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# Universality and quantum effects in one-component critical fluids

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Non-universal scale transformations of the physical fields are extended to pure quantum fluids and used to calculate susceptibility, specific heat and the order parameter along the critical isochore of  $\text{He}^3$  near its liquid-vapor critical point. Within the so-called preasymptotic domain, where the Wegner expansion restricted to the first term of confluent corrections to scaling is expected valid, the results show agreement with the experimental measurements and recent predictions, either based on the minimal-substraction renormalization and the massive renormalization schemes within the  $\Phi_{d=3}^4$  ( $n = 1$ )-model, or based on the crossover parametric equation of state for Ising-like systems.

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## 1. INTRODUCTION

It is well known that the thermodynamic quantities of real pure fluids close to their gas-liquid critical point (CP) follow the asymptotic power-law behavior predicted for the 3D Ising-like universality class in the *asymptotic* critical domain where  $\kappa \ll \Lambda_0$  [1]. The distance to the critical point is here measured by the parameter  $\kappa$ , related to the inverse correlation length  $\xi^{-1}$ .  $\xi$  characterizes the spatial extent of the diverging fluctuations of the local density which is related to the order parameter density of the gas-liquid transition.  $\Lambda_0$  is a (nonuniversal) finite wave-number characterizing a discrete microscopic structure of a fluid with spacing  $\Lambda_0^{-1}$ . So that the critical thermophysical behavior of the fluid properties occurs when  $\xi\Lambda_0 \gg 1$ . Asymptotically close to the critical point, this microscopic parameter  $\Lambda_0$  which characterizes each pure fluid, becomes unimportant when the thermodynamic properties become singular. It means that all the pure fluids in their asymptotic critical domain obey to the two-scale universality associated to hyperscaling. Their properties can then be described by the same reduced equation of state (e.o.s.) and the same correlation functions, using only two dimensionless parameters which are two fluid-dependent parameters, in conformity with the two-scale-factor universality of the 3D Ising-like universality class.

However, it is also now well established that away from this asymptotic critical region, the properties of real pure fluids can deviate from hyperscaling. This deviation can take origin on crossover phenomenon which reflects a competition between universality and nonuniversality when  $\xi\Lambda_0 \gtrsim 1$ . This crossover problem has been investigated in considerable details, mainly in the classical-to-critical crossover framework of field theory [2]. The resulting field theoretical crossover functions describe the crossover behavior of the  $\phi_{d=3}^4$  ( $n = 1$ ) model in the uni-

versality class  $\mathcal{O}$  ( $n = 1$ ) in three dimensions ( $n = 1$  is the dimension of the order parameter density for the critical transition, and  $d = 3$  is the space dimension of the system). A better understanding of non-universal behavior linked to finite values (although large) of the correlation length is then accounted for by a restricted summation of the Wegner expansion [3], which introduces one additional system-dependent parameter to characterize the preasymptotic domain [4, 5] [as discussed in [5] the singular power laws expressed at the first-order of the Wegner expansion are expected to be valid within the preasymptotic domain]. Moreover, the values of the adjustable parameters can then be dependent on the approximations needed by each particular renormalization scheme. As a practical result, the microscopic length, the crossover parameter, as well as the two asymptotic scale factors, enter in a larger set of *adjustable* parameters, including obviously an extended set of *measurable* critical point coordinates. Therefore the exact nature of the two asymptotic scale factors for the fluid physical fields, still remains an implicit open question. This is still the object of a debating situation [6], due to the fact that *fluid variables* have no definite critical scaling dimensionality *at finite distance* to the critical point.

The asymptotic existence of such two scale factors proper to the one-component fluid subclass, were initially postulated in [7, 8] on a phenomenological basis supporting the asymptotic results of the massive renormalization scheme [4, 5, 9, 10] of the field theory framework. It was hypothesized that the complete information to estimate asymptotic singular fluid behaviors is provided by the *experimental critical point location*, i.e. by a minimal parameter set, noted  $Q_{c,a\bar{p}}^{\min}$  [11], composed of four (generalized) critical coordinates (the subscript  $c$  refers to a property defined at the critical point, while the subscript  $\bar{p}$  refers to a property normalized per particle). This minimal set defines the critical point location on the equilibrium phase surface of equation  $\Phi(p, v_{\bar{p}}, T) = 0$ , where  $p$  is the pressure,  $v_{\bar{p}} = \frac{V}{N}$  is the volume per particle, and  $T$  is the temperature ( $N$  is the total number of particles occupying a total volume  $V$ ). The generalized coordinates

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are composed by three usual critical point coordinates and one preferred direction of the tangent plane to the phase surface. Using xenon as a standard critical fluid [7, 8, 10], it was then proposed to perform adequate scale dilatations of the two relevant physical variables for each one-component fluid. Applying such a scale dilatation method, we were able to “renormalize” (i.e. rescale) the physical singular behavior of any one-component fluid on the corresponding “master” (i.e. unique) singular behavior, where master (i.e. constant amplitudes) features with respect to the one-component fluid subclass are conform to universal features with respect to the complete 3D Ising like universality class.

Specifically, this initial substantiation of master scaling is based on the explicit choice of the same metric factor for thermodynamics *and* correlations. That permits an unambiguous definition of the microscopic length  $\Lambda_0^{-1}$  proportional to a critical length scale factor  $\alpha_c = \left(\frac{k_B T_c}{p_c}\right)^{\frac{1}{d}}$  made from an appropriate combination of the critical temperature  $T_c$  and pressure  $p_c$  coordinates [ $k_B$  is the Boltzmann constant and  $d = 3$ ]. From well-known shorted-range of the Lennard-Jones (LJ) like molecular interactions in one-component fluids [12], characterized by the equilibrium position  $r_e^{LJ}$  between two interacting particles, we have  $\Lambda_0^{-1} = \alpha_c \cong 2r_e^{LJ}$  [7], ignoring then the possible contribution of quantum effects on  $\Lambda_0^{-1}$ . In this paper, using the recent experimental measurements near the critical point of  ${}^3He$  [13, 14], we extend this scale dilatation method (SDM) to the quantum fluid case. This extension is based on a phenomenological modification of the non-quantum renormalized critical behavior, which is only valid at the critical temperature. Since experimental values of the minimum critical set already contain their actual contribution of quantum effects, we expect that remaining part of quantum effects only affect the microscopic length  $(\Lambda_0)^{-1}$ , in such a relative way that  $(\Lambda_0 \Lambda_{qe}^*)^{-1} = \alpha_c$  [see below our Eq. (14)]. The adjustable dimensionless parameter  $\Lambda_{qe}^*$  is here introduced in order to maintain the master features observed for the one-component fluid subclass. Therefore, in addition to the minimal set of four critical parameters, the renormalized variables need to use only one supplementary well-defined dimensionless parameter  $\Lambda_{qe}^*$ , whose value is, either fluid-independent ( $\Lambda_{qe}^* = 1$ ) in the absence of quantum effects, or quantum-fluid-dependent ( $\Lambda_{qe}^* > 1$ ) in the presence of quantum effects, without violating the asymptotic universal features of the 3D Ising-like universality class.

The paper is organized as follows. In section 2 we recall the basic elements of the scale dilatation method and we introduce its extension to account for quantum effects on the microscopic length scale. In section 3 we consider the fitting results [14] obtained by Zhong et al for  ${}^3He$  to discuss our estimated value of the adjustable parameter  $\Lambda_{qe}^*({}^3He) = 1.11009$ . Before to conclude, the section 4 gives a brief comparison with three crossover modelling of  ${}^3He$  critical properties.

## 2. SCALE DILATATION OF THE FLUID PHYSICAL VARIABLES

### 2.1. The minimal set $Q_{c,a_{\bar{p}}}^{min}$ of four scale factors

As recalled in our introduction, the basic idea [7, 8] of the scale dilatation method relies on a simple thermodynamic assertion concerning the thermodynamic information provided by the critical point location on the fluid phase surface of equation  $\Phi(p, v_{\bar{p}}, T) = 0$  [11]. The minimum of information needed to predict singular thermodynamic behavior of a pure fluid is given by:

(1) the three critical coordinates  $T_c$ ,  $p_c$ , and  $v_{\bar{p},c}$  of the liquid-vapor critical point;

(2) the two preferred directions which define the position of the tangent plane to the phase surface at the critical point (both needed in order to characterize the linearized asymptotic approach towards the critical point along two well-defined thermodynamic paths). One direction is common to all pure fluids (since  $\left(\frac{\partial p}{\partial v_{\bar{p}}}\right)_{T=T_c} = \left(\frac{\partial T}{\partial v_{\bar{p}}}\right)_{p=p_c} = 0$ ), and only the second direction,

$$\gamma'_c = \left(\frac{\partial p}{\partial T}\right)_{v_{\bar{p}}=v_{\bar{p},c}} = \left(\frac{dp_{sat}}{dT}\right)_{T=T_c} \quad (1)$$

is characteristic of each pure fluid.  $p_{sat}$  is the saturation pressure in the non homogeneous domain. We note

$$Q_{c,a_{\bar{p}}}^{min} = \{T_c, p_c, v_{\bar{p},c}, \gamma'_c\} \quad (2)$$

this minimal set made of four critical parameters.

From thermodynamic principles, this topological information concerns all the incipient equilibrium states very close to the unstable single critical point.

