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# ASYMPTOTIC DIRECTION FOR RANDOM WALKS IN RANDOM ENVIRONMENTS

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**ABSTRACT.** In this paper we study the property of asymptotic direction for random walks in random i.i.d. environments (RWRE). We prove that if the set of directions where the walk is transient is non empty and open, the walk admits an asymptotic direction. The main tool to obtain this result is the construction of a renewal structure with cones. We also prove that RWRE admits at most two opposite asymptotic directions.

**RESUMÉ:** Dans cet article, nous étudions la propriété de direction asymptotique pour les marches aléatoires en milieu aléatoire i.i.d. (RWRE). Nous prouvons que si l'ensemble des directions dans lesquelles la marche est transiente est un ouvert non vide, la marche admet une direction asymptotique. La construction d'une structure de renouvellement avec cônes est le principal outil pour la preuve de ce résultat. Nous montrons aussi qu'une RWRE admet au plus 2 directions asymptotiques opposées.

## 1. INTRODUCTION AND RESULTS

In this paper, we give a characterization of random walks in random i.i.d environments having an asymptotic direction. We first describe the model that we will use. Fix a dimension  $d \geq 1$  (but think more particularly of the case where  $d \geq 2$  because this work becomes obvious when  $d = 1$ ). Let  $\mathcal{P}_+$  denote the  $(2d - 1)$  dimensional simplex,  $\mathcal{P}_+ = \{x \in [0, 1]^{2d}, \sum_{i=1}^{2d} x_i = 1\}$ . An environment  $\omega$  in  $\mathbb{Z}^d$  is an element of  $\Omega := \mathcal{P}_+^{\mathbb{Z}^d}$ . For any environment  $\omega$ ,  $P_{x,\omega}$  denotes the Markov chain with state space  $\mathbb{Z}^d$  and transition given by

$$P_{x,\omega}(X_0 = x) = 1 \quad \text{and}$$

$$P_{x,\omega}(X_{n+1} = z + e | X_n = z) = \omega_z(e) \quad (z \in \mathbb{Z}^d, e \in \mathbb{Z}^d \text{ s.t. } |e| = 1, n \geq 0)$$

where  $|\cdot|$  denotes the Euclidean norm in  $\mathbb{Z}^d$ .

For any law  $\mu$  on  $\mathcal{P}_+$ , we define a random environment  $\omega$  in  $\mathbb{Z}^d$ , random variable on  $\Omega$  with law  $\mathbb{P} := \mu^{\otimes \mathbb{Z}^d}$ . For any  $x$  in  $\mathbb{Z}^d$  and any fixed  $\omega$ , the law  $P_{x,\omega}$  is called quenched law. The annealed law  $P_x$  is defined on  $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$  by the semi-product

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$P_x := \mathbb{P} \times P_{x,\omega}$ . In this article, the law  $\mu$  will verify the assumption of strict ellipticity

$$\forall e \in \mathbb{Z}^d \text{ s.t. } |e| = 1, \mathbb{P} - a.s. \quad \mu(\omega_0(e) > 0) = 1$$

which is weaker than the usual uniform ellipticity (see remark 1).  $S^{d-1}$  denotes the unit circle for the Euclidean norm. For any  $\ell$  in  $\mathbb{R}^d$ , we define the set  $A_\ell$  of transient trajectories in direction  $\ell$

$$A_\ell = \left\{ \lim_{n \rightarrow +\infty} X_n \cdot \ell = +\infty \right\}$$

and for any  $\nu$  in  $S^{d-1}$ ,  $B_\nu$  is defined as the set of trajectories having  $\nu$  for asymptotic direction

$$B_\nu = \left\{ \lim_{n \rightarrow +\infty} \frac{X_n}{|X_n|} = \nu \right\}$$

This model is well studied in the one dimensional case where many sharp properties of the walk are known. However in higher dimensions the behavior of the walk is much less well-understood. One of the oldest question was asked by Kalikow in [3]: does  $P_0(A_\ell)$  follow a 0 – 1 law? In the same article he found a weaker but very useful 0 – 1 law for  $P_0(A_\ell \cup A_{-\ell})$  which was explicitly stated in [7]. Zerner answers positively to Kalikow’s question in [11], but only in the case  $d = 2$ . In higher dimensions, this is still an unanswered question and this is one of the main obstacle to solve another fundamental question: the derivation of a law of large numbers. Many recent articles are concerned with this topic. Generalizing [4] to higher dimensions, Sznitman and Zerner construct in [7] a renewal structure in order to derive a law of large numbers under a strong drift condition (Kalikow’s condition). Zerner proves in [10] that the walk satisfies a law of large numbers without assuming any conditions, but only for dimension 2. For higher dimensions, Sznitman improves on sufficient conditions to obtain a law of large numbers([5] and [6]). In this paper, we describe the class of walks having a unique asymptotic direction under the annealed law (theorem 1). This means that the walk is transient and escapes to infinity in a direction which has a deterministic almost surely direction. In fact, if the walk admits an asymptotic direction, it also follows a law of large numbers (see remark 2) but in the non-ballistic case the asymptotic direction gives an interesting information of the behavior of the walk which is not contained in the law of large numbers. The main difficulty to obtain an asymptotic direction for a transient walk is to control the fluctuations of the walk in the hyperplane transverse to transience direction. One way to control those fluctuations is to introduce the following assumption.

