result of [LP2] in the characteristic 3 case. As an application, we recover if $p = 3$, and we show if $p = 5$, that there is a degree $2p - 2$ surface S is $|2\Theta_1|$ such that the equality of divisors in $|2\Theta_1|$

$$\widetilde{V}^{-1}(\mathrm{Kum}_X) = \mathrm{Kum}_{X_1} + 2S$$

holds scheme-theoretically. The computations have been carried out using Maple 9 and Magma.

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2 The (coarse) moduli space M_X

2.1 Preliminaries

Let k be an algebraically closed field of odd characteristic. Let X be a proper and smooth curve of genus 2 over k and let J (resp. J^1) be its Jacobian variety (resp. the moduli space of degree one line bundles over X, which is a principal homogeneous space under the action J). The embedding $X \hookrightarrow J^1$ defined by $\mathcal{O}(\Delta)$, where Δ is the diagonal $\Delta \subseteq X \times X$, defines a theta divisor Θ.

Let M_X be the (coarse) moduli space of semi-stable rank 2 vector bundles with trivial determinant over X. The set of its closed points is the set of S-equivalence classes $[E]$ of semi-stable rank 2 vector bundles E with trivial determinant over X. Note that M_X is endowed with a natural action of $J[2]$ defined set-theoretically by $(\tau, [E]) \mapsto [E \otimes \tau]$. If E is strictly semi-stable, i.e., semi-stable but non-stable, there is a degree 0 line bundle j such that E is an extension $0 \to j^{-1} \to E \to j \to 0$, i.e., E is S-equivalent to $j \oplus j^{-1}$. If $\mathcal{L}$ is the Poincaré's bundle over $X \times J$, one can consider the family $\mathcal{E} = \mathcal{L} \oplus \mathcal{L}^{-1}$, parameterized by J, of semi-stable rank 2 bundles with trivial determinant over X. Because of the coarse moduli property, there is a unique $b : J \to M_X$ such that, for any j in J, $b(j) = [\mathcal{E}_{|X \times \{j\}}] = [j \oplus j^{-1}]$. Note that the semi-stable boundary in M_X, i.e., the image of b, is (globally) $J[2]$-invariant and that b is $J[2]$-equivariant.

When $k = \mathbb{C}$ (but this result extends straightforwardly to any algebraically closed field with characteristic different from 2, and, with a little more work (see [LP1] for a sketch of proof) to an algebraically closed field of characteristic 2), it has been shown ([NR]) that there is a canonical isomorphism $D : M_X \xrightarrow{\sim} \mathbb{P}H^0(J^1, \mathcal{O}(2\Theta))$ that maps a semi-stable bundle E over X to the reduced (hence linearly equivalent to 2Θ) divisor over J^1 with support the set $\{\xi \in J^1 / h^0(X, E \otimes \xi) \geq 1\}$ and recall by the way that any divisor

linearly equivalent to 2Θ is completely determined by its support ([NR], Proposition 6.4). In particular, this divisor is reducible if and only if it is of the form $D[j \oplus j^{-1}]$. In other words, D maps the semi-stable boundary of M_X onto the set of reducible divisors of $|2\Theta|$, namely $D[j \oplus j^{-1}] = T_j^*\Theta + T_{j^{-1}}^*\Theta$, where T_j (resp. $T_{j^{-1}}$) is the isomorphism $J^1 \xrightarrow{\sim} J^1$ that corresponds to j (resp. j^{-1}) via the action of J on J^1 .

Choose once for all an effective theta characteristic κ_0 of X (that is to say one of the 6 Weierstrass points of X) and consider the corresponding isomorphism $J \xrightarrow{\sim} J^1$ given by $j \mapsto j \otimes \kappa_0$. We still denote by Θ the divisor of J obtained as the inverse image of Θ by means of this isomorphism. Its support is the set $\{\zeta \in J/H^0(\zeta \otimes \kappa_0) \geq 1)$ and using the Riemann-Roch theorem, Θ is symmetric. It defines the canonical principal polarization on J. Using this isomorphism, we obtain an isomorphism

$$D : M_X \xrightarrow{\sim} \mathbb{P}H^0(J, \mathcal{O}(2\Theta)) = |2\Theta|$$

that maps $[E]$ to the (unique) divisor in $|2\Theta|$ with support $\{\zeta \in J/h^0(E \otimes \kappa_0 \otimes \zeta) \geq 1\}$. Of course, the semi-stable boundary is M_X still coincides with the set of reducible divisors in $|2\Theta|$.

Now, we recall the basic facts of the theory of theta group schemes associated to an ample line bundle L over an abelian variety A over k, as presented in [Mu2], Section 1. Let $\mathcal{G}(L)$ (resp. $K(L)$) be the group scheme (resp. the finite group scheme) such that, for any k-scheme S,

$$\mathcal{G}(L)(S) = \{(x, \gamma) |\, x \in A(S),\ \gamma : L \xrightarrow{\sim} T_x^*L\} \ \ (\text{resp. } K(L)(S) = \{x \in A(S) |\, T_x^*L \cong L\})$$

We suppose that $K(L)$ is reduced-reduced, in which case L is said to be of separable type. The theta group $\mathcal{G}(L)$ is a central extension

$$1 \to \mathbb{G}_m \to \mathcal{G}(L) \to K(L) \to 0$$

Note that $\mathcal{G}(L)$ has a natural action on $H^0(A, L)$ and that this action has weight 1 in the sense that, being given λ in $\mathbb{G}_m$ and s in $H^0(A, L)$, $\lambda.s = \lambda \mathrm{Id}(s)$. More generally, if $\mathcal{G}(L)$ acts on a vector space W, one says that this action has weight r if, being given λ in $\mathbb{G}_m$ and s in $H^0(A, L)$, one has $\lambda.s = \lambda^r \mathrm{Id}(s)$. As an example, one can consider the space of r-symmetric powers $\mathrm{Sym}^r H^0(A, L)$ on which $\mathcal{G}(L)$ has a natural action of weight r.

Theorem 2.1 *The vector space $H^0(A, L)$ is the unique (up to isomorphism) irreducible representation of weight 1 of $\mathcal{G}(L)$.*

<u>Proof</u> : Combine the Proposition 3 and the Theorem 2 of [Mu2], Section 1.

Being given α, β in $K(L)$, $\widetilde{\alpha}$, $\widetilde{\beta}$ in $\mathcal{G}(L)$ above α and β respectively, the invertible scalar $\widetilde{\alpha}\widetilde{\beta}\widetilde{\alpha}^{-1}\widetilde{\beta}^{-1}$ depends only on the choice of α and β and it induces a skew-symmetric bilinear form $e^L : K(L) \times K(L) \to \mathbb{G}_m$ that is non degenerate because $\mathbb{G}_m$ is the center of $\mathcal{G}(L)$.

Now, let $\pi : A \to B$ be a separable isogeny between abelian varieties and suppose that $L \cong \pi^* M$ where M is an ample line bundle over B. Because L descends on B, there is an action of $\ker \pi$ on L covering the action of $\ker \pi$ on A, i.e., a lifting $\ker \pi \xrightarrow{\sim} \widetilde{K} \subseteq \mathcal{G}(L)$ and $M \cong (\pi_* L)^{\widetilde{K}}$. In particular, $\ker \pi$ must be an isotropic subgroup of $K(L)$ with respect to e^L. Furthermore, γ in $\mathcal{G}(L)$ induces an isomorphism $\pi_*(\gamma)$ of $\mathcal{G}$ if and only if γ lies in the centralizer $Z(\widetilde{K})$ of $\widetilde{K}$ in $\mathcal{G}(L)$ and, when it is the case, $\pi_*(\gamma)$ depends only on the class of γ in $Z(\widetilde{K})/\widetilde{K}$. More precisely,

Theorem 2.2 *(1) Suppose that $\pi : A \to B$ is a separable isogeny between abelian varieties and let L be an ample line bundle over A. Then, there is a one-to-one correspondence between :*

- *The set of isomorphism classes of line bundles M over B such that $L \cong \pi^* M$. Such an M is necessarily ample.*
- *The set of homomorphisms $\ker \pi \to \mathcal{G}(L)$ lifting the inclusion $\ker \pi \hookrightarrow A$.*

(2) Being given a homomorphism $\ker \pi \to \widetilde{K} \subseteq \mathcal{G}(L)$ as in (1) and letting $M \cong (\pi_* L)^{\widetilde{K}}$ be the corresponding ample line bundle over B, there are canonical isomorphisms

$$\mathcal{G}(M) \cong Z(\widetilde{K})/\widetilde{K} \text{ and } K(M) \cong (\ker \pi)^{\perp}/\ker \pi$$

where $(\ker \pi)^{\perp} = \{\alpha \in K(L) |\, e^L(\alpha, \beta) = 1 \text{ for any } \beta \in \ker \pi\}$.
In particular, if M is a principal polarization, $\ker \pi$ is a maximal isotropic subgroup of $K(L)$.

<u>Proof</u> : (1) [Mu1], Section 23, Theorem 2. The fact that M is necessarily ample comes from [Mu1], Section 6, Proposition 6 and the finiteness of $K(M)$.
(2) [Mu2], Section 1, Theorem 4 and [Mu1], Section 23, Theorem 4 for the last assertion.

Returning to our situation, consider the ample line bundle $\mathcal{O}(2\Theta)$ over J and the associated theta group scheme $\mathcal{G}(\mathcal{O}(2\Theta))$. Its natural action on $H^0(J, \mathcal{O}(2\Theta))$ induces an action of $J[2]$ onto $|2\Theta|$ and the morphism

$$\mathrm{K}_X : J \to |2\Theta|,\ j \mapsto T_j^*\Theta + T_{-j}^*\Theta$$

which obviously factors through the quotient of J under the involution $j \mapsto j^{-1}$, is $J[2]$-equivariant. The following lemma is an obvious consequence of the definitions :

Lemma 2.3 *The diagram*

$$\begin{tikzcd} & M_X \arrow[d, "D"] \\ J \arrow[ur, "b"] \arrow[r, "\mathrm{K}_X"'] & {|2\Theta|} \end{tikzcd}$$

of $J[2]$-equivariant morphisms is commutative.

Let H be the hyperplan in $|2\Theta|$ consisting in those effective divisors passing through the origin of J. The corresponding line bundle is $\mathcal{O}_{\mathbb{P}^3}(1)$. We let Δ be the reduced divisor over M_X with support $D^{-1}(H)$. The associated invertible sheaf $\mathcal{O}(\Delta)$ is isomorphic to $D^*(\mathcal{O}_{\mathbb{P}^3}(1))$ and one has

$$H^0(M_X,\, \mathcal{O}(\Delta)) \cong H^0(|2\Theta|,\, \mathcal{O}(1)) \cong H^0(J,\, \mathcal{O}(2\Theta))$$

Furthermore, $b^*(\mathcal{O}(\Delta)) \cong \mathcal{O}(2\Theta)$. It can be shown that $\mathcal{O}(\Delta)$ is a determinant line bundle is the following sense : Let S be a k-scheme S and let $\mathcal{E}$ be a rank 2 vector bundle with trivial determinant over $X \times S$ such that $\mathcal{E}(s) := \mathcal{E}_{|X\times\{s\}}$ is semi-stable for any closed point s in S. Because of the coarse moduli property, there is a map $\varphi : S \to M_X$ such that, for any closed point $s \in S$, one has $\varphi(s) = [\mathcal{E}(s)]$. Denote by q the second projection $X \times S \to S$ and suppose moreover that $h^0(X,\, \mathcal{E}(s) \otimes \kappa_0)) = 0$ (which implies that $h^1(X,\, \mathcal{E}(s) \otimes \kappa_0) = 0$) for s generic in S. Therefore, $q_*(\mathcal{E} \boxtimes \kappa_0) = 0$ and one can find a locally free resolution

$$0 \to E_0 \xrightarrow{u} E_1 \to R^1 q_*(\mathcal{E} \boxtimes \kappa_0) \to 0$$

The map u is generically bijective and the section $\det(u)$ defines an effective divisor on S that does not depend on the choice of the projective resolution, and that coincides with $\varphi^{-1}(\Delta)$ (notice that $\operatorname{Supp} \varphi^{-1}(\Delta) = \{s \in S |\, h^1(X,\, \mathcal{E}(s) \otimes \kappa_0) \geq 1\}$). The associated line bundle $(\det R^1 q_*(\mathcal{E} \boxtimes \kappa_0))^{-1}$ therefore coincides with $\varphi^*(\mathcal{O}(\Delta))$.

Let $\varphi_{2\Theta} : J \to \mathbb{P}H^0(J,\, \mathcal{O}(2\Theta))^* = |2\Theta|^*$ be the canonical morphism associated to $\mathcal{O}(2\Theta)$. It is of course $J[2]$-equivariant. One has the following lemma due to Wirtinger.

Lemma 2.4 (Wirtinger) *There is a non degenerate bilinear pairing on the vector space $W^* := H^0(J,\, \mathcal{O}(2\Theta))^*$ such that the induced isomorphism $B_W : W^* \xrightarrow{\sim} W$ makes the following diagram commutative*

$$\begin{tikzcd} & |2\Theta|^* \arrow[d, "\mathbb{P}B_W"] \\ J \arrow[ur, "\varphi_{2\Theta}"] \arrow[r, "\mathrm{K}_X"'] & |2\Theta| \end{tikzcd}$$

This pairing is well-defined up to a non zero constant.

<u>Proof</u> : : We only sketch the proof that can be found in ([Mu3]). The key idea is to consider the isogeny $\xi : J \times J \to J \times J$ given by $(x,\, y) \mapsto (x+y,\, x-y)$, the kernel of which is the finite group $J[2]$, embedded diagonally in $J \times J$. One proves that the pullback $\xi^*(\mathcal{O}(\Theta) \boxtimes \mathcal{O}(\Theta))$ of the principal polarization for $J \times J$ is isomorphic to $\mathcal{O}(2\Theta) \boxtimes \mathcal{O}(2\Theta)$. Hence, there is a unique maximal level subgroup $\widetilde{H}$ of the Heisenberg group $\mathcal{G}(\mathcal{O}(2\Theta) \boxtimes \mathcal{O}(2\Theta))$ such

that the $\widetilde{H}$-invariant part of $\xi_*(\mathcal{O}(2\Theta) \boxtimes \mathcal{O}(2\Theta))$ is isomorphic to $\mathcal{O}(\Theta) \boxtimes \mathcal{O}(\Theta)$. If θ is "the" non-zero element of $H^0(J, \mathcal{O}(\Theta))$, $\xi^*(\theta \boxtimes \theta)$ is "the" non-zero $\widetilde{H}$-invariant element of $H^0(J \times J, \mathcal{O}(2\Theta) \boxtimes \mathcal{O}(2\Theta)) \cong W \otimes W$ and the corresponding bilinear form on W^* is non degenerate because the action of $\mathcal{G}(\mathcal{O}(2\Theta))$ over W is irreducible.$\square$

2.5. *Remark.* Note that Wirtinger's result remains true whenever J is a principally polarized abelian variety and Θ is a symmetric representative for this polarization. In the sequel, we will always identify $|2\Theta|^* = \mathbb{P}H^0(J, \mathcal{O}(2\Theta))^*$ and $|2\Theta| = \mathbb{P}H^0(J, \mathcal{O}(2\Theta)))$ using the isomorphism $\mathbb{P}B_W$.

The diagram in the lemma above is $J[2]$-equivariant as well. Thus, gathering these preliminary results, we obtain the following proposition (also found in [B]):

Proposition 2.6 *The diagram*

$$\begin{tikzcd}
 & & M_X \arrow[dd, "\wr\ D"] \\
J \arrow[urr, "b"] \arrow[r, "\varphi_{2\Theta}"] \arrow[drr, "\mathrm{K}_X"'] & |2\Theta|^* \arrow[ur, "\wr"] \arrow[dr, "\wr\mathbb{P}B_W"] & \\
 & & |2\Theta|
\end{tikzcd}$$

of $J[2]$-equivariant morphisms is commutative.

2.2 Choosing a Theta structure

Most of the material in that section comes from [Mu2], Section 1.

Let us define the two groups

$$\begin{cases} H = (\mathbb{Z}/2\mathbb{Z})^2 \\ \hat{H} = \mathrm{Hom}((\mathbb{Z}/2\mathbb{Z})^2,\, k^*) \end{cases}$$

We identify H and $\hat{H}$ by means of the bilinear form $(x, y) \mapsto (-1)^{{}^t x.y}$ and the canonical evaluation map $H \times \hat{H} \to k^*$ maps (x, y^*) to $y^*(x) = (-1)^{{}^t y.x}$, where y corresponds to y^* via the above mentioned identification $\hat{H} \xrightarrow{\sim} H$. Define the Heisenberg group $\mathcal{H}$ as the set $k^* \times H \times \hat{H}$, endowed with the group structure given by

$$(t, x, x^*)(s, y, y^*) = (sty^*(x), x + y, x^* + y^*)$$

Denote by $E : (H \times \hat{H}) \times (H \times \hat{H}) \to k^*$ the non degenerate bilinear form defined by the commutator in $\mathcal{H}$, namely

$$E((x,\, x^*),\, (y,\, y^*)) = [(1,\, x,\, x^*),\, (1,\, y,\, y^*)] = x^*(y)y^*(x)$$

It makes $H \times \hat{H}$ self-dual, the sub-groups H and $\hat{H}$ are isotropic with respect to E and the isomorphism of H induced by the restriction of E to $(H \times \{0\}) \times (\{0\} \times \hat{H})$ is the identity.

Let e_2 denote the bilinear pairing defined onto $J[2]$ by the commutator in the theta group $\mathcal{G}(\mathcal{O}(2\Theta))$. As it is non degenerate, it induces an isomorphism

$$\phi : H \times \hat{H} \xrightarrow{\sim} J[2]$$

symplectic with respect to E and e_2, namely a Göpel system for $J[2]$.

Because H (resp. $\hat{H}$) is isotropic, one can choose a lifting $H \xrightarrow{\sim} \widetilde{H}$ (resp. $\hat{H} \xrightarrow{\sim} \widetilde{\hat{H}}$) in $\mathcal{G}(\mathcal{O}(2\Theta))$. There is a unique isomorphism

$$\widetilde{\phi} : \mathcal{H} \xrightarrow{\sim} \mathcal{G}(\mathcal{O}(2\Theta))$$

that induces identity on the centers, that maps H (resp. $\hat{H}$) onto $\widetilde{H}$ (resp. $\widetilde{\hat{H}}$). Such an isomorphism is a theta structure on $\mathcal{G}(\mathcal{O}(2\Theta))$ and we choose one once for all. In particular, $W := H^0(J,\, \mathcal{O}(2\Theta))$ is an irreducible representation $U : \mathcal{H} \to \mathrm{GL}(W)$ of weight 1.

Given such a representation, one can construct a basis $\{X_x |\, x \in H\}$, defined up to a multiplicative scalar, satisfying the following properties :

$$y.X_x = X_{x+y} \text{ for any } x,\, y \in H,\ \ x^*.X_x = x^*(x)X_x \text{ for any } x,\, x^* \in H \times \hat{H}.$$

Namely, the subgroup $\hat{H} \subset \mathcal{H}$ being abelian, the induced representation on W is completely reducible and splits in a direct sum of dimensional 1 subspaces indexed by the group of character of $\hat{H}$, i.e., by H. Let X_{00} be a non-zero element of $W^{\hat{H}}$ and set $X_x = (1,\, x,\, 0).X_{00}$ for any $x \in H$. Then, for any $x^* \in \hat{H}$, the equalities

$$(1, 0,\, x^*)X_x = (1, 0,\, x^*)(1, x, 0)X_{00} = x^*(x)(1,\, x, 0)(1, 0,\, x^*)X_{00} = x^*(x)X_x$$

show that X_x spans the subspace W_x on which $(1, 0,\, x^*)$ acts like $x^*(x)Id$. Because the canonical evaluation map $(x,\, x^*) \mapsto x^*(x)$ is non degenerate, the family $\{X_x |\, x \in H\}$ is free and, since it contains 4 elements, we can conclude.

We call $\{X_{00},\, X_{01},\, X_{10},\, X_{11}\}$ the theta basis of W (associated to the theta structure $\phi : \mathcal{H} \xrightarrow{\sim} \mathcal{G}(\mathcal{O}(2\Theta))$). In terms of that basis, one can give an explicit description of $U : \mathcal{H} \to \mathrm{GL}(W)$: Set

$$\alpha_{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \qquad \alpha_{10} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \alpha_{11} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\beta_{01} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \qquad \beta_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \qquad \beta_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For convenience, set also $\alpha_{00} = \beta_{00} = Id$ and notice that $\alpha_{11} = \alpha_{01}\alpha_{10} = \alpha_{10}\alpha_{01}$ and $\beta_{11} = \beta_{01}\beta_{10} = \beta_{10}\beta_{01}$. Then $U_{(t,\,x,\,x^*)} = t\beta_{x^*}\alpha_x$ for any $(t,\, x,\, x^*) \in \mathcal{H}$. These six matrices are order 2 elements of $\mathrm{GL}(W)$ with determinant 1 and the same statements hold for any product $\beta_{x^*}\alpha_x$ with $(x,\, x^*)$ non-zero in $H \times \hat{H}$.

2.7. *Remark.* Note that the isomorphism B_W (Lemma 2.2) allows us to identify W and W^*. We let $\{x_\bullet\}$ be the corresponding dual basis of W^*. It gives a set of homogeneous coordinates for $\mathbb{P}W = |2\Theta|$, canonical in the sense that it depends only on the choice of the Göpel system $H \times \hat{H} \xrightarrow{\sim} J[2]$.

2.3 The Kummer quartic surface

Let Kum_X denote the image of the map $\mathrm{K}_X : J \to |2\Theta|$ appearing in Lemma 2.3. We have the following lemma

Lemma 2.8 *Let X be a genus 2 curve and let J be its Jacobian. The map $\mathrm{K}_X : J \to |2\Theta|$ identifies with the quotient of J under the action of $\{\pm\}$. Its image is a reduced, irreducible, $J[2]$-invariant quartic in $|2\Theta|$ with 16 nodes and no other singularities, i.e., a* Kummer surface.