From these four coordinates we can calculate unequivocally the following four fluid characteristic parameters,

$$(\beta_c)^{-1} = k_B T_c \sim [energy] \quad (3)$$

$$\alpha_c = \left(\frac{k_B T_c}{p_c}\right)^{\frac{1}{d}} \sim [length] \quad (4)$$

$$Z_c = \frac{p_c m_{\bar{p}}}{\rho_c k_B T_c} \quad (5)$$

$$Y_c = \left(\gamma'_c \frac{T_c}{P_c}\right) - 1 \quad (6)$$

where  $\rho_c = \left(\frac{N}{V}\right)_c m_{\bar{p}}$  is the critical density of the system made of particles of known individual mass  $m_{\bar{p}}$ .  $(\beta_c)^{-1}$  of Eq. (3) fixes the energy unit at the macroscopic scale.  $\alpha_c$  of Eq. (4) fixes the length unit at the macroscopic scale. The two Eqs. (3,4) are sufficient to make dimensionless

all the thermodynamic and correlation functions of pure fluids [7, 8].  $Z_c$  of Eq. (5) is the critical compression factor. We then introduce the useful compression factor  $Z = \frac{-J(T,V,\mu_{\bar{p}})}{k_B T}$  as the dimensionless opposite form of the total Grand potential  $J(T, V, \mu_{\bar{p}}) = -p(T, \mu_{\bar{p}}) \times V$ , expressed in terms of its three natural intensive variables  $T$ ,  $V$ , and  $\mu_{\bar{p}}$  [For the total system,  $\mu_{\bar{p}}$  is the chemical potential per particle, i.e. the intensive variable conjugated to  $N$ , independent to  $p$  and  $T$ , respectively, which are the two other independent intensive variables, conjugated to  $V$  and  $S$ , respectively]. From the experimental phase surface of equation  $\Phi(p, v_{\bar{p}}, T) = 0$  it is easy to construct another practical phase surface of equation  $\Phi(Z, \rho, T) = 0$ . In such a representation of the fluid equilibrium states, the characteristics numbers  $Z_c$  of Eq. (5) and  $Y_c Z_c$  made of the product between Eqs. (5) and (6), read as follows

$$Z_c = - \left[ \left( \frac{\partial Z}{\partial \rho} \right)_{\varpi} \right]_{CP} = - \left[ \left( \frac{\partial Z}{\partial \rho} \right)_{LVE} \right]_{CP} \quad (7)$$

$$Y_c Z_c = \left[ \left( \frac{\partial Z}{\partial T} \right)_{\rho} \right]_{CP} \quad (8)$$

where all the derivatives refer to their values for the critical point coordinates, while  $\varpi$  means any isocline at constant (critical) value of one intensive variable  $\varpi$  among  $T$ ,  $p$ , or  $\mu_{\bar{p}}$ , and  $LVE$  means the liquid vapor equilibrium line. Therefore, the two characteristic numbers  $Z_c$  and  $Y_c Z_c$  are the two ‘‘preferred’’ critical directions [15] at the critical point of the phase surface, for the critical isotherm path and the critical isochore path, respectively.

From basic modeling of a binary effective interaction characterized by a minimum energy well depth  $\varepsilon_m^{LJ}$  at the pair equilibrium position  $r_m^{LJ}$  between two particles, we obtain,  $(\beta_c)^{-1} \cong \varepsilon_m^{LJ}$  and  $\alpha_c \cong 2 r_m^{LJ}$ , where  $\varepsilon_m^{LJ}$  and  $r_m^{LJ}$  are the respective natural units for energy and length, at the microscopic scale. Here the subscript  $LJ$  stands for a *short-ranged* Lennard-Jones-like *potential* [12]. It follows that  $\alpha_c$  measures the mean extension range of the attractive dispersion forces and

$$v_{c,I} = (\alpha_c)^d \quad (9)$$

is the critical volume of the microscopic *critical interaction cell*. In such a configuration, the inverse of the critical compression factor takes clear physical meaning since

$$\frac{1}{Z_c} = \frac{v_{c,I}}{v_{\bar{p},c}} = n_{c,I}^* \quad (10)$$

is the *number of fluid particles filling the interaction cell at criticality*, i.e. for  $T = T_c$ ,  $n = n_c$ , and  $\Lambda_0 \xi = \infty$  [ $v_{\bar{p},c} = (\frac{V}{N})_{PC}$  is the critical volume per particle, and  $n(n_c)$  is the (critical) number density]. From that result, to formulate dimensionless thermodynamics in terms of normalization per particle (subscript  $\bar{p}$ ), or in terms of

normalization per critical interaction cell (subscript  $I$ ), appears easy. As an immediate consequence, from Eqs. (10) and (8),  $\frac{1}{Z_c}$  and  $Y_c$  are two characteristics numbers of the critical interaction cell.

The next step consists to postulate that the two numbers  $\{Z_c; Y_c\}$ , defined by the two Eqs. (5) and (6), are the remaining pair of dimensionless characteristic parameters at the *scale of the critical interaction volume*, whatever the selected one-component fluid. In addition, it is admitted that  $Z_c$  is the characteristic factor of the scaling at the critical point and along the critical isotherm, while  $Y_c$  is the characteristic factor of the scaling along the critical isochore. Rewriting Eq. (2) as

$$Q_{c,a_{\bar{p}}}^{min} = \left\{ (\beta_c)^{-1}, \alpha_c, Z_c, Y_c \right\}_{CIC} \quad (11)$$

we can expect that the complete information is made from four scale factors which characterize the critical interaction cell (subscript  $CIC$ ). Then, as initially proposed in [7, 8], the master singular behavior of the correlation functions at exact criticality and along the critical isochore permits one to link unequivocally their associated asymptotic amplitudes  $\hat{D}$  and  $\xi_0^+$  [16], to  $Z_c$  and  $Y_c$ , respectively, providing simultaneously the hyperscaling [8].

## 2.2. Quantum effects on the scale dilatation of physical fluid variables

The scheme given in [8] also requires that the inverse microscopic wave number  $\Lambda_0^{-1}$  is proportional to the characteristic length scale  $\alpha_c$ . Now, owing to the short ranged molecular interaction in light pure fluids [17], the influence of quantum mechanical effects changes appreciably the shape of the Lennard-Jones-like potential, slightly increasing the range of this interparticle potential [12]. This qualitative evidence was demonstrated by introducing an effective potential, which is then a temperature-dependent quantity [18, 19, 20]. The quantum effects increase as temperature decreases. However, due to the formal analogy with the FT renormalization scheme, our rescaling is basically defined for the critical asymptotic domain, i.e. only when  $T \cong T_c$ . Moreover, since the use of the actual critical parameter already includes quantum effects, the remaining additional quantum effect, for  $T \cong T_c$ , acts only through the relative modification of the microscopic length at  $T_c$ . In the absence of theoretical support to do this modification, we propose to normalize its contribution with respect to the microscopic inverse wave number defined for non-quantum fluids.

This contribution is expected low and then limited to a small additive departure from unity. This additive value, noted  $\lambda_c$ , can then include the two main phenomenological characteristics of quantum particles :

i) their low mass and size, accounted for using proportionality to the ratio  $\frac{\Lambda_{T_c,c}}{\alpha_c}$  between the critical thermal

wavelength,

$$\Lambda_{T,c} = \frac{h_P}{(2\pi m_{\bar{p}} k_B T_c)^{\frac{1}{2}}}$$

(where  $h_P$  is the Planck constant), and our microscopic critical range  $\alpha_c$  of the interaction;

ii) their statistics (like bosons, fermions, etc.), accounted for by introducing a supplementary free parameter, noted  $\lambda_{q,f}$ .

So that, we characterize the quantum corrections by the following non-dimensional factor

$$\Lambda_{qe}^* = 1 + \lambda_c \quad (12)$$

with

$$\lambda_c = \lambda_{q,f} \frac{\Lambda_{T,c}}{\alpha_c} \quad (13)$$

$\lambda_c \geq 0$  is then the measure of the relative modification of the shape and range of molecular interaction due to the quantum effects.

Since the quantum effects increase slightly the range of the molecular interaction, we postulate that the *corrected* microscopic wave number now reads

$$\Lambda_0 \Lambda_{qe}^* = \frac{1}{\alpha_c} \quad (14)$$

(in a non-quantum fluid, our previous relation was  $\Lambda_0 = \frac{1}{\alpha_c}$ , implicitly). We expect that the rescaled *quantum*-fluid correlation length  $\ell_{qf}^*$  presents the master divergence previously defined for all the non-quantum one-component fluids [8, 21]. Then,

$$\ell_{qf}^* = \Lambda_0 \xi = (\Lambda_{qe}^*)^{-1} \xi^* \quad (15)$$

with

$$\xi^* = \frac{\xi}{\alpha_c} \quad (16)$$

Similarly, any rescaled singular thermodynamic property of the quantum-fluid can be formulated from dimensional analysis, in order to account for its proper  $\Lambda_{qe}^*$  contribution within  $v_{c,I}$ , which maintains valid the previous master hypotheses made for the non-quantum fluid subclass.

Therefore, thanks to the formal analogy between the scale dilatation method [8] and the basic hypotheses of the renormalization group approach [22, 23], all the above quantum corrections are intrinsically accounted for according our renormalization scheme, provided that the transformations (dilatations) of the two relevant physical fields are made throughout the following analytical relations

$$\mathcal{T}_{qf}^* \equiv \mathcal{T}^* = Y_c \Delta \tau^* \quad (17)$$

$$\mathcal{H}_{qf}^* = (\Lambda_{qe}^*)^2 \mathcal{H}^* = (\Lambda_{qe}^*)^2 (Z_c)^{-\frac{d}{2}} \Delta h^* \quad (18)$$

Consequently, the dilatation of the physical order parameter density reads as follow

$$\mathcal{M}_{qf}^* = \Lambda_{qe}^* \mathcal{M}^* = \Lambda_{qe}^* (Z_c)^{\frac{d}{2}} \Delta m^* \quad (19)$$

In Eqs. (17) to (19),

$$\Delta \tau^* = k_B \beta_c (T - T_c) \quad (20)$$

$$\Delta h^* = \beta_c (\mu_{\bar{p}} - \mu_{\bar{p},c}) \quad (21)$$

while,

$$\Delta m^* = (n - n_c) (\alpha_c)^d \quad (22)$$

$\mu_{\bar{p},c}$  is the critical chemical potential per particle. Obviously, in Eqs. (17) to (19),  $\mathcal{T}^*$ ,  $\mathcal{H}^*$  and  $\mathcal{M}^*$  are the renormalized variables already defined for non-quantum fluids [8, 21].