**Assumption.**  $\ell$  in  $\mathbb{R}_*^d$  verifies assumption (H) if there exists a neighborhood  $\mathcal{V}$  of  $\ell$  such that

$$\forall \ell' \in \mathcal{V}, \quad P_0(A_{\ell'}) = 1 \quad (H)$$

When (H) holds, we will note  $\mathcal{V}$  the neighborhood given by the assumption.

The main purpose of this article is to prove the following theorem.

**Theorem 1.** *The following three statements are equivalent*

i) *There exists a non empty open set  $\mathcal{O}$  of  $\mathbb{R}^d$  such that*

$$\forall \ell \in \mathcal{O}, \quad P_0(A_\ell) = 1$$

ii)  $\exists \nu \in S^{d-1}$  s.t.

$$P_0 - a.s., \quad \frac{X_n}{|X_n|} \rightarrow \nu$$

iii)  $\exists \nu \in \mathbb{R}_*^d$  s.t.  $\forall \ell \in \mathbb{R}^d$

$$\ell \cdot \nu > 0 \implies P_0(A_\ell) = 1$$

Using arguments similar to those applied in the proof of theorem 1, we also show

**Proposition 1.** *If  $\nu$  and  $\nu'$  are two distinct vectors in  $S^{d-1}$  such that  $P_0(B_\nu)P_0(B_{\nu'}) > 0$  then  $\nu' = -\nu$*

An obvious consequence of this proposition is the following corollary.

**Corollary 1.** *Under  $P_0$ , there is at most two asymptotic directions, in this case these two potential directions are opposite each other.*

The proofs will be given in the second part of this paper. We finish this section with some notations which will be useful in the proofs. Denote by  $\theta_n$  the time shift ( $n$  natural number is the argument) and by  $t_x$  the spatial shift ( $x$  in  $\mathbb{Z}^d$  is the argument). For any fixed  $\ell$  in  $\mathbb{R}_*^d$ , we let  $T_u$  be the hitting time of the open half-space  $\{x, x \cdot \ell > u\}$ :

$$T_u = \inf\{n > 0, X_n \cdot \ell > u\}$$

and  $D^\ell$  the return time of the walk behind the starting point:

$$D^\ell = \inf\{n \geq 0, X_n \cdot \ell \leq X_0 \cdot \ell\}$$

Notice that these two definitions are quite different from those used in [7].

We complete  $\ell$  into an orthogonal basis  $(e_2, \dots, e_d)$ , such that for every  $i$  in  $\llbracket 2, d \rrbracket$ ,  $|e_i| = 1$ . For all  $i \in \llbracket 2, d \rrbracket$  we define the following two vectors:

$$\ell'_i(\alpha) = \ell + \alpha e_i \quad \text{and} \quad \ell'_{-i}(\alpha) = \ell - \alpha e_i$$

For all positive real  $\alpha$  we can define the convex cone  $C(\alpha)$  by

$$C(\alpha) = \bigcap_{i=2}^d \{x, x \cdot \ell'_i(\alpha) \geq 0 \quad \text{and} \quad x \cdot \ell'_{-i}(\alpha) \geq 0\}$$

We also define the exit time  $D_\alpha^\ell$  of the cone  $C(\alpha)$

$$D_\alpha^\ell = \inf\{n \geq 0, X_n \notin C(\alpha)\}$$

## 2. PROOFS

*Proof of theorem 1.* The first step of the proof is the following lemma, where it is proved that under  $(H)$  the walk has a positive probability never to exit a cone  $C(\alpha)$  for  $\alpha$  small enough.

**Lemma 1.** *Let  $\ell$  be a vector in  $\mathbb{R}^d$  satisfying  $(H)$  then, for any choice of an orthogonal basis  $(\ell, e_2, \dots, e_d)$  with  $|e_i| = 1$  for any  $i$  in  $\llbracket 2, d \rrbracket$ , there exist some  $\alpha_0 > 0$  such that,*

$$\forall \alpha \leq \alpha_0 \quad P_0(D_\alpha^\ell = \infty) > 0 \tag{1}$$

*Proof.* Fix a basis satisfying the assumption of the lemma. We will first show that

$$P_0(\cdot | D^\ell = \infty) - a.s., \quad \exists \alpha_1 \text{ random } s.t. \quad D_{\alpha_1}^\ell = \infty \quad (2)$$