<u>Proof</u> : Because K_X coincides with $\varphi_{2\Theta}$ (Lemma 2.4) and because of the Riemann-Roch Theorem for abelian varieties (see, e.g., [Mu1]), one has

$$\deg(\varphi_{2\Theta}).\deg(\mathrm{Kum}_X) = (2\Theta)^2 = 8$$

Therefore, $\deg(\mathrm{Kum}_X) = 4$. Because J is an irreducible abelian surface, Kum_X is reduced. Thus, it is a quartic and K_X coincides with the quotient $J \to J/\{\pm\}$ (see [GD], Proposition 4.23 for details). It is therefore finite, surjective and separable, generically 2-1 hence of degree 2 (see, e.g., [Mu1], Section 7) and Kum_X is irreducible. The map $\mathrm{K_X}$ is furthermore generically étale and it ramifies only at the 2-torsion points of J thus the singular locus of Kum_X is contained in the image of $J[2]$. An easy calculation in the formal completion of the local ring at a point of $J[2]$ shows that it is a node (see [GD], Note 4.16). The fact that it is $J[2]$-invariant is clear. $\square$

2.9. *Remark.* Let us consider a principally polarized abelian surface A and let Θ be a symmetric representative for that polarization. Either A is an irreducible abelian variety

or A is the product of two elliptic curves. In the former case, one can show that Θ is a non singular genus 2 curve the Jacobian of which is isomorphic to A and this is the situation of the lemma above. In the latter case, Θ is the union of two elliptic curves meeting transversally in one point and the morphism $\varphi_{2\Theta} : A \to |2\Theta|^*$ associated to $\mathcal{O}(2\Theta)$ makes the following diagram commutative

$$\begin{tikzcd} E_1 \times E_2 \arrow[d] \arrow[dr, "\varphi_{2\Theta}"] & \\ \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, "\text{Segre embedding}"'] & |2\Theta|^* \cong \mathbb{P}^3 \end{tikzcd}$$

where the map $E_1 \times E_2 \to \mathbb{P}^1 \times \mathbb{P}^1$ is the product of the canonical maps $E_i \to \mathbb{P}^1$ ($i = 1, 2$), which identifies with the quotient $E_i \to E_i/\{\pm\}$. The image of the Segre embedding is a non singular quadric in $\mathbb{P}^3$.

2.10. *Remark.* Conversely, it corresponds to any quartic surface S in $\mathbb{P}^3$ with the properties described in the lemma (i.e., a Kummer surface) a principally polarized abelian surface A whose principal polarization is a non-singular curve ([GD], Proposition 4.22), hence a genus 2 curve the Jacobian of which is isomorphic to A (see the previous Remark). In other words, one recovers the moduli space of proper and smooth curves of genus 2 over k (see [H], Chapter IV, Ex. 2.2 for another (much more basic) description).

Lemma 2.11 *(1) In the coordinate system $\{x_\bullet\}$ defined above, there are scalars k_{00}, k_{01}, k_{10} and k_{11} such that the equation defining the Kummer quartic surface* Kum_X *is*

$$S + 2k_{00}P + k_{01}Q_{01} + k_{10}Q_{10} + k_{11}Q_{11} \tag{2.1}$$

where

$$S = x_{00}^4 + x_{01}^4 + x_{10}^4 + x_{11}^4, \qquad P = x_{00}x_{01}x_{10}x_{11},$$
$$Q_{01} = x_{00}^2x_{01}^2 + x_{10}^2x_{11}^2, \qquad Q_{10} = x_{00}^2x_{10}^2 + x_{01}^2x_{11}^2, \qquad Q_{11} = x_{00}^2x_{11}^2 + x_{01}^2x_{10}^2.$$

(2) These scalars k_{00}, k_{01}, k_{10} and k_{11} satisfy the cubic relationship

$$4 + k_{01}k_{10}k_{11} - k_{01}^2 - k_{10}^2 - k_{11}^2 + k_{00}^2 = 0 \tag{2.2}$$

and one has

$$\begin{cases} k_{01} \neq \pm 2,\ k_{10} \neq \pm 2,\ k_{11} \neq \pm 2, \\ k_{01} + k_{10} + k_{11} + 2 \pm k_{00} \neq 0, \\ k_{01} + k_{10} - k_{11} - 2 \pm k_{00} \neq 0, \\ k_{01} - k_{10} + k_{11} - 2 \pm k_{00} \neq 0, \\ -k_{01} + k_{10} + k_{11} - 2 \pm k_{00} \neq 0 \end{cases} \tag{2.3}$$

Proof : (1) Using the fact that the Kummer surface in $|2\Theta|$ is $J[2]$-invariant, the element of $\mathrm{Sym}^4 W^*$ that generates its ideal is invariant (up to scalar) under the natural action (of weight -4) of $\mathcal{H}$ on $\mathrm{Sym}^4 W^*$. Thus ([GD], Theorem 2.20), one can look for the equation of Kum_X under the form (2.1).
(2) We let $(\vartheta_{00} : \vartheta_{01} : \vartheta_{10} : \vartheta_{11})$ be the homogeneous coordinates of the image $\varphi_{2\Theta}(0)$ of the origin of J in $|2\Theta|^*$, i.e., the generalized theta constants in classical terminology of theta functions. Note that the orbit of that point under the action of $J[2]$ consists in 16 distinct points. Because these points are the points of a 16-6 configuration (see [GD], Section 1), the corresponding conditions on the $\vartheta_\bullet$ are equivalent to the following :

$$\begin{array}{lll} \vartheta_{00}\vartheta_{01} \neq \pm\vartheta_{10}\vartheta_{11}, & \vartheta_{00}\vartheta_{10} \neq \pm\vartheta_{01}\vartheta_{11}, & \vartheta_{00}\vartheta_{11} \neq \pm\vartheta_{01}\vartheta_{10}, \\ \vartheta_{00}^2+\vartheta_{01}^2 \neq \vartheta_{10}^2+\vartheta_{11}^2, & \vartheta_{00}^2+\vartheta_{10}^2 \neq \vartheta_{01}^2+\vartheta_{11}^2, & \vartheta_{00}^2+\vartheta_{11}^2 \neq \vartheta_{01}^2+\vartheta_{10}^2, \\ \vartheta_{00}^2+\vartheta_{01}^2+\vartheta_{10}^2+\vartheta_{11}^2 \neq 0 & & \end{array}$$

The fact that $\varphi_{2\Theta}(0)$ is a node means that the four partial derivatives of the equation (2.1) vanish at $(\vartheta_{00} : \vartheta_{01} : \vartheta_{10} : \vartheta_{11})$. It gives a linear system of four equations (depending on the $\vartheta_\bullet$) that the $k_\bullet$ must satisfy. Solving the system, we can express the $k_\bullet$ in terms of the $\vartheta_\bullet$. Namely, one has

$$k_{01} = -\frac{\vartheta_{00}^4+\vartheta_{01}^4-\vartheta_{10}^4-\vartheta_{11}^4}{\vartheta_{00}^2\vartheta_{01}^2-\vartheta_{10}^2\vartheta_{11}^2}, \qquad k_{10} = -\frac{\vartheta_{00}^4-\vartheta_{01}^4+\vartheta_{10}^4-\vartheta_{11}^4}{\vartheta_{00}^2\vartheta_{10}^2-\vartheta_{01}^2\vartheta_{11}^2},$$
$$k_{11} = -\frac{\vartheta_{00}^4-\vartheta_{01}^4-\vartheta_{10}^4+\vartheta_{11}^4}{\vartheta_{00}^2\vartheta_{11}^2-\vartheta_{01}^2\vartheta_{10}^2},$$

and k_{00} can then be computed directly from (2.1). A few more calculations ([GD], Lemma 2.21) show that one can eliminate the $\vartheta_\bullet$ from the expressions of the $k_\bullet$ to find the cubic relationship

$$4 + k_{01}k_{10}k_{11} - k_{01}^2 - k_{10}^2 - k_{11}^2 + k_{00}^2 = 0$$

From this equation, we obtain the following four equations

$$\begin{array}{rcl} (k_{01}+2)(k_{10}+2)(k_{11}+2) & = & (k_{01}+k_{10}+k_{11}+2-k_{00})(k_{01}+k_{10}+k_{11}+2+k_{00}) \\ (k_{01}-2)(k_{10}-2)(k_{11}+2) & = & (k_{01}+k_{10}-k_{11}-2-k_{00})(k_{01}+k_{10}-k_{11}-2+k_{00}) \\ (k_{01}-2)(k_{10}+2)(k_{11}-2) & = & (k_{01}-k_{10}+k_{11}-2-k_{00})(k_{01}-k_{10}+k_{11}-2+k_{00}) \\ (k_{01}+2)(k_{10}-2)(k_{11}-2) & = & (-k_{01}+k_{10}+k_{11}-2-k_{00})(-k_{01}+k_{10}+k_{11}-2+k_{00}) \end{array}$$

Note that neither

$$k_{01} - 2 = -\left[\frac{(\vartheta_{00}^2+\vartheta_{01}^2-\vartheta_{10}^2-\vartheta_{11}^2)(\vartheta_{00}^2+\vartheta_{01}^2+\vartheta_{10}^2+\vartheta_{11}^2)}{\vartheta_{00}^2\vartheta_{01}^2-\vartheta_{10}^2\vartheta_{11}^2}\right]$$

nor

$$k_{01} + 2 = -\left[\frac{(\vartheta_{00}^2-\vartheta_{01}^2+\vartheta_{10}^2-\vartheta_{11}^2)(\vartheta_{00}^2-\vartheta_{01}^2-\vartheta_{10}^2+\vartheta_{11}^2)}{\vartheta_{00}^2\vartheta_{01}^2-\vartheta_{10}^2\vartheta_{11}^2}\right]$$

can be zero. By symmetry, one has $k_{10} \neq \pm 2$ and $k_{11} \neq \pm 2$ as well. Together with the four equations deduced from (2.2), one can conclude. $\square$

2.4 Invariant lines in M_X

We are interested in those linear subspaces of $|2\Theta|$ on which a non zero τ in $J[2]$ acts like identity. Let $\tau = (x,\, x^*)$ be an order 2 element in $J[2]$.

A lifting $(t,\, x,\, x^*)$ of τ in $\mathcal{H}$ has order 2 if and only if $t^2 = x^*(x)$. Choose once for all a square root i of -1 in k and let $\widetilde{\tau}$ be $(\mu,\, x,\, x^*)$ with $\mu = 1$ (resp. $\mu = i$) if $x^*(x) = 1$ (resp. $x^*(x) = -1$). The corresponding element $\mu\beta_{x^*}\alpha_x$ in $\mathrm{GL}(W)$ has order 2 and determinant $\mu^4 = 1$. As it cannot be the identity matrix (because $\tau \neq 0$), W splits in the direct sum

$$W = W^{\widetilde{\tau}} \oplus W^{-\widetilde{\tau}}$$

of two 2-dimensional spaces of eigenvectors, associated to the eigenvalues $+1$ and -1 of $\widetilde{\tau}$ respectively.
We construct a basis adapted to this decomposition, that will be useful in the computations of the Section 4.

Proposition 2.12 *Let τ be any non-zero element of $J[2]$ and let $\widetilde{\tau}$ be the order 2 element of $\mathcal{H}$ defined above. There is a basis $\{\Lambda_0(\tau),\, \Lambda_1(\tau),\, \bar{\Lambda}_0(\tau),\, \bar{\Lambda}_1(\tau)\}$ for W that splits into bases $\{\Lambda_0(\tau),\, \Lambda_1(\tau)\}$ and $\{\bar{\Lambda}_0(\tau),\, \bar{\Lambda}_1(\tau)\}$ for $W^{\widetilde{\tau}}$ and $W^{-\widetilde{\tau}}$ respectively. Furthermore, one can find a theta structure, i.e., an automorphism of Heisenberg groups $\rho : \mathcal{H} \to \mathcal{H}$, mapping $(1, 00, 10)$ to $\widetilde{\tau}$, such that this basis coincides with the theta basis corresponding to the representation $\mathcal{H} \xrightarrow{\rho} \mathcal{H} \xrightarrow{U} \mathrm{GL}(W)$.*

<u>Proof</u> : Denote again by $\widetilde{\tau}$ the element $U_{\widetilde{\tau}}$ of $\mathrm{GL}(W)$ and by 1 the identity matrix. As $\widetilde{\tau}^2 = 1$, it is easily seen that

$$p_\tau^+ = (\widetilde{\tau} + 1)/2 \ \text{ and } \ p_\tau^- = (1 - \widetilde{\tau})/2)$$

is rank 2 projectors of the linear space W, and that their images are $W^{\widetilde{\tau}}$ and $W^{-\widetilde{\tau}}$ respectively. One can extract a basis of $W^{\widetilde{\tau}}$ (resp. $W^{-\widetilde{\tau}}$) from the family $\{p_\tau^+(X_\bullet)\}$ (resp. $\{p_\tau^-(X_\bullet)\}$). Let us distinguish whether $x = 00$ or not.

If x is zero, one has

$$p_\tau^+(X_z) = \frac{1 + x^*(z)}{2} X_z = \left| \begin{array}{ll} X_z & \text{if } x^*(z) = 1 \\ 0 & \text{if } x^*(z) = -1 \end{array} \right.$$

and

$$p_\tau^-(X_z) = \frac{1 - x^*(z)}{2} X_z = \left| \begin{array}{ll} 0 & \text{if } x^*(z) = 1 \\ X_z & \text{if } x^*(z) = -1 \end{array} \right.$$

Therefore, we let $\Lambda_\bullet(\tau)$ (resp. $\bar{\Lambda}_\bullet(\tau)$) be the non zero elements $p_\tau^+(X_\bullet)$ (resp. $p_\tau^-(X_\bullet)$)) with corresponding lexical order. In other words, we permute the elements of the basis $\{X_\bullet\}$. Notice that one always has $\Lambda_0(\tau) = X_{00}$ and let $\{z_1,\, z_2,\, z_3\}$ be the permutation of $\{01,\, 10,\, 11\}$ such that

$$\Lambda_1(\tau) = X_{z_1},\ \bar{\Lambda}_0(\tau) = X_{z_2},\ \bar{\Lambda}_1(\tau) = X_{z_3}$$

Because $z_1 + z_2 + z_3 = 00$, the bijection $\rho_H : H \to H$ well-defined by the conditions $\rho_H(00) = 00$, $\rho_H(01) = z_1$ and $\rho_H(10) = z_2$ is a group automorphism of H. Furthermore, there is a unique automorphism $\rho_{\hat{H}}$ of $\hat{H}$ such that the automorphism

$$H \times \hat{H} \xrightarrow{\rho_H \times \rho_{\hat{H}}} H \times \hat{H}$$

is symplectic with respect to E. The automorphism ρ of $\mathcal{H}$ acting like $\rho_H \times \rho_{\hat{H}}$ on $H \times \hat{H}$ and acting like 1 on the centers is the one we are looking for. Indeed, $X_{00} = \Lambda_0(\tau)$ is invariant under the action of $\hat{H}$ and one has

$$\rho_H(01).\Lambda_0(\tau) = \Lambda_1(\tau),\ \rho_H(10).\Lambda_0(\tau) = \bar{\Lambda}_0(\tau),\ \rho_H(11).\Lambda_0(\tau) = \bar{\Lambda}_1(\tau)$$

thus the basis $\{\Lambda_0(\tau),\, \Lambda_1(\tau),\, \bar{\Lambda}_0(\tau),\, \bar{\Lambda}_1(\tau)\}$ is the theta basis corresponding to the representation $\mathcal{H} \xrightarrow{\rho} \mathcal{H} \xrightarrow{U} \mathrm{GL}(W)$.

If x is non zero, $\widetilde{\tau} = (\mu,\, x,\, x^*)$ with $\mu^2 = x^*(x)$ and one has

$$p_\tau^+(X_z) = \frac{1}{2}(X_z + \mu x^*(x+z) X_{x+z})$$

Thus, the elements of the family $\{p_\tau^+(X_\bullet)\}$ are pairwise colinear. In particular, we find that $p_\tau^+(X_x) = \mu p_\tau^+(X_{00})$. We let $\Lambda_0(\tau)$ be $p_\tau^+(X_{00})$ and we let z_0 be the first (for lexical order) non-zero element of H such that

$$\{\Lambda_0(\tau),\, p_\tau^+(X_{z_0})\}$$

is a basis for $W^{\widetilde{\tau}} = \operatorname{Im} p_\tau^+$. This z_0 is necessarily different from 00 and x (more precisely, $z_0 = 10$ if $x = 01$ and $z_0 = 01$ in the two other cases). We set

$$\Lambda_1(\tau) = p_\tau^+(X_{z_0}),\ \bar{\Lambda}_0(\tau) = p_\tau^-(X_{00}),\ \bar{\Lambda}_1(\tau) = p_\tau^-(X_{z_0})$$

and we obtained the announced basis of W. Taking $\widetilde{\gamma} = (t,\, y,\, y^*)$ in $\mathcal{H}$, one has

$$\widetilde{\gamma}\,(\Lambda_0(\tau)) = \frac{1 + x^*(y) y^*(x) \widetilde{\tau}}{2}(\widetilde{\gamma}(X_{00})) = \frac{t y^*(y)}{2}(X_y + \mu(x^* + y^*)(x) X_{x+y})$$

If $\widetilde{\gamma}\,(\Lambda_0(\tau)) = \Lambda_0(\tau)$, then either $y = 0$ or $y = x$. In the former case, one must have $t = 1$ and $y^*(x) = 1$. There is a unique non-zero element in $\hat{H}$ (say z_1^*) fulfilling this condition and because x and z_0 generate H, one must have $z_1^*(z_0) = -1$. In the latter case, one must have $t = \mu$ and $y^*(x) = x^*(x)$. The two elements of $\hat{H}$ fulfilling this condition are x^* and $x^* + z_1^*$.
Consider the abelian subgroup $\{1,\, (1, 00,\, z_1^*),\, \widetilde{\tau},\, (\mu,\, x,\, x^* + z_1^*)\}$ of $\mathcal{H}$. By construction, it fixes $\Lambda_0(\tau)$. Still by construction, one has

$$\widetilde{\tau}(\Lambda_1(\tau)) = \Lambda_1(\tau),\quad \widetilde{\tau}(\bar{\Lambda}_0(\tau)) = -\bar{\Lambda}_0(\tau),\quad \widetilde{\tau}(\bar{\Lambda}_1(\tau)) = -\bar{\Lambda}_1(\tau)$$

Finally, because $E((00,\, z_1^*),\, \tau) = z_1^*(x) = 1$ and $z_1^*(z_0) = -1$, one obtains

$$(1, 00,\, z_1^*)(\Lambda_1(\tau)) = -\Lambda_1(\tau), \quad (1, 00,\, z_1^*)(\bar{\Lambda}_0(\tau)) = \bar{\Lambda}_0(\tau), \quad (1, 00,\, z_1^*)(\bar{\Lambda}_1(\tau)) = -\bar{\Lambda}_1(\tau)$$

Now, choose two elements z_2^* and z_3^* in $\hat{H}$, not necessarily different, such that $z_2^*(x) = x^*(z_0)$ and $z_3^*(x) = -1$. It is easily checked that, on the one hand,

$$(z_2^*(z_0),\, z_0,\, z_2^*)(\Lambda_0(\tau)) = z_2^*(z_0)^2 \frac{1 + x^*(z_0) z_2^*(x) \widetilde{\tau}}{2} X_{z_0} = \Lambda_1(\tau)$$

and

$$(1, 00,\, z_3^*)(\Lambda_0(\tau)) = \frac{1 + z_3^*(x)\widetilde{\tau}}{2} X_{00} = \bar{\Lambda}_0(\tau)$$

and that, on the other hand,

$$[(z_2^*(z_0),\, z_0,\, z_2^*),\, (1, 00,\, z_1^*)] = z_1^*(z_0) = -1, \;\; [(z_2^*(z_0),\, z_0,\, z_2^*),\, \widetilde{\tau}] = x^*(z_0) z_2^*(x) = 1$$

and

$$[(1, 00,\, z_3^*),\, (1, 00,\, z_1^*)] = 1, \;\; [(1, 00,\, z_3^*),\, \widetilde{\tau}] = z_3^*(x) = -1$$

where $[.,.]$ is the commutator is $\mathcal{H}$. Hence, one can define an automorphism $\rho : \mathcal{H} \xrightarrow{\sim} \mathcal{H}$ by setting

$$\begin{array}{ll} \rho((1,00,01)) = (1, 00,\, z_1^*), & \rho((1,00,10)) = \widetilde{\tau}, \\ \rho((1,01,00)) = (z_2^*(z_0),\, z_0,\, z_2^*), & \rho((1,10,00)) = (1, 00,\, z_3^*), \end{array}$$

and by asking that the map induced on the centers is 1. $\square$

2.13. *Remark.* The bases $\{X_{00},\, X_{01},\, X_{10},\, X_{11}\}$ and $\{\Lambda_0(\tau),\, \Lambda_1(\tau),\, \bar{\Lambda}_0(\tau),\, \bar{\Lambda}_1(\tau)\}$ are the same for $\tau = (0010)$.

Corollary 2.14 *(1) Given τ in $J[2] \setminus \{0\}$, there are two (disjoint) τ-invariant lines in $|2\Theta|$. The τ-invariant locus in $|2\Theta|$ is globally invariant under the action of $J[2]$ and a element α in $J[2]$ permutes the connected components if and only if $e_2(\alpha,\, \tau) = -1$.*
(2) If $\Delta(\tau)$ is a line of (1), there are coordinates $\{\lambda_0,\, \lambda_1\}$ such that $\Delta(\tau) \cap \mathrm{Kum}_X$ consists in four reduced points with homogeneous coordinates

$$(a : b), \;\; (a : -b), \;\; (b : a) \text{ and } (b : -a)$$

These scalars don't depend on the choice of the τ-invariant line $\Delta(\tau)$.