### 2.3. Master and physical singular behavior

Because such transformations of the physical fields in the FT framework have a range of validity including (at least) the first correction-to-scaling [24], our rescaled thermodynamic and correlation functions should conform to the two-term (leading and first-confluent) asymptotic description of singularities within the preasymptotic domain. For example when  $\mathcal{T}^*$  goes to zero along the critical isochore, the critical behavior of any rescaled singular property  $\mathcal{P}_{qf}^*$  reads

$$\mathcal{P}_{qf}^* = \mathcal{Z}_{\mathcal{P}}^{\pm} |\mathcal{T}^*|^{-x} \left[ 1 + \mathcal{Z}_{\mathcal{P}}^{1,\pm} |\mathcal{T}^*|^{\Delta} + \mathcal{O}(|\mathcal{T}^*|^{2\Delta}) \right] \quad (23)$$

where  $x$  and  $\Delta$  are the associated universal critical exponents [25]. The subscript  $+$  is for the homogenous domain  $\mathcal{T}^* > 0$  (i.e.  $T > T_c$ ), and the subscript  $-$  is for the non-homogeneous domain  $\mathcal{T}^* < 0$  (i.e.  $T < T_c$ ). The leading amplitudes  $\mathcal{Z}_{\mathcal{P}}^{\pm}$ , and the first confluent amplitudes  $\mathcal{Z}_{\mathcal{P}}^{1,\pm}$ , are *master* (constant) numbers for all pure fluids. Their respective values are obtained using xenon as a standard critical fluid [21], and accounting as closely as possible for up-to-date estimates [24, 25] of universal asymptotic critical quantities (exponents and amplitudes combinations).

When the generalized critical parameters of a pure fluid are known, there is an immediate practical interest to reverse the use of the scale dilatation method. In fact, the basic advantage of this method is its ability to calculate all the amplitudes appearing in the singular divergences expressed at first-order of the Wegner expansion in  $\Delta \tau^*$ . For  $\Delta \tau^* \rightarrow 0^{\pm}$ , the critical behavior of the physical property  $P$  of the selected pure fluid is represented by the two-term equation

$$P = P_0^{\pm} |\Delta \tau^*|^{-x} \left[ 1 + P_1^{\pm} |\Delta \tau^*|^{\Delta} + \mathcal{O}(|\Delta \tau^*|^{2\Delta}) \right] \quad (24)$$

where  $P_0^\pm$  and  $P_1^\pm$  are the leading and the first confluent amplitudes. All the values of  $P_0^\pm$  and  $P_1^\pm$  can then be estimated when the basic set of critical parameters is known for a selected pure fluid, using each unequivocal relation linking the physical quantity to its renormalized one (see Section 3 below, and the Table I, columns 6 and 7).

### 3. THE HELIUM 3 CASE

#### 3.1. Notations

The scale dilation method is now applied to the description of the isothermal susceptibility, specific heat, and coexisting density measurements [14] along the critical isochore of  $^3He$ . We complete these measurements by the estimation of the correlation length inferred from the two-scale-factor universality. Let us introduce the corresponding notations for:

i) the master singular behaviors

$$\ell_{qf}^* = \mathcal{Z}_\ell^\pm (\mathcal{T}^*)^{-\nu} \left[ 1 + \mathcal{Z}_\ell^{1,\pm} (\mathcal{T}^*)^\Delta \right] \quad (25)$$

$$\chi_{qf}^* = \mathcal{Z}_\chi^\pm (\mathcal{T}^*)^{-\gamma} \left[ 1 + \mathcal{Z}_\chi^{1,\pm} (\mathcal{T}^*)^\Delta \right] \quad (26)$$

$$\mathcal{C}_{qf}^* = \mathcal{Z}_c^\pm (\mathcal{T}^*)^{-\alpha} \left[ 1 + \mathcal{Z}_c^{1,\pm} (\mathcal{T}^*)^\Delta \right] + \quad (27)$$

$$\mathcal{M}_{qf}^* = \mathcal{Z}_M (\mathcal{T}^*)^\beta \left[ 1 + \mathcal{Z}_M^1 (\mathcal{T}^*)^\Delta \right] \quad (28)$$

The universal values of the critical exponents  $\nu$ ,  $\gamma$ ,  $\alpha$ , and  $\beta$ , estimated by Guida et al [25] are given in column 2 of Table 1.  $\Delta = 0.502 (\pm 0.004)$  [25] is the lowest value of the confluent exponent. The master (i.e. constant) values of the leading ( $\mathcal{Z}_\ell^+$ ,  $\mathcal{Z}_\chi^+$ ,  $\mathcal{Z}_c^+$ , and  $\mathcal{Z}_M$ ) and confluent amplitudes ( $\mathcal{Z}_\ell^{1,+}$ ,  $\mathcal{Z}_\chi^{1,+}$ ,  $\mathcal{Z}_c^{1,+}$ , and  $\mathcal{Z}_M^1$ ) are given in columns 3 and 4 (respectively), of Table 1. The master correlation length  $\ell_{qf}^*$  of Eq. (23) provides a direct comparison from the size of the critical fluctuations to the range of molecular interaction, in order to control that the basic condition  $\ell_{qf}^* \gg 1$  for critical phenomena understanding is valid. That provides also a criteria to define the master extension of the preasymptotic domain for the one-component fluid subclass [21].

ii) the physical singular behaviors

$$\xi = \xi_0^\pm |\Delta\tau^*|^{-\nu} \left[ 1 + a_\xi^\pm |\Delta\tau^*|^\Delta \right] \quad (29)$$

$$\chi_\rho^* = \Gamma^\pm |\Delta\tau^*|^{-\gamma} \left[ 1 + a_\chi^\pm |\Delta\tau^*|^\Delta \right] \quad (30)$$

$$\Delta c_{V,\rho}^* = \frac{A^\pm}{\alpha} |\Delta\tau^*|^{-\alpha} \left[ 1 + \alpha a_\chi^\pm |\Delta\tau^*|^\Delta \right] + B_{cr}^* \quad (31)$$

$$\Delta \tilde{\rho}_{LV} = B |\Delta\tau^*|^\beta \left[ 1 + a_m |\Delta\tau^*|^\Delta \right] \quad (32)$$

$\xi$  of Eq. (29) is the correlation length, i.e. the actual size of the critical fluctuations of the order parameter density. The Eqs. (30) to (32) are written with usefull variables of fluid related critical phenomena [26], which needs a complementary analysis made in the next subsection, to precise the normalization of the thermodynamics and the role of the energy and length scale units given by Eqs. (3) and (4), respectively.

#### 3.2. Thermodynamic properties

##### 3.2.1. The isothermal susceptibility

Considering a mass unit of the fluid as in the standard thermodynamic presentation of *specific* properties, the susceptibility  $\chi_\rho = \left( \frac{\partial \rho}{\partial \mu_\rho} \right)_T = \rho \left( \frac{\partial \rho}{\partial p} \right)_T \sim [kg^2 J^{-1} m^{-3}]$  is expressed in units of  $\frac{\rho_c^2}{p_c}$ , while the subscript  $\rho$  recalls for the thermodynamic normalization per mass unit. Therefore, in Eq. (30),  $\chi_\rho^* = \chi_\rho \frac{p_c}{\rho_c^2} = \kappa_T \left( \frac{p}{\rho_c} \right)^2 p_c = (\tilde{\rho})^2 \kappa_T^*$ , with  $\kappa_T = \frac{1}{\rho} \left( \frac{\partial \rho}{\partial p} \right)_T$  and  $\kappa_T^* = p_c \kappa_T$ .  $\mu_\rho = \frac{\mu_T}{m_T}$  is the chemical potential per mass unit, dual from the (mass) density  $\rho$ .  $\kappa_T$  is the isothermal compressibility.  $\tilde{\rho} = \frac{\rho}{\rho_c}$  is the practical dimensionless form of the density, which differs by a factor  $Z_c$  from the dimensionless form  $\rho^* = \rho \frac{(\alpha_c)^d}{m_T}$  obtained with our length unit  $\alpha_c$  [7, 8]. We note that the above susceptibility  $\chi_\rho$  also differs from the susceptibility  $\chi_{\bar{p}} = \left( \frac{\partial n}{\partial \mu_{\bar{p}}} \right)_T = n \left( \frac{\partial n}{\partial p} \right)_T \sim [J m^3]^{-1}$  where the subscript  $\bar{p}$  recalls for the thermodynamic normalization *per particle*. Expressing  $\chi_{\bar{p}}$  in coherent [i.e. using Eqs. (3) and (4)] units of  $\frac{\beta c}{(\alpha_c)^d}$ , we obtain  $\chi_{\bar{p}}^* = \left( \frac{1}{Z_c} \right)^2 \chi_\rho^* = (n_c^*)^2 \kappa_T^*$  [using  $n^* = n(\alpha_c)^d$ ]. However, pressure ( $\sim \frac{[energy]}{[volume]}$ ) appears appropriately expressed in units of  $p_c = \frac{(\beta c)^{-1}}{(\alpha_c)^d}$ , within the both (practical and coherent) dimensionless formulations.