Since  $\mathcal{V}$  is an open set, there exist some  $\alpha_2 > 0$  such that for every  $i \in \llbracket 2, d \rrbracket$ :

$$\ell'_i(\alpha_2) \in \mathcal{V} \text{ and } \ell'_{-i}(\alpha_2) \in \mathcal{V}$$

For these  $(2d - 2)$  directions, we use the renewal structure described in section 1 of [7]. The choice of the parameter  $a$  in this structure has no importance and can be done arbitrarily. Remember that, for any fixed direction  $\ell$ , the first renewal time  $\tau^\ell$  of [7] is the first time the walk reaches a new record in direction  $\ell$ , and later never backtracks.

**Remark 1.** In [7], as in further references, uniform ellipticity is assumed. When we quote these articles, we have verified that this stronger assumption is not necessary or can be relaxed as in [11].

Using (H) we obtain that for each  $i \in \llbracket 2, d \rrbracket$ ,

$$P_0(A_{\ell'_i(\alpha_2)}) = P_0(A_{\ell'_{-i}(\alpha_2)}) = 1$$

From proposition 1.2 in [7]:

$$\tau^{\ell'_2(\alpha_2)} \vee \dots \vee \tau^{\ell'_d(\alpha_2)} \vee \tau^{\ell'_{-2}(\alpha_2)} \vee \dots \vee \tau^{\ell'_{-d}(\alpha_2)} < \infty \quad P_0 - a.s.$$

Using a proof very close from proposition 1.2 in [7] (see also theorem 3 in [3]) we obtain  $P_0(D^\ell = \infty) > 0$  and so:

$$\tau^{\ell'_2(\alpha)} \wedge \dots \wedge \tau^{\ell'_d(\alpha)} \wedge \tau^{\ell'_{-2}(\alpha)} \wedge \dots \wedge \tau^{\ell'_{-d}(\alpha)} < \infty \quad P_0(\cdot | D^\ell = \infty) - a.s. \quad (3)$$

We now define the following variables:

$$\begin{aligned} N &= \inf\{n_0 \geq 1, \forall n \geq n_0, X_n \in C(\alpha_2)\} \quad (\inf \emptyset = +\infty) \\ C &= \inf_{1 \leq n \leq N} X_n \cdot \ell \\ M &= \sup_{1 \leq n \leq N} \sum_{i=2}^d |X_n \cdot e_i|^2 \end{aligned}$$

From (3), it is clear that:

$$\begin{aligned} P_0(\cdot | D^\ell = \infty) - a.s., \quad N &< \infty, \\ C > 0 \quad \text{and} \quad M &< \infty \end{aligned}$$

We now define  $\alpha_1 = \frac{C}{\sqrt{M}} \wedge \alpha_2$  (notice that  $\alpha_1$  is random), using Cauchy-Schwarz inequality for  $n \leq N$  and the definition of  $N$  and  $C(\alpha)$  for  $n \geq N$ , we obtain:

$$\forall i \in \llbracket 2, d \rrbracket, \forall n \geq 0, \quad X_n \cdot \ell'_i(\alpha_1) = X_n \cdot \ell + \alpha_1(X_n \cdot e_i) \geq 0$$

which ends the proof of (2).

It is clear that

$$\alpha < \alpha' \quad \text{implies} \quad C(\alpha') \subset C(\alpha)$$

and so

$$\lim_{\alpha \rightarrow 0} P_0(\{D_\alpha^\ell = \infty\}) = P_0\left(\bigcup_{\alpha > 0} \{D_\alpha^\ell = \infty\}\right)$$

From (2), we have

$$\bigcup_{\alpha > 0} \{D_\alpha^\ell = \infty\} \stackrel{P_0\text{-a.s.}}{=} \{D^\ell = \infty\}$$

Since  $P_0(D^\ell = \infty) > 0$ , this concludes the proof of lemma 1.  $\square$

We will now construct a renewal structure in the same spirit as in [7] or [1]. The idea is to define a time where the walk reaches a new record in the direction  $\ell$  and never goes out of a cone (also oriented in direction  $\ell$ ) after. In [7], the walk moves from one slab to the next one, here, as in [1] or [2], the walk will move from one cone to the next one.