<u>Proof</u> : Because of the proposition, it is enough to prove the corollary for a particular τ so let τ be the element (0010) of $J[2]$ and let $\widetilde{\tau} = (1,\, 00,\, 10)$ in $\mathcal{H}$. We will denote by $\Delta^+(\tau)$ (resp. $\Delta^-(\tau)$) the projective line in $|2\Theta|$ corresponding to $W^{\widetilde{\tau}}$ (resp. $W^{-\widetilde{\tau}}$). If X is any eigenvector of $\widetilde{\tau}$ and if $\widetilde{\alpha}$ is any element in $\mathcal{H}$ with class α in $H \times \hat{H}$, one has

$$\widetilde{\tau}(\widetilde{\alpha}(X)) = e_2(\alpha,\, \tau) = \widetilde{\alpha}(\widetilde{\tau}(X))$$

which proves (1).

The ideal (x_{10}, x_{11}) of $\Delta^+(\tau)$ coincides with the kernel of the dual map $W^* \twoheadrightarrow (W^{\tilde{\tau}})^*$. Denote by λ_0 (resp. λ_1) the image of x_{00} (resp. x_{01}) in $(W^{\tilde{\tau}})^*$. The equation (2.1) of Kum_X maps to

$$\lambda_0^4 + \lambda_1^4 + k_{01}\lambda_0^2\lambda_1^2 = 0$$

in $\mathrm{Sym}^4\,(W^{\tilde{\tau}})^*$ and this quartic generates the ideal of the scheme-theoretic intersection $\Delta^+(\tau) \cap \mathrm{Kum}_X$. Hence, because $k_{01} \neq \pm 2$ (Lemma 2.11.(2)), $\Delta^+(\tau) \cap \mathrm{Kum}_X$ consists in four reduced points with homogeneous coordinates

$$(a : b : 0 : 0), \quad (b : a : 0 : 0), \quad (a : -b : 0 : 0), \quad (b : -a : 0 : 0).$$

where a and b are pairwise different non zero scalars satisfying the equality $k_{01} = -\dfrac{b^4 + a^4}{a^2b^2}$.

Because Kum_X is $J[2]$-invariant and because α_{10} maps X_{00} to X_{10} and X_{01} to X_{11}, it is clear that one would have similar results concerning $\Delta^-(\tau)$. $\square$

We let $\omega(\tau)$ be the unique scalar such that the equation of the Kummer surface restricts to

$$\lambda_0(\tau)^4 + \lambda_1(\tau)^4 + \omega(\tau)\lambda_0(\tau)^2\lambda_1(\tau)^2 = 0 \tag{2.4}$$

on $\Delta(\tau)$. The equations (2.3) exactly tell us that $\omega(\tau) \neq \pm 2$ for any τ in $J[2] \setminus \{0\}$ and we gather their expression in terms of the $k_\bullet$, computed thanks to the bases constructed in the Proposition 2.12, in the following chart.

$x \setminus x^*$	00	01	10	11
00	$\star$	k_{10}	k_{01}	k_{11}
01	$\dfrac{2(k_{00}+k_{10}+k_{11})}{2+k_{01}}$	$\dfrac{2(-k_{00}+k_{10}-k_{11})}{2-k_{01}}$	$\dfrac{2(-k_{00}+k_{10}+k_{11})}{2+k_{01}}$	$\dfrac{2(k_{00}+k_{10}-k_{11})}{2-k_{01}}$
10	$\dfrac{2(k_{00}+k_{01}+k_{11})}{2+k_{10}}$	$\dfrac{2(-k_{00}+k_{01}+k_{11})}{2+k_{10}}$	$\dfrac{2(-k_{00}+k_{01}-k_{11})}{2-k_{10}}$	$\dfrac{2(k_{00}+k_{01}-k_{11})}{2-k_{10}}$
11	$\dfrac{2(k_{00}+k_{01}+k_{10})}{2+k_{11}}$	$\dfrac{2(k_{00}+k_{01}-k_{10})}{2-k_{11}}$	$\dfrac{2(-k_{00}+k_{01}-k_{10})}{2-k_{11}}$	$\dfrac{2(-k_{00}+k_{01}+k_{10})}{2+k_{11}}$

(2.5)

2.15. *Remark.* Let $\Delta(\tau)$ be a τ-invariant line in $|2\Theta|$. We will prove in the next section that this line identifies with the quotient $P/\{\pm\}$, where P is the Prym variety associated to τ, which is an elliptic curve in that case. Furthermore, the points of the intersection $\Delta(\tau) \cap \mathrm{Kum}_X$ are the Weierstrass points of P. This gives another proof of the fact that $\omega(\tau) \neq \pm 2$ for any τ in $J[2] \setminus \{0\}$.

3 Prym varieties

Most of the material in that section has been adapted to our situation from the very clear exposition found in [Mu3].

3.1 Etale double cover

Choose a non-zero element τ of $J[2]$. One can construct an étale double cover $\pi : \widetilde{X} \to X$ as follows : An isomorphism $\phi : \tau \otimes \tau \xrightarrow{\sim} \mathcal{O}_X$ allows us to give the direct sum $\mathcal{O}_X \oplus \tau$ a structure of $\mathcal{O}_X$-algebra with product

$$(a,\, l)(b,\, m) = (ab + \phi(lm),\, am + bl)$$

and we set $\widetilde{X} := \mathbf{Spec}(\mathcal{O}_X \oplus \tau)$, which is of genus 3 (Hurwitz). Of course, it does not depend (up to isomorphism) on the choice of the isomorphism $\phi : \tau \otimes \tau \xrightarrow{\sim} \mathcal{O}_X$. Denote by $\widetilde{J}$ the Jacobian of $\widetilde{X}$. If $\widetilde{J}^{\,2}$ stands for the moduli space of degree 2 line bundles over $\widetilde{X}$, there is a canonical theta divisor $\widetilde{\Theta} \subseteq \widetilde{J}^{\,2}$. As π is étale, Hurwitz's formula assures that the canonical divisor $\widetilde{K}$ on $\widetilde{X}$ coincides with the pull-back $\pi^* K$ of the canonical divisor on X. Therefore, $\pi^*\kappa_0$ is a theta characteristic on $\widetilde{X}$ and pulling-back $\widetilde{\Theta}$ by the isomorphism $\widetilde{J} \xrightarrow{\sim} \widetilde{J}^2$ defined by $j \mapsto j \otimes \pi^*\kappa_0$), we obtain a symmetric divisor (still denoted $\widetilde{\Theta}$) that represents the canonical polarization of $\widetilde{J}$.

We have homomorphisms $\pi^* : J \to \widetilde{J}$ and $\mathrm{Nm} : \widetilde{J} \to J$, the latter being deduced from the push-forward of divisors via π. We easily check that the composite $\mathrm{Nm}.\pi^*$ is multiplication by 2. Therefore, $\ker \pi^* \subseteq J[2]$ and it is easily seen to be equal to $< \tau >$. Furthermore, using divisors, one can show (see, e.g., [H], Chapter IV, Ex. 2.6) that, for any $j \in \widetilde{J}$,

$$\det(\pi_* j) \cong \det(\pi_* \mathcal{O}_{\widetilde{X}}) \otimes \mathrm{Nm}(j) \cong \tau \otimes \mathrm{Nm}(j) \tag{3.1}$$

The homomorphisms π^* and Nm are dual one to each other, i.e., the following diagrams (equivalent by duality) commute

$$\begin{tikzcd} \widetilde{J} \arrow[r, "\lambda_{\widetilde{\Theta}}", "\sim"'] & \widehat{\widetilde{J}} \\ J \arrow[u, "\pi^*"] \arrow[r, "\lambda_\Theta", "\sim"'] & \widehat{J} \arrow[u, "\widehat{\mathrm{Nm}}"'] \end{tikzcd} \qquad \begin{tikzcd} \widetilde{J} \arrow[r, "\lambda_{\widetilde{\Theta}}", "\sim"'] \arrow[d, "\mathrm{Nm}"] & \widehat{\widetilde{J}} \arrow[d, "\widehat{\pi^*}"] \\ J \arrow[r, "\lambda_\Theta", "\sim"'] & \widehat{J} \end{tikzcd} \tag{3.2}$$

where $\lambda_\Theta : J \xrightarrow{\sim} \widehat{J}$ (resp. $\lambda_{\widetilde{\Theta}} : \widetilde{J} \xrightarrow{\sim} \widehat{\widetilde{J}}$) is the isomorphism of abelian varieties defined by $j \mapsto \mathcal{O}(T_j^*\Theta - \Theta)$ (resp. $j \mapsto \mathcal{O}(T_j^*\widetilde{\Theta} - \widetilde{\Theta})$). This implies that the third diagram

(3.3)

$$\begin{tikzcd} \widetilde{J} \arrow[r, "\lambda_{\widetilde{\Theta}}", "\sim"'] & \widehat{\widetilde{J}} \arrow[d, "\widehat{\pi^*}"] \\ J \arrow[u, "\pi^*"] \arrow[r, "2\lambda_\Theta"] & \widehat{J} \end{tikzcd}$$

is also commutative. One checks that, from a set-theoretical point of view,

$$\begin{aligned} (\pi^*)^{-1}(\text{Supp } \widetilde{\Theta}) &= \{j \in J | h^0(\widetilde{X}, \pi^*(j \otimes \kappa_0) \geq 1\} \\ &= \{j \in J | h^0(X, \pi_* \mathcal{O}_{\widetilde{X}} \otimes j \otimes \kappa_0) \geq 1\} \\ &= \{j \in J | h^0(X, (j \otimes \kappa_0) \oplus (\tau \otimes j \otimes \kappa_0)) \geq 1\} = \text{Supp } \Theta \cup \text{Supp } T_\tau^* \Theta \end{aligned}$$

Therefore, the divisor $(\pi^*)^{-1}(\widetilde{\Theta})$ is defined and the commutativity of the latter diagram assures that it is algebraically equivalent to 2Θ.

3.1. *Remark.* Note that this result holds for any representative of the principal polarization $\lambda_{\widetilde{\Theta}}$, i.e., for any j in $\widetilde{J}$, as soon as the divisor $(\pi^*)^{-1}(T_j^* \widetilde{\Theta})$ is defined, it is algebraically equivalent to 2Θ.

Let us introduce the set

$$S_\tau = \{z \in J / z^2 = \tau\} \subseteq J[4]$$

which is a principal homogeneous space under $J[2]$. For any z in S_τ, $\pi^*(z)$ belongs to $\widetilde{J}[2]$ for $\pi^*(z)^2 = \pi^*(z^2) = \pi^*(\tau) = 0$. Let $\widetilde{\Theta}_z$ be the effective divisor $T^*_{\pi^*(z)} \widetilde{\Theta}$, which is a symmetric representative for the principal polarization $\lambda_{\widetilde{\Theta}}$. From a set-theoretical point of view,

$$\begin{aligned} (\pi^*)^{-1}(\text{Supp } \widetilde{\Theta}_z) &= \{j \in J | h^0(\widetilde{X}, \pi^*(j \otimes z \otimes \kappa_0) \geq 1\} \\ &= \{j \in J | h^0(X, (j \otimes z \otimes \kappa_0) \oplus (z^{-1} \otimes j \otimes \kappa_0)) \geq 1\} \\ &= \text{Supp } T_z^* \Theta \cup \text{Supp } T_{-z}^* \Theta \end{aligned}$$

The divisor $(\pi^*)^{-1}(\widetilde{\Theta}_z)$ is therefore defined and algebraically equivalent to 2Θ. Note furthermore that π^* being a degree 2 map onto its image, we have, in terms of divisors, the equality $(\pi^*)^{-1}(\widetilde{\Theta}_z) = T_z^* \Theta + T_{-z}^* \Theta$ and the latter is linearly equivalent to 2Θ.

3.2 Prym varieties

Denote by P the abelian variety $\ker(\text{Nm} : \widetilde{J} \to J)^0$, and choose z in S_τ. Denote by σ the homomorphism

$$J \times P \to \widetilde{J};\ (x, q) \mapsto \pi^*(x) + q$$

One easily sees that σ has finite kernel $K_\sigma \subseteq J[2] \times P[2]$. Thus, it is a separable isogeny and P is an abelian variety of dimension 1, i.e., an elliptic curve.

Lemma 3.2 *(1) The composite*

$$J \times P \xrightarrow{\sigma} \widetilde{J} \xrightarrow{\lambda_{\widetilde{\Theta}_z}} \widehat{\widetilde{J}} \xrightarrow{\hat{\sigma}} \hat{J} \times \hat{P}$$

defines a polarization for $J \times P$ that splits into a product $2\lambda_\Theta \times \rho$, where $\rho : P \to \hat{P}$ is a polarization for P.
(2) There is a isomorphism

$$\varphi : \tau^\perp / < \tau > \xrightarrow{\sim} P[2]$$

where $\tau^\perp = \{\alpha \in J[2] |\ e_2(\alpha, \tau) = 1\}$, symplectic with respect to $e_{2,J}$ and $e_{2,P}$.
In particular, $\rho = 2\lambda_P$, where $\lambda_P : P \xrightarrow{\sim} \hat{P}$ is a principal polarization.

<u>Proof</u> : The results in that lemma can be found in [Mu3], Sections 2 and 3. We only sketch Mumford's arguments.
(1) The polarization defined in the lemma may be viewed as a 2×2 matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \rho \end{pmatrix}$, where α, β, γ and ρ are the expected homomorphisms of abelian varieties. Because of the diagram (3.3) and the Remark 3.1 in the previous subsection, $\alpha = 2\lambda_\Theta$. Because any polarization is symmetric, $\beta = \hat{\gamma}$ and because $\widetilde{J} \xrightarrow{\lambda_{\widetilde{\Theta}_z}} \hat{\widetilde{J}} \xrightarrow{\widehat{\pi^*}} \hat{J}$ is zero by the very definition of P (see the diagram (3.2)), $\beta = 0$. Therefore, the considered polarization splits into the product $2\lambda_\Theta \times \rho$, where $\rho : P \to \hat{P}$ is a polarization for P.
(2) On the one hand, because $K_\sigma \cap (\{0\} \times P[2]) = \{0\}$, one can find a subgroup $K \subset J[2]$, containing $\ker \pi^* =< \tau >$, and an injective homomorphism $\varphi : K/ < \tau > \to P[2]$ such that K_σ identifies with K by means of the map

$$K \xrightarrow{\sim} K_\sigma,\ \alpha \mapsto (\alpha, \varphi(\bar{\alpha}))$$

where $\bar{\alpha}$ is the class of α via the quotient map $H \twoheadrightarrow H/ < \tau >$. It is clear that

(a) $\varphi(K/ < \tau >) \subset \ker \rho \subset P[2]$.

On the other hand, because $\lambda_{\widetilde{\Theta}_z}$ is a principal polarization, K_σ is a maximal isotropic subgroup of $\ker(2\lambda_\Theta \times \rho) \cong J[2] \times \ker(\rho)$, with respect to the product bilinear form $e_{2,J} \times e_\rho$ (see Theorem 2.2.(2)). Thus,

(b) $\mathrm{card}(K)^2 = \mathrm{card}(K_\sigma)^2 = \mathrm{card}(J[2])\mathrm{card}(\ker \rho)$,

(c) for any α, β in K, $e_{2,X}(\alpha, \beta) e_\rho(\varphi(\bar{\alpha}), \varphi(\bar{\beta})) = 1$.

Use (c) to obtain the inclusion $< \tau > \subseteq K^\perp$ and combine (a) and (b) to show that it is an equality. Thus, because P is an elliptic curve, the inclusions of (a) are equalities and (b) assures that the corresponding isomorphism is symplectic.
It ends the proof of the lemma since the fact that $\ker(\rho) = P[2]$ implies that ρ is twice the principal polarization of P. $\square$

Choose a symmetric divisor Ξ representing the principal polarization λ_P. The line bundle $\mathcal{O}(2\Xi)$ is canonical and because the polarization defined above splits, one has

$$\sigma^*(\mathcal{O}(\widetilde{\Theta}_z)) \cong \mathcal{O}(2\Theta) \boxtimes \mathcal{O}(2\Xi)$$

Consider the Heisenberg group $\mathcal{G}(\mathcal{O}(2\Theta)\boxtimes\mathcal{O}(2\Xi))$ associated to that ample line bundle over $J\times P$. It is a central extension

$$1\to\mathbb{G}_m\to\mathcal{G}(\mathcal{O}(2\Theta)\boxtimes\mathcal{O}(2\Xi))\to J[2]\times P[2]\to 1$$

and it is isomorphic to $\mathcal{G}(\mathcal{O}(2\Theta))\times\mathcal{G}(\mathcal{O}(2\Xi))/((t,\,t^{-1}),\,t\in\mathbb{G}_m)$. Because of the Theorem 2.2.(1), there is a unique level subgroup $\widetilde{K}_\sigma\subset\mathcal{G}(\mathcal{O}(2\Theta)\boxtimes\mathcal{O}(2\Xi))$, isomorphic to $K_\sigma=\ker\sigma$, such that

$$\sigma_*(\mathcal{O}(2\Theta)\boxtimes\mathcal{O}(2\Xi))^{\widetilde{K}_\sigma}\cong\mathcal{O}(\widetilde{\Theta}_z)$$

Denote by $\widetilde{\tau}$ the image of τ via the lifting $K_\sigma\xrightarrow{\sim}\widetilde{K}_\sigma$.

Proposition 3.3 *(1) There is an isomorphism (well defined up to a multiplicative scalar)*

$$\chi:H^0(J,\,\mathcal{O}(2\Theta))^{\widetilde{\tau}}\xrightarrow{\sim}H^0(P,\,\mathcal{O}(2\Xi))^*\tag{3.4}$$

(2) The application $\delta_{\tau,\,z}:P\to|2\Theta|$ that maps a point q in P to the well-defined divisor $(\pi^*)^{-1}(T_q^*\widetilde{\Theta}_z)$ is a morphism that factors as the composite

$$P\xrightarrow{\varphi_{2\Xi}}\mathbb{P}H^0(P,\,\mathcal{O}(2\Xi))^*\xrightarrow{\sim}\mathbb{P}H^0(J,\,\mathcal{O}(2\Theta))^{\widetilde{\tau}}\subset|2\Theta|$$

where the (well-defined) isomorphism is deduced from χ.

<u>Proof</u> : The results of that proposition can be found in [Mu3], Sections 4 and 5, and we sketch Mumford's arguments again.
(1) Consider the homomorphism $\pi^*:J\to\widetilde{J}$. Its kernel is $<\tau>$ and its image identifies with $J/<\tau>$. We let i_1 be the (separable) isogeny $J\to J/<\tau>$ and $\sigma:J\times P\to\widetilde{J}$ factors as

$$J\times P\xrightarrow{i_1\times\mathrm{Id}}J/<\tau>\times P\xrightarrow{i_2}\widetilde{J}$$

One has (Theorem 2.2)

$$i_2^*(\mathcal{O}(\widetilde{\Theta}_z))\cong((i_1\times\mathrm{Id})_*(\mathcal{O}(2\Theta)\boxtimes\mathcal{O}(2\Xi)))^{\widetilde{\tau}}\cong(i_{1*}\mathcal{O}(2\Theta))^{\widetilde{\tau}}\boxtimes\mathcal{O}(2\Xi)$$

and, letting $Z(\widetilde{\tau})$ be the centralizer of $\widetilde{\tau}$ in $\mathcal{G}(\mathcal{O}(2\Theta))$, one has the isomorphism of Heisenberg groups (of weight 1 in the sense that it induces the identity on the centers $\mathbb{G}_m$)

$$\mathcal{G}((i_{1*}\mathcal{O}(2\Theta))^{\widetilde{\tau}})\cong Z(\widetilde{\tau})/<\widetilde{\tau}>$$

Furthermore, the space $H^0(J,\,\mathcal{O}(2\Theta))^{\widetilde{\tau}}$ is the unique (up to isomorphism) irreducible representation of weight 1 of the Heisenberg group $Z(\widetilde{\tau})/<\widetilde{\tau}>$.
The symplectic isomorphism $\varphi:\tau^\perp/<\tau>\xrightarrow{\sim}P[2]$ constructed in the Lemma 3.2 allows us to construct an isomorphism of Heisenberg groups (of weight -1 in the sense hat it induces $\lambda\mapsto\lambda^{-1}$ on the centers)

$$\widetilde{\varphi}:Z(\widetilde{\tau})/<\widetilde{\tau}>\to\mathcal{G}(\mathcal{O}(2\Xi))$$

Indeed, since $e_{2,X}(\alpha,\beta) = e_{2,P}(\varphi(\bar{\alpha}),\varphi(\bar{\beta}))^{-1}$ for any α, β in $\tau^{\perp}$ (see (c) in the proof of the Lemma 3.2), one can find an homomorphism $\widetilde{\varphi}$ such that the following diagram commutes

$$\begin{array}{ccccccc} 1 \to & \mathbb{G}_m & \to & Z(\widetilde{\tau})/<\widetilde{\tau}> & \to & \tau^{\perp}/<\tau> & \to 1 \\ & \wr\downarrow \lambda^{-1} & & \wr\downarrow \widetilde{\varphi} & & \wr\downarrow \varphi & \\ 1 \to & \mathbb{G}_m & \to & \mathcal{G}(\mathcal{O}(2\Xi)) & \to & P[2] & \to 1 \end{array}$$

and since φ is an isomorphism, $\widetilde{\varphi}$ ditto. Let $H^0(P,\mathcal{O}(2\Xi))^*$ be the dual vector space of $H^0(P,\mathcal{O}(2\Xi))$. It is the unique irreducible representation of weight -1 of $\mathcal{G}(\mathcal{O}(2\Xi))$ and the latter isomorphism of Heisenberg groups, together with the Theorem 2.1, assures that we have an isomorphism (unique up to scalar as it can be shown using Schur's lemma)

$$\chi : H^0(J,\mathcal{O}(2\Theta))^{\widetilde{\tau}} \xrightarrow{\sim} H^0(P,\mathcal{O}(2\Xi))^*$$

