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# ON EXPONENTIAL OBSERVABILITY ESTIMATES FOR THE HEAT SEMIGROUP IN SHORT AND INFINITE TIME

LUC MILLER

## 1. INTRODUCTION

The natural setting for the problem to be discussed is on manifolds, but all the statements can be understood, and are already interesting, when the domain  $M$  is a smooth bounded open set of  $\mathbb{R}^d$  with the flat metric so that the distance is  $\text{dist}(x, y) = |y_1 - x_1|^2 + \dots + |y_d - x_d|^2$  and the Laplacian is  $\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_d^2}$ , always considered with Dirichlet condition on the boundary  $\partial M$ . We shall refer to this setting as *the Euclidean case*.

Although it can be skipped, for completeness we now describe the general setting. Let  $(M, g)$  be a smooth connected compact  $d$ -dimensional Riemannian manifold with metric  $g$  and smooth boundary  $\partial M$ . When  $\partial M \neq \emptyset$ ,  $M$  denotes the interior and  $\bar{M} = M \cup \partial M$ . Let  $\text{dist} : \bar{M} \rightarrow \mathbb{R}_+$  denote the distance function. Let  $\Delta$  denote the (negative) Dirichlet Laplacian on  $L^2(M)$  (with domain  $H_0^1(M) \cap H^2(M)$ ).

The *observation region*  $\Omega$ , is a non-empty open subset of  $M$  such that  $\Omega \neq M$ . Unless mentioned otherwise, the range of the time  $T$  is  $(0, \infty)$  and the range of the initial state  $u_0$  is  $L^2(M)$ . The corresponding solution of the Cauchy problem for the (forward) heat equation is denoted by  $u(T, x) = (e^{T\Delta}u_0)(x)$ , in short:  $u = e^{T\Delta}u_0$  is the (relative) temperature on  $\mathbb{R}_+ \times M$ .

In this note, we make some remarks about the following observability inequality from  $\Omega$  of the final state at time  $T$ : for all small times  $T$  (hence for all times  $T$ ),

$$(1) \quad \forall u_0, \quad \int_M |e^{T\Delta}u_0|^2 dx \leq K \int_0^T \int_\Omega |e^{t\Delta}u_0|^2 dx dt \quad \text{with } K = Ce^{A/T}.$$

When  $K$  is an unspecified constant, this inequality is interesting from various points of view. If  $u$  is always zero on  $\Omega$  then it implies that  $u$  is zero everywhere on  $M$  at the final time  $T$ , which implies by backward uniqueness that  $u$  is always zero everywhere. Thus (1) is a unique continuation estimate. Moreover, by the duality in [DR77], the existence of a constant  $K$  such that (1) holds is equivalent to the ability of steering the heat flux from any  $u_0$  to zero in time  $T$  by a square integrable source supported in  $\Omega$  at a *cost*  $K$  (hence the optimal  $K$  does not increase with  $T$ ). This property is called *null-controllability* or exact controllability to zero. Its validity in this context was proved a decade ago in [LR95, Èma95].

Indeed (1) specifies how the cost  $K = Ce^{A/T}$  depends on  $T$ . The first such *exponential cost estimate* are due to Seidman (cf. [Sei84] and the survey [Sei05]). As far as I know, the best results about the validity of this estimate are threefold and use different methods. In the Euclidean case, (1) was proved in [FCZ00] by global Carleman estimates with singular weights as in [Èma95]. Under the geometrical optics condition on  $\Omega$  (i.e.  $L_\Omega < \infty$  with the notation of theorem 4), (1) was deduced in [Mil04b] by the control transmutation method (in short CTM, cf. section 2.2) from the observability of the wave group in [BLR92]. In the general setting, a

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slightly weaker exponential cost estimate was proved in [Mil05c] by the control strategy of [LR95] as implemented in the [LZ98]: for all  $\beta > 1$ , there are positive constants  $A_\beta$  and  $C_\beta$  such that (1) holds with  $K = C_\beta e^{A_\beta/T^\beta}$ .