From lemma 1, we know that we can choose  $\alpha$  small enough so that

$$P_0(D_\alpha^\ell) > 0$$

We define now the two stopping time sequences  $(S_k)_{k \geq 0}$  and  $(R_k)_{k \geq 1}$ , and the sequence of successive maxima  $(M_k)_{k \geq 0}$

$$S_0 = 0, \quad R_0 = D_\alpha^\ell, \quad M_0 = \sup\{\ell \cdot X_n, 0 \leq n \leq R_0\}$$

And for all  $k \geq 0$ :

$$S_k = T_{M_k}, \quad R_k = D_\alpha^\ell \circ \theta_{S_{k+1}} + S_{k+1}, \quad M_{k+1} = \sup\{\ell \cdot X_n, 0 \leq n \leq R_{k+1}\}$$

$$K = \inf\{k \geq 0, S_k < \infty, R_k = \infty\}$$

On the set  $K < \infty$ , we also define:

$$\tau_1 = S_K$$

The random time  $\tau_1$  is called the first cone renewal time, and will not be confused with  $\tau^\ell$  introduced above. Under assumption (H),

$$S_0 \leq R_0 < S_1 \leq R_1 < \dots < S_n \leq R_n < \dots \leq \infty \quad (4)$$

**Proposition 2.**

$$P_0 - a.s. \quad K < \infty$$

*Proof.* For all  $k \geq 1$ ,

$$\begin{aligned} P_0(R_k < \infty) &= \mathbb{E}[E_{0,\omega}[S_k < \infty, D_\alpha \circ \theta_{X_{S_k,\omega}} < \infty]] \\ &= \sum_{x \in \mathbb{Z}^d} \mathbb{E}[E_{0,\omega}[S_k < \infty, X_{S_k} = x, D_\alpha \circ \theta_{X_{S_k,\omega}} < \infty]] \end{aligned}$$

Using Markov property we obtain,

$$\sum_{x \in \mathbb{Z}^d} \mathbb{E}[E_{0,\omega}[S_k < \infty, X_{S_k} = x] E_{x,\omega}[D_\alpha < \infty]]$$

For every  $x$  in  $\mathbb{Z}^d$ , the variables  $E_{0,\omega}[S_k < \infty, X_{S_k} = x]$  and  $E_{x,\omega}[D_\alpha^\ell < \infty]$  are respectively  $\sigma\{\omega(y, \cdot), \ell \cdot y < \ell \cdot x\}$  and  $\sigma\{\omega(y, \cdot), y \in t_x \circ C(\alpha)\}$  measurable. As this two  $\sigma$ -fields are independent,

$$\begin{aligned} P_0(R_k < \infty) &= \sum_{x \in \mathbb{Z}^d} E_0[S_k < \infty, X_{S_k} = x] E_x[D_\alpha < \infty] \\ &= P_0(S_k < \infty) P_0(D_\alpha < \infty) \\ &= P_0(R_{k-1} < \infty) P_0(D_\alpha < \infty) \end{aligned}$$

By induction, we obtain,

$$P_0(R_k < \infty) = P_0(D_\alpha < \infty)^{k+1}$$

In view of (4), this concludes the proof.  $\square$

We now define a sequence of renewal time  $(\tau_k)_{k \geq 1}$  by the following recursive relation:

$$\tau_{k+1} = \tau_1(X_\cdot) + \tau_k(X_{\tau_1+\cdot} - X_{\tau_1}) \quad (5)$$

Using proposition (2), we have:

$$\forall k \geq 0, \quad \tau_k < \infty$$

**Proposition 3.**

*$((X_{\tau_1 \wedge \cdot}, \tau_1), ((X_{(\tau_1+\cdot) \wedge \tau_2} - X_{\tau_1}), \tau_2 - \tau_1), \dots, ((X_{(\tau_k+\cdot) \wedge \tau_{k+1}} - X_{\tau_k}), \tau_{k+1} - \tau_k))$  are independent variables and for  $k \geq 1$ ,  $((X_{(\tau_k+\cdot) \wedge \tau_{k+1}} - X_{\tau_k}), \tau_{k+1} - \tau_k)$  are distributed like  $((X_{\tau_1 \wedge \cdot}, \tau_1)$  under  $P_0(\cdot | D_\alpha^\ell = \infty)$*

The proof is similar to those of corollary 1.5 in [7] and will not be repeated here.

For the classical renewal structure, Zerner proved that  $E_0[X_{\tau_1} \cdot \ell]$  is finite and computes its value. We provide here the same result but for a renewal structure with cones.

Fix a direction  $\ell$  with integer coordinates  $(a_1, \dots, a_d)$  such that their greatest common divisor,  $\gcd(a_1, \dots, a_d) = 1$ . Assume that  $(H)$  is satisfied for  $\ell$ . Complete  $\ell$  in an orthogonal basis  $(\ell, e_2, \dots, e_d)$  such that for every  $i$  in  $[[2, d]]$ ,  $|e_i| = 1$ . By lemma 1, we can choose  $\alpha$  small enough so that  $P_0(D_\alpha^\ell) > 0$  and construct the associated renewal structure that is described above.

**Lemma 2.**

$$E_0[X_{\tau_1} \cdot \ell | D_\alpha^\ell = \infty] = \frac{1}{P_0(D_\alpha^\ell = \infty)}$$

*Proof.* This proof follows an unpublished argument of M. Zerner but can be found in lemma 3.2.5 p265 of [9]. Since  $\gcd(a_1, \dots, a_d) = 1$ , we have  $\{x \cdot \ell, x \in \mathbb{Z}\} = \mathbb{Z}$ .