(2) Let s_0 be a non zero section of $H^0(\widetilde{J},\mathcal{O}(\widetilde{\Theta}_z))$. Its pull-back $\sigma^*(s_0)$ is the unique (up to scalar) $\widetilde{K}_\sigma$-invariant element of the space of global sections

$$H^0(J\times P,\mathcal{O}(2\Theta)\boxtimes\mathcal{O}(2\Xi)) \cong H^0(J,\mathcal{O}(2\Theta))\otimes H^0(P,\mathcal{O}(2\Xi))$$

Let q be a point of P. The zero locus of $\sigma^*(s_0)_{|J\times\{q\}}$ identifies set-theoretically with the inverse image $(\pi^*)^{-1}(\mathrm{Supp}\,(T_q^*\widetilde{\Theta}_z))$. Suppose that is proper subset of J. Then,

$$(\pi^*)^{-1}(T_q^*\widetilde{\Theta}_z)$$

is a well-defined divisor on J, that is algebraically equivalent to 2Θ (see the Remark 3.1). Because one has taken q in P, it is in fact linearly equivalent to 2Θ. Indeed, the line bundle associated to the algebraically trivial divisor $(\pi^*)^{-1}(T_q^*\widetilde{\Theta}_z)-(\pi^*)^{-1}(\widetilde{\Theta}_z)$ corresponds to the point $\widehat{\pi^*}(\lambda_{\widetilde{\Theta}}(q))$ in $\hat{J}$. Because of the diagrams (3.2), P identifies with $\lambda_{\widetilde{\Theta}}^{-1}(\ker(\widehat{\pi^*})^0)$, and we find that

$$(\pi^*)^{-1}(T_q^*\widetilde{\Theta}_z) \sim_{\mathrm{lin}} (\pi^*)^{-1}(\widetilde{\Theta}_z) \sim_{\mathrm{lin}} 2\Theta$$

Thus $\sigma^*(s_0)$ determines a rational map

$$\delta_{\tau,z} : P \dashrightarrow |2\Theta|$$

that maps q to the divisor $\delta_{\tau,z}(q) = (\pi^*)^{-1}(T_q^*\widetilde{\Theta}_z)$ when defined.
Choose a basis $\{\Lambda_0,\Lambda_1\}$ of $H^0(J,\mathcal{O}(2\Theta))^{\widetilde{\tau}}$. The isomorphism χ allows us to construct a basis $\{\Gamma_0,\Gamma_1\}$ of $H^0(P,\mathcal{O}(2\Xi))$, uniquely defined up to a multiplicative scalar, and satisfying the conditions

$$\chi(\Lambda_i)(\Gamma_j) = \delta_{ij} \text{ for any } i,\, j = 0 \text{ or } 1$$

By construction, $\sum_i \Lambda_i\boxtimes\Gamma_i$ is $\widetilde{K}_\sigma$-invariant, hence equal to $\sigma^*(s_0)$ after suitable normalization. Therefore, the restriction $\sigma^*(s_0)_{|J\times\{q\}}$ coincides with the section

$$\sum_i \Gamma_i(q).\Lambda_i$$

of $H^0(J, \mathcal{O}(2\Theta))$, that is non zero since the linear system $|2\Xi|$ is base-point free. Thus the rational map $\delta_{\tau,z}$ is actually a morphism $P \to |2\Theta|$. Furthermore, the corresponding linear map $H^0(J, \mathcal{O}(2\Theta))^* \to H^0(P, \mathcal{O}(2\Xi))$ obviously factors as

$$H^0(J, \mathcal{O}(2\Theta))^* \twoheadrightarrow (H^0(J, \mathcal{O}(2\Theta))^{\tilde{\tau}})^* \xrightarrow{\sim} H^0(P, \mathcal{O}(2\Xi))$$

where the surjection is the dual of the inclusion and the isomorphism is deduced from χ, hence the proposition. $\square$

3.3 Prym varieties and M_X

For any q in P, one can construct a semi-stable rank 2 vector bundle with trivial determinant over X, namely

$$\pi_* q \otimes z, \text{ where } z \in S_\tau = \{z \in J |\, z^2 = \tau\}$$

Indeed, the corresponding sheaf is locally free because π is flat, it has determinant

$$\mathrm{Nm}(q) \otimes \tau \otimes z^2 \cong \mathcal{O}_X$$

(see the isomorphism (3.1)) and if there were an invertible sub-sheaf $L \subset \pi_* q \otimes z$ with non negative degree, the projection formula would give a non zero map $\pi^* L \to q \otimes \pi^*(z)$, which is contradictory since $\deg \pi^* L > \deg(q \otimes \pi^*(z)) = 0$.
Taking the universal line bundle $\mathcal{L}$ over $\widetilde{X} \times P$ and considering

$$((\pi \times \mathrm{Id})_* \mathcal{L}) \boxtimes z$$

as a family of semi-stable rank 2 vector bundles with trivial determinant over X, parameterized by P, we obtain, using the universal property of M_X, a morphism

$$d_{\tau,z} : P \to M_X$$

depending on τ and z.

Lemma 3.4 *The following diagram is commutative.*

$$\begin{tikzcd} & M_X \arrow[d, "D"] \\ P \arrow[ur, "d_{\tau,z}"] \arrow[r, "\delta_{\tau,z}"'] & |2\Theta| \end{tikzcd}$$

The intersection $\delta_{\tau,z}(P) \cap \mathrm{Kum}_X$ in $|2\Theta|$ is the image $\delta_{\tau,z}(P[2])$ of $P[2]$.

Proof : For any q in P, the set $\mathrm{Supp}\,(\delta_{\tau,\,z}(q))$ is by definition the set

$$\mathrm{Supp}\,((\pi^*)^{-1}(T_q^*\widetilde{\Theta}_z)) = \{j \in J|\, h^0(\widetilde{X},\, \pi^*(j \otimes z \otimes \kappa_0) \otimes q) \geq 1\}$$

Because of the adjunction formula, it coincides with the set

$$\{j \in J|\, h^0(X,\, (\pi_* q \otimes z) \otimes j \otimes \kappa_0) \geq 1\}$$

which is precisely the support of the divisor $D([\pi_* q \otimes z])$. As the divisors linearly equivalent to 2Θ are determined by their support (see [NR], Proposition 6.4), the diagram commutes.

Suppose $\delta_{\tau,\,z}(q)$ is in $\delta_{\tau,\,z}(P) \cap \mathrm{Kum}_X$. It means that $\pi_* q \otimes z$ is a non-stable bundle. If L is a degree 0 invertible sub-sheaf of $\pi_* q \otimes z$, then the projection formula gives a non zero morphism of invertible sheaves

$$\pi^* L \to q \otimes \pi^* z$$

over $\widetilde{X}$ and $q \otimes \pi^*(z \otimes L^{-1})$ must be the trivial sheaf. Thus, with the notations of Section 3.2, $z \otimes L^{-1}$ must be an element of $K = \tau^\perp$ and q must be an element of $P[2]$, corresponding to the class $\overline{z \otimes L^{-1}}$ of $z \otimes L^{-1}$ in $\tau^\perp / < \tau >$ via the symplectic isomorphism φ of the Lemma 3.2. As φ is surjective, we are done. $\square$

3.5. *Remark.* One can ask what happens if one takes $z' \neq z$ in S_τ. In particular, we shall compare $\delta_{\tau,\,z}$ and $\delta_{\tau,\,-z}$ in the next section. Let α in $J[2]$ be the difference $z' - z$ and consider the maps $d_{\tau,\,z'}$ and $\delta_{\tau,\,z'}$. The map $d_{\tau,\,z'}$ (resp. $\delta_{\tau,\,z'}$) coincides with the composite of $d_{\tau,\,z}$ (resp. $\delta_{\tau,\,z}$) with the automorphism of M_X (resp. $|2\Theta|$) induced by the action of α. More precisely, the unique (up to scalar) isomorphism $\mathcal{O}(\widetilde{\Theta}_{z'}) \xrightarrow{\sim} T^*_{\pi^*(\alpha)}\mathcal{O}(\widetilde{\Theta}_z)$ provides an isomorphism $\Psi(\alpha)$ defined as the composite

$$\begin{tikzcd}
\mathcal{O}(2\Theta) \boxtimes \mathcal{O}(2\Xi) \arrow[rrr, dashed, "\Psi(\alpha)"] \arrow[d, "\widetilde{\alpha} \otimes 1"', "\wr"] & & & \mathcal{O}(2\Theta) \boxtimes \mathcal{O}(2\Xi) \\
T^*_\alpha \mathcal{O}(2\Theta) \boxtimes \mathcal{O}(2\Xi) \arrow[r, "\sim"] & T^*_\alpha(\sigma^*\mathcal{O}(\widetilde{\Theta}_z)) \arrow[r, "\sim"] & \sigma^*(T^*_{\pi^*(\alpha)}\mathcal{O}(\widetilde{\Theta}_z)) \arrow[r, "\sim"] & \sigma^*\mathcal{O}(\widetilde{\Theta}_{z'}) \arrow[u, "\wr"']
\end{tikzcd}$$

where $\widetilde{\alpha}$ is any lifting of α in $\mathcal{G}(\mathcal{O}(2\Theta))$ and where we use the splitting of $\sigma^*(\mathcal{O}(\widetilde{\Theta}_z)$ into $\mathcal{O}(2\Theta) \boxtimes \mathcal{O}(2\Xi)$. Now, the automorphism of $\mathcal{G}(\mathcal{O}(2\Theta) \boxtimes \mathcal{O}(2\Theta))$ defined by

$$\gamma \mapsto \Psi(\widetilde{\alpha}) \circ \gamma \circ \Psi(\widetilde{\alpha})^{-1}$$

depends only on the choice of z and z' and it gives an isomorphism $\widetilde{K}_\sigma \xrightarrow{\sim} \widetilde{K}'_\sigma$, where $\widetilde{K}'_\sigma$ is the unique lifting of K_σ in the theta group scheme $\mathcal{G}(\mathcal{O}(2\Theta) \boxtimes \mathcal{O}(2\Theta))$ such that $\sigma_*(\mathcal{G}(\mathcal{O}(2\Theta) \boxtimes \mathcal{O}(2\Theta)))^{\widetilde{K}'_\sigma} \cong \mathcal{O}(\widetilde{\Theta}_{z'})$. Let γ_J be the image of γ through the quotient map $\mathcal{G}(\mathcal{O}(2\Theta) \boxtimes \mathcal{O}(2\Xi)) \to J[2] \times P[2] \to J[2]$. We find that

$$\Psi(\widetilde{\alpha}) \circ \gamma \circ \Psi(\widetilde{\alpha})^{-1} = e_{2,X}(\alpha,\, \gamma_J)\,.\gamma$$

If $\widetilde{\tau}$ (resp. $\widetilde{\tau}'$) is the image of τ via the isomorphism $K_\sigma \xrightarrow{\sim} \widetilde{K}_\sigma$ (resp. $K_\sigma \xrightarrow{\sim} \widetilde{K}'_\sigma$), one has

$$\widetilde{\tau}' = \Psi(\widetilde{\alpha}) \circ \widetilde{\tau} \circ \Psi(\widetilde{\alpha})^{-1} = e_{2,X}(\alpha, \tau) \,.\widetilde{\tau}$$

and one finds that if α is not in $\tau^\perp$, $\delta_{\tau,z}$ and $\delta_{\tau,z'}$ don't map P onto the same projective line in $|2\Theta|$. If α belongs to $\tau^\perp$, then $\widetilde{K}_\sigma = \widetilde{K}'_\sigma$ and the automorphism of $\widetilde{K}_\sigma$ induced by $\gamma \mapsto \Psi(\widetilde{\alpha}) \circ \gamma \circ \Psi(\widetilde{\alpha})^{-1}$ only depends on the class of α in $\tau^\perp / < \tau > \cong P[2]$. In other words, the morphisms $\delta_{\tau,z}$ and $\delta_{\tau,z'}$ differ from the involutional translation on P corresponding to the class of α in $P[2]$. In particular, one has $\delta_{\tau,-z} = \delta_{\tau,z}$.

3.6. *Remark.* Recall (Corollary 2.14) that we are able to give the homogeneous coordinates of the points in $\delta_{\tau,z}(P) \cap \mathrm{Kum}_X$ in $|2\Theta|$ in terms of the coefficients of the Kummer surface (see the equation (2.4) and the chart (2.5)). As $\delta_{\tau,z}$ coincides with the canonical map $\varphi_{2\Xi}$, this set is precisely the set of ramification points of the canonical map $\varphi_{2\Xi}$. Therefore, we are able to characterize the elliptic curves arising that way.

4 The generalized Verschiebung $V : M_{X_1} \dashrightarrow M_X$

4.1 Review of Theta groups in characteristic p

The curve X is now supposed to have p-rank 2, i.e., to be an ordinary genus 2 curve. Because of the Remark 2.10, one knows that a general Kummer surface Kum_X in $\mathbb{P}^3$ is associated to such a curve.

Let X_1 be the p-twist of X. Denote by i the semi-k-linear isomorphism $X_1 \to X$ and by F the relative Frobenius $X \to X_1$, which is radicial and flat. Notice that there is a canonical bijection from the set of Weierstrass points of X to the set of Weierstrass points of X_1. Thus, the choice of the effective theta characteristic κ_0 for X determines an effective theta characteristic $i^*(\kappa_0)$ for X_1, still denoted by κ_0.
Denote by J_1 the p-twist of J and let $F : J \to J_1$ (resp. $i : J_1 \to J$) the relative Frobenius (resp. the semi-k-linear isomorphism) which is flat. The abelian variety J_1 coincides with the Jacobian of X_1 and κ_0 enables us, as before, to give a symmetric representative Θ_1 for the principal polarization on J_1. It is easily seen that $\Theta_1 = i^*\Theta$.

Let $\hat{G}$ denote the kernel of $F : J \to J_1$. It is a local group scheme and as J is an ordinary abelian variety, $\hat{G}$ is the local part of the group scheme $J[p]$ of p-torsion points of J. Denote by G the reduced part of $J[p]$. The relative Frobenius maps G isomorphically onto the kernel of the isogeny

$$J_1 \cong J/\hat{G} \to J/J[p] \cong J$$

which is separable and of degree $p^g = p^2$, and is called the Verschiebung V. It maps a degree 0 line bundle ζ_1 over X_1 to the degree 0 line bundle $F^*\zeta_1$ over X. Note that both

composites

$$J \xrightarrow{F} J_1 \xrightarrow{V} J \text{ and } J_1 \xrightarrow{V} J \xrightarrow{F} J_1$$

are multiplication by p.

The finite group $J_1[p]$ is self-dual and consequently, we have the Göpel system

$$J_1[p] \cong G \times \hat{G} \cong (\mathbb{Z}/p\,\mathbb{Z})^g \times (\mu_p)^g$$

Consider the line bundle $\mathcal{O}(p\,\Theta_1)$ over J_1. Its automorphism group $\mathcal{G}(\mathcal{O}(p\,\Theta_1))$ which can be obtained as a central extension

$$1 \to \mathbb{G}_m \to \mathcal{G}(\mathcal{O}(p\,\Theta_1)) \to J_1[p] \to 0$$

and $\mathcal{O}(p\,\Theta_1)$ is no longer of separable type. Nevertheless, [Sek] proves that the main results about theta groups (recalled in Theorems 2.2 and 2.3) extend to line bundles of non-separable type. We gather some useful results given in [LP1] in the

Lemma 4.1 *(1) There are the three isomorphisms $\mathcal{O}(p\,\Theta_1) \cong V^*\mathcal{O}(\Theta)$, $\mathcal{O}(p\,\Theta) \cong F^*\mathcal{O}(\Theta_1)$ and $\mathcal{O}(\Theta_1) \cong i^*\mathcal{O}(\Theta)$.*
(2) The restrictions of $\mathcal{G}(\mathcal{O}(p\,\Theta))$ to both G and $\hat{G}$ are canonically split. Therefore, the decomposition $J[p] \cong G \times \hat{G}$ is symplectic (with respect to e_p).
(3) There exists a basis $\{X_g\}_{g\in G}$ of $H^0(J, \mathcal{O}(p\,\Theta))$, unique up to a multiplicative scalar, which satisfies the following relations

$$a.X_g = X_{a+g} \quad \alpha.X_g = e_p(\alpha,\, g)X_g \quad \forall a,\, g \in G,\, \alpha \in \hat{G}$$

(4) For any $g \in G$, there is a unique Y_g in $H^0(J_1, \mathcal{O}(p\,\Theta_1))$ such that $X_g^p = F^*Y_g$. The family $\{Y_g\}_{g\in G}$ is a basis of $H^0(J_1, \mathcal{O}(p\,\Theta_1))$ that corresponds to the basis $\{X_g\}_{g\in G}$ via $i^* : H^0(J, \mathcal{O}(p\,\Theta)) \to H^0(J_1, \mathcal{O}(p\,\Theta_1))$.

<u>Sketch of Proof</u> (Complete proofs can be found in [LP1]) : One needs to define a splitting

$$G \hookrightarrow \mathcal{G}(\mathcal{O}(p\,\Theta_1))$$

for the central extension above. Because G is reduced, it is enough to find it at the level of k-point and because k^* is divisible, this can be worked out. Furthermore, because the skew-symmetric form

$$e_p : J_1[p] \times J_1[p] \to \mathbb{G}_m$$

(associated to the commutator in $\mathcal{G}(\mathcal{O}(p\,\Theta_1))$) takes its value in μ_p, this splitting is unique (another one would differ from the first one by a morphism $G \to \mu_p$ and $\mu_p(k) = \{1\}$). Therefore, the analog of Theorem 2.3.(1) in the non separable case assures the existence and the uniqueness of a line bundle M over J such that $\mathcal{O}(p\,\Theta_1) \cong V^*M$.
One can show that M defines a principal polarization and that the isomorphism

$$V^*F^*\mathcal{O}(\Theta_1) \cong \mathcal{O}(p^2\,\Theta_1) \xrightarrow{\sim} V^*M^{\otimes p}$$

obtained in taking p-powers commutes with the action of G, hence descends onto an isomorphism

$$F^*\mathcal{O}(\Theta_1) \xrightarrow{\sim} M^{\otimes p}$$

Furthermore, if D is the unique effective divisor on J such that $L = \mathcal{O}(D)$, on has, set-theoretically,

$$V^{-1}(\mathrm{Supp}(D)) = \bigcup_{g\in G} T_g^*(\mathrm{Supp}(\Theta_1))$$

Because Θ_1 is symmetric, $V^{-1}(\mathrm{Supp}(D))$ is symmetric and finally M is a symmetric representative for the principal polarization on J. Thus, the difference between M and $\mathcal{O}(\Theta)$ lies in $J[p\,] \cap J[2] = \{0\}$ and $M \cong \mathcal{O}(\Theta)$. The third isomorphism announced by (1) is tautological (see our definition of Θ_1).
(2) is the Lemma 2.3 of [LP1].
(3) can be obtained in a very similar way as in the construction of the basis $\{X_\bullet\}$ in Section 2.2, taking of the fact that the invertible sheaf $\mathcal{O}(p\,\Theta)$ is of non separable type.
(4) Each vector X_g^p is invariant under the action of $\hat{G}$ hence of the form F^*Y_g with Y_g in $H^0(J_1, \mathcal{O}(p\,\Theta_1))$. Since k is divisible, the family $\{Y_g\}$ is free, hence a basis. $\square$

4.2. *Remark.* These results remain true, *mutatis mutandis*, for any ordinary principally polarized abelian variety (see [LP1]).

4.2 Extending Verschiebung to $|2\Theta_1|$

Let E_1 be a semi-stable bundle with trivial determinant over X_1. Then, F^*E_1 is a rank 2 vector bundle with trivial determinant over X, which may not be semi-stable since the pull-back by Frobenius destabilizes some vector bundles ([R]). Nevertheless, F induces a rational map $V : M_{X_1} \dashrightarrow M_X$. If E_1 is the non stable bundle $j \oplus j^{-1}$, its pull-back $F^*j \oplus (F^*j)^{-1}$ is semi-stable but non stable and we find that the following diagram is commutative

$$\begin{tikzcd} J_1 \arrow[r, "V"] \arrow[d, "b_1"'] & J \arrow[d, "b"] \\ M_{X_1} \arrow[r, dashed, "V"] & M_X \end{tikzcd} \tag{4.1}$$

It is a diagram of $J[2]$-equivariant morphisms in the following sense : On the one hand, because p is odd, $[p\,]$ induces identity on $J[2]$. Therefore, $F : J[2] \to J_1[2]$ and $V : J_1[2] \to J[2]$ are isomorphisms, inverse one to each other. Thus, one can define an action of $J[2]$ on both J_1 and M_{X_1}, compatible with the maps involved in the diagram.
In particular, the indeterminacy locus of V does not meet the Kummer surface. Thus, it is a finite set $\mathcal{I}$ and we let U be the Zariski open subset $M_{X_1} \setminus \mathcal{I}$.

Let $\mathcal{O}(\Delta_1)$ be the determinant line bundle over M_{X_1}. It has been shown ([LP2]) that

Proposition 4.3 *There is an isomorphism* $V_2^*(\mathcal{O}(\Delta)) \cong (\mathcal{O}(\Delta_1)^{\otimes p})_{|U}$.

Because $\mathcal{I}$ is a finite set, they obtain that the rational map V is given by degree p polynomials.

Using the Proposition 2.6, one obtains a rational map $\widetilde{V} : |2\Theta_1|^* \dashrightarrow |2\Theta|^*$ and the vertical arrows in (4.1) become the canonical maps $\varphi_{2\Theta_1}$ and $\varphi_{2\Theta}$. Thus, the pull-back by the Verschiebung $V^* : H^0(J, \mathcal{O}(2\Theta)) \to H^0(J_1, \mathcal{O}(2p\,\Theta_1))$ factors as the composite

$$H^0(J, \mathcal{O}(2\Theta)) \xrightarrow{\widetilde{V}^*} \mathrm{Sym}^p H^0(J_1, \mathcal{O}(2\Theta_1)) \to H^0(J_1, \mathcal{O}(2p\,\Theta_1)) \tag{4.2}$$

where the last arrow is the canonical evaluation map.