This note reviews the known bounds on the optimal rate  $A$  in (1) (section 2) and other similar cost estimates (section 3), and relates (1) to several of the following variants considered in [FCZ00, Zua01] in the Euclidean case (theorem 1). The method of global Carleman estimates leads more naturally to the following integrated inequality with singular weight:

$$(2) \quad \forall u_0, \quad \int_0^T \int_M e^{-\tilde{A}/t} |e^{t\Delta} u_0|^2 dx dt \leq \tilde{C} \int_0^T \int_\Omega |e^{t\Delta} u_0|^2 dx dt,$$

This is proved in proposition 6.1 of [FCZ00]. Among open problems, it is stated in [Zua01] (equation 4.3) that the following variant for infinite time still holds:

$$(3) \quad \forall u_0, \quad \int_0^\infty \int_M e^{-A/t} |e^{t\Delta} u_0|^2 dx dt \leq C_\infty \int_0^\infty \int_\Omega |e^{t\Delta} u_0|^2 dx dt .$$

Remark 6.1 of [FCZ00] extracts from the proof of theorem 6.1 the following inequality for fixed  $T$ , which is sharper than (1), at least when  $T \geq B$ :

$$(4) \quad \forall u_0, \quad \int_M |e^{-B\sqrt{-\Delta}} u_0|^2 dx \leq K' \int_0^T \int_\Omega |e^{t\Delta} u_0|^2 dx dt \quad \text{with } K' = C' e^{A'/T}.$$

Replacing the  $L^2$  norm of the final state in (1) by its norm in a Sobolev space of real order  $s$  yields the following inequality, better for positive  $s$ :

$$(5) \quad \forall u_0 \in H^s(M), \quad \|e^{T\Delta} u_0\|_{H^s}^2 \leq K_s \int_0^T \int_\Omega |e^{t\Delta} u_0|^2 dx dt \quad \text{with } K_s = C_s e^{A_s/T}.$$

We prove in this note that (1) for small times implies its four variants (2), (3), (4) and (5), with rates  $A$ ,  $\tilde{A}$ ,  $A'$  and  $A_s$  which are roughly the same everywhere. More precisely, section 4 proves:

**Theorem 1.** *For  $B > \sqrt{2\tilde{A}}$ ,  $A' > A$ ,  $s \in \mathbb{R}$  and  $A_s > A$  ( $A_s = A$  if  $s \leq 0$ ), if the final time observability inequality (1) holds for all  $T \leq T_0$ , then*

- i. *the integrated inequality (2) holds for all  $T \leq T_0$  with  $\tilde{A} = A$ , and  $\tilde{C} = CT$ ,*
- ii. *the infinite time inequality (3) holds with  $C_\infty = CT_0(1 + e^{A/T_0})$ ,*
- iii. *the sharp inequality (4) holds for all  $T$ .*
- iv. *the Sobolev inequality (5) holds for all  $T$ .*

*Conversely, for  $A > \tilde{A}$ , if the integrated inequality (2) holds for all  $T \leq T_0$ , then the final time inequality (1) holds for all  $T$ .*

Even in the Euclidean case and for fixed  $T$ , theorem 1 simplifies the proof of (4) (proposition 6.1 in [FCZ00] already uses (1) but also goes back to the global Carleman inequality). The fast cost estimate in (4) seems to be new:

**Corollary 2.** *Under the geometrical optics condition on  $\Omega$  or in the Euclidean case, there are positive constants  $B$ ,  $A'$  and  $C'$  such that:*

$$\forall T, \forall u_0, \quad \int_M |e^{-B\sqrt{-\Delta}} u_0|^2 dx \leq C' e^{A'/T} \int_0^T \int_\Omega |e^{t\Delta} u_0|^2 dx dt .$$

## 2. BOUNDS ON THE OPTIMAL RATE $A$ IN (1)

**2.1. Lower bounds.** It is proved in [Mil04b] that (1) for all small  $T$  implies

$$(6) \quad A \geq \sup_{y \in M} \text{dist}(y, \bar{\Omega})^2 / 2 .$$

The proof relies on Varadhan's formula for the heat kernel in small time (cf. [Var67]), which requires very low smoothness assumptions as proved in [Nor97]. This improves on the former lower bound in the Euclidean case stated in section 4.1 of [Zua01] which was based on a construction made in the proof of Theorem 6.2 in [FCZ00]:  $A \geq \sup_{\overline{B_\rho} \subset M \setminus \overline{\Omega}} \rho^2/4$ , where the supremum is taken over balls  $B_\rho$  of radius  $\rho$ .