##### 3.2.2. The heat capacity at constant volume

The total heat capacity at constant volume  $C_V \sim [J K^{-1}]$  of the fluid mass  $M$  is divided by the total fluid volume  $V$  to have a unit of  $\rho c_{V,\rho}$ , where  $c_{V,\rho} = \frac{C_V}{M} \sim [J kg^{-1} K^{-1}]$  is the specific heat at constant volume.

$\mathcal{P}_{gf}^*$	$x$	$\mathcal{Z}_p^\pm$	$\mathcal{Z}_p^{1,\pm}$	$P$	$P_0^\pm$	$P_1^\pm$
$\ell_{gf}^{*,+}$	$\nu = 0.6304 \pm 0.0013$	$\mathcal{Z}_\ell^+ = 0.570365$	$\mathcal{Z}_\ell^{1,+} = 0.37685$	$\xi$	$\xi_0^+ = \alpha_c \Lambda_{qe} (Y_c)^{-\nu} \mathcal{Z}_\ell^+$	$a_\ell^+ = \mathcal{Z}_\ell^{1,+} (Y_c)^\Delta$
$\chi_{gf}^{*,+}$	$\gamma = 1.2397 \pm 0.0013$	$\mathcal{Z}_\chi = 0.119$	$\mathcal{Z}_\chi^{1,+} = 0.555$	$\chi^*$	$\Gamma^+ = (\Lambda_{qe}^*)^{d-2} (Z_c)^{-1} (Y_c)^{-\gamma} \mathcal{Z}_\chi^+$	$a_\chi^+ = \mathcal{Z}_\chi^{1,+} (Y_c)^\Delta$
$\mathcal{C}_{gf}^{*,+}$	$\alpha = 0.1088 \pm 0.0039$	$\mathcal{Z}_C^+ = 0.105656$	$\mathcal{Z}_C^{1,+} = .52310$	$\Delta c_V^*$	$\frac{A^+}{\alpha} = (\Lambda_{qe}^*)^{-d} (Y_c)^{2-\alpha} \mathcal{Z}_C^+$	$a_C^+ = \mathcal{Z}_C^{1,+} (Y_c)^\Delta$
$\mathcal{M}_{gf}^*$	$-\beta = -0.3258 \pm 0.0014$	$\mathcal{Z}_M = 0.468$	$\mathcal{Z}_M^1 = 0.4995$	$\Delta \rho_{LV}^*$	$B = (\Lambda_{qe}^*)^{-1} (Z_c)^{-\frac{1}{2}} (Y_c)^\beta \mathcal{Z}_M$	$a_M = \mathcal{Z}_M^1 (Y_c)^\Delta$

Table I: Parameters for the master critical behavior of the correlation length, the susceptibility, the specific heat and the liquid-gas coexisting density along the critical isochore of pure fluid. Exponent values on column 2, amplitude ratios values, and  $\Delta = 0.502 \pm 0.004$ , are from [24, 25].

The dimensionless specific heat  $c_{V,\rho}^*$  is then obtained expressing the total heat capacity in units of  $\frac{p_c V}{T_c}$ , so that  $c_{V,\rho}^* = \rho c_V \frac{T}{p_c}$ . Therefore, in Eq. (31), the singular specific heat  $\Delta c_{V,\rho}^*$  ( $\Delta \tau^*$ ) is such that the total specific heat  $c_{V,\rho}^*(T^*)$  as a function of  $T^* = \frac{T}{T_c}$  reads as follows

$$c_{V,\rho}^*(T^*) = \Delta c_{V,\rho}^*(\Delta \tau^*) + C_{B,\rho}^*(T^*) \quad (33)$$

In Eq. (31),  $B_{cr}^*$  is a critical constant while, in Eq. (33),  $C_{B,\rho}^*(T^*)$  is the regular background reflecting the analytical part of the free energy. In our coherent formulation of the particle properties, the heat capacity per particle  $c_{V,\bar{p}} = \frac{C_V}{N} \sim [JK^{-1}]$  have the (universal)  $k_B$  dimension. As a matter of fact, the heat capacity per particle is the unique measurable thermodynamic property which can be made dimensionless only using the Boltzmann factor  $k_B$ , i.e. *without reference* to  $\alpha_c$  and  $(\beta_c)^{-1}$ . Therefore, when the singular heat capacity at constant volume, normalized per particle, obeys the asymptotic power law

$$\Delta c_{V,\bar{p}} = \frac{A_{0,\bar{p}}^\pm}{\alpha} |\Delta \tau^*|^{-\alpha} \left[ 1 + \mathcal{O} \left\{ |\Delta \tau^*|^\Delta \right\} \right] \quad (34)$$

along the critical isochore, *one (+ or -) among the two dimensionless amplitudes  $\frac{A_{0,\bar{p}}^\pm}{k_B}$  is mandatorily a characteristic fluid-particle-dependent number* (the two amplitudes being related by the universal ratio  $\frac{A_{0,\bar{p}}^+}{A_{0,\bar{p}}^-} \approx 0.537$  for  $d = 3$  [25]). However, hyperscaling features impose that the same length scale is used in thermodynamic and correlation functions. For example, in the case of an “incompressible” 3D Ising-system of the lattice spacing  $a_{Ising}$ , the singular part of the heat capacity normalized by  $k_B$  can be expressed in unit of  $(a_{Ising})^d$  [27] [the extensive nature of the total number of particle is then implicitly accounted for in a crystallized solid system since the total volume is proportional to the cell lattice volume containing a fixed number of particles]. Similarly, in the case of the compressible one-component fluid, that needs to express normalized heat capacity per particle in unit of  $(\alpha_c)^d$  (ignoring in this simple dimensional analysis the quantum effects on the microscopic wavelength). The number of particles within the critical interaction cell being  $\frac{1}{Z_c}$ , we thus define the singular part of the heat capacity for the volume of the critical interaction

cell as follows

$$\Delta c_{V,I}^* = \frac{1}{Z_c} \Delta c_{V,\bar{p}}^* \quad (35)$$

which  $\Delta c_{V,\bar{p}}^* = \frac{\Delta c_{V,\bar{p}}}{k_B}$ . Accordingly,  $\frac{1}{Z_c}$  takes equivalent microscopic nature of the coordination number in the lattice description of the three dimensional Ising systems, while  $\alpha_c$  takes equivalent microscopic nature of the lattice spacing  $a_{Ising}$ . Now, for comparison with the notations used in fluid-related critical phenomena where all the thermodynamic potentials are divided by the total fluid volume, we also introduce the heat capacity at constant volume, *for a fluid in a container of unit volume*,  $\Delta c_{V=1} = \frac{\Delta c_{V,\bar{p}}}{v_{\bar{p},c}}$  (labelled here with the subscript  $V = 1$ ). Expressed in our above unit length scale [Eq. (4)], the associated dimensionless form reads

$$\Delta c_{V=1}^* = \frac{\Delta c_{V,\bar{p}}}{k_B} \times \frac{1}{v_{\bar{p},c} (\alpha_c)^d} = \frac{\Delta c_{V=1}}{k_B} \times \frac{1}{(\alpha_c)^d} \quad (36)$$

Obviously,  $\Delta c_{V=1}^*$  is identical to

i) the previous dimensionless form  $\Delta c_{V,\rho}^* = \frac{\Delta c_V}{V} \times \frac{T_c}{p_c}$  of the total singular heat capacity  $\Delta c_V = N \Delta c_{V,\bar{p}}$  of the constant total fluid volume  $V$ , filled with  $N$  (fixed) particles and,

ii) the our dimensionless form  $\Delta c_{V,I}^* = \Delta c_{V,\bar{p}}^* \times \frac{1}{Z_c}$  of the singular heat capacity of the microscopic interacting volume  $v_{c,I}$  [Eq. (9)], filled with  $\frac{1}{Z_c}$  (fixed) particles. In this latter situation, we have an explicit comprehension of the extensive nature of the two independent variables  $V$  and  $N$  for compressible fluids. Specially, we note here the importance of the thermodynamic normalization for better understanding of the scaling nature of the critical amplitudes, such as in Eq. (31) for example. Considering the hyperscaling law  $2 - \alpha = d\nu$ , throughout the universal quantity made by the product

$$(\Delta \tau^*)^2 \times \frac{\Delta c_{V,\rho}^*}{(\alpha_c)^d} \times \xi^d = \text{universal quantity}$$

and rewriting this product such as

$$(\Delta \tau^*)^2 \times \frac{1}{Z_c} \times (\Delta c_{V,\bar{p}}^*) \times \left( \frac{\xi}{\alpha_c} \right)^d = \left( R_\xi^\pm \right)^d$$

[with  $R_\xi^+ \approx 0.2696$  and  $R_\xi^- \approx 0.169$ , for  $d = 3$  [25]], we can easily demonstrate that the universal amplitude

combination

$$\left(R_\xi^\pm\right)^d = \frac{1}{Z_c} \left(\frac{A_{0,\bar{p}}^\pm}{k_B}\right) \left(\frac{\xi_0^\pm}{\alpha_c}\right)^d \quad (37)$$

contains the two independent extensive features (volume and number of particles) of the fluid system at the scale of the critical interaction cell. In such a situation, the dimensioned leading amplitudes  $A_{0,\bar{p}}^\pm$  (associated to a particle property), and  $\xi_0^\pm$  (associated to the microscopic wavelength), have a well-understood physical meaning with respect to the universal features of the universality class. As an essential consequence, the universal feature of any singular free energy must then be expressed in terms of the unique remaining energy scale  $(\beta_c)^{-1}$ . We will return below (see §.3.3) on this important remark to account for quantum effects in the master singular behavior of the one-component fluid subclass.