For all  $i > 0$ ,

$$\begin{aligned} P_0(\{\exists k \geq 1, X_{\tau_k} \cdot \ell = i\}) &= \sum_{x/\ell \cdot x = i} \mathbb{E} [E_{0,\omega} [\{X_{T_i} = y, D_\alpha^\ell \circ \theta_{T_i} = \infty\}]] \\ &= \sum_{x/\ell \cdot x = i} \mathbb{E} [E_{0,\omega} [X_{T_i} = x] E_{x,\omega} [D_\alpha^\ell = \infty]] \end{aligned} \quad (6)$$

$$= P_0(D_\alpha^\ell = \infty) \quad (7)$$

We used the strong Markov property in (6).

In (7), we notice that  $E_{0,\omega}(X_{T_i} = x)$  is  $\sigma\{\omega(y, \cdot), \ell \cdot y < \ell \cdot x\}$  measurable and  $E_{x,\omega}(D_\alpha^\ell = \infty)$  is  $\sigma\{\omega(y, \cdot), y \in t_y \circ C(\alpha)\}$  measurable and that those two  $\sigma$ -fields are independent. We will now compute the same value in another way.

$$\begin{aligned} \lim_{i \rightarrow \infty} P_0(\{\exists k \geq 1, X_{\tau_k} \cdot \ell = i\}) &= \lim_{i \rightarrow \infty} P_0(\{\exists k \geq 2, X_{\tau_k} \cdot \ell = i\}) \\ &= \lim_{i \rightarrow \infty} \sum_{n \geq 1} P_0(\{\exists k \geq 2, (X_{\tau_k} - X_{\tau_1}) \cdot \ell = i - n, X_{\tau_1} \cdot \ell = n\}) \\ &= \lim_{i \rightarrow \infty} \sum_{n \geq 1} P_0(\{\exists k \geq 2, (X_{\tau_k} - X_{\tau_1}) \cdot \ell = i - n\}) P_0(X_{\tau_1} \cdot \ell = n) \end{aligned}$$

Notice also that the first equality is true because  $P_0(X_{\tau_1} \cdot \ell > i) \rightarrow 0 (i \rightarrow \infty)$ . Using now the renewal theorem (corollary 10.2 p76 in [8]) we obtain

$$\lim_{i \rightarrow \infty} P_0(\{\exists k \geq 2, (X_{\tau_k} - X_{\tau_1}) \cdot \ell = i - n\}) = \frac{1}{E_0[(X_{\tau_2} - X_{\tau_1}) \cdot \ell]}$$

The dominated convergence theorem leads to

$$\lim_{i \rightarrow \infty} P_0(\{\exists k \geq 1, X_{\tau_k} \cdot \ell = i\}) = \frac{1}{E_0[(X_{\tau_2} - X_{\tau_1}) \cdot \ell]}$$

Comparing this result with (7), we easily obtain lemma 2.  $\square$

We have now all the tools to prove theorem 1. We will first use the two lemmas to prove that  $i)$  implies  $ii)$ .

We choose  $\ell$  with rational coordinates in the open set  $\mathcal{O}$ . It is clear that  $\ell$  satisfies assumption  $(H)$ . Actually, we can also assume, without loss of generality, that  $\ell$  has integer coordinates and that their greatest common divisor is 1. Indeed, there is  $\lambda$  rational such that  $\lambda\ell$  has integer coordinates with greatest common divisor equal to 1, and of course,  $\lambda\ell$  also satisfies  $(H)$ .

We complete  $\ell$  into an orthogonal basis  $(e_2, \dots, e_d)$ , such that for every  $i$  in  $\llbracket 2, d \rrbracket$ ,  $|e_i| = 1$ . Using lemma 1, we choose  $\alpha$  small enough so that  $P_0(D_\alpha^\ell = \infty) > 0$ . We can now use the renewal structure with cones and we have from lemma 2 that:

$$E_0[X_{\tau_1} \cdot \ell | D_\alpha^\ell = \infty] = \frac{1}{P_0(D_\alpha^\ell = \infty)} < \infty$$

From the definition of the cone renewal structure,

$$\exists c(\alpha) \in \mathbb{R}^+ \text{ s.t. } P_0(\cdot | D_\alpha^\ell = \infty) - a.s., \quad |X_{\tau_1}| \leq c(\alpha) X_{\tau_1} \cdot \ell$$



and so using lemma 2,

$$E_0 [|X_{\tau_1}| | D_\alpha^\ell = \infty] < \infty \quad (8)$$

We can now apply the law of large numbers, and obtain

$$\frac{X_{\tau_k}}{k} \xrightarrow{P_0\text{-a.s.}} E_0 [X_{\tau_1} | D_\alpha^\ell = \infty]$$