For finite times  $T$ , the lack of observability at a better cost is only due to the finite linear combinations of the eigenmodes corresponding to frequencies lower than a threshold of order  $1/T$ . To state this result from [Mil04b] more precisely <sup>1</sup>, we introduce the spectral data:  $(\omega_k)_{k \in \mathbb{N}^*}$  is a nondecreasing sequence of nonnegative real numbers and  $(e_k)_{k \in \mathbb{N}^*}$  is an orthonormal basis of  $L^2(M)$  such that  $e_k$  is an eigenvector of  $-\Delta$  with eigenvalue  $\omega_k^2$ , i.e.:

$$(7) \quad -\Delta e_k = \omega_k^2 e_k \quad \text{and} \quad e_k = 0 \text{ on } \partial M .$$

**Theorem 3** ([Mil04b]). *Let  $d \in (0, \sup_{y \in M} \text{dist}(y, \overline{\Omega})$ ). If (1) holds for all small  $T$  and for any  $u_0$  in the linear span of  $\{e_k\}_{\omega_k \leq d/T}$ , then  $A \geq d^2/2$ .*

**2.2. Upper bounds.** In view of theorem 1, upper bounds on the optimal rate  $A$  in (1) imply upper bounds on the optimal rates in (2), (3), (4) and (5).

**Theorem 4** ([Mil04b]). *Let  $L_\Omega$  be the length of the longest generalized geodesic<sup>2</sup> in  $\overline{M}$  which does not intersect  $\Omega$ . For all  $A > (2(36/37)L_\Omega)^2$  there is a positive constant  $C$  such that (1) holds for all  $u_0$  and  $T$ .*

The same bound is immediately deduced, by Theorem 1.6 in [Mil05d], for the heat semigroup on the product manifold  $M \times \tilde{M}$  observed from  $\Omega \times \tilde{M}$ , where  $\tilde{M}$  denotes another smooth complete  $\tilde{n}$ -dimensional Riemannian manifold (e.g. an infinite strip observed from any infinite strip in the interior). To the best of my knowledge, there are no better upper bound of the optimal rate in the literature.

When comparing theorem 4 to the lower bound in (6), one should bear in mind that  $L_\Omega$  is always greater than  $2 \sup_{y \in M} \text{dist}(y, \overline{\Omega})$  (as the length of a generalized geodesic through  $y$  which does not intersect  $\Omega$  is always greater than  $2 \text{dist}(y, \overline{\Omega})$ ) and can be infinitely so. But, for some simple geometries, theorem 4 implies an upper bound of the optimal rate in terms of  $\sup_{y \in M} \text{dist}(y, \overline{\Omega})$  as well, e.g.:

**Corollary 5.** *In the Euclidean case, if  $M$  is a ball and  $\Omega$  is a small enough neighborhood of its boundary then for all  $A > 16 \sup_{y \in M} \text{dist}(y, \overline{\Omega})^2$  there is a  $C > 0$  such that (1) holds for all  $T$ .*

Theorem 4 is deduced from the observability of the wave group (cf. [BLR92]) by the *Control Transmutation Method*, in short CTM. This method applies to control problems the guiding principle in the kernel estimates method of [CGT82]: systems with finite propagation speed yield geometrical information in small times about systems with similar generators but without propagation speed. Here, it consists in constructing a time kernel  $k$ , coined "the fundamental controlled solution", which

<sup>1</sup>Theorem 3 is not explicitly stated in [Mil04b], but it is roughly explained after theorem 2.1 in [Mil04b]. Moreover, theorem 3 for the Schrödinger group instead of the heat semigroup is proved by the same method and explicitly stated in [Mil04c].