### 3.2.3. The (dual) densities and chemical potentials

We finally consider the non homogeneous domain below  $T_c$ , where the practical dimensionless form of the symmetrized order parameter density [see eq. (32)] is defined by

$$\Delta\tilde{\rho}_{LV}(|\Delta\tau^*|) = \frac{\Delta\rho_{LV}(|\Delta\tau^*|)}{2\rho_c} = \frac{\rho_L - \rho_V}{2\rho_c} \quad (38)$$

$\rho_L$  ( $\rho_V$ ) is the liquid (vapor) density of one coexisting phase. Such a dimensionless form occurs from the usefull variable  $\tilde{\rho} = \frac{\rho}{\rho_c}$  (see above), leading to consider the quantity

$$\Delta\tilde{\rho} = \frac{\rho - \rho_c}{\rho_c} = \tilde{\rho} - 1 \quad (39)$$

as a practical order parameter density, and the quantity

$$\Delta\tilde{\mu}_\rho = \frac{\mu_\rho - \mu_{\rho,c}}{\mu_{\rho,c}} = \tilde{\mu}_\rho - 1 \quad (40)$$

as a practical ordering field.  $\mu_{\rho,c} = \frac{\mu_{\bar{p}}}{m_{\bar{p}}}$  is the specific chemical potential at the critical point.  $\tilde{\mu}_\rho = \frac{\mu_\rho}{\mu_{\rho,c}}$  is the practical dimensionless form of the chemical potential, which differs by a factor  $(\beta_c \mu_{\bar{p},c})^{-1}$  from our dimensionless form  $\mu_{\bar{p}}^* = \beta_c \mu_{\bar{p}}^*$  obtained with our energy unit  $(\beta_c)^{-1}$ . From comparison between the two definitions of the order parameter density by Eq. (22) and (39), we obtain

$$\begin{aligned} \Delta\tilde{\rho} &= Z_c \Delta m^* \\ \Delta\tilde{\rho}_{LV} &= Z_c \Delta m_{LV}^* \end{aligned} \quad (41)$$

where

$$\Delta m_{LV}^* = (n_L - n_V) (\alpha_c)^d \quad (42)$$

The main conclusive remark to note using these dual variables  $\Delta\tilde{\rho}$  and  $\Delta\tilde{\mu}_\rho$ , of respective Eqs. (39) and (40), concerns the implicit addition of a new length scale factor  $\tilde{\alpha}_c = \left(\frac{m_{\bar{p}}}{\rho_c}\right)^{\frac{1}{d}}$  and a new energy scale factor  $(\tilde{\beta}_c)^{-1} = m_{\bar{p}} \mu_{\rho,c}$ . As a consequence, the nonuniversal nature of each leading amplitude is a complex combination of the interrelated dimensionned scale factors and of the two scale factors associated to universal scaling in fluids.

### 3.3. Two-scale-factor universality and quantum effects

In addition to Eqs. (25) to (32), we now introduce:

i) the renormalized singular free energy density  $\mathcal{A}_{qf}^*(\mathcal{T}^*, \mathcal{M}_{qf}^*)$  which, along the isocline  $\mathcal{M}_{qf}^* = 0$ , asymptotically behaves as

$$\mathcal{A}_{qf}^*(\mathcal{T}^*) = Z_{\mathcal{A}}^\pm (\mathcal{T}^*)^{2-\alpha} \left[1 + \mathcal{O}\left\{(\mathcal{T}^*)^\Delta\right\}\right] \quad (43)$$

with respect to the master thermal field  $\mathcal{T}^*$  going to zero. Correspondingly, the thermodynamics definitions of the renormalized properties of present interest are  $\mathcal{H}_{qf}^*(\mathcal{T}^*, \mathcal{M}_{qf}^*) = \left(\frac{\partial \mathcal{A}_{qf}^*}{\partial \mathcal{M}_{qf}^*}\right)_{\mathcal{T}^*}$ ,  $\chi_{qf}^*(\mathcal{T}^*, \mathcal{M}_{qf}^*) = \left(\frac{\partial \mathcal{H}_{qf}^*}{\partial \mathcal{H}_{qf}^*}\right)_{\mathcal{T}^*}$ ,  $\frac{\mathcal{C}_{qf}^*(\mathcal{T}^*, \mathcal{M}_{qf}^*)}{\mathcal{T}^*} = -\left(\frac{\partial^2 \mathcal{A}_{qf}^*}{\partial \mathcal{T}^{*2}}\right)_{\mathcal{M}_{qf}^*=0}$ , [with  $Z_{\mathcal{A}}^\pm = \frac{Z_c^\pm}{\alpha(1-\alpha)(2-\alpha)}$ ];

ii) the singular part  $\Delta a_\rho(T, \rho) = \frac{\Delta A_\rho}{V}$  of the Helmholtz free energy density, where temperature  $T$  and (practical) density  $\rho = \frac{Nm_{\bar{p}}}{V}$  are the two selected variables to describe a fluid maintained in a container of constant total volume  $V$ . In our case where order parameter density is related to the (natural) number density  $n = \frac{N}{V}$ , we note  $\Delta a(T, n) = \frac{\Delta A}{V}$  this singular part of the Helmholtz free energy density. Due to the appropriate dimensionless form of the pressure mentionned above, both the usefull dimensionless form  $\Delta a_\rho^*(\Delta\tau^*, \Delta\tilde{\rho}) = \frac{\Delta A_\rho}{V} \times \frac{1}{\rho_c}$  and the natural dimensionless form  $\Delta a^*(\Delta\tau^*, \Delta m^*) = \beta_c \Delta A \times \frac{(\alpha_c)^d}{V}$  are identical, except the use of two distinct reduced forms  $\Delta\tilde{\rho}$  and  $\Delta m^*$  of the order parameter density. Along the critical isochore  $\Delta\tilde{\rho} = \Delta m^* = 0$ , the singular part of the free energy behaves as

$$\begin{aligned} \Delta a_\rho^*(\Delta\tau^*) &\equiv \Delta a^*(\Delta\tau^*) \\ &= A'^\pm |\Delta\tau^*|^{2-\alpha} \left[1 + \mathcal{O}\left\{|\Delta\tau^*|^\Delta\right\}\right] \end{aligned} \quad (44)$$

The basic thermodynamic definitions of the physical properties are:  $\Delta\tilde{\mu}_\rho(\Delta\tau^*, \Delta\tilde{\rho}) = \left(\frac{\partial \Delta a_\rho^*}{\partial \Delta\tilde{\rho}}\right)_{\Delta\tau^*}$ , or  $\Delta\mu_{\bar{p}}^*(\Delta\tau^*, \Delta m^*) = \left(\frac{\partial \Delta a^*}{\partial \Delta m^*}\right)_{\Delta\tau^*}$ ;  $\chi_\rho^*(\Delta\tau^*, \Delta\tilde{\rho}) = \left(\frac{\partial \Delta\tilde{\rho}}{\partial \Delta\mu_\rho^*}\right)_{\Delta\tau^*}$ , or  $\chi_{\bar{p}}^*(\Delta\tau^*, \Delta m^*) = \left(\frac{\partial \Delta a^*}{\partial \Delta\mu_{\bar{p}}^*}\right)_{\Delta\tau^*}$ ; and



$$\frac{\Delta c_{V,\rho}^*(\Delta\tau^*,\Delta\bar{\rho})}{T^*} = \left( \frac{\partial^2 \Delta \tilde{a}_\rho}{\partial (T^*)^2} \right)_{\Delta\bar{\rho}}, \text{ or } \frac{\Delta c_V^*(\Delta\tau^*,\Delta m^*)}{T^*} = \left( \frac{\partial^2 \Delta a^*}{\partial (T^*)^2} \right)_{\Delta\bar{\rho}};$$

It is thus easy to obtain the relations reported in the columns 6 and 7 of Table I, using the above basic thermodynamic definitions of the renormalized and physical variables. That also provides a comprehensive understanding of the quantum effect correction to master singular behavior.

As a matter of fact, following the argument first proposed by Widom [28], the renormalized energy associated with the spontaneous density fluctuations that extend over a distance  $\ell_{qf}^*$  must be of the order  $(\beta_c)^{-1}$ , leading to a renormalized free energy density of order  $[\beta_c (\alpha_c)^d]^{-1}$ . Along the critical isochore, this energy will be associated to  $\mathcal{A}_{qf}^*(T^*)$  of Eq. (43). The product  $\mathcal{A}_{qf}^*(T^*) \times (\ell_{qf}^*)^d$  being a universal quantity, the relative quantum correction to the renormalized singular free energy reads

$$\mathcal{A}_{qf}^*(T^*) = (\Lambda_{qe}^*)^d \mathcal{A}^*(T^*) \quad (45)$$

due to the Eq. (15) for  $\ell_{qf}^*$ .  $\mathcal{A}^*(T^*)$  must be the renormalized singular free energy already defined for non-quantum fluids such as

$$\mathcal{A}^* = \beta_c (\alpha_c)^d \times \frac{\Delta A}{V} \quad (46)$$

Therefore, from the comparison between the leading terms of the renormalized and the physical second derivatives of the singular free energy densities with respect to their associated thermal fields, we obtain

$$A^\pm = \frac{1}{Z_c} \frac{A_{0,\bar{p}}^\pm}{k_B} = (\Lambda_{qe}^*)^{-d} (Y_c)^{2-\alpha} Z_c^\pm \quad (47)$$