As  $|E_0 [X_{\tau_1} | D_\alpha^\ell = \infty]| > 0$ ,

$$\frac{X_{\tau_k}}{|X_{\tau_k}|} \xrightarrow{P_0\text{-a.s.}} \frac{E_0 [X_{\tau_1} | D_\alpha^\ell = \infty]}{|E_0 [X_{\tau_1} | D_\alpha^\ell = \infty]|} \stackrel{(def)}{=} \nu \quad (9)$$

To complete the proof, we have to control the behavior of the walk between the renewal times. For each natural  $n$ , we introduce the index  $k(n)$  such that,

$$\tau_{k(n)} \leq n < \tau_{k(n)+1}$$

Recall that if  $(Z_n)$  is an i.i.d. sequence of variables with finite expectation, the application of the strong law of large numbers to  $(Z_{n+1} - Z_n)_n$  shows that  $\frac{Z_n}{n}$  converges almost surely to 0. We apply this remark to the sequence  $(\sup_n |X_{n \wedge \tau_{k+1}} - X_{\tau_k}|)_k$  and obtain:

$$\frac{\sup_n |X_{n \wedge \tau_{k+1}} - X_{\tau_k}|}{k} \xrightarrow{P_0\text{-a.s.}} 0 \quad (k \rightarrow \infty) \quad (10)$$

Using equation (9) and (10), we study the convergence in *ii*),

$$\frac{X_n}{|X_n|} = \frac{X_n - X_{\tau_{k(n)}}}{|X_n|} + \frac{X_{\tau_{k(n)}}}{|X_n|} \quad (11)$$

By proposition 2 and (5),

$$P_0\text{-as} \quad k(n) \rightarrow \infty \quad (n \rightarrow \infty)$$

As  $|X_n| \geq k(n)$ , (10) leads to:

$$\frac{X_n - X_{\tau_{k(n)}}}{|X_n|} \xrightarrow{P_0\text{-a.s.}} 0 \quad (12)$$

To control the second term in (11), we simply write

$$\frac{|X_{\tau_{k(n)}}|}{k(n)} \leq \frac{|X_n|}{k(n)} \leq \frac{|X_{\tau_{k(n)}}|}{k(n)} + \frac{|X_n - X_{\tau_{k(n)}}|}{k(n)}$$

Using (9) and (10), we obtain,

$$\frac{|X_n|}{k(n)} \xrightarrow{P_0\text{-a.s.}} |E_0 [X_{\tau_1} \cdot \ell | D_\alpha^\ell = \infty]|$$

We finally obtain the desired convergence :

$$\frac{X_n}{|X_n|} \xrightarrow{P_0\text{-a.s.}} \nu = \frac{E_0 [X_{\tau_1} | D_\alpha^\ell = \infty]}{|E_0 [X_{\tau_1} | D_\alpha^\ell = \infty]|}$$

The end of the proof of theorem 1 is easy: it is obvious that *iii*) implies *i*) and so we just have to show that *ii*) implies *iii*).

Let  $\ell$  be a direction such that  $\ell \cdot \nu > 0$ . It is known since [7] (lemma 1.1) that

$P_0(A_\ell \cup A_{-\ell})$  follows a 0 – 1 law under assumption of strict uniform ellipticity, but we use here proposition 3 in [11] where the same result is proved under the weaker assumption of strict ellipticity.

If  $P_0(A_\ell \cup A_{-\ell}) = 0$ , it is known that the walk under  $P_0$  oscillates,

$$\limsup_{n \rightarrow \infty} X_n \cdot \ell = -\liminf_{n \rightarrow \infty} X_n \cdot \ell = +\infty \quad P_0 - a.s.$$

This is not possible in view of *ii*) and so  $P_0(A_\ell \cup A_{-\ell}) = 1$ .

But because of *ii*),  $P_0(A_{-\ell}) = 0$ , and we can conclude

$$P_0(A_\ell) = 1$$

□

**Remark 2.** *From the proof of theorem 1, we know that if a walk has an asymptotic direction, we can construct a renewal structure with cones and  $E[X_{\tau_1}]$  is finite. We can then easily derive a law of large numbers, namely*

$$\frac{X_n}{n} \xrightarrow{+ \infty} \frac{E[|X_{\tau_1}| | D_\alpha^\ell = \infty]}{E[\tau_1 | D_\alpha^\ell = \infty]} \quad P_0 - a.s.$$

*However this limit can be null (if and only if  $E[\tau_1 | D_\alpha^\ell = \infty] = +\infty$ ) and, in this case, the asymptotic direction is an interesting information about the walk behavior.*