<sup>2</sup>In this context, the generalized geodesics are continuous trajectories  $t \mapsto x(t)$  in  $\overline{M}$  which follow geodesic curves at unit speed in  $M$  (so that on these intervals  $t \mapsto \dot{x}(t)$  is continuous); if they hit  $\partial M$  transversely at time  $t_0$ , then they reflect as light rays or billiard balls (and  $t \mapsto \dot{x}(t)$  is discontinuous at  $t_0$ ); if they hit  $\partial M$  tangentially then either there exists a geodesic in  $M$  which continues  $t \mapsto (x(t), \dot{x}(t))$  continuously and they branch onto it, or there is no such geodesic curve in  $M$  and then they glide at unit speed along the geodesic of  $\partial M$  which continues  $t \mapsto (x(t), \dot{x}(t))$  continuously until they may branch onto a geodesic in  $M$ . The meaning of the geometrical optics condition  $L_\Omega < \infty$ , due to Bardos-Lebeau-Rauch in [BLR92], is discussed at length in [Mil02].

transforms the control function for the wave group in time  $L$  into an input function for the heat semigroup in time  $T$ : the norms of  $k$  must be estimated explicitly in terms of  $L$  and  $T$  only. Contrary to Russell's harmonic analysis method in [Rus73], it does not use information on the spectrum and it extends to the most general abstract setting (cf. [Mil04a]).

### 3. A CONJECTURE AND RELATED RESULTS

**3.1. Conjecture and open problems.** Combining the upper and lower bounds for the optimal rate  $A$  in (1) in the simple example of corollary 5 ( $M$  is a Euclidean ball and  $\Omega$  is a small enough neighborhood of its boundary) yields

$$\alpha := A \left( \sup_{y \in M} \text{dist}(y, \bar{\Omega}) \right)^{-2} \in [1/2, 16) .$$

Since we believe that there is no solution of the heat equation which is more singular than the heat kernel, it is natural to conjecture that the lower bound (6) is also an upper bound : the optimal rate  $A$  such that (1) holds for small  $T$  is  $\sup_{y \in M} \text{dist}(y, \bar{\Omega})^2/2$  for any  $(M, g)$  and  $\Omega \neq M$  (i.e.  $\alpha = 1/2$ ).

If  $K(T)$  denotes the optimal cost in (1) for fixed  $T$ , then the function  $K : (0, \infty) \rightarrow (0, \infty)$  does not increase (as a result of the semigroup property or the duality with null-controllability), but this is not enough to ensure that  $\lim_{T \rightarrow 0} T \ln K(T)$  exists. The existence of this limit is part of the conjecture but could possibly be established independently. Until then, the optimal rate can only be defined as<sup>3</sup>  $A^* = \limsup_{T \rightarrow 0} T \ln K(T)$ .

Theorem 1 roughly says that the ‘‘optimal rates’’  $A$  and  $A'$  in (1) and (2) are equal<sup>4</sup>. It does not say whether the ‘‘optimal rates’’  $A$  in (1) and (3) are also equal (it roughly says that the ‘‘optimal rate’’ is not greater in (3) than in (1)).

Other related open problems shall appear in [Zua05].

**3.2. Boundary observability and window problems.** For steering the temperature to zero with the temperature on  $\Gamma \subset \partial M$  as input, the corresponding observability inequality of the final state from  $\Gamma$  is similar to (1):

$$(8) \quad \forall u_0, \quad \int_M |e^{T\Delta} u_0|^2 dx \leq K_\nu \int_0^T \int_\Gamma |\partial_\nu u|^2 dx dt \quad \text{with } K_\nu = C_\nu e^{A_\nu/T} ,$$

where  $\partial_\nu$  denotes the Neumann derivative at the boundary.

When  $M$  is a Euclidean segment and  $\Gamma$  is one endpoint, (8) is an inequality on sums of exponentials coined a ‘‘window problem’’ in [SAI00]. A well trodden path in the harmonic analysis of this problem is to construct a Riesz basis of bi-orthogonal functions. This reduces by the Paley-Wiener theorem to the construction of entire functions with zeros and growth conditions. Proving exponential cost estimates in this setting is a non-classical aspect of this problem deeply studied in [SAI00]. We refer to [SAI00, Sei05, Mil04b] for more details and references.