In addition to the explicit  $Y_c$  and  $\Lambda_{qe}^*$  dependences of the leading dimensionless amplitude  $A^\pm$ , the above Eqs. (47), also show the role of the particle number  $\frac{1}{Z_c}$  such as the multiplicative factor to the leading particle amplitude  $A_{0,\bar{p}}^\pm \sim [k_B]$ . That provides understanding of the master (i.e. unique) singular behaviors of the one-component fluid subclass in terms of the master (i.e. constant) properties of the critical interaction cell of any one-component fluid. Similarly, from the comparison between the leading terms of the renormalized and the physical correlation lengths, we obtain

$$\xi_0^\pm = \alpha_c \Lambda_{qe}^* (Y_c)^{-\nu} Z_\xi^\pm \quad (48)$$

In Eqs. (47) and (48),  $A_{0,\bar{p}}^\pm \sim [k_B]$  and  $\xi_0^\pm \sim [length]$  have the appropriate  $Q_c^{min}$  and  $\Lambda_{qe}^*$  dependences to satisfy the universal amplitude combination of Eq. (37). These two equations (47) and (48), or more generally, all the relations given in the column 6 of Table I, also demonstrate that the estimation of the adjustable parameter  $\lambda_{q,f}$ , introduced throughout the Eqs. (12) and (13), is

unequivocally made from the leading power law behavior of any property, when  $Q_c^{min}$  is known. That provides a very sensitive test of the above phenomenological approach to account for quantum effects, provided that the same length scale  $\alpha_c$  and the same energy scale  $(\beta_c)^{-1}$  are used for thermodynamic and correlation functions at  $T \cong T_c$ . In such a coherent thermodynamic normalization, the relative quantum modification [proportional to  $(\Lambda_{qe}^*)^d$ ] of the energy within the critical interaction cell is correlated to the relative quantum modification of the microscopic wave number [proportional to  $\Lambda_{qe}^*$ ]. We thus provide the microscopic quantum mechanical modification which complement the Widom's [28] and Staufer et al's [29] macroscopic argument, when it is expected that the free energy associated to fluctuations of size  $\xi$  were solely responsible for the singular contribution of thermodynamic potentials and correlation functions.

### 3.4. ${}^3He$ results

For the fermionic quantum fluid  ${}^3He$ , the  $Q_c^{min}$  set is composed of the following critical coordinates  $T_c = 3.315546 K$ ,  $p_c = 1.14724 10^5 Pa$ ,  $\rho_c = 41.45 kg m^{-3}$ , and  $\gamma_c = 1.1759 10^5 Pa K^{-1}$  [14]. Using Eqs. (4) to (6), the values of the four scale factors are  $(\beta_c)^{-1} = 4.5776 10^{-23} J$ ,  $\alpha_c = 7.362 10^{-10} m$ ,  $Y_c = 2.39837$ ,  $Z_c = 0.301284$ . By  $\chi^2$ -optimization only using the susceptibility data above and below  $T_c$  in the range  $|\Delta\tau^*| < 5 10^{-3}$ , with  $\frac{\Gamma^+}{\Gamma^-} = \frac{Z^+}{Z^-} = 4.79$  [25], the adjustable parameter  $\lambda_{q,f}$  takes the numerical value  $\lambda_{q,{}^3He} = 0.146423$ , leading to  $\Lambda_{qe}^* = 1.11009$ . For the specific case of the heat capacity, the additional critical ( $B_{cr}^*$ ) and background ( $C_B^*$ ) terms are treated as one single adjustable constant ( $B_{cr}^* + C_B^*$ ).

The main results are illustrated in Fig. 1 where the comparison is made to the recent published experimental data (black points in Fig. 1) of Zhong et al [14]. In order to simplify the comparison, the same corresponding scaled data by the asymptotic power law term  $(\Delta\tau^*)^{-x}$  were used for susceptibility and heat capacity above  $T_c$ , and for coexisting liquid vapor densities below  $T_c$ , which improves the sensitivity of the relative representation from the asymptotic amplitude values. Obviously, that provides a simultaneous significative test of the quantum effect contribution since, among the four leading amplitudes  $\Gamma^+$ ,  $A^+$ ,  $\Gamma^-$ , and  $B$ , only one is readily sufficient to define the unequivocal  $\Lambda_{qe}^*$ -dependence. Moreover, to illustrate the first confluent term contribution associated to the scale dilatation method, the full (red) lines and the dot-dashed (blue) lines in Fig. 1 correspond to the respective first-order Wegner expansions obtained from Table I, and from the Zhong et al [14] initial fit, using the minimal subtraction renormalization (labelled *MSR*) scheme (see also the corresponding numerical values of the amplitudes listed in Table II). For the three selected properties, the predicted singular behavior fits well the experimental results and matches

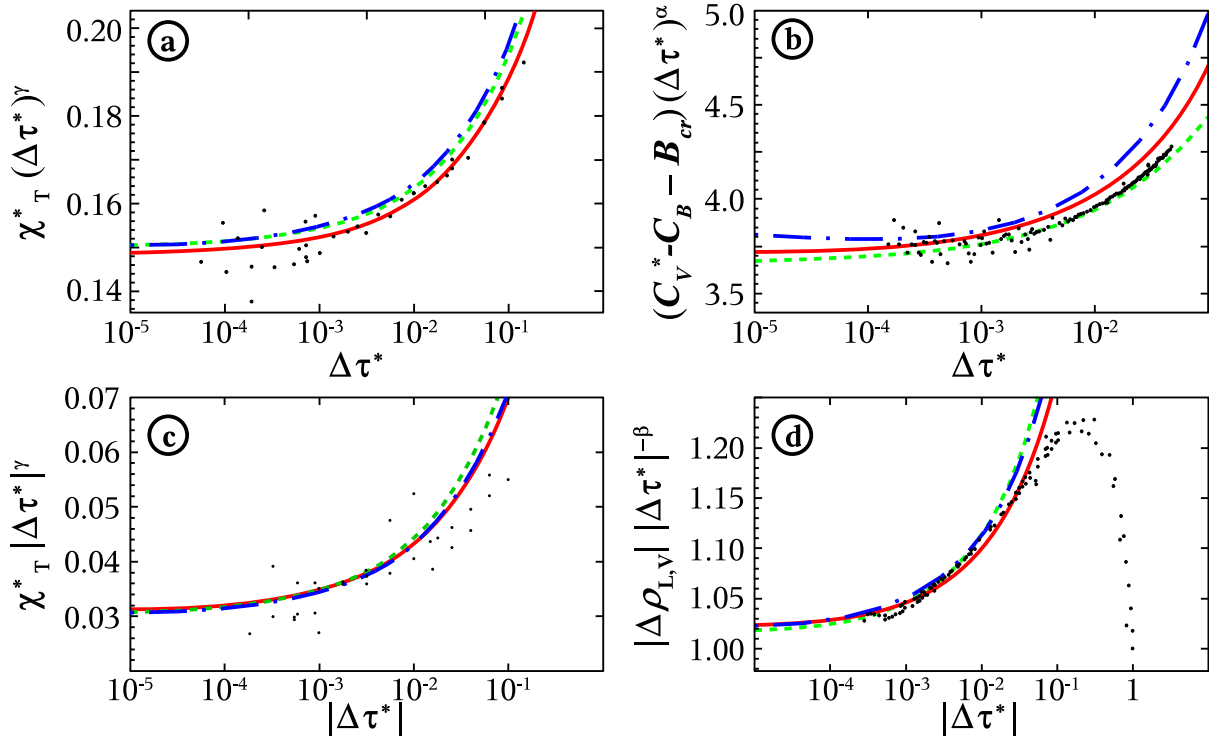


Figure 1: Asymptotic two-term predictions compared to the  ${}^3\text{He}$  measurements (black points). The solid (red) lines are the actual predictions by the dilated scale method (see Eq. (24) and column 4 of Table II). The dot-dashed (blue) lines are the two-term prediction obtained from the best fit by the MSR  $\Phi_{d=3}^4(1)$ -model [14] (Eq. (24); column 2 of Table II). The dashed (green) lines are the two-term prediction obtained from CPM model [36] (Eq. (24); column 3 of Table II). (a) Susceptibility measurements for  $T > T_c$  (corresponding to upper part of Fig. 1 in [14]). (b) Specific heat measurements for  $T > T_c$ , where the small difference in the additional constant term  $C_B + B_{cr}$  is accounted in the vertical scale (see also the lower part of Fig. 3 in [14]). (c) Susceptibility measurements for  $T < T_c$  (see also the lower part of Fig. 1 in [14]). (d) Liquid-gas coexisting density measurements (see also Fig. 4 in [14]).

Amplitude	MSR [14]	MSR [30]	MR6[30]	MR7[30]	CPM [36]	CPM [30]	This work
$\xi_0^+ \left(\overset{\circ}{A}\right)$	$2.71 \pm 0.02$				$2.68 \pm 0.04$		2.68541
$a_\xi^+$	$0.732 \pm 0.007$						0.58474
$\Gamma^+$	$0.150 \pm 0.007$	$0.147 \pm 0.001$	$0.146 \pm 0.001$	$0.148 \pm 0.001$	$0.150 \pm 0.002$	$0.153 \pm 0.001$	0.148247
$\Gamma^-$	$0.0303 \pm 0.0015$	$0.0299 \pm 0.0003$	$0.0308 \pm 0.0001$	$0.0310 \pm 0.0001$		$0.0310 \pm 0.0002$	0.030953
$a_\chi^+$	$0.98 \pm 0.08$	$1.10 \pm 0.01$	$1.13 \pm 0.01$	$1.17 \pm 0.01$	$0.941 \pm 0.007$	$0.81 \pm 0.01$	0.860931
$a_\chi^-$	$4.29 \pm 0.34$	$4.83 \pm 0.05$	$3.58 \pm 0.05$	$5.30 \pm 0.07$		$4.17 \pm 0.07$	4.01366
$\frac{A^+}{\alpha}$	$3.73 \pm 0.45$	$3.76 \pm 0.05$	$3.72 \pm 0.01$	$3.84 \pm 0.02$	$3.548 \pm 0.031$	$3.63 \pm 0.02$	3.71132
$\frac{A^-}{\alpha}$	$6.97 \pm 0.83$	$7.03 \pm 0.10$	$6.883 \pm 0.026$	$7.149 \pm 0.027$	$6.823 \pm 0.01$	$6.935 \pm 0.04$	6.90948
$\alpha a_C^+$	$1.2 \pm 0.1$	$0.99 \pm 0.01$	$1.13 \pm 0.01$	$1.07 \pm 0.01$	$0.712 \pm 0.006$	$0.61 \pm 0.01$	0.810892
$\alpha a_C^-$	$1.1 \pm 0.1$	$0.92 \pm 0.01$	$1.17 \pm 0.01$	$0.83 \pm 0.01$	$0.593 \pm 0.012$	$0.74 \pm 0.01$	0.59712
$B_{cr} + C_B$	$-1.65 \pm 0.85$	$-1.67 \pm 0.13$	$-1.64 \pm 0.04$	$-1.81 \pm 0.04$	$-0.96 \pm 1.0$	$-1.23 \pm 0.05$	-1.40
$B$	$1.020 \pm 0.006$	$1.021 \pm 0.003$	$1.008 \pm 0.004$	$1.039 \pm 0.004$	1.0047	$1.028 \pm 0.004$	1.02134
$a_M^-$	$0.91 \pm 0.02$	$0.91 \pm 0.01$	$1.001 \pm 0.023$	$0.218 \pm 0.003$	0.8441	$0.73 \pm 0.01$	0.77484