Recall (Section 2.2) that we have chosen a theta structure $\mathcal{H} \xrightarrow{\sim} \mathcal{G}(\mathcal{O}(2\Theta))$ and that $W := H^0(J, \mathcal{O}(2\Theta))$ is the unique (up to isomorphism) irreducible representation of $\mathcal{H}$ (of weight 1). We let W_1 denote the vector space

$$W_1 := H^0(J_1, \mathcal{O}(2\Theta_1))$$

It is, analogously, the unique irreducible representation of the Theta group $\mathcal{G}(\mathcal{O}(2\Theta_1))$.

Lemma 4.4 *(1) One can endow* $\mathrm{Sym}^p W_1$ *and* $H^0(J_1, \mathcal{O}(2p\,\Theta_1))$ *with an action (of weight* p*) of* $\mathcal{G}(\mathcal{O}(2\Theta_1))$.
(2) The evaluation map $\mathrm{Sym}^p W_1 \to H^0(J_1, \mathcal{O}(2p\,\Theta_1))$ *is* $\mathcal{G}(\mathcal{O}(2\Theta_1))$*-equivariant for these actions.*

<u>Proof</u> : (1) The homomorphism

$$\varepsilon_p : \mathcal{G}(\mathcal{O}(2\,\Theta_1)) \to \mathcal{G}(\mathcal{O}(2p\,\Theta_1))$$

that maps an isomorphism $\gamma : \mathcal{O}(2\,\Theta_1) \xrightarrow{\sim} T_x^*\mathcal{O}(2\,\Theta_1)$ to the isomorphism

$$\gamma^{\otimes p} : \mathcal{O}(2p\,\Theta_1) \xrightarrow{\sim} T_x^*\mathcal{O}(2p\,\Theta_1)$$

fits into the commutative diagram

$$\begin{array}{ccccccc}
0 \to \mathbb{G}_m & \longrightarrow & \mathcal{G}(\mathcal{O}(2\Theta_1)) & \longrightarrow & J_1[2] & \longrightarrow & 0 \\
p\text{-power} \downarrow & & \downarrow \varepsilon_p & & \downarrow \text{inclusion} & & \\
0 \to \mathbb{G}_m & \longrightarrow & \mathcal{G}(\mathcal{O}(2p\,\Theta_1)) & \longrightarrow & J_1[2p\,] & \longrightarrow & 0
\end{array}$$

It gives the action of $\mathcal{G}(\mathcal{O}(2\Theta_1))$ onto $H^0(J_1, \mathcal{O}(2p\,\Theta_1))$. The other case is straightforward.
(2) The evaluation map $W_1 \otimes \mathcal{O}_{X_1} \to \mathcal{O}(2\Theta_1)$ is of course $\mathcal{G}(\mathcal{O}(2\Theta_1))$-equivariant and taking its p-symmetric power, one obtains, at the level of global sections, the canonical map $\mathrm{Sym}^p W_1 \to H^0(J_1, \mathcal{O}(2p\,\Theta_1))$, which is still $\mathcal{G}(\mathcal{O}(2\Theta_1))$-equivariant for the induced actions on both spaces. These are the ones of (1), hence the lemma. $\square$

Lemma 4.5 *(1) There is an homomorphism of theta groups $\mathcal{H} \to \mathcal{G}(\mathcal{O}(2\Theta_1))$ (of weight p). Therefore, $\mathcal{H}$ has an action of weight p^2 on $\mathrm{Sym}^p W_1$.*
(2) The map $\widetilde{V}^* : W \to \mathrm{Sym}^p W_1$ is injective and $\mathcal{H}$-equivariant, up to a multiplicative scalar.

<u>Proof</u> : (1) On the one hand, the isomorphism of sheaves $V^*\mathcal{O}(2\Theta) \cong \mathcal{O}(2p\,\Theta_1)$ induces (by pull-back) a homomorphism $V^* : \mathcal{H} \to \mathcal{G}(\mathcal{O}(2p\,\Theta_1)$ (of weight 1). On the other hand, for any $\gamma : \mathcal{O}(2p\,\Theta_1) \xrightarrow{\sim} T_x^*\mathcal{O}(2p\,\Theta_1)$, there is a unique isomorphism $\rho : \mathcal{O}(2\Theta_1) \xrightarrow{\sim} T_{p\,x}^*\mathcal{O}(2\Theta_1)$ such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}(2p^2\,\Theta_1) \arrow[r, "\gamma^{\otimes p}"] & T_x^*\mathcal{O}(2p^2\,\Theta_1) \\
{[p\,]^*\mathcal{O}(2\Theta_1)} \arrow[u, "\wr"'] \arrow[r, "{[p\,]^*\rho}"] & {[p\,]^*T_{p\,x}^*\mathcal{O}(2\Theta_1)} \arrow[u, "\wr"']
\end{tikzcd}$$

where the vertical isomorphisms come from the fact that $\mathcal{O}(2\Theta_1)$ is a symmetric line bundle. We let $\eta_p : \mathcal{G}(\mathcal{O}(2p\,\Theta_1)) \to \mathcal{G}(\mathcal{O}(2\,\Theta_1))$ be the homomorphism that maps γ to ρ. It fits into the commutative diagram of central extensions

$$\begin{tikzcd}
0 \to \mathbb{G}_m \arrow[r] \arrow[d, "p\text{-power}"'] & \mathcal{G}(\mathcal{O}(2p\,\Theta_1)) \arrow[r] \arrow[d, "\eta_p"] & J_1[2p\,] \arrow[r] \arrow[d, "{[p\,]}"] & 0 \\
0 \to \mathbb{G}_m \arrow[r] & \mathcal{G}(\mathcal{O}(2\Theta_1)) \arrow[r] & J_1[2] \arrow[r] & 0
\end{tikzcd}$$

Now, we consider the composite homomorphism

$$\mathcal{H} \xrightarrow{V^*} \mathcal{G}(\mathcal{O}(2p\,\Theta_1)) \xrightarrow{\eta_p} \mathcal{G}(\mathcal{O}(2\Theta_1))$$

It has weight p. Using the latter and the natural action (of weight p) of $\mathcal{G}(\mathcal{O}(2\Theta_1))$ on $\mathrm{Sym}^p W_1$, one obtains an action of weight p^2 of $\mathcal{H}$ on $\mathrm{Sym}^p W_1$.
(2) Let

$$\mathcal{G}(\mathcal{O}(2p\,\Theta_1))_2$$

be the maximal subgroup of $\mathcal{G}(\mathcal{O}(2p\,\Theta_1))$ lying above $J_1[2]$, viewed as a sub-group of $J_1[2p\,]$. It is obviously the image of the homomorphism ε_p and since $[p\,]$ acts trivially on $J_1[2]$, η_p restricts to an homomorphism

$$\mathcal{G}(\mathcal{O}(2p\,\Theta_1))_2 \to \mathcal{G}(\mathcal{O}(2\Theta_1))$$

with kernel μ_p. The composite homomorphism $\varepsilon_p \circ \eta_p$ fits into the commutative diagram of central extensions

$$\begin{tikzcd}
0 \to \mathbb{G}_m \arrow[r] \arrow[d, "p^2\text{-power}"'] & \mathcal{G}(\mathcal{O}(2p\,\Theta_1))_2 \arrow[r] \arrow[d, "\varepsilon_p \circ \eta_p"] & J_1[2] \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \to \mathbb{G}_m \arrow[r] & \mathcal{G}(\mathcal{O}(2p\,\Theta_1))_2 \arrow[r] & J_1[2] \arrow[r] & 0
\end{tikzcd}$$

Now, since $V^*(\mathcal{H}) \subseteq \mathcal{G}(\mathcal{O}(2p\,\Theta_1))_2$, the map $V^* : W \to H^0(J_1,\, \mathcal{O}(2p\,\Theta_1))$ is $\mathcal{H}$-equivariant (up to a multiplicative scalar) when one endows $H^0(J_1,\, \mathcal{O}(2p\,\Theta_1))$ with the action $\mathcal{H}$ induced by the composite $\varepsilon_p \circ \eta_p \circ V^*$. As this map is non-zero, it is injective and the map $\widetilde{V}^*$ is injective as well. The evaluation map must induce an isomorphism between the image of $\widetilde{V}^*$ and the image of $\widetilde{V}^*$, that is $\mathcal{H}$-equivariant (Lemma 4.5) and $\widetilde{V}^*$ is $\mathcal{H}$-equivariant (up to a multiplicative scalar) as well. $\square$

4.6. *Remark.* The reason why $\widetilde{V}^*$ is not $\mathcal{H}$-equivariant is that the action of $\mathcal{H}$ on the spaces W and $\mathrm{Sym}^p\, W_1$ do not have the same weight, in contradiction with the k-linearity of $\widetilde{V}^*$. This obstruction vanishes when one considers the induced action of the subgroup $\hat{H} \subseteq \mathcal{H}$ (resp. $H \subseteq \mathcal{H}$) on both spaces.

4.7. *Remark.* One notices that taking $\rho = i^*\gamma : \mathcal{O}(2\Theta_1) \to T^*_{\alpha_1}\mathcal{O}(2\Theta_1)$ makes the following diagram commutative

$$\begin{array}{ccc} \mathcal{O}(2p^2\,\Theta_1) & \xrightarrow{(V^*\,\gamma)^{\otimes p}} & T^*_{\alpha_1}\mathcal{O}(2p^2\Theta_1) \\ \uparrow \wr & & \uparrow \wr \\ [p\,]^*\mathcal{O}(2\Theta_1) & \xrightarrow{[p\,]^*\rho} & [p\,]^*T^*_{\alpha_1}\mathcal{O}(2\Theta_1) \end{array}$$

Namely, one has, using the fact that F^*_{abs} is the p-power, that

$$[p\,]^*i^*\gamma = V^*(F^*i^*)\gamma = V^*(\gamma^{\otimes p}) = (V^*\,\gamma)^{\otimes p}$$

Therefore, the homomorphism $\eta_p \circ V^*$ (of weight p) coincides with the homomorphism $i^* : \mathcal{H} \to \mathcal{G}(\mathcal{O}(2\Theta_1))$ induced by the pull-back by the quasi-isomorphism i^*.

4.8. *Remark.* Using the map $\eta_p \circ V^*$, one finds that, for any two elements $\bar{\alpha}$ and $\bar{\beta}$ in $J[2]$, $e_2(F(\bar{\alpha}),\, F(\bar{\beta})) = e_2(\bar{\alpha},\, \bar{\beta})^p$. Because $J[2]$ is reduced and because e_2 takes its values in μ_2,we find that F (hence V) is a symplectic isomorphism. This implies that the Göpel system $J[2] \cong H \times \hat{H}$ determines a Göpel system $J_1[2] \cong H \times \hat{H}$, that a theta structure $\mathcal{H} \xrightarrow{\sim} \mathcal{G}(2\Theta)$ determines a theta structure $\mathcal{H}_1 \xrightarrow{\sim} \mathcal{G}(2\Theta_1)$ (where $\mathcal{H}_1 := \mathcal{H} \otimes_{F_{\mathrm{abs}},\,k} k$), and that the basis $\{X_\alpha\}_{\alpha \in H}$ determines a basis $\{Y_{\alpha_1}\}_{\alpha_1 \in H_1}$, compatible in the sense that $Y_{\alpha_1} = i^*X_{V(\alpha_1)}$.

Proposition 4.9 *Let X be a smooth and proper ordinary curve of genus 2 over an algebraically closed field of characteristic $p \geq 3$. Then, the generalized Verschiebung $V : M_{X_1} \dashrightarrow M_X$ is completely determined by an irreducible sub-representation of the $\mathcal{G}(\mathcal{O}(2\Theta))$ in $\mathrm{Sym}^p\, H^0(J_1,\, \mathcal{O}(2\Theta_1))^*$ (isomorphic to $H^0(J,\, \mathcal{O}(2\Theta))^*$ as a vector space).*

<u>Proof</u> : As previously, we identify the space W (resp. W_1) with its dual W^* (resp. W_1) by means of the isomorphism of the Lemma 2.4, and we let $\{x_\bullet\}$ (resp. $\{y_\bullet\}$) be the basis

of W^* (resp. W_1^*) dual to $\{X_\bullet\}$ (resp. $\{Y_\bullet\}$) constructed in the Section 2.2 (resp. in the Remark 4.8). Denote by V_i ($i = 00$, 01, 10 and 11) the degree p homogeneous polynomials (see the Proposition 4.3 above, due to [LP2]) such that the rational map $\widetilde{V} : |2\Theta_1| \dashrightarrow |2\Theta|$ corresponding to V is given by

$$\begin{array}{rccc} \widetilde{V} : & |2\Theta_1| & \dashrightarrow & |2\Theta| \\ & (y_i) & \mapsto & (V_i(\underline{y})) \end{array} \tag{4.3}$$

Using the Lemma 4.5 and the Remark 4.6, we find that V_{00} is invariant under the action of the subgroup $\hat{H}$ of $\mathcal{H}$. It coincides therefore, up to a multiplicative scalar, with $\widetilde{V}(x_{00})$ and upon normalizing suitably V_{00}, one can suppose that

$$\widetilde{V}^*(x_{00}) = V_{00}$$

By construction again, $\widetilde{V}^*$ is H-equivariant thus one obtains V_i (for $i = 01$, 10, 11) as the transform of V_{00} under the action of the unique element of H that maps x_{00} to x_i. $\square$

4.3 Prym varieties and Frobenius

Let τ be a non zero element of $J[2]$ and let $\pi : \widetilde{X} \to X$ be the corresponding étale double cover. The base change induced by the Frobenius morphism on the base field gives an étale double cover $\pi_1 : \widetilde{X}_1 \to X_1$ corresponding to a non zero τ_1 of $J[2]$ which is the image of τ under the isomorphism $J[2] \to J_1[2]$ defined earlier. The following lemma is well-known (see [SGA1]) :

Lemma 4.10 *If F is the relative Frobenius, the following diagram*

$$\begin{tikzcd} \widetilde{X} \arrow[r, "F"] \arrow[d, "\pi"'] & \widetilde{X}_1 \arrow[d, "\pi_1"] \\ X \arrow[r, "F"] & X_1 \end{tikzcd}$$

is cartesian.

<u>Proof</u> : The diagram is certainly commutative, because of Frobenius functoriality, thus there is a unique $h : \widetilde{X} \to \widetilde{X}_1 \times_{X_1} X$ that makes the following diagram commutative

$$\begin{tikzcd} \widetilde{X} \arrow[dr, "h"] \arrow[drr, bend left, "F"] \arrow[ddr, bend right, "\pi"'] & & \\ & \widetilde{X}_1 \times_{X_1} X \arrow[r, "pr_1"] \arrow[d, "pr_2"] \arrow[dr, phantom, "\square"] & \widetilde{X}_1 \arrow[d, "\pi_1"] \\ & X \arrow[r, "F"] & X_1 \end{tikzcd}$$

As π and π_1 are étale and proper, we find that pr_2, hence h, are étale and proper as well, hence a finite étale covering. Now, F being radicial, h is radicial as well and therefore, an isomorphism. $\square$

As above, write $\widetilde{J}_1$ for the Jacobian variety of $\widetilde{X}_1$. It coincides with the p-twist of $\widetilde{J}$ and we can define the relative Frobenius and the Verschiebung. Denote again by $\widetilde{\Theta}_1$ the Theta divisor obtained as the pull-back of the canonical Theta divisor on $\widetilde{J}_1{}^2$ by means of the Theta characteristic $\pi_1^*(\kappa_0)$ (where κ_0 has to be understood here as the effective Theta characteristic of X_1 we have constructed).

Lemma 4.11 *The morphisms $\pi^* : J \to \widetilde{J}$ and $\mathrm{Nm} : \widetilde{J} \to J$ commute with V. In other words, the two following diagrams are commutative*

$$\begin{tikzcd} \widetilde{J}_1 \arrow[r, "V"] & \widetilde{J} \\ J_1 \arrow[u, "\pi_1^*"] \arrow[r, "V"] & J \arrow[u, "\pi^*"'] \end{tikzcd} \qquad \begin{tikzcd} \widetilde{J}_1 \arrow[r, "V"] \arrow[d, "\mathrm{Nm}"'] & \widetilde{J} \arrow[d, "\mathrm{Nm}"] \\ J_1 \arrow[r, "V"] & J \end{tikzcd}$$

<u>Proof</u> : The commutation of the left-hand diagram is a straightforward consequence of the commutation of the diagram in the Lemma 4.10. For the right-hand one, one takes an element $j \in \widetilde{J}_1$ and sees, using the Lemma 4.10 again, that $F^*(\pi_* j) \cong \pi_*(F^* j)$. Thus, using (3.1), one can write that

$$\begin{aligned} \mathrm{Nm}(V(j)) = \det(F^* j) \otimes \tau &\cong F^*(\det(\pi_* j)) \otimes \tau \\ &\cong F^*(\det(\pi_* j) \otimes \tau_1) = V(\mathrm{Nm}(j)) \qquad \square \end{aligned}$$

Proposition 4.12 *The following diagram*

$$\begin{tikzcd} J \times P \arrow[r, "\sigma"] & \widetilde{J} \\ J_1 \times P_1 \arrow[u, "V \times V"] \arrow[r, "\sigma_1"] & \widetilde{J}_1 \arrow[u, "V"'] \end{tikzcd}$$

is commutative.
Furthermore, σ induces an isomorphism $J[p] \times P[p] \xrightarrow{\sim} \widetilde{J}[p]$. In particular, if J is an ordinary abelian variety, then $\widetilde{J}$ is ordinary if and only if P is ordinary.

<u>Proof</u> : Because of the right-hand diagram in the previous lemma, the Prym variety

$$P_1 := \ker(\mathrm{Nm})^0 \subseteq \widetilde{J}_1$$

coincides with the p-twist of $P := \ker(\mathrm{Nm})^0 \subseteq \widetilde{J}$ and it is mapped by V onto P. Furthermore, the restriction $V_{|P_1} : P_1 \to P$ being the pull-back of particular line bundles over $\widetilde{X}_1$ by the relative Frobenius, it coincides with the Verschiebung $V : P_1 \to P$ for P. Therefore,

the commutation of the diagram in the proposition follows from the left-hand commutative diagram in the lemma and from the fact that V is a homomorphism.
One knows that $\ker\sigma \subseteq J[2] \times P[2]$ and, as $A[p\,] \cap A[2] = \{0\}$ for any abelian variety, we have the isomorphism announced. Recalling that X was supposed to be ordinary, the last assertion follows from the induced isomorphism

$$J[p\,]_{\mathrm{red}} \times P[p\,]_{\mathrm{red}} \xrightarrow{\sim} \widetilde{J}[p\,]_{\mathrm{red}}$$

on the reduced parts of these group schemes. $\square$

The following result enables us to apply the rsults gathered in the Lemma 4.1 to both $\widetilde{J}$ and P for any sufficiently general curve X.

Proposition 4.13 (B. Zhang) *Let X be a general, proper and smooth connected curve over an algebraically closed field of characteristic p and let $f : Y \to X$ be an étale cover with abelian Galois group G. Then Y is ordinary.*

<u>Proof</u> : [Zh]

Let us investigate a bit further in the relationship between P and P_1. Choose an element z in $S_\tau = \{z \in J|\, z^2 = \tau\} \subset J[4]$. In particular, it determines the image of the map $\delta_{\tau,\,z} : P \to |2\Theta|$, namely, one of the two τ-invariant projective line in $|2\Theta|$.

As $[p\,] : J \to J$ induces $[-1]^{\frac{p-1}{2}}$ on $J[4]$, F and V induce isomorphisms between $J[4]$ and $J_1[4]$, and we let z_1 be the isomorphic image of z via F. Thus, $F^*z_1 = V(z_1) = (-1)^{\frac{p-1}{2}}\, z$ and as

$$F^*((\pi_1)_*(q_1) \otimes z_1) \cong \pi_*(F^*(q_1)) \otimes F^*z_1$$

for any q_1 of P_1 (Lemma 4.10), one sees (Lemma 3.4 and Remark 3.5) that the following diagram commutes

$$\begin{tikzcd} P \arrow[r, "d_{\tau,\,z}"] & M_X \\ P_1 \arrow[u, "V"] \arrow[r, "d_{\tau_1,\,z_1}"] & M_{X_1} \arrow[u, dashed, "V"'] \end{tikzcd} \tag{4.4}$$

Let Ξ (resp. Ξ_1) be a symmetric representative for the principal polarization of P (resp. P_1). Upon choosing Ξ and Ξ_1 suitably, one can ask that $\mathcal{O}(p\,\Xi_1) \cong V^*\mathcal{O}(\Xi)$ (Lemma 4.1). Let $\varphi_{2\Xi}$ (resp. $\varphi_{2\Xi_1}$) be the canonical map

$$P \to \mathbb{P}H^0(P,\, \mathcal{O}(2\Xi))^* \text{ (resp. } P_1 \to \mathbb{P}H^0(P_1,\, \mathcal{O}(2\Xi_1))^*)$$

Because V commutes with $[-1]$, it induces a map $\widetilde{V}$ such that the following diagram commutes

$$\begin{array}{ccc} P_1 & \xrightarrow{\quad V \quad} & P \\ {\scriptstyle \varphi_{2\Xi_1}}\downarrow & & \downarrow{\scriptstyle \varphi_{2\Xi}} \\ \mathbb{P}H^0(P_1,\, \mathcal{O}(2\Xi_1)) & \xrightarrow{\widetilde{V}} & \mathbb{P}H^0(P,\, \mathcal{O}(2\Xi)) \end{array}$$

In other words, $V^* : H^0(P,\, \mathcal{O}(2\Xi)) \to H^0(P_1,\, \mathcal{O}(2p\,\Xi_1))$ factors as the composite

$$H^0(P,\, \mathcal{O}(2\Xi)) \xrightarrow{\widetilde{V}^*} \mathrm{Sym}^p\, H^0(P_1,\, \mathcal{O}(2\Xi_1)) \to H^0(P_1,\, \mathcal{O}(2p\,\Xi_1)) \tag{4.5}$$

Let $\widetilde{\tau}$ (resp $\widetilde{\tau}_1$) be the lifting of τ (resp. τ_1) in $\mathcal{H}$ (resp. $\mathcal{H}_1$) corresponding to our choice of z in S_τ (resp. to $z_1 = F(z)$ in S_{τ_1}). Consider the basis $\{\Lambda_0,\, \Lambda_1\}$ of $W^{\widetilde{\tau}}$ constructed in the Proposition 2.12 and construct the basis $\{\Gamma_0,\, \Gamma_1\}$ of $H^0(P,\, \mathcal{O}(2\Xi))$ by means of the isomorphism χ, as it is done in the proof of the Proposition 3.3. Upon normalizing χ_1 suitably, the two bases

$$\{\Lambda_0^{(p)},\, \Lambda_1^{(p)}\} \text{ (of } W_1^{\widetilde{\tau}_1}) \text{ and } \{\Gamma_0^{(p)},\, \Gamma_1^{(p)}\} \text{ (of } H^0(P_1,\, \mathcal{O}(2\Xi_1)))$$

obtained via i^* correspond one to the other via χ_1.
Now, let Q_0, Q_1 be the degree p homogeneous polynomials such that

$$\widetilde{V}^*(\Gamma_i) = Q_i(\underline{\Gamma}^{(p)}) \tag{4.6}$$

These polynomials depend only on the elliptic curve P, hence on τ, and can be explicitly computed (see Section 5). The diagram 4.4, together with the Proposition 3.3.(2), gives the following commutative diagram

$$\begin{array}{ccccc} \mathbb{P}H^0(P,\, \mathcal{O}(2\Xi)) & \xrightarrow{\quad\sim\quad} & \mathbb{P}W^{\widetilde{\tau}} & \subset & |2\Theta| \\ {\scriptstyle \widetilde{V}}\uparrow & & & & \uparrow{\scriptstyle V} \\ \mathbb{P}H^0(P_1,\, \mathcal{O}(2\Xi_1)) & \xrightarrow{\quad\sim\quad} & \mathbb{P}W_1^{\widetilde{\tau}_1} & \subset & |2\Theta_1| \end{array} \tag{4.7}$$

Therefore, there corresponds to $\widetilde{V}$ a morphism $\mathbb{P}W_1^{\widetilde{\tau}_1} \to \mathbb{P}W^{\widetilde{\tau}}$ still denoted by $\widetilde{V}$ and letting $\{\lambda_\bullet\}$ (resp. $\{\lambda_\bullet^{(p)}\}$) be the basis of $(W^{\widetilde{\tau}})^*$ (resp. $(W_1^{\widetilde{\tau}_1})^*$), dual to the basis $\{\Lambda_\bullet\}$ (resp. $\{\Lambda_\bullet^{(p)}\}$, obtained thanks to the Lemma 2.4, one has $\widetilde{V}(\lambda_i) = Q_i(\underline{\lambda}^{(p)})$ for $i = 0,\, 1$.