In this context,  $L = \sup_{y \in M} \text{dist}(y, \bar{\Gamma})$  is the length of the segment  $M$ . The best upper bound obtained so far by this method is (cf. [Mil04b]): for  $A_\nu > 2\alpha_* L^2$ , (8) holds for all  $T$ , where  $\alpha_* = 2 \left( \frac{36}{37} \right)^2 < 2$ . Any improvement of the value of  $\alpha_*$  in this result, and in the analogous result where the Neumann derivative is removed in (8), will improve theorem 4 to  $A > 2\alpha_* L_\Omega^2$ . N.b. in the CTM which deduces this theorem from the boundary observability estimate on the segment there is a loss of a factor 4 since  $L_\Gamma = 2L$  on the segment.

<sup>3</sup>If  $A > A^*$  then (1) holds for small  $T$ , and conversely, if (2) holds for small  $T$  then  $A \geq A^*$ . Whether (1) holds for small  $T$  when  $A = A^*$  is an open problem.

<sup>4</sup>Theorem 1 proves:  $\tilde{A} > A^*$  implies ‘‘(2) holds for small  $T$ ’’ implies  $\tilde{A} \geq A^*$ .

The CTM has been extended in [Mil04a] to the observability (by unbounded operators) of holomorphic semigroups generated by the generator of a cosine operator function. Since [BLR92] proved the boundary observability for (the real part of) the wave group  $\cos(t\sqrt{-\Delta})$ , which is the model of all cosine operator evolutions, theorem 4 still holds when (1) and  $\Omega$  are replaced by (8) and  $\Gamma$  (this is theorem 6.1 in [Mil04a]).

By theorem 1.5 in [Mil05d], these two estimates of the cost in (8) for problems in dimension one and greater extend to the problems that can be deduced from them by a tensor product, e.g. the better one dimensional result extends to an infinite strip observed from one of the boundary lines.

**3.3. Other evolution systems.** The heat kernel method used to prove the lower bound in theorem 3 and the control transmutation method (CTM) used to prove the upper bound in theorem 4 were adapted to the interior observability of the Schrödinger group in [Mil04c] (n.b. this is the observation-control system to which a transmutation method was first applied in [Phu01]). Thanks to a new necessary and sufficient condition for the observability of unitary groups by unbounded operators, coined a “resolvent observability estimate” (this theorem 5.1 in [Mil05a] is the analogue of the Hautus test for finite dimensional control systems, cf. [RW94]), the CTM has been extended to this abstract setting. Thus it allows to deduce from [BLR92] exponential observability estimates from the boundary for the Schrödinger group (theorem 10.2 in [Mil05a]).

The slightly weaker exponential cost estimates mentioned in the introduction i.e.  $K = C_\beta e^{A_\beta/T^\beta}$  for any  $\beta > 1$  and some  $A_\beta > 0$  and  $C_\beta > 0$ , were generalized by the same method to the system of thermoelastic plates without rotary inertia (in the Euclidean case, with hinged mechanical boundary conditions and Dirichlet thermal boundary condition) observed from  $\Omega$  by either the mechanical or thermal component (cf. [Mil05e]), and to the plate equation with square root damping observed from  $\Omega$  (in the Euclidean case, with hinged boundary conditions, cf. [Mil05b] where the CTM was also adapted to this system and yields  $\beta = 1$  under the geometrical optics condition on  $\Omega$ ). The same method was applied to more general abstract linear elastic systems with structural damping in [Mil05b] and yields various ranges for  $\beta$  depending on the strength of the damping. It also applies to anomalous diffusions generated by the fractional Laplacian  $-(-\Delta)^p$ , for  $p > 1/2$ , where it yields  $\beta > 1/(2p - 1)$  (cf. [Mil05c]).

The exponential cost estimates in [Mil05b, Mil05e] use earlier *polynomial cost estimates* proved in [Tri03, AL03b, AL03a] in the case  $\Omega = M$  which we have excluded at the very beginning of this note because (1) holds with  $K = C/T^{1/2}$  when  $\Omega = M$ . Triggiani, Lasiecka and Avalos proved cost estimates of the form  $K = C/T^p$ ,  $p \geq 1/2$ , where  $p$  is related to the strength of the damping. These estimates are similar to the optimal cost estimates for finite dimensional control systems proved in [Sei88] which we now describe. Let  $A$  be an  $n \times n$  matrix defining a system of linear differential equations in  $\mathbb{R}^n$ , and let  $B$  be the  $m \times n$  matrix which prescribes the  $m$  observed coordinates in  $\mathbb{R}^n$ . The observability inequality is:

$$(9) \quad \forall x_0 \in \mathbb{R}^n, \quad \|x_0\|^2 \leq K \int_0^T \|Be^{tA}x_0\|^2 dt .$$