Table II: Calculated values for critical amplitudes of  ${}^3\text{He}$  with  $T_c = 3.315546 \text{ K}$ ,  $p_c = 1.14724 \cdot 10^5 \text{ Pa}$ ,  $\rho_c = 41.45 \text{ kg m}^{-3}$ ,  $\gamma'_c = 1.1759 \cdot 10^5 \text{ Pa K}^{-1}$ , and  $\lambda_{q,{}^3\text{He}} = 0.146423$  (see text). Using Eqs. (4), (6) and the definition of  $\Lambda_{qe}^*$  from Eqs. (12) to (14), the values of the (five) characteristics parameters for  ${}^3\text{He}$  are  $(\beta_c)^{-1} = 4.5776 \cdot 10^{-23} \text{ J}$ ,  $\alpha_c = 7.362 \cdot 10^{-10} \text{ m}$ ,  $Y_c = 2.39837$ ,  $Z_c = 0.301284$ , and  $\Lambda_{qe}^* = 1.11009$ .

the theoretical predictions of the minimal-substraction renormalization scheme.

More generally, as shown in Table II, the two-term asymptotical results obtained with the scale dilatation method are in good agreement with the two-term parametric modeling recently obtained by Zhong and Barmatz [30], based on three different theoretical models. Two of these models are issued from the two main field-theoretical renormalization schemes that treat classical-to-critical crossover phenomena, namely the minimal-substraction renormalization scheme of Doms and co-workers [31, 32, 33, 34, 35], and the massive renormalization scheme of Bagnuls and Bervillier [4, 5, 9, 24], only applied to the primary critical path corresponding to the homogeneous and non homogeneous domain along the critical isochore. The third model, namely the crossover parametric model, proposed by Agayan and coworkers [36, 37], is a complete parametric equation of state issued from a phenomenological crossover transformation of a classical Landau expansion of the singular free energy [38, 39, 40]. Although it is phenomenological, this crossover Landau model was successfully applied to several one-component fluids. A previous comparison of the results obtained by the crossover Landau model and the scale dilatation method was already made in the case of seven non-quantum fluids. In Table II are reported,

i) columns 2 and 3 labelled *MSR*, the results obtained by Zhong et al [14] and Zhong and Barmatz [30] from the minimal-substraction renormalization scheme;

ii) columns 4 and 5 labelled *MR6* and *MR7*, the results obtained by Zhong and Barmatz [30] from the massive renormalization scheme in the sixth- [5, 9] and seventh-loop [24] series;

iii) columns 6 and 7 labelled *CPM*, the results obtained by Agayan et al [36] and Zhong and Barmatz [30] from the crossover parametric model;

iv) column 8, the results obtained in this work applying the scale dilatation method.

The excellent agreement between the amplitude values permits to discuss now the introduction of the adjustable parameters in the modeling, and to explain why only two adjustable parameters in the models are significant with respect to the fit quality, as concluded by Zhong and Barmatz [30].

## 4. ${}^3\text{He}$ CRITICAL MODELLING

### 4.1. The two renormalization schemes along the critical isochore

As clearly mentioned in the Appendix D of the reference [14], the three free parameters of the *MSR*-model originate from the undetermined integration constants  $z_\phi$ ,  $z_a$ , and  $z_\mu$ , associated to the flow equations of their respective  $Z_\phi(u)$ ,  $Z_r(u)$ , and  $Z_u(u)[Z_\phi(u)]^{-2}$  field theoretical functions (here we have adopted the Zhong et al notation for  $z_a$  and  $z_\mu$ , adding  $z_\phi$  as being the undeter-

mined integration constant to solve Eq. (7) of reference [14]). These integration constants are system dependent and can be obtained by fitting experimental data to the theory. However, considering uniquely the critical isochore, the given set composed by the explicit adjustable parameters (such as  $\{u, \mu, a\}$  in the MSR model case [13, 14]) or calculated parameters (such as the leading amplitudes and  $t_0$  in the MSR model case [13, 14]) result in complicated scaled forms of combinations between  $z_\phi$ ,  $z_a$ , and  $z_\mu$ . Specifically, to account correctly for the  $z_\phi$  system dependence needs to use several properties. The susceptibility fitting results reported in Figure 2 of Ref. [14], where only two ( $\mu$  and  $t_0$ ) among the three scaled parameters ( $\mu$ ,  $a$  and  $t_0$ ) have the expected power law dependence on  $1 - \frac{u}{u^*}$  near the fixed point ( $u = u^*$ ), should be also due to a non-representative test of one asymptotical scaled form. A preliminary comparison of the functional forms of the leading amplitudes obtained from the MSR model and the scale dilatation method for the case of the non-quantum fluid subclass, suggests for example that the true independent scaled factors of each physical system are then such as

$$\frac{z_a}{(z_\mu)^{\zeta_r^*}} \propto Y_c \quad (49)$$

and

$$\frac{z_\phi}{(z_\mu)^{\zeta_\phi^*}} \propto Z_c \quad (50)$$

In Eqs. (49) and (50),  $\zeta_r^* = \zeta_r(u^*) = 1 - \frac{1}{\nu}$  and  $\zeta_\phi^* = \zeta_\phi(u^*) = -\eta$  are the respective values of the field theoretical functions at the Ising fixed point  $u = u^*$  (see Zhong et al's [14] notations). This suggestion should be useful for a possible rescaling of the leading amplitudes which gives better evidence for the two asymptotical parameters which are readily independent in the modeling from the minimal-substraction renormalization scheme.

The two-term master asymptotical behavior obtained from the scale dilatation method can be described [41] by the massive renormalization scheme of Bagnuls and Bervillier, thanks to its formal analogy to the basic analytical hypotheses of the renormalization [22, 23]. Using a similar approach which introduces one common (i.e.  $P^*$ -independent) crossover parameter  $\vartheta_{3\text{He}}$ , and adjustable prefactors  $\mathbb{P}_{0,3\text{He}}^\pm$  for each dimensionless property  $P^*$ , we obtain the following values  $\mathbb{L}_{0,3\text{He}}^+ = 1.2925$ ,  $\mathbb{X}_{0,3\text{He}}^+ = 1.818$ ,  $\mathbb{C}_{0,3\text{He}}^+ = 2.1503$ , and  $\mathbb{M}_{0,3\text{He}}^\pm = 1.0894$ , for the leading prefactors of the correlation length, the susceptibility, the heat capacity and the coexistence curve, respectively. These four leading parameters are interrelated by the following combinations  $\mathbb{L}_{0,3\text{He}}^+ \left(\mathbb{C}_{0,3\text{He}}^+\right)^{\frac{1}{2}} = 1$  and  $\frac{\mathbb{X}_{0,3\text{He}}^+}{\left(\mathbb{M}_{0,3\text{He}}^\pm\right)^2} \left(\mathbb{L}_{0,3\text{He}}^+\right)^{-d} = 1$ , so that only two of them are independent, by virtue of the two scale factor universality. The estimated value of the crossover parameter is  $\vartheta_{3\text{He}} = 0.0113$ . The mean crossover functions [42]

will be used in a future work to implement the master estimation of their free parameters from the four scale factors defined by  $Q_c^{min}$ .