4.4 Determining the equations of the Verschiebung

Proposition 4.14 *Let λ_0 be the image of x_{00} via $W^* \twoheadrightarrow (W^{\widetilde{\tau}})^*$. Via the canonical surjection*

$$\mathrm{Sym}^{\,p}\, W_1^* \to \mathrm{Sym}^{\,p}\, (W_1^{\widetilde{\tau}_1})^*$$

the element V_{00} of $\mathrm{Sym}^{\,p}\, W_1^*$ (defined at (4.3)) maps to

$$\widetilde{V}^*(\lambda_0) = Q_0(\lambda_\bullet^{(p)})$$

(where Q_0 is the polynomial defined at (4.6)).

<u>Proof</u> : Let $K_{V\times V} := J_1[p\,]_{\rm red} \times P_1[p\,]_{\rm red}$ denote the kernel of $V\times V$ and let $f : J_1 \times P_1 \to \widetilde{J}$ be the diagonal map $\sigma \circ (V \times V) = V \circ \sigma_1$ in the diagram of the Proposition 4.12. Its kernel K_f is isomorphic to $K_{\sigma_1} \times K_{V\times V}$ and we let

$$\vartheta : K_f \xrightarrow{\sim} \widetilde{K}_f \subseteq \mathcal{G}(\mathcal{O}(2p\,\Theta_1) \boxtimes \mathcal{O}(2p\,\Xi_1))$$

be the unique level structure such that

$$f_*(\mathcal{O}(2p\,\Theta_1) \boxtimes \mathcal{O}(2p\,\Xi_1))^{\widetilde{K}_f} \cong \mathcal{O}(\widetilde{\Theta}_z)$$

Let s_0 (resp. s_1) be the unique (up to scalar) non zero section of $\mathcal{O}(\widetilde{\Theta}_z)$ (resp. $\mathcal{O}(\widetilde{\Theta}_{z_1})$. Upon normalizing suitably s_1, one has

$$V^*(s_0)^p = [p\,]^*(s_1)$$

or, equivalently, $s_1 = i^*(s_0)$ (see Remark 4.7 for an analogous fact). The morphism $\delta_{\tau,\,z}$ (resp. $\delta_{\tau_1,\,z_1}$) is defined by $\sigma^*(s_0)$ (resp. $\sigma^*(s_1)$) (see Proposition 3.3).
The pull-back $f^*(s_0)$ has to be the unique non zero section (up to scalar) of

$$H^0(J_1,\, \mathcal{O}(2p\,\Theta_1)) \otimes H^0(P_1,\, \mathcal{O}(2p\,\Xi_1))$$

invariant under the action of $\widetilde{K}_f$ and it induces an arrow

$$H^0(J_1,\, \mathcal{O}(2p\,\Theta_1))^* \to H^0(P_1,\, \mathcal{O}(2p\,\Xi_1))$$

Now, insert this map in the following diagram

$$\begin{tikzcd}
H^0(J,\, \mathcal{O}(2\Theta))^* \arrow[r] \arrow[d] & \mathrm{Sym}^{\,p}\,(H^0(J_1,\, \mathcal{O}(2\Theta_1)))^* \arrow[r] \arrow[d] & H^0(J_1,\, \mathcal{O}(2p\,\Theta_1))^* \arrow[dd] \\
\big(H^0(J,\, \mathcal{O}(2\Theta))^{\widetilde{\tau}}\big)^* \arrow[d, "\wr"] & \mathrm{Sym}^{\,p}\,\big(H^0(J_1,\, \mathcal{O}(2\Theta_1))^{\widetilde{\tau}_1}\big)^* \arrow[d, "\wr"] & \\
H^0(P,\, \mathcal{O}(2\Xi)) \arrow[r] & \mathrm{Sym}^p H^0(P_1,\, \mathcal{O}(2\Xi_1)) \arrow[r] & H^0(P_1,\, \mathcal{O}(2p\,\Xi_1))
\end{tikzcd}$$

where

- the top line is the factorization (4.2) of $V^* : H^0(J,\, \mathcal{O}(2\Theta))^* \to H^0(J_1,\, \mathcal{O}(2p\,\Theta_1))^*$
- the bottom line is the factorization (4.5) of $V^* : H^0(J,\, \mathcal{O}(2\Xi)) \to H^0(J_1,\, \mathcal{O}(2p\,\Xi_1)))$
- the isomorphisms are deduced from χ and χ_1 (see Proposition 3.3)

- the surjections are deduced from the canonical restriction maps.

It is enough to show that the left-hand square is commutative. Note that the composite map in the left-hand column is the one induced by $\sigma^*(s_0)$. Therefore, the big square is commutative for $f^*(s_0) = (V \times V)^*(\sigma^*(s_0))$. Note as well that the composite map in the middle column is the one induced by $\sigma_1^*(s_1)$ by taking symmetric p-powers. Because

$$\sigma_1^*((s_1)^p) = \sigma_1^*(V^*(s_0)) = (V \times V)^*(\sigma^*(s_0)) = f^*(s_0)$$

the right-hand square commutes as well. The following lemma ends the proof. $\square$

Lemma 4.15 *The evaluation map* $\mathrm{Sym}^p H^0(P_1, \mathcal{O}(2\Xi_1)) \to H^0(P_1, \mathcal{O}(2p\,\Xi_1))$ *is injective.*

<u>Proof</u> : On the one hand, as the evaluation map $H^0(P_1, \mathcal{O}(2\Xi_1)) \otimes \mathcal{O}_{P_1} \to \mathcal{O}(2\Xi_1)$ corresponds to the canonical map $\varphi_{2\Xi_1}$, the map

$$\mathrm{Sym}^p H^0(P_1, \mathcal{O}(2\Xi_1)) \otimes \mathcal{O}_{P_1} \to \mathcal{O}(2p\,\Xi_1)$$

corresponds to the composite

$$P_1 \xrightarrow{\varphi_{2\Xi_1}} \mathbb{P}^1 \xrightarrow{\rho_p} \mathbb{P}^p$$

where ρ_p is the p-uple embedding. On the other hand, because $\mathcal{O}(2p\,\Xi_1)$ is very ample for any $p > 2$, the map

$$H^0(P_1, \mathcal{O}(2p\,\Xi_1)) \otimes \mathcal{O}_{P_1} \to \mathcal{O}(2p\,\Xi_1)$$

corresponds to the embedding $P_1 \hookrightarrow \mathbb{P}H^0(P_1, \mathcal{O}(2p\,\Xi_1))^*$. Now, the evaluation map

$$\mathrm{Sym}^p H^0(P_1, \mathcal{O}(2\Xi_1)) \to H^0(P_1, \mathcal{O}(2p\,\Xi_1))$$

induces a map

$$\mathbb{P}H^0(P_1, \mathcal{O}(2p\,\Xi_1))^* \to \mathbb{P}^p$$

and the former is injective if and only if the latter is non-degenerate. But all these maps fit into the commutative diagram

$$\begin{array}{ccc} P_1 & \longrightarrow & \mathbb{P}H^0(P_1, \mathcal{O}(2p\,\Xi_1))^* \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \longrightarrow & \mathbb{P}^p \end{array}$$

and the required injectivity comes from the fact that the image of the p-uple embedding is non-degenerate. $\square$

This enables us to state the main result of this paper :

Theorem 4.16 *Let X be a smooth, proper, curve of genus 2, sufficiently general, over an algebraically closed field of characteristic $p = 3$, 5 or 7. The generalized Verschiebung $V : M_{X_1} \dashrightarrow M_X$ is completely determined by its restriction to the projective lines that are invariant under the action of a non zero element of $J[2]$.*

Corollary 4.17 *For $p = 3$, 5, 7, one can compute the homogeneous degree p polynomials V_i ($i = 00$, 01, 10 and 11) such that*

$$\begin{array}{rccc} \widetilde{V}: & |2\Theta_1| & \dashrightarrow & |2\Theta| \\ & y_i & \mapsto & V_i(\underline{y}) \end{array}$$

4.4.1 Proof of the Theorem 4.16 : A digression in combinatorial algebra.

Using Remark 3.5 and Lemma 4.5, we find that the direct sum $\mathrm{Sym}^p\,(W_1^{\widetilde{\tau}_1})^* \oplus \mathrm{Sym}^p\,(W_1^{-\widetilde{\tau}_1})^*$ (which depends only on the choice of τ) is endowed with an action of $\mathcal{H}$ (of weight p^2) and the canonical map $\mathrm{Sym}^p\, W_1^* \to \mathrm{Sym}^p\,(W_1^{\widetilde{\tau}_1})^* \oplus \mathrm{Sym}^p\,(W_1^{-\widetilde{\tau}_1})^*$ is equivariant for the action of $\mathcal{H}$ on both spaces. Taking all order 2 elements of $J[2]$ together, we find a morphism of $\mathcal{H}$-representations

$$R_P : \mathrm{Sym}^{\,p}\, W_1^* \to BG := \bigoplus_{\tau \in J[2]\setminus\{0\}} \mathrm{Sym}^p\,(W_1^{\widetilde{\tau}_1})^* \oplus \mathrm{Sym}^p\,(W_1^{-\widetilde{\tau}_1})^*$$

Because of the Proposition 4.14, one knows the image of the irreducible sub-representation $W^* \subset \mathrm{Sym}^{\,p}\, W_1^*$ that determines V (see Proposition 4.9) in BG and one can ask whether or not these data allow us to determine completely this sub-representation.

A necessary condition is that the above map R_P is injective. It cannot be the case for large p since

$$\dim\ (\mathrm{Sym}^{\,p}\, W_1^*) = \left(\begin{array}{c} p+3 \\ 3 \end{array}\right) \sim \frac{p^3}{6} \text{ for large } p$$

and

$$\dim BG = 30(p+1)$$

More precisely, it cannot be injective for any prime $p > 7$.

As the map R_P is $\mathcal{H}$-equivariant, it is injective if and only if its restriction to the subspace $(\mathrm{Sym}^p\, W_1^*)^{\hat{H}}$ is. One has the

Lemma 4.18 *The reunion of the two families*

$$A(p) = \left\{ A_{\underline{f}} = y_{00} \left[\prod_{i \in H} y_i^{f_i} \right]^2,\ \textit{with } |\underline{f}| = \frac{p-1}{2} \right\}$$

and

$$B(p) = \left\{ B_{\underline{f}} = y_{01} y_{10} y_{11} \left[\prod_{i \in H} y_i^{f_i} \right]^2,\ \textit{with } |\underline{f}| = \frac{p-3}{2} \right\}$$

(where $\underline{f}$ is, in both cases, a multi-index $(f_{00},\, f_{01},\, f_{10},\, f_{11})$ with $|\underline{f}| = \sum f_i$) is a basis for the space $(\mathrm{Sym}^p\, W_1^*)^{\hat{H}}$.
In particular, there are scalars $a_{\underline{f}}$ and $b_{\underline{f}}$ such that

$$V_{00} = \widetilde{V}^*(x_{00}) = \sum_{|\underline{f}|=\frac{p-1}{2}} a_{\underline{f}} A_{\underline{f}} + \sum_{|\underline{f}|=\frac{p-3}{2}} b_{\underline{f}} B_{\underline{f}} \tag{4.8}$$

<u>Proof</u> : The subspace $(\mathrm{Sym}^p\, W_1^*)^{\hat{H}}$ is generated by the free family of monomials $\prod_{i\in H} y_i^{e_i}$ with $e_{01} + e_{10} \equiv e_{01} + e_{11} \equiv e_{10} + e_{11} \equiv 0$ (mod. 2) and we can divide it into the two families of the lemma. $\square$

Let $\tau = (x,\, x^*)$ be an non zero element of $J[2]$ and $\widetilde{\tau} = (\mu,\, x,\, x^*)$ with $\mu = 1$ if $x^*(x) = 1$ and $\mu = i$ if $x^*(x) = -1$. We will use the basis $\{\Lambda_0(\tau),\, \Lambda_1(\tau)\}$ (resp. $\{\bar{\Lambda}_0(\tau),\, \bar{\Lambda}_1(\tau)\}$) of $W^{\widetilde{\tau}}$ (resp. $W^{-\widetilde{\tau}}$) constructed in the Proposition 2.12. Taking account of the fact that we are working over the p-twist J_1 of J, one can give the images of the $y_\bullet$ via the restriction maps

$$W_1^* \to (W_1^{\widetilde{\tau}_1})^* \text{ (resp. } W_1^* \to (W_1^{-\widetilde{\tau}_1})^*)$$

for every τ, in terms of the $\lambda_\bullet^{(p)}(\tau_1)$ (resp. the $\bar{\lambda}_\bullet^{(p)}(\tau_1)$) and one can then deduce the images of the $A_{\underline{f}}$ and the $B_{\underline{f}}$ in

$$\mathrm{Sym}^p\,(W_1^{\widetilde{\tau}_1})^* \text{ (resp. } \mathrm{Sym}^p\,(W_1^{-\widetilde{\tau}_1})^*)$$

For sake of readability, we shall write $\lambda_\bullet$ (resp. $\bar{\lambda}_\bullet$) instead of $\lambda_\bullet^{(p)}(\tau_1)$ (resp. $\bar{\lambda}_\bullet^{(p)}(\tau_1)$).

If $x = 00$, the $A_{\underline{f}}$ map to 0 in $\mathrm{Sym}^p\,(W_1^{-\widetilde{\tau}_1})^*$ for y_{00} maps to 0 in $(W_1^{-\widetilde{\tau}_1})^*$, and the $B_{\underline{f}}$ map to 0 in both $\mathrm{Sym}^p\,(W_1^{\widetilde{\tau}_1})^*$ and $\mathrm{Sym}^p\,(W_1^{-\widetilde{\tau}_1})^*$ for at least one of the three y_{01}, y_{10} and y_{11} maps to 0 in the corresponding space $(W_1^{\widetilde{\tau}_1})^*$ or $(W_1^{-\widetilde{\tau}_1})^*$. In the following chart, we have gathered the images of the $A_{\underline{f}}$ in $\mathrm{Sym}^p\,(W_1^{\widetilde{\tau}_1})^*$.

τ	$A_{\underline{f}}$
0001	$\lambda_0^{2f_{00}+1}\lambda_1^{f_{10}}$ if $f_{01} = f_{11} = 0$, 0 else.
0010	$\lambda_0^{2f_{00}+1}\lambda_1^{f_{01}}$ if $f_{10} = f_{11} = 0$, 0 else.
0011	$\lambda_0^{2f_{00}+1}\lambda_1^{f_{11}}$ if $f_{01} = f_{10} = 0$, 0 else.

(4.9)

If $x \neq 00$, upon identifying $\lambda_\bullet$ and $\bar{\lambda}_\bullet$, the $A_{\underline{f}}$ (resp. the $B_{\underline{f}}$) have the same images in both $\mathrm{Sym}^p\,(W_1^{\widetilde{\tau}_1})^*$ and $\mathrm{Sym}^p\,(W_1^{-\widetilde{\tau}_1})^*$. Straightforward calculations give the results gathered in the following chart (4.10) on the next page.

Now, write an element of $(\mathrm{Sym}^p W_1^*)^{\widetilde{\hat{H}}}$ under the form

$$\sum_{|\underline{f}|=\frac{p-1}{2}} a_{\underline{f}} A_{\underline{f}} + \sum_{|\underline{f}|=\frac{p-3}{2}} b_{\underline{f}} B_{\underline{f}}$$

and suppose that it is in the kernel of R_P.
Because of the computations summed up in (4.9), one has $a_{\underline{f}} = 0$ as soon as two of the f_{01}, f_{10}, f_{11} are zero.

τ	$A_{\underline{f}}$	$B_{\underline{f}}$
0100	$\lambda_0^{1+2(f_{00}+f_{01})}\lambda_1^{2(f_{10}+f_{11})}$	$\lambda_0^{1+2(f_{00}+f_{01})}\lambda_1^{2(f_{10}+f_{11}+1)}$
0101	$(-1)^{f_{01}+f_{11}}\lambda_0^{1+2(f_{00}+f_{01})}\lambda_1^{2(f_{10}+f_{11})}$	$(-1)^{1+f_{01}+f_{11}}\lambda_0^{1+2(f_{00}+f_{01})}\lambda_1^{2(f_{10}+f_{11}+1)}$
0110	$\lambda_0^{1+2(f_{00}+f_{01})}\lambda_1^{2(f_{10}+f_{11})}$	$-\lambda_0^{1+2(f_{00}+f_{01})}\lambda_1^{2(f_{10}+f_{11}+1)}$
0111	$(-1)^{f_{01}+f_{11}}\lambda_0^{1+2(f_{00}+f_{01})}\lambda_1^{2(f_{10}+f_{11})}$	$(-1)^{f_{01}+f_{11}}\lambda_0^{1+2(f_{00}+f_{01})}\lambda_1^{2(f_{10}+f_{11}+1)}$
1000	$\lambda_0^{1+2(f_{00}+f_{10})}\lambda_1^{2(f_{01}+f_{11})}$	$\lambda_0^{1+2(f_{00}+f_{10})}\lambda_1^{2(f_{01}+f_{11}+1)}$
1001	$\lambda_0^{1+2(f_{00}+f_{10})}\lambda_1^{2(f_{01}+f_{11})}$	$-\lambda_0^{1+2(f_{00}+f_{10})}\lambda_1^{2(f_{01}+f_{11}+1)}$
1010	$(-1)^{f_{10}+f_{11}}\lambda_0^{1+2(f_{00}+f_{10})}\lambda_1^{2(f_{01}+f_{11})}$	$(-1)^{1+f_{10}+f_{11}}\lambda_0^{1+2(f_{00}+f_{10})}\lambda_1^{2(f_{01}+f_{11}+1)}$
1011	$(-1)^{f_{10}+f_{11}}\lambda_0^{1+2(f_{00}+f_{10})}\lambda_1^{2(f_{01}+f_{11})}$	$(-1)^{f_{10}+f_{11}}\lambda_0^{1+2(f_{00}+f_{10})}\lambda_1^{2(f_{01}+f_{11}+1)}$
1100	$\lambda_0^{1+2(f_{00}+f_{11})}\lambda_1^{2(f_{01}+f_{10})}$	$\lambda_0^{1+2(f_{00}+f_{11})}\lambda_1^{2(f_{01}+f_{10}+1)}$
1101	$(-1)^{f_{10}+f_{11}}\lambda_0^{1+2(f_{00}+f_{11})}\lambda_1^{2(f_{01}+f_{10})}$	$(-1)^{f_{10}+f_{11}}\lambda_0^{1+2(f_{00}+f_{11})}\lambda_1^{2(f_{01}+f_{10}+1)}$
1110	$(-1)^{f_{10}+f_{11}}\lambda_0^{1+2(f_{00}+f_{11})}\lambda_1^{2(f_{01}+f_{10})}$	$(-1)^{1+f_{10}+f_{11}}\lambda_0^{1+2(f_{00}+f_{11})}\lambda_1^{2(f_{01}+f_{10}+1)}$
1111	$\lambda_0^{1+2(f_{00}+f_{11})}\lambda_1^{2(f_{01}+f_{10})}$	$-\lambda_0^{1+2(f_{00}+f_{11})}\lambda_1^{2(f_{01}+f_{10}+1)}$

(4.10)

Because of the computations summed up in the chart (4.10) above, one finds that, for any $0 \leq k \leq \dfrac{p-1}{2}$,

$$\sum_{f_{00}+f_{01}=k \text{ and } f_{01}+f_{11} \text{ even}} a_{\underline{f}} = 0; \qquad \sum_{f_{00}+f_{01}=k \text{ and } f_{01}+f_{11} \text{ odd}} a_{\underline{f}} = 0$$

$$\sum_{f_{00}+f_{10}=k \text{ and } f_{10}+f_{11} \text{ even}} a_{\underline{f}} = 0; \qquad \sum_{f_{00}+f_{10}=k \text{ and } f_{10}+f_{11} \text{ odd}} a_{\underline{f}} = 0$$

$$\sum_{f_{00}+f_{11}=k \text{ and } f_{10}+f_{11} \text{ even}} a_{\underline{f}} = 0; \qquad \sum_{f_{00}+f_{11}=k \text{ and } f_{10}+f_{11} \text{ odd}} a_{\underline{f}} = 0$$

and we have analogous results for the $b_{\underline{f}}$ (with $0 \leq k \leq \dfrac{p-3}{2}$).