Kalman proved that (9) holds if and only if there is an integer  $p < n$  such that the  $n \times nm$  block matrix  $\{B^*, A^*B^*, \dots, A^{*p}B^*\}$  is of rank  $n$  (the star denotes transposed matrices). Seidman proved that, as  $T$  tends to zero, the optimal cost in (9) satisfies  $K \sim C/T^{1/2+p}$  where  $p$  is the smallest integer satisfying Kalman’s rank condition.

We are still longing for such a complete result regarding infinite dimensional control systems, at least for distributed systems with infinite propagation speed such as the heat semigroup.

#### 4. PROOF OF THEOREM 1

**i.** For  $T \leq T' \leq T_0$ , multiplying (1) by  $e^{-A/T}$ , bounding  $\int_0^T$  from above by  $\int_0^{T'}$ , then integrating  $T$  over  $(0, T')$  yields (2) with  $T = T'$ ,  $\tilde{C} = CT'$ ,  $A = \tilde{A}$ .

**ii.** This point results from the previous one and the following lemma.

**Lemma 6.** *If (1) and (2) hold, then (3) holds with  $C_\infty = \tilde{C} + CT e^{A/T}$ .*

*Proof.* Since  $e^{-A/t} \leq 1$  and  $t \mapsto \|e^{t\Delta} u_0\|_{L^2(M)}$  does not increase: for all  $n \in \mathbb{N}^*$ ,

$$\int_{nT}^{(n+1)T} \int_M e^{-A/t} |e^{t\Delta} u_0|^2 dx dt \leq T \int_M |e^{nT\Delta} u_0|^2 dx \leq CT e^{A/T} \int_{(n-1)T}^{nT} \int_\Omega |e^{t\Delta} u_0|^2 dx dt,$$

where (1) with  $u_0$  replaced by  $e^{(n-1)T\Delta} u_0$  is used in the last step. Summing up over  $n \geq 1$ , yields:

$$\int_T^\infty \int_M e^{-A/t} |e^{t\Delta} u_0|^2 dx dt \leq CT e^{A/T} \int_0^\infty \int_\Omega |e^{t\Delta} u_0|^2 dx dt.$$

Adding this inequality to (2) yields (3) with  $C_\infty = \tilde{C} + CT e^{A/T}$ .  $\square$

**iii.** This point results from the previous one and the following lemma.

**Lemma 7.** *For  $B > \sqrt{2A}$  and  $A' > A$ , there is a  $C' > 0$  such that for all  $T$ :*

$$(10) \quad \forall u_0, \quad \int_M |e^{-B\sqrt{-\Delta}} u_0|^2 dx \leq C' e^{A'/T} \int_0^T \int_M e^{-A/t} |e^{t\Delta} u_0|^2 dx dt.$$

*Proof.* Let  $T_0 > 0$ . Writing  $u_0 = \sum_k c_k e_k$  with  $\sum_k |c_k|^2$  in the eigenbasis (7) yields:

$$(11) \quad \int_M |e^{-B\sqrt{-\Delta}} u_0|^2 dx = \sum_k e^{-2B\omega_k} |c_k|^2,$$

$$(12) \quad \int_0^T \int_M e^{-A/t} |e^{t\Delta} u_0|^2 dx dt = \sum_k I_A(T, 2\omega_k^2) |c_k|^2,$$

$$\text{with } I_A(T, \lambda) = \int_0^T e^{-\lambda t - A/t} dt = \sqrt{\lambda/A} \int_0^{T\sqrt{A/\lambda}} e^{-\sqrt{A\lambda}(s+1/s)} ds.$$