**Remark 3.** *The class of walks admitting an asymptotic direction is exactly the class of transient walks (in a direction  $\ell$ ) such that  $E[|X_{\tau_\ell}| | D^\ell = \infty] < \infty$ . For one of the inclusion just notice that  $\tau_1$  is also one of the hyperplane renewal time. For the other one, use the end of the proof of *i*) implies *ii*) in theorem 1 (from (8)) with  $\tau^\ell$  instead of  $\tau_1$ .*

**Remark 4.** *It is not easy to check *i*) (or (H)). However we can find in previous papers some examples of walks admitting an asymptotic direction and then satisfying *i*). Of course all criteria assuring a ballistic law of large numbers work like Kalikow's condition in [3] or more recently condition (T') of Sznitman in [6]. The condition (T) $_\gamma$  in [6] also works and it is not proved that it implies a law of large numbers. More generally, the previous remark gives criteria to describe exhaustively the class of walks admitting an asymptotic direction.*

*Proof of Proposition 1.* Suppose that the proposition is false and call  $\nu$  and  $\nu'$  two vectors of  $S^{d-1}$  non opposite such that  $P_0(B_\nu)P_0(B_{\nu'}) > 0$ , then we will show that

$$\exists \nu_0 \text{ s.t. } P_0(B_{\nu_0} | B_\nu \cup B_{\nu'}) = 1 \quad (13)$$

what establishes, of course, a contradiction.

**Remark 5.** *If  $\nu \in S^{d-1}$  is such that  $P_0(B_\nu) > 0$  then,*

$$\forall \ell \in \mathbb{R}^d \text{ s.t. } \ell \cdot \nu > 0 \quad P_0(A_\ell | B_\nu) = 1 \quad (14)$$

*Indeed, it is clear from 0 – 1 law that  $P_0(A_\ell \cup A_{-\ell}) = 1$  (just notice that the walk does not oscillate along direction  $\ell$  on  $B_\nu$ , set with positive probability). But as  $B_\nu \subset \{\exists N \text{ s.t. } \forall n \geq N, X_n \cdot \ell > 0\}$ , it is clear that  $P_0$  almost surely,  $B_\nu \subset A_\ell$ .*

$E = \{x, x \cdot \nu > 0\} \cap \{x, x \cdot \nu' > 0\}$  is a non empty open set and from (14),

$$\forall \ell \in E, \quad P_0(A_\ell | B_\nu \cup B_{\nu'}) = 1 \quad (15)$$

Fix a vector  $\ell_0$  in  $E$ , it is obvious that there exists a neighborhood  $\mathcal{V}$  of  $\ell_0$  such that,

$$\forall \ell \in \mathcal{V}, \quad P_0(A_\ell | B_\nu \cup B_{\nu'}) = 1$$

This last equation reminds the assumption (H) and the proof of (13) will be adapted from that of theorem 1. Before beginning, notice that if  $P_0(B_\nu \cup B_{\nu'}) = 1$ , we can easily conclude using theorem 1, but the proof is not that obvious if  $P_0(B_\nu \cup B_{\nu'}) < 1$ .

The first step is to show that for  $\alpha$  small enough  $P_0(D_\alpha^{\ell_0}) > 0$ . We will use a proof very close from those of lemma 1 replacing the probability  $P_0$  by  $P_0(\cdot | B_\nu \cup B_{\nu'})$ . We have to prove that

$$P_0(D^{\ell_0} = \infty | B_\nu \cup B_{\nu'}) > 0 \quad (16)$$

Suppose that  $P_0(D^{\ell_0} = \infty, B_\nu \cup B_{\nu'}) = 0$ . Using invariance translation, it is clear that for any  $x$  in  $\mathbb{Z}^d$ ,  $P_x(D^{\ell_0} = \infty, B_\nu \cup B_{\nu'}) = 0$  which means

$$\forall x \in \mathbb{Z}^d, \mathbb{P} - as, \quad P_{x,\omega}(D^{\ell_0} = \infty, B_\nu \cup B_{\nu'}) = 0$$

and we obtain

$$\mathbb{P} - as, \forall x \in \mathbb{Z}^d, \quad P_{x,\omega}(D^{\ell_0} = \infty, B_\nu \cup B_{\nu'}) = 0$$

We note  $(D^{\ell_0})^n$  the  $n$ -th backtrack time of the walk, defined by the following recursive relation

$$\begin{aligned} (D^{\ell_0})^1 &= D^{\ell_0} \\ (D^{\ell_0})^n &= D^{\ell_0} \circ \theta_{(D^{\ell_0})^{n-1}} \quad \forall n \geq 2 \end{aligned}$$