Thus, we have reduced our problem to the following combinatorial situation : Being given a set of scalars $a_{\underline{f}}$, where the $\underline{f}$ are four-letters multi-index $\underline{f}$ with $|\underline{f}| = r$, satisfying the $6(r+1)$ relations stated above, are these scalars meant to be zero ? If they are for $r = \dfrac{p-3}{2}$ but are not for $r = \dfrac{p-1}{2}$, does the indetermination only concern the $a_{\underline{f}}$ for which two of the f_{01}, f_{10}, f_{11} are zero (the indetermination would therefore vanish because of the additional data deduced from (4.9)) ?

In fact, the first question has a positive answer for $r = 0, 1, 2, 3$ (we leave the proof of this assertion to the reader) and the theorem follows.

5 Computing the equations of V_2 for small p

5.1 Multiplication by p on an elliptic curve

Let k be an algebraically closed field of characteristic $p \geq 3$ and let (E, q_0) be an elliptic curve.

Let us recall briefly how the group law of E can be recovered from the geometry of the curve (see, e.g., [Sil] for further references on that question). The sheaf $\mathcal{O}(q_0)$ gives a principal polarization thus $\mathcal{O}(3q_0)$ is very ample and determines an embedding $E \hookrightarrow \mathbb{P}^2$. Three points q_1, q_2 and q_3 on the curve lie on the same projective line in $\mathbb{P}^2$ if and only if $\mathcal{O}(q_1 + q_2 + q_3) \cong \mathcal{O}(3q_0)$. On the other hand, any projective line in $\mathbb{P}^2$ intersects with E in three points (counted with multiplicities). It is easily seen that E is isomorphic to its Jacobian variety by means of $q \mapsto \mathcal{O}(q - q_0)$ and the group law on the latter gives the group law on the former. Namely, one sets $q_1 + q_2 = -q_3$, where q_3 is the unique point in E such that $\mathcal{O}(q_1 + q_2 + q_3) \cong \mathcal{O}(3q_0)$.

The projection $\mathbb{P}^2 \to \mathbb{P}^1$ from the point q_0 induces the canonical map $E \to \mathbb{P}^1$, which is a ramified double cover, and the choice of a suitable rational coordinate x on the projective line allows us to give a birational model

$$y^2 = x(x-1)(x-\mu)$$

of E, with μ different from 0 and 1.

One can determine explicitly the group law over E in intersecting this plane curve with lines. Namely, one has the following duplication and addition formulae found in [Sil] (III, §2). For convenience, we let P be the polynomial $x(x-1)(x-\mu)$ and we let P' be its derivative.

Duplication formula : Let q_1 be a k-point on the curve with coordinates (x_1, y_1) and let q_2, with coordinates (x_2, y_2), be $[2]\,(q_1)$. The opposite of the latter is the unique point of E (different from q_1) lying on the tangent line to E at the point q_1. This tangent line

has equation $y = \alpha x + \beta$ with

$$\alpha = \frac{P'(x_1)}{2y_1} = \frac{3x_1^2 - 2(\mu+1)x_1 + \mu}{2y_1} \quad \text{and} \quad \beta = \frac{2P(x_1) - x_1 P'(X_1)}{2y_1} = \frac{x_1(\mu - x_1^2)}{2y_1}$$

Thus, one has

$$\left\{ \begin{array}{rcl} x_2 & = & \alpha^2 + (\mu+1) - 2x_1 = \dfrac{(P'(x_1))^2 - 4(2x_1 - (\mu+1))P(x_1)}{4P(x_1)} \\ & = & \dfrac{1}{4}\dfrac{(x_1^2 - \mu)^2}{x_1(x_1-1)(x_1-\mu)} \\ \\ y_2 & = & -(\alpha x_2 + \beta) \end{array} \right. \tag{5.1}$$

<u>Addition formula</u> : Let q_1 and q_2 be two k-points on the curve with coordinates $(x_i,\, y_i)$ $(i = 1,\, 2)$ and let q_3, with coordinates $(x_3,\, y_3)$, be the sum $q_1 + q_2$. Suppose that $q_1 \neq \pm q_2$, i.e., that $x_1 \neq x_2$. The unique line passing through q_1 and q_2 has equation

$$y = \alpha x + \beta \ \text{ with } \alpha = \frac{y_2 - y_1}{x_2 - x_1} \ \text{ and } \ \beta = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}$$

Thus, the third intersection point of that line and the plane curve being $-(q_1 + q_2)$, q_3 has coordinates

$$\left\{ \begin{array}{rcl} x_3 & = & \alpha^2 + (\mu+1) - (x_1 + x_2) \\ & = & \left(\dfrac{y_2 - y_1}{x_2 - x_1}\right)^2 + (\mu+1) - (x_1 + x_2) \\ y_3 & = & -(\alpha x_3 + \beta) \end{array} \right. \tag{5.2}$$

Combining these two formulae, we are theoretically able to give the coordinates $(x_n,\, y_n)$ of the point $q_n = [n](q_1)$, in terms of x_1 and y_1, at least for a general point q_1. Note that two opposite points of E collapse in $\mathbb{P}^1$, i.e., that the canonical $E \to \mathbb{P}^1$ is a quotient under the action of $\{\pm\}$ and that the branched points $0,\, 1,\, \infty$ and μ of that map are precisely the order 2 points of E. As the action of $\{\pm\}$ commutes with multiplication by p, the latter induces a map $\mathbb{P}^1 \to \mathbb{P}^1$. It has total degree p^2 and separable degree p. Hence, if we let z be $1/x$, so that $\{x,\, z\}$ is a basis for $H^0(E,\, \mathcal{O}(2q_0)) \cong H^0(\mathbb{P}^1,\, \mathcal{O}(1))$, one can find two homogeneous polynomials of degree p (say D and N) such that the map induced by $[p\,]$ on $\mathbb{P}^1$ is given by

$$\begin{array}{rcl} \mathbb{P}^1 & \to & \mathbb{P}^1 \\ (x\,:\,z) & \mapsto & (N(x^p,\, z^p)\,:\, D(x^p,\, z^p)) \end{array}$$

If E_1 is the p-twist of E, the map $\mathbb{P}^1 \to \mathbb{P}^1$, induced by the separable part $V : E_1 \to E$ of multiplication by p, is therefore given by

$$(x^{(p)}\,:\,z^{(p)}) \ \mapsto \ (N(x^{(p)},\, z^{(p)})\,:\, D(x^{(p)},\, z^{(p)})) \tag{5.3}$$

where $x^{(p)}$ and $z^{(p)}$ are the p-twisted coordinates of $\mathbb{P}^1$ corresponding to x and z respectively.

Division Polynomials : In the case $p \geq 5$, [Sil] gives, as an exercise (Ex. 3.7.), the following formulae, that are more convenient to implement when trying to determine the polynomials N and D using a computer. Take an elliptic curve

$$y'^2 = x'^3 + Ax' + B \tag{5.4}$$

Define ψ_m in $\mathbb{Z}[A,\, B,\, x',\, y']$ inductively by :

$$\begin{cases} \psi_1 = 1, \\ \psi_2 = 2y', \\ \psi_3 = 3x'^4 + 6Ax'^2 + 12Bx' - A^2, \\ \psi_4 = 4y'(x'^6 + 5Ax'^4 + 20Bx'^3 - 5A^2x'^2 - 4ABx' - 8B^2 - A^3), \\ \psi_{2m+1} = \psi_{m+2}\psi_m^3 - \psi_{m-1}\psi_{m+1}^3 \quad (m \geq 2) \\ 2y'\psi_{2m} = \psi_m(\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2) \quad (m \geq 2). \end{cases} \tag{5.5}$$

Define furthermore

$$\phi_m = x'\psi_m^2 - \psi_{m+1}\psi_{m-1} \tag{5.6}$$

Then, using the equation (5.4), one checks that, for m odd, the polynomials ϕ_m and ψ_m (of degree m^2 and m^2-1 respectively) lie in fact in $\mathbb{Z}[A,\, B,\, x']$, and that the map $\mathbb{P}^1 \to \mathbb{P}^1$ induced by $[p]$ is given by

$$\begin{array}{rcl} \mathbb{P}^1 & \to & \mathbb{P}^1 \\ (x' : z') & \mapsto & \Big(z'^{p^2}\, \phi_p(x'/z') : z'^{p^2}\, \psi_p(x'/z')\Big) \end{array}$$

where z' is the rational coordinate of $\mathbb{P}^1$ defined by $z' = 1/x'$. We let N' (resp. D') be the homogeneous degree p polynomial such that $N'(x'^p,\, z'^p) = z'^{p^2}\, \phi_p(x'/z')$ (resp. $D'(x'^p,\, z'^p) = z'^{p^2}\, \phi_p(x'/z')$). As above, the map $\mathbb{P}^1 \to \mathbb{P}^1$ induced by $V : E_1 \to E$ is therefore given by

$$\begin{array}{rcl} \mathbb{P}^1 & \to & \mathbb{P}^1 \\ (x'^{(p)} : z'^{(p)}) & \mapsto & (N'(x'^{(p)},\, z'^{(p)}) : D'(x'^{(p)},\, z'^{(p)})) \end{array}$$

where $x'^{(p)}$ and $z'^{(p)}$ are the p-twisted coordinates of $\mathbb{P}^1$ corresponding to x' and z' respectively.

One finds a coordinates change transforming the elliptic curve $y^2 = x(x-1)(x-\mu)$ in the (isomorphic) elliptic curve $y'^2 = x'^3 + Ax + B$, e.g.,

$$x' = x - \frac{\mu+1}{3}z, \quad z' = z,$$

and the scalars A and B are therefore

$$A = \frac{\mu^2 - \mu + 1}{3}, \quad B = \frac{(\mu+1)^3 - 3(\mu^3+1)}{27}.$$

Finally, one obtains the polynomials N and D defined in (5.3) as follows :

$$\begin{cases} N(x^{(p)},\, z^{(p)}) = N'\left(x^{(p)} - \dfrac{\mu^p+1}{3} z^{(p)},\, z^{(p)}\right) + \dfrac{\mu+1}{3}\, D'\left(x^{(p)} - \dfrac{\mu^p+1}{3} z^{(p)},\, z^{(p)}\right) \\ D(x^{(p)},\, z^{(p)}) = D'\left(x^{(p)} - \dfrac{\mu^p+1}{3^p} z^{(p)},\, z^{(p)}\right) \end{cases} \tag{5.7}$$

Using the Proposition 2.12, the Corollary 2.13 and the Lemma 3.4, we find that there is a unique linear automorphism of $\mathbb{P}^1$, i.e., an element of $PGL(k,\, 2)$, which maps $(a : b)$ to 0, $(a : -b)$ to 1 and and $(b : a)$ to ∞. It maps $(b : -a)$ to $(\mu : 1)$ with

$$\mu = \left(\frac{b^2+a^2}{2ab}\right)^2 = \frac{2-\omega(\tau)}{4}$$

If x and z are the corresponding rational coordinates of $\mathbb{P}^1$, one has

$$x = -\frac{1}{a}\lambda_0 + \frac{1}{b}\lambda_1, \quad z = \frac{-2a}{a^2+b^2}\lambda_0 + \frac{2b}{a^2+b^2}\lambda_1,$$

and the elliptic curve P has equation $y^2 = x(x-1)(x-\mu)$.

If we let $\lambda_0^{(p)}$ and $\lambda_1^{(p)}$ be the p-twisted coordinates of $\mathbb{P}^1$ corresponding to λ_0 and λ_1, and if we denote by Q_0 and Q_1 the homogeneous polynomials of degree p such that the map $\mathbb{P}^1 \to \mathbb{P}^1$ induced by $V : E_1 \to E$ is given by

$$(\lambda_0^{(p)} : \lambda_1^{(p)}) \mapsto (Q_0(\lambda_0^{(p)},\, \lambda_1^{(p)}) : Q_1(\lambda_0^{(p)},\, \lambda_1^{(p)}))$$

then one has, writing $\lambda_\bullet$ instead of $\lambda_\bullet^{(p)}$ for sake of readability

$$\begin{aligned} Q_0(\lambda_0,\, \lambda_1) \;=\; & \frac{2ba}{a^2+b^2} N\left(-\frac{1}{a^p}\lambda_0 + \frac{1}{b^p}\lambda_1,\, \frac{-2a^p}{(a^2+b^2)^p}\lambda_0 + \frac{2b^p}{(a^2+b^2)^p}\lambda_1\right) \\ & -\frac{1}{b} D\left(-\frac{1}{a^p}\lambda_0 + \frac{1}{b^p}\lambda_1,\, \frac{-2a^p}{(a^2+b^2)^p}\lambda_0 + \frac{2b^p}{(a^2+b^2)^p}\lambda_1\right) \end{aligned} \tag{5.8}$$

5.1. *Remark.* Note that we are only interested in Q_0, onto which V_{00} restricts. Furthermore, Q_1 can be obtained from Q_0 under the action of a suitable element of $P[2]$.

5.2. *Remark.* The final result Q_0 should not depend on a and b but only on the constant $\omega(\tau) = -\dfrac{a^4+b^4}{a^2b^2}$.

Lemma 5.3 *With the notations given above, one has*
• $p = 3$.

$$Q_0(\lambda_0,\, \lambda_1) = \lambda_0^3 - \omega\lambda_0\lambda_1^2 \tag{5.9}$$

• $p = 5$.

$$Q_0 = \lambda_0^5 + \omega(\omega^2+2)\lambda_0^3\lambda_1^2 + (\omega^2+2)\lambda_0\lambda_1^4 \tag{5.10}$$

• $p = 7$.

$$Q_0 = \lambda_0^7 - 2\omega(\omega^4-1)\lambda_0^5\lambda_1^2 + \omega^2(\omega^2-1)(\omega^2-2)\lambda_0^3\lambda_1^4 - \omega(\omega^2-1)\lambda_0\lambda_1^6 \tag{5.11}$$

<u>Proof</u> : In the case $p = 3$, one computes x_3 directly using formulae (5.1) and (5.2). One finds

$$x_3 = \frac{x_1^9 + 2\mu(\mu+1)x_1^6 + \mu^2(\mu+1)^2x_1^3}{((\mu+1)x_1^3+\mu^2)^2}$$

Thus, one has

$$N(x,\, z) = x(x+\mu(\mu+1)z)^2 \text{ and } D(x,\, z) = z((\mu+1)x+\mu^2 z)^2$$

This result is consistent with the fact that, in characteristic 3, the unique supersingular elliptic curve has parameter μ equal to -1 (see [H], Chapter IV, Example 4.23.1). A straightforward application of the formula (5.8) gives the formula (5.9).
In the cases $p = 5$ and 7, the computations cannot be worked out by hand. We use Maple 9 to compute the division polynomials defined in (5.5) and (5.6), then we apply the formula (5.7) to find

$$N(x,z) = x\left[x^2 - \mu(\mu+1)(\mu^2-\mu+1)xz + \mu^4(\mu^2-\mu+1)z^2\right]^2$$

and

$$D(x,z) = z\left[(\mu^2-\mu+1)\left[x^2-\mu^2(\mu+1)xz\right] + \mu^6 z^2\right]^2$$

when $p = 5$ and

$$\begin{aligned} N(x,z) \;=\; & x\left[x^3 + 2\mu(\mu+1)(\mu-2)(\mu-4)(\mu^2+3\mu+1)x^2z \right.\\ & \left. +\mu^4(\mu+1)^2\mu-2)(\mu-4)(\mu^2+1)xz^2 + \mu^9(\mu+1)(\mu-2)(\mu-4)z^3\right]^2 \end{aligned}$$

and

$$D(x,z) = z\left[(\mu+1)(\mu-2)(\mu-4)\left[x^3+\mu^2(\mu+1)(\mu^2+1)x^2z+\mu^6(\mu^2+3\mu+1)xz^2\right]+\mu^{12}z^3\right]^2$$

when $p = 7$. These results are consistent with the fact that, in characteristic 5 (resp. 7), the only supersingular elliptic curves have parameter μ equal to j or $-j$ with $j^3 = 1$ (resp. -1, 2 or 4) (see [H], Chapter IV, Example 4.23.2 (resp. 4.23.3)). Applying the formula (5.8), we obtain the formulae (5.10) and (5.11). □

5.2 Equations of V for $p = 3$

Let V_{00} be the $\widetilde{\hat{H}}$-invariant element of the sub-representation determining V. It can be written under the form (4.8) (see Lemma 4.18) which is, for $p = 3$,

$$a_{00}y_{00}^3 + a_{01}y_{00}y_{01}^2 + a_{10}y_{00}y_{10}^2 + a_{11}y_{00}y_{11}^2 + by_{01}y_{10}y_{11}$$

We do not need to determine the Prym varieties for every non zero τ in $J[2]$. Doing it in the cases $\tau = 0001$, $\tau = 0010$, $\tau = 0011$ and $\tau = 0100$ is enough.
We fix $a_{00} = 1$ and we obtain, using the formula (5.9) and the expression (given in the chart (2.5)) of the needed $\omega(\tau)$ in terms of the coefficients $k_\bullet$ of the Kummer surface Kum_X, the following :

$$V_{00}(\underline{y}) = y_{00}^3 + 2k_{01}y_{00}y_{01}^2 + 2k_{10}y_{00}y_{10}^2 + 2k_{11}y_{00}y_{11}^2 + 2k_{00}y_{01}y_{10}y_{11}$$

Then, one can deduce the V_i ($i = 01, 10, 11$) by permuting suitably the coordinate functions $y_\bullet$ in V_{00} (see Proposition 4.9).

Notice that V_i is the partial (with respect to y_i) of a quartic surface

$$S + 2k_{00}P + k_{10}Q_{01} + k_{01}Q_{10} + k_{11}Q_{11}$$

(with

$$S = y_{00}^4 + y_{01}^4 + y_{10}^4 + y_{11}^4; \qquad P = y_{00}y_{01}y_{10}y_{11};$$
$$Q_{01} = y_{00}^2y_{01}^2 + y_{10}^2y_{11}^2; \qquad Q_{10} = y_{00}^2y_{10}^2 + y_{01}^2y_{11}^2; \qquad Q_{11} = y_{00}^2y_{11}^2 + y_{01}^2y_{10}^2.)$$

isomorphic to Kum_X. Thus, one recovers the second assertion of the :

Theorem 5.4 (Laszlo-Pauly) *Let X be a smooth and projective curve of genus 2 over an algebraically closed filed of characteristic* 3.
(1) There is an embedding $\alpha : \mathrm{Kum}_X \hookrightarrow |2\Theta_1|$ such that the equality of divisors in $|2\Theta_1|$

$$\widetilde{V}^{-1}(\mathrm{Kum}_X) = \mathrm{Kum}_{X_1} + 2\alpha(\mathrm{Kum}_X)$$

holds scheme-theoretically.
(2) The cubic equations of $\widetilde{V}$ are given by the 4 partial derivatives of the quartic equation of the Kummer surface $\alpha(\mathrm{Kum}_X) \subseteq |2\Theta_1|$. In other words, $\widetilde{V}$ is the polar map of the surface $\alpha(\mathrm{Kum}_X)$.

<u>Proof</u> : [LP2], Theorem 6.1.

The inverse image $\widetilde{V}^{-1}(\mathrm{Kum}_X)$ can be computed explicitly in our situation as it is defined by the ideal generated by the pull-back $\widetilde{V}^*(K)$ of the equation (2.1) of Kum_X, more precisely by its image via the k-linear homogeneous ring map of degree p

$$\widetilde{V}^* : \mathrm{Sym}W^* \to \mathrm{Sym}W_1^*$$

In other words, a few more computations enable us to recover the first assertion of the Theorem. Namely, one knows (see the diagram 4.1) that the equation K_1 of Kum_{X_1} divides $\widetilde{V}^*(K)$. Let Q be the exact quotient $\widetilde{V}^*(K)/K_1$. Using Magma, one computes the square root of Q (e.g., as the greatest common divisor of the partial derivative $\dfrac{\partial Q}{\partial y_{11}}$ and Q). This homogeneous polynomial furthermore coincides with K_X.

Note by the way that the base locus $\mathcal{I}$ of the rational map $\widetilde{V} : |2\Theta_1| \dashrightarrow |2\Theta|$ is tautologically contained in the zero locus of $\widetilde{V}^*(K)$. As $\widetilde{V}$ restricts to a morphism on Kum_{X_1}, $\mathcal{I}$ is contained in the zero locus of $Q = \widetilde{V}^*(K)/K_1$ and one checks that it is actually contained in the zero locus of A. In other words, $\mathcal{I}$ is a reduced zero dimensional sub-scheme of $\alpha(\mathrm{Kum}_X)$ which coincides furthermore with its singular locus.

5.5. *Remark.* Notice that this theorem is true for *any* curve X (in particular, with no particular assumptions concerning its p-rank) whereas our calculations only give the result for a sufficiently general ordinary curve.