Henceforth, we keep the same notation  $\varepsilon$  and  $C_\varepsilon$  meaning ‘‘for all small  $\varepsilon > 0$ , there is  $C_\varepsilon > 0$  independent of  $\lambda$  and  $T$  such that...’’ although their value may change. For  $T\sqrt{A/\lambda} > 1$ , we may bound the last integral from below by  $\int_{1-\varepsilon}^1 \dots ds$ , hence:

$$I_A(T, \lambda) \geq \varepsilon \sqrt{\lambda/A} e^{-\sqrt{A\lambda}(1-\varepsilon+1/(1-\varepsilon))} \geq C_\varepsilon e^{-2(1+\varepsilon)\sqrt{A\lambda}}.$$

For  $T\sqrt{A/\lambda} \leq 1$ , i.e.  $\lambda \geq AT^2$ , we have  $I_A(T, \lambda) \geq I_A(T, AT^2)$ , hence:

$$I_A(T, \lambda) \geq e^{-AT^3} \int_{(1-\varepsilon)T}^T e^{-A/t} dt \geq e^{-AT^3} \varepsilon T e^{-A/((1-\varepsilon)T)} \geq C_\varepsilon e^{-(1+\varepsilon)A/T},$$

where  $T \leq T_0$  was used in the last step. Joining the two cases yields:

$$\forall \lambda > 0, \forall T \leq T_0, \quad I_A(T, \lambda) \geq C_\varepsilon e^{-(1+\varepsilon)(2\sqrt{A\lambda}+A/T)}.$$

With (11) and (12), this proves: for  $T_0 > 0$ ,  $B > \sqrt{2A}$  and  $A' > A$ , there is a  $C'' > 0$  such that: for all  $T \leq T_0$ , (10) holds with  $C'$  replaced by  $C''$ . Now with  $C' = C'' e^{A'/T_0}$ , (10) holds for all  $T$ , which completes the proof of the lemma.  $\square$

iv. For negative  $s$ , since  $L^2(M)$  is continuously embedded in  $H^s(M)$ , (1) implies (5) with  $A_s = A$  by density. Let  $s > 0$  from now on. Since  $e^{t\Delta}$  is an analytic semigroup, it satisfies the smoothing property:  $S_s := \sup_{t>0} \|t^s e^{t\Delta}\|_{\mathcal{L}(L^2; H^s)} < \infty$ . Let  $K(T)$  and  $K_s(T)$  denote the optimal costs  $K$  and  $K_s$  in (1) and (5). For all  $\varepsilon \in (0, 1)$  and  $T$ ,  $K_s(T) \leq S_s(\varepsilon T)^{-s} K((1-\varepsilon)T)$ . Since  $\varepsilon$  is arbitrarily small, for all  $A_s > A$ , there is a  $C'_s$  such that: for any  $T_0$ , if  $K(T) \leq C e^{A/T}$  holds for all  $T \leq T_0$  then  $K_s(T) \leq C'_s e^{A_s/T}$  holds for all  $T \leq T_0$ . Therefore, with  $C_s = C'_s e^{A_s/T_0}$ : if (1) holds for all  $T \leq T_0$ , then (5) holds for all  $T$ .

**Converse.** The last statement of theorem 1 results from the following lemma.

**Lemma 8.** *For all  $A > \tilde{A}$ , there is a  $C > 0$  such that:*

$$\forall T, \forall u_0, \quad \int_M |e^{T\Delta} u_0|^2 dx \leq C e^{A/T} \int_0^T \int_M e^{-\tilde{A}/t} |e^{t\Delta} u_0|^2 dx dt .$$

*Proof.* Let  $\varepsilon \in (0, 1)$ . Bounding  $\int_0^T$  from below by  $\int_{(1-\varepsilon)T}^T$  yields:

$$\forall u_0, \quad \int_0^T \int_M e^{-\tilde{A}/t} |e^{t\Delta} u_0|^2 dx dt \geq (1-\varepsilon) T e^{-\tilde{A}/((1-\varepsilon)T)} \int_M |e^{T\Delta} u_0|^2 dx ,$$

since  $t \mapsto e^{-A/t}$  does not decrease and  $t \mapsto \|e^{t\Delta} u_0\|_{L^2(M)}$  does not increase. Since  $\varepsilon$  is arbitrarily small, this completes the proof of the lemma.  $\square$

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