$\mathbb{P} - as$ , for any  $n \geq 1$ ,

$$\begin{aligned} &P_{0,\omega}((D^{\ell_0})^n < \infty, (D^{\ell_0})^{n+1} = \infty, B_\nu \cup B_{\nu'}) \\ &= \sum_{x \in \mathbb{Z}^d} P_{0,\omega}((D^{\ell_0})^n < \infty, X_{(D^{\ell_0})^n} = x) P_{x,\omega}(D^{\ell_0} = \infty, B_\nu \cup B_{\nu'}) \\ &= 0 \end{aligned} \quad (17)$$

We obtain (17) writing that for any  $x$  in  $\mathbb{Z}^d$  and natural  $m$

$$\begin{aligned} &\{(D^{\ell_0})^n = m, X_{(D^{\ell_0})^n} = x\} \cap \{B_\nu \cup B_{\nu'}\} \\ &= \{(D^{\ell_0})^n = m, X_{(D^{\ell_0})^n} = x\} \cap \left\{ \lim_{n > m} \frac{X_n - x}{|X_n - x|} = +\infty \right\} \end{aligned} \quad (18)$$

Here is the only difference with the proof of (1.16) in proposition 1.2 in [7]. Remember that

$$P_{0,\omega}(D^{\ell_0} = \infty, B_\nu \cup B_{\nu'}) = 0$$

and notice that as  $B_\nu \cup B_{\nu'} \subset A_{\ell_0}$ ,

$$P_{0,\omega}(B_\nu \cup B_{\nu'}, (D^{\ell_0})^k < \infty, \forall k \geq 1) = 0$$

We obtain

$$\mathbb{P} - as, \quad P_{0,\omega}(B_\nu \cup B_{\nu'}) = 0$$

$P_0(B_\nu \cup B_{\nu'}) = 0$  contradicts the assumption and this concludes the proof of (16). Following the proof of lemma 1 with  $P_0(\cdot|B_\nu \cup B_{\nu'})$  instead of  $P_0$ , we obtain

$$\exists \alpha_0 \text{ s.t. } \forall \alpha \leq \alpha_0, \quad P_0(D_\alpha^{\ell_0}) > 0$$

The second step is to show that on  $A_{\ell_0}$  (this is much easier than on  $B_\nu \cup B_{\nu'}$ ), the walk admits a unique asymptotic direction. We define the same renewal structure with cones as in the proof of theorem 1. It is easy to adapt the results in [7] from proposition 1.2 to corollary 1.5, we obtain the same results but with variables and random times associated to the renewal structure with cones and not with hyperplanes. We also adopt the same notation,  $Q_0$  to denote the probability measure  $P_0(\cdot|A_{\ell_0})$ . We can then adapt lemma 2 and obtain

**Lemma 3.**

$$E_{Q_0}[X_{\tau_1} \cdot \ell | D_\alpha^\ell = \infty] = \frac{1}{P_0(D_\alpha^\ell = \infty)}$$

The end of the proof now follows exactly that of theorem 1 except that  $Q_0$  replaces  $P_0$ . We obtain the existence of  $\nu_0$  satisfying (13).  $\square$

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## REFERENCES

- [1] Francis Comets and Ofer Zeitouni. A law of large numbers for random walks in random mixing environments. *Ann. Probab.*, 32(1B):880–914, 2004.
- [2] Francis Comets and Ofer Zeitouni. Gaussian fluctuations for random walks in random mixing environments. *Israel Journal of Mathematics*, 148:87–114, 2005.
- [3] Steven A. Kalikow. Generalized random walk in a random environment. *Ann. Probab.*, 9(5):753–768, 1981.
- [4] Harry Kesten. A renewal theorem for random walk in a random environment. In *Probability (Proc. Sympos. Pure Math., Vol. XXXI, Univ. Illinois, Urbana, Ill., 1976)*, pages 67–77. Amer. Math. Soc., Providence, R.I., 1977.
- [5] Alain-Sol Sznitman. On a class of transient random walks in random environment. *Ann. Probab.*, 29(2):724–765, 2001.
- [6] Alain-Sol Sznitman. An effective criterion for ballistic behavior of random walks in random environment. *Probab. Theory Related Fields*, 122(4):509–544, 2002.
- [7] Alain-Sol Sznitman and Martin Zerner. A law of large numbers for random walks in random environment. *Ann. Probab.*, 27(4):1851–1869, 1999.
- [8] Hermann Thorisson. *Coupling, stationarity, and regeneration*. Probability and its Applications (New York). Springer-Verlag, New York, 2000.
- [9] Ofer Zeitouni. Random walks in random environment. In *Lectures on probability theory and statistics*, volume 1837 of *Lecture Notes in Math.*, pages 189–312. Springer, Berlin, 2004.
- [10] Martin P. W. Zerner. A non-ballistic law of large numbers for random walks in i.i.d. random environment. *Electron. Comm. Probab.*, 7:191–197 (electronic), 2002.
- [11] Martin P. W. Zerner and Franz Merkl. A zero-one law for planar random walks in random environment. *Ann. Probab.*, 29(4):1716–1732, 2001.