5.3 Equations of V for $p = 5$

Proposition 5.6 *Let X be a general proper and smooth curve of genus 2 over an algebraically closed field of characteristic 5. There are coordinates $\{x_\bullet\}$ and $\{y_\bullet\}$ for $|2\Theta|$ and $|2\Theta_1|$ respectively such that the Kummer surface Kum_X in $|2\Theta|$ has an equation of the form (2.1) and such that, if the polynomials $(V_\bullet)$ define $\widetilde{V} : |2\Theta_1| \dashrightarrow |2\Theta|$, $(y_i) \mapsto (V_i(\underline{y}))$, then*

$$\begin{aligned} V_{00} \;=\; & y_{00}^5 + a_{1100}y_{00}^3y_{01}^2 + a_{1010}y_{00}^3y_{10}^2 + a_{1001}y_{00}^3y_{11}^2 + a_{0200}y_{00}y_{01}^4 + a_{0110}y_{00}y_{01}^2y_{10}^2 \\ & + a_{0101}y_{00}y_{01}^2y_{11}^2 + a_{0020}y_{00}y_{01}^4 + a_{0011}y_{00}y_{10}^2y_{11}^2 + a_{0002}y_{00}y_{11}^4 \\ & + b_{00}y_{00}^2y_{01}y_{10}y_{11} + b_{01}y_{01}^3y_{10}y_{11} + b_{10}y_{01}y_{10}^3y_{11} + b_{11}y_{01}y_{10}y_{11}^3 \end{aligned}$$

with

$$\begin{array}{lll} a_{1100} = k_{01}(k_{01}^2+2), & a_{1010} = k_{10}(k_{10}^2+2), & a_{1001} = k_{11}(k_{11}^2+2), \\ a_{0200} = (k_{01}^2+2), & a_{0020} = (k_{10}^2+2), & a_{0002} = (k_{11}^2+2), \end{array}$$

$$\begin{aligned} a_{0110} &= 3k_{11}(k_{00}^2+k_{11}^2) + k_{01}k_{10}(1-k_{11}^2), \\ a_{0101} &= 3k_{10}(k_{00}^2+k_{10}^2) + k_{01}k_{11}(1-k_{10}^2), \\ a_{0011} &= 3k_{01}(k_{00}^2+k_{01}^2) + k_{10}k_{11}(1-k_{01}^2), \\ b_{00} &= 3k_{00}(k_{00}^2+1) + k_{00}k_{01}k_{10}k_{11}, \end{aligned}$$

$$b_{01} = k_{00}(k_{01}+3k_{10}k_{11}),\ b_{10} = k_{00}(k_{10}+3k_{01}k_{11}),\ b_{11} = k_{00}(k_{11}+3k_{01}k_{10})$$

where the $k_\bullet$ are the coefficients of the equation (2.1) of Kum_X. The V_i ($i = 01, 10, 11$) can be deduced from V_{00} by a suitable permutation of the coordinate functions $y_\bullet$, namely the unique pairwise permutation that exchanges y_{00} and y_i.

Proof : Define

$$\alpha_\tau = \omega(\tau)(\omega(\tau)^2 + 2)$$
$$\beta_\tau = \omega(\tau)^2 + 2$$

so that the formula (5.10) can be written

$$\lambda_0^5 + \alpha_\tau \lambda_0^3 \lambda_1^2 + \beta_\tau \lambda_0 \lambda_1^4$$

Using the equation (4.8) of the Lemma 4.10, normalized by the condition $a_{2000} = 1$, one can look for V_{00} under the form given in the proposition.
Using the data (4.9) for $\tau = 0001$, one finds that the two equations

$$\lambda_0^5 + a_{1010}\lambda_0^3\lambda_1^2 + a_{0020}\lambda_0\lambda_1^4$$

and

$$\lambda_0^5 + \alpha_{0001}\lambda_0^3\lambda_1^2 + \beta_{0001}\lambda_0\lambda_1^4$$

coincide up to a multiplicative scalar. Therefore, one obtains

$$a_{1010} = \alpha_{0001}, \qquad a_{0020} = \beta_{0001},$$

Similarly, using the data (4.9) for $\tau = 0010$ and 0011 respectively, one finds

$$\begin{array}{ll} a_{1010} = \alpha_{0001}, & a_{0020} = \beta_{0001}, \\ a_{1100} = \alpha_{0010}, & a_{0200} = \beta_{0010}, \\ a_{1001} = \alpha_{0011}, & a_{0002} = \beta_{0011}. \end{array}$$

Now, using the data (4.10) for $\tau = 0100$, one finds that the the two equations

$$(1+a_{1100}+a_{0200})\lambda_0^5+(a_{1010}+a_{1001}+a_{0110}+a_{0101}+b_{00}+b_{01})\lambda_0^3\lambda_1^2+(a_{0020}+a_{0011}+a_{0002}+b_{10}+b_{11})\lambda_0\lambda_1^4$$

and

$$\lambda_0^5 + \alpha_{0100}\lambda_0^3\lambda_1^2 + \beta_{0100}\lambda_0\lambda_1^4$$

coincide up to a multiplicative scalar. Therefore, one obtains

$$\begin{cases} a_{1010} + a_{1001} + a_{0110} + a_{0101} + b_{00} + b_{01} = (1 + a_{1100} + a_{0200})\alpha_{0100} \\ a_{0020} + a_{0011} + a_{0002} + b_{10} + b_{11} = (1 + a_{1100} + a_{0200})\beta_{0100} \end{cases}$$

Similarly, using the data (4.10) for $\tau = 0101$, 0110 and 0111 respectively, one finds

$$\begin{cases} a_{1010} - a_{1001} - a_{0110} + a_{0101} - b_{00} + b_{01} = (1 - a_{1100} + a_{0200})\alpha_{0101} \\ a_{0020} - a_{0011} + a_{0002} - b_{10} + b_{11} = (1 - a_{1100} + a_{0200})\beta_{0101} \end{cases}$$
$$\begin{cases} a_{1010} + a_{1001} + a_{0110} + a_{0101} - b_{00} - b_{01} = (1 + a_{1100} + a_{0200})\alpha_{0110} \\ a_{0020} + a_{0011} + a_{0002} - b_{10} - b_{11} = (1 + a_{1100} + a_{0200})\beta_{0110} \end{cases}$$
$$\begin{cases} a_{1010} - a_{1001} - a_{0110} + a_{0101} + b_{00} - b_{01} = (1 - a_{1100} + a_{0200})\alpha_{0111} \\ a_{0020} - a_{0011} + a_{0002} + b_{10} - b_{11} = (1 - a_{1100} + a_{0200})\beta_{0111} \end{cases}$$

Combining these results, one can express the $a_\bullet$ as well as the $b_\bullet$ in terms of the α_τ and the β_τ. Finally, we use the data (2.5) to express the $\alpha_\bullet$ and the $\beta_\bullet$ in terms of the coefficients of Kum_X and Maple 9 gives expressions that, up to a multiple of the equation (2.2) between the $k_\bullet$, are those stated in the Proposition. $\square$

Using Magma to exploit these formulae, one can show the following corollary :

Corollary 5.7 *There is a degree $2p-2=8$ hypersurface S is $|2\Theta_1|$ such that the equality of divisors in $|2\Theta_1|$*

$$\widetilde{V}^{-1}(\mathrm{Kum}_X) = \mathrm{Kum}_{X_1} + 2S$$

holds scheme-theoretically.

<u>Proof</u> : Define the field L as the extension

$$\mathbb{F}_p(k_{01},\, k_{10},\, k_{11})[k_{00}]/(k_{00}^2 - k_{01}^2 - k_{10}^2 - k_{11}^2 + k_{01}k_{10}k_{11} + 4)$$

of the prime field $\mathbb{F}_p$, and define R as the L vector space generated by y_{00}, y_{01}, y_{10} and y_{11}. The homogeneous polynomials V_{00}, V_{01}, V_{10} and V_{11} define a L-linear ring homomorphism $\widetilde{V}^* : R \to R$ (defined by $\widetilde{V}^*(y_i) = V_i$). Letting K (resp. K_1) be the equation (2.1) of the Kummer surface Kum_X in $|2\Theta|$ (resp. Kum_{X_1} in $|2\Theta_1|$), Magma checks that K_1 divides $\widetilde{V}^*(K)$. Letting Q be the exact quotient $\widetilde{V}^*(K)/K_1$, Magma checks that it is a square. $\square$

5.4 Equations of V for $p = 7$

Proposition 5.8 *Let X be a general proper and smooth curve of genus 2 over an algebraically closed field of characteristic 7. There are coordinates $\{x_\bullet\}$ and $\{y_\bullet\}$ for $|2\Theta|$ and $|2\Theta_1|$ respectively such that the Kummer surface Kum_X in $|2\Theta|$ has an equation of the form (2.1) and such that, if the polynomials $(V_\bullet)$ define $\widetilde{V} : |2\Theta_1| \dashrightarrow |2\Theta|$, $(y_i) \mapsto (V_i(\underline{y}))$, then*

$$\begin{aligned}
V_{00} \;=\;& y_{00}^7 + a_{2100}y_{00}^5y_{01}^2 + a_{2010}y_{00}^5y_{10}^2 + a_{2001}y_{00}^5y_{11}^2 + a_{1200}y_{00}^3y_{01}^4 + a_{1110}y_{00}^3y_{01}^2y_{10}^2 \\
&+a_{1101}y_{00}^3y_{01}^2y_{11}^2 + a_{1020}y_{00}^3y_{10}^4 + a_{1011}y_{00}^3y_{10}^2y_{11}^2 + a_{1002}y_{00}^3y_{11}^4 \\
&+a_{0300}y_{00}y_{01}^6 + a_{0210}y_{00}y_{01}^4y_{10}^2 + a_{0201}y_{00}y_{01}^4y_{11}^2 + a_{0120}y_{00}y_{01}^2y_{10}^4 + a_{0111}y_{00}y_{01}^2y_{10}^2y_{11}^2 \\
&+a_{0102}y_{00}y_{01}^2y_{11}^4 + a_{0030}y_{00}y_{10}^6 + a_{0021}y_{00}y_{10}^4y_{11}^2 + a_{0012}y_{00}y_{10}^2y_{11}^4 + a_{0003}y_{00}y_{11}^6 \\
&+b_{2000}y_{00}^4y_{01}y_{10}y_{11} + b_{1100}y_{00}^2y_{01}^3y_{10}y_{11} + b_{1010}y_{00}^2y_{01}y_{10}^3y_{11} + b_{1001}y_{00}^2y_{01}y_{10}y_{11}^3 \\
&+b_{0200}y_{01}^5y_{10}y_{11} + b_{0110}y_{01}^3y_{10}^3y_{11} + b_{0101}y_{01}^3y_{10}y_{11}^3 \\
&+b_{0020}y_{01}y_{10}^5y_{11} + b_{0011}y_{01}y_{10}^3y_{11}^3 + b_{0002}y_{01}y_{10}y_{11}^5
\end{aligned}$$

with

$$\begin{array}{lll}
a_{2100} = -2k_{01}(k_{01}^4 - 1), & a_{2010} = -2k_{10}(k_{10}^4 - 1), & a_{2001} = -2k_{11}(k_{11}^4 - 1), \\
a_{1200} = k_{01}^2(k_{01}^2 - 1)(k_{01}^2 - 2), & a_{1020} = k_{10}^2(k_{10}^2 - 1)(k_{10}^2 - 2), & a_{1002} = k_{11}^2(k_{11}^2 - 1)(k_{11}^2 - 2), \\
a_{0300} = -k_{01}(k_{01}^2 - 1), & a_{0030} = -k_{10}(k_{10}^2 - 1), & a_{0003} = -k_{11}(k_{11}^2 - 1),
\end{array}$$

$$
\begin{aligned}
a_{0111} &= 3k_{00}^6 + 2(k_{01}^6 + k_{10}^6 + k_{11}^6) + 2(k_{00}^4 + k_{01}^4 + k_{10}^4 + k_{11}^4) + k_{00}^2(k_{00}^2 + 4)k_{01}k_{10}k_{11} \\
&\quad +(4k_{00}^2 + 1)k_{01}^2k_{10}^2k_{11}^2 - 2k_{00}^2(k_{01}^4 + k_{10}^4 + k_{11}^4 + 4) - 2k_{01}k_{10}k_{11}(k_{01}^2k_{10}^2k_{11}^2 - 1) + 1, \\
\\
a_{0210} &= 3k_{01}^2k_{10}k_{11}^4 + 3k_{01}^3k_{11}^3 - 2k_{01}k_{10}^2k_{11}^3 - k_{01}k_{11}^5 + 4k_{10}^3k_{11}^2 + 2k_{10}k_{11}^4 + k_{01}^3k_{11} + k_{01}k_{10}^2k_{11} \\
&\quad +k_{01}^2k_{10} + 4k_{10}^3 - 2k_{10}k_{11}^2 + 3k_{01}k_{11} - 2k_{10}, \\
a_{0201} &= 3k_{01}^2k_{11}k_{10}^4 + 3k_{01}^3k_{10}^3 - 2k_{01}k_{11}^2k_{10}^3 - k_{01}k_{10}^5 + 4k_{11}^3k_{10}^2 + 2k_{11}k_{10}^4 + k_{01}^3k_{10} + k_{01}k_{11}^2k_{10} \\
&\quad +k_{01}^2k_{11} + 4k_{11}^3 - 2k_{11}k_{10}^2 + 3k_{01}k_{10} - 2k_{11}, \\
a_{0120} &= 3k_{10}^2k_{01}k_{11}^4 + 3k_{10}^3k_{11}^3 - 2k_{10}k_{01}^2k_{11}^3 - k_{10}k_{11}^5 + 4k_{01}^3k_{11}^2 + 2k_{01}k_{11}^4 + k_{10}^3k_{11} + k_{10}k_{01}^2k_{11} \\
&\quad +k_{10}^2k_{01} + 4k_{01}^3 - 2k_{01}k_{11}^2 + 3k_{10}k_{11} - 2k_{01}, \\
a_{0102} &= 3k_{11}^2k_{01}k_{10}^4 + 3k_{11}^3k_{10}^3 - 2k_{11}k_{01}^2k_{10}^3 - k_{11}k_{10}^5 + 4k_{01}^3k_{10}^2 + 2k_{01}k_{10}^4 + k_{11}^3k_{10} + k_{11}k_{01}^2k_{10} \\
&\quad +k_{11}^2k_{01} + 4k_{01}^3 - 2k_{01}k_{10}^2 + 3k_{11}k_{10} - 2k_{01}, \\
a_{0021} &= 3k_{10}^2k_{11}k_{01}^4 + 3k_{10}^3k_{01}^3 - 2k_{10}k_{11}^2k_{01}^3 - k_{10}k_{01}^5 + 4k_{11}^3k_{01}^2 + 2k_{11}k_{01}^4 + k_{10}^3k_{01} + k_{10}k_{11}^2k_{01} \\
&\quad +k_{10}^2k_{11} + 4k_{11}^3 - 2k_{11}k_{01}^2 + 3k_{10}k_{01} - 2k_{11}, \\
a_{0012} &= 3k_{11}^2k_{10}k_{01}^4 + 3k_{11}^3k_{01}^3 - 2k_{11}k_{10}^2k_{01}^3 - k_{11}k_{01}^5 + 4k_{10}^3k_{01}^2 + 2k_{10}k_{01}^4 + k_{11}^3k_{01} + k_{11}k_{10}^2k_{01} \\
&\quad +k_{11}^2k_{10} + 4k_{10}^3 - 2k_{10}k_{01}^2 + 3k_{11}k_{01} - 2k_{10}, \\
\\
a_{1110} &= k_{11}(2(k_{00}^4 - k_{00}^2k_{11}^2 - k_{11}^4) + 2k_{00}^2 + k_{11}^2) + k_{01}k_{10}(3k_{11}^4 - k_{00}^4 + 2k_{00}^2 - k_{11} + 2) \\
&\quad +k_{01}^2k_{10}^2k_{11}(4k_{11}^2 + k_{00}^2 + 2) + 4k_{01}^3k_{10}^3k_{11}^2, \\
a_{1101} &= k_{10}(2(k_{00}^4 - k_{00}^2k_{10}^2 - k_{10}^4) + 2k_{00}^2 + k_{10}^2) + k_{01}k_{11}(3k_{10}^4 - k_{00}^4 + 2k_{00}^2 - k_{10} + 2) \\
&\quad +k_{01}^2k_{11}^2k_{10}(4k_{10}^2 + k_{00}^2 + 2) + 4k_{01}^3k_{11}^3k_{10}^2, \\
a_{1011} &= k_{01}(2(k_{00}^4 - k_{00}^2k_{01}^2 - k_{01}^4) + 2k_{00}^2 + k_{01}^2) + k_{11}k_{10}(3k_{01}^4 - k_{00}^4 + 2k_{00}^2 - k_{01} + 2) \\
&\quad +k_{11}^2k_{10}^2k_{01}(4k_{01}^2 + k_{00}^2 + 2) + 4k_{11}^3k_{10}^3k_{01}^2, \\
\\
b_{2000} &= k_{00}(2k_{00}^4 + k_{01}^4 + k_{10}^4 + k_{11}^4 + 2k_{00}^2(k_{01}k_{10}k_{11} - 2) + 4k_{01}^2k_{10}^2k_{11}^2), \\
b_{0200} &= k_{00}(4k_{00}^2 + 2k_{01}^2 + 1 + 4k_{10}^2k_{11}^2), \\
b_{0020} &= k_{00}(4k_{00}^2 + 2k_{10}^2 + 1 + 4k_{01}^2k_{11}^2), \\
b_{0002} &= k_{00}(4k_{00}^2 + 2k_{11}^2 + 1 + 4k_{01}^2k_{10}^2), \\
\\
b_{1100} &= -k_{00}(k_{01}(k_{01}^4 + (k_{00}^2 + 3)(k_{01}^2 + 2k_{00}^2 + 4)) - 3(k_{01}^2 + 4)(k_{01}^2 - 2k_{00}^2 - 2)k_{10}k_{11} \\
&\quad -2k_{01}(k_{01}^2 - 2)k_{10}^2k_{11}^2, \\
b_{1010} &= -k_{00}(k_{10}(k_{10}^4 + (k_{00}^2 + 3)(k_{10}^2 + 2k_{00}^2 + 4)) - 3(k_{10}^2 + 4)(k_{10}^2 - 2k_{00}^2 - 2)k_{01}k_{11} \\
&\quad -2k_{10}(k_{10}^2 - 2)k_{01}^2k_{11}^2, \\
b_{1001} &= -k_{00}(k_{11}(k_{11}^4 + (k_{00}^2 + 3)(k_{11}^2 + 2k_{00}^2 + 4)) - 3(k_{11}^2 + 4)(k_{11}^2 - 2k_{00}^2 - 2)k_{01}k_{10} \\
&\quad -2k_{11}(k_{11}^2 - 2)k_{01}^2k_{10}^2, \\
\\
b_{0110} &= k_{00}(k_{11}(5k_{11}^4 + (k_{00}^2 + 2)(3k_{11}^2 + 1)) - 3(k_{11}^2 + 2)(k_{11}^2 + 1)k_{01}k_{10}, \\
b_{0101} &= k_{00}(k_{10}(5k_{10}^4 + (k_{00}^2 + 2)(3k_{10}^2 + 1)) - 3(k_{10}^2 + 2)(k_{10}^2 + 1)k_{01}k_{11}, \\
b_{0011} &= k_{00}(k_{01}(5k_{01}^4 + (k_{00}^2 + 2)(3k_{01}^2 + 1)) - 3(k_{01}^2 + 2)(k_{01}^2 + 1)k_{10}k_{11},
\end{aligned}
$$

where the $k_\bullet$ are the coefficients of the equation (2.1) of Kum_X*. The V_i ($i = 01, 10, 11$) can be deduced from V_{00} by a suitable permutation of the coordinate functions $y_\bullet$, namely the unique pairwise permutation that exchanges y_{00} and y_i. .*

Proof : We make the same kind of calculations as in the case of the proof of the Proposition 5.6, except that we use the formula 5.11 instead of the formula 5.10 and that we must use more data to determine all the coefficients involved. □

The computer the author used to carry out these computations was not powerful enough to obtain the same result as in the case $p = 5$. Actually, one could only check that the equation of the twisted Kummer surface Kum_{X_1} did divide the image of the the equation on the Kummer surface Kum_X via the ring homomorphism defined by the above equations.

6 Further questions

Question 1 : Notice (Remark 5.5) that the Theorem 6.1 of [LP2] holds for any proper and smooth curve of genus 2 over an algebraically closed field of characteristic 3. Therefore, one can ask if the formulae given in Propositions 5.6 and 5.8 remain valid for any smooth and proper curve in characteristic 5 and 7.

One way to answer this question is, being given any curve X over k, to consider a family of genus 2 curve $\mathcal{X}$ over $\operatorname{Spec} k[[t]]$ with sufficiently general generic fiber and special fiber isomorphic to X and to study how do the equations specialize (see [LP2] or [Du] for examples of that method in characteristic 2). For that purpose, we would need a more refined description of the moduli space $\mathfrak{M}_K(k)$ of Kummer surfaces in $\mathbb{P}^3_k$, or equivalently (see Remark 2.10), of the moduli space of smooth and proper curve over k.

Note that, combining the chart 2.5 giving the expressions of the $\omega(\tau)$ in terms of the $k_\bullet$ and the classification of supersingular elliptic curve in small characteristic (see [H], Chapter IV, Examples 4.23.1 and followings), one can determine the open subset of $\mathfrak{M}_K(k)$ the closed points of which correspond to curves with Prym varieties associated to étale double cover that all are ordinary.

Question 2 : One would like to use the formulae given is Proposition 5.6 and 5.8 to say if, as in the characteristic 3 case, the generalized Verschiebung base locus is reduced in characteristics 5 and 7 (see [LnP] for a discussion on that topic). We would like to study the singular locus of the surface S in Corollary 5.7, with the aim of showing that it is zero dimensional and comparing it with the base locus of the Verschiebung. We plan to use the computers of the MEDICIS at the Ecole Polytechnique to carry out these computations.

Question 3 : The most interesting question arisen by this work is how can we recover the results obtained by computational means in a more geometric way. We plan to return to this latter question in a future work.

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Laurent Ducrohet
Université Pierre et Marie Curie
Analyse algébrique, UMR 7586
4, place Jussieu
75252 Paris Cedex 05 France
e-mail: ducrohet@math.jussieu.fr