#### 4.2. The crossover parametric model of the e.o.s.

The crossover parametric model is issued from the crossover Landau model (CLM) of the e.o.s. based on a phenomenological transformation of a classical Landau expansion of the singular free energy of the fluid as a function of the local order parameter density. In such a modeling, the simplest crossover description involves three free parameters, made of the two coupling constants  $a_0$  and  $u_0$  and one gradient prefactor  $c_0$  (see for example [36] for notations). After transformation of variables and coefficients, the three initial system-dependent coefficients  $a_0$ ,  $u_0$ , and  $c_0$ , are replaced by two dimensionless asymptotic scaling parameters (noted  $c_t$  and  $c_\rho$  in the general CLM approach) and one dimensionless crossover parameter (noted  $g$ ). However, from the field theory framework, any description of a 3D Ising like system with “finite” cutoff, needs to maintain the appropriate interdependence between the nonuniversal parameters, specially the microscopic wavelength  $\Lambda_0 \sim [length]^{-1}$  and the coupling constant  $u_0 \sim [length]$ . Introducing then a common *arbitrary* length scale unit permits to replace the product  $u_0\Lambda_0$  by the product  $u\Lambda$  of the corresponding dimensionless wavelength  $\Lambda$  and coupling constant  $u$ . The convenient normalization  $\bar{u} = \frac{u}{u^*}$ , where  $u^*$  corresponds to the universal value at the non Gaussian fixed point, leads to an arbitrary choice for the dimensionless microscopic wavelength  $\Lambda$  and the dimensionless coupling parameter  $\bar{u}$ , provided that  $\bar{u}\Lambda$  remains finite in order to account for theoretical infinite cutoff approximation,  $\Lambda \rightarrow \infty$  and  $\bar{u} \rightarrow 0$ . In this infinite-cutoff limit where  $g$  is related to the Ginzburg number [40], the crossover behavior is then universal by rescaling the thermal field like variable using a single crossover parameter (such as  $g = \frac{(\bar{u}\Lambda)^2}{c_t} = \Delta\tau_X^*$ , or such as the crossover temperature  $t_X = c_t\Delta\tau_X^*$ , equivalently [40]), However, at the general symmetrical fourth-order (with only two independent coupling quantities  $a_0$  and  $u_0$ ) of the phenomenological transformation of the classical Landau expansion of the singular free energy, the crossover behavior is governed by the two dimensionless parameters  $g$  and  $\bar{u}$ . In such a situation all the dimensionless quantities are canonical constants, provided one have defined a microscopic characteristic length scale for each fluid. That provides implicit connection between  $\Lambda$  and  $\bar{u}$ , or equivalently between  $\Lambda$  and, for example  $c_t$ , when the explicit  $g = \frac{(\bar{u}\Lambda)^2}{c_t}$  dependence is accounted for, as mentionned above. As a consequence, the only way to monitor the asymptotic critical behavior of the crossover Landau model is to change  $\bar{u}$ , or equivalently  $c_t$ . We recall that, in a previous analysis of the corresponding results for the case of seven non-quantum fluids [21], we

have shown that

$$c_t(\bar{u}\Lambda) = Y_c \times f_t(\bar{u}\Lambda) \quad (51)$$

and

$$c_\rho(\bar{u}\Lambda) = (Z_c)^{\frac{1}{2}} \times f_\rho(\bar{u}\Lambda) \quad (52)$$

are unequivocally well-related to our scale factors  $Y_c$  and  $Z_c$ , respectively. In Eqs. (51) and (52),  $f_t(\bar{u}\Lambda)$  and  $f_\rho(\bar{u}\Lambda)$  are two appropriate universal power laws of the product  $\bar{u}\Lambda$ , uniquely. In the following, we will also provide one possible estimation of the coupling constants  $a_0(g)$  and  $u_0(g)$  from  $Y_c$  and  $Z_c$ , now using the system-dependent coefficients of the crossover parametric model.

The *three-parameter* crossover parametric model contains two asymptotic scaling parameters, noted  $l_0$  and  $m_0$ , and again the crossover parameter  $g$ . A comparison between definitions of asymptotic amplitudes  $\Gamma^+$  and  $B$  leads to the following relations,

$$l_0 = \frac{3.38317}{\mathcal{Z}_\chi} \frac{\mathcal{Z}_\mathcal{M}}{3.28613} (\Lambda_{qe}^*)^{-2} (Z_c)^{\frac{1}{2}} (Y_c)^{\beta+\gamma} \quad (53)$$

and

$$m_0 = \frac{\mathcal{Z}_\mathcal{M}}{3.28613} (\Lambda_{qe}^*)^{-1} (Z_c)^{-\frac{1}{2}} (Y_c)^\beta \quad (54)$$

(see our Table I and Table III of Ref. [36] for details). Our direct estimation of the two free values  $l_0 = 7.0929$  and  $m_0 = 0.3108$  from Eqs. (53) and (54), are in close-agreement with the values  $l_0 = 6.89 \pm 0.12$  and  $m_0 = 0.306 \pm 0.01$ , deduced from the fitting procedure of Agayan et al (see Ref. [36]), and with the values  $l_0 = 6.902 \pm 0.012$  and  $m_0 = 0.3128 \pm 0.0004$ , recently obtained by Zhong and Barmatz [30] in their recent comparison of theoretical models of crossover behavior. Moreover, as previously mentionned, from the identification of the leading amplitudes given in Table I of Ref. [36], calculated using, either the crossover Landau model, or the crossover parametric model, it is now easy to show that the two coupling constants  $a_0$  and  $u_0$  are related to  $Y_c$  and  $Z_c$  (and  $\Lambda_{qe}^*$ , obviously), by the following relations

$$a_0(g) = (\Lambda_{qe}^*)^{-1} Z_c (Y_c)^\gamma f_{a_0}(g) \quad (55)$$

and

$$u_0(g) = \Lambda_{qe}^* (Z_c)^2 (Y_c)^{2\beta-\gamma} f_{u_0}(g) \quad (56)$$

In Eqs. (55) and (56),  $f_{a_0}(g)$  and  $f_{u_0}(g)$  are two appropriate universal power laws of the crossover parameter  $g$ .

The first confluent amplitude for the susceptibility obtained from the crossover parametric model reads  $\Gamma_1^+ = g_\chi^+ g^{-\Delta_s} (1 - \bar{u})$ , with  $g_\chi^+ = 0.590$ ,  $\Delta_s = 0.51$ , and  $g = \frac{(\bar{u}\Lambda)^2}{c_t}$  (see Table III of Ref. [36]). The identification with our corresponding amplitude  $a_\chi^+ = \mathcal{Z}_\chi^{1,+} (Y_c)^\Delta$  (Table 1), gives

$$g_\chi^+ \left( \frac{\bar{u}\Lambda}{(c_t)^{\frac{1}{2}}} \right)^{-2\Delta_s} (1 - \bar{u}) = \mathcal{Z}_\chi^{1,+} (Y_c)^\Delta \quad (57)$$

demonstrating unequivocal relation between  $g^{\frac{1}{2}} = \frac{\bar{u}\Lambda}{(c_t)^{\frac{1}{2}}}$  and  $Y_c$ . However, the rescaled coupling constant  $\bar{u}$  remains dependent, on the one hand, to the correlation between the three adjustable dimensionless parameters  $c_t$ ,  $\bar{u}$ , and  $\Lambda$  of the model, and on another hand, to the master value  $\mathcal{Z}_\chi^{1,+} = 0.555$  initially estimated from the analysis of the isothermal compressibility data of xenon. That implies the implicit introduction of one characteristic microscopic length which must take a unique “thermodynamic” definition [by Eq. (4)], whatever the selected one-component fluid. In that “normalized” situation, our present value  $a_\chi^+ = 0.861$  for  ${}^3\text{He}$ , results in good agreement with for example the values  $\Gamma_1^+ = 0.941 \pm 0.007$  [36] and  $\Gamma_1^+ = 0.81 \pm 0.01$  [30] obtained from data fitting with the crossover parametric model (see below for more details on the uncertainty associated to the  $\Gamma_1^+$  determination). Accounting for the arbitrary relation  $\frac{\Lambda}{(c_t)^{\frac{1}{2}}} = \pi$  adopted by the authors of Ref. [36], our calculated value  $\bar{u} = 0.18075$  from Eq. (57), yields to  $g^{\frac{1}{2}} = 0.5678$ , which compares favourably to  $g^{\frac{1}{2}} = 0.528 \pm 0.003$  obtained from the data fitting performed by Agayan et al [36]. Accounting for the arbitrary relation  $\frac{\Lambda}{(c_t)^{\frac{1}{2}}} = \frac{\pi}{\sqrt{6}}$  adopted by the authors of Ref. [30], with  $\Lambda$  fixed at unity (yielding to  $g^{\frac{1}{2}} = u^* = 0.472$ ), our calculated value  $\bar{u} = 0.35187$  from (57) (yielding to  $g^{\frac{1}{2}} = 0.4513$ ), compares favourably to  $\bar{u} = 0.368 \pm 0.004$  obtained from the Zhong et al [30] data fitting. The  $\sim 10\%$  residual difference between these two estimations of the fluid-dependent parameters, reflects the small differences between theoretical values of universal exponents and amplitude combinations, added to the uncertainty in the direct estimation of the confluent amplitude, the latter one being greater than 10% (for example, using “equivalent” crossover Landau modeling of the same  ${}^3\text{He}$  experimental data, the resulting values are  $\Gamma_1^+ = 0.946 \pm 0.006$  and  $\Gamma_1^+ = 1.000 \pm 0.028$  in Ref. [37],  $\Gamma_1^+ = 0.941 \pm 0.007$  in Ref. [36], and  $\Gamma_1^+ = 0.81 \pm 0.01$  in Ref. [30], while using the minimal-subtraction renormalization scheme, the resulting values are  $\Gamma_1^+ = 1.01 \pm 0.08$  in Ref. [13],  $\Gamma_1^+ = 0.98 \pm 0.08$  or  $\Gamma_1^+ = 1.13 \pm 0.01$  in Ref. [14], and  $\Gamma_1^+ = 1.10 \pm 0.01$  in Ref. [30], leading to the practical “mean” value  $\Gamma_1^+ = 0.97 \pm 0.16$ ) (see also Table II). Nevertheless, this agreement confirms our previous analyses [10, 21] of the confluent correction to scaling for the one-component fluid subclass satisfying to the classical-to-critical crossover description along the

ideal RG trajectory [43, 44].

## 5. CONCLUSIONS

The present study in terms of the dilated physical fields for quantum fluids adds only one well-defined adjustable parameter, which accounts for microscopic quantum effects only asymptotically close to the critical point ( $T \cong T_c$ ). The adjustable parameter is introduced in a phenomenological manner which maintains universal feature of the singular free energy in a appropriate microscopic volume. Since our selected standard fluid is xenon, we provide here a complementary new light to the recent discussions [13, 45] about the definitions of the crossover temperature  $t_X$  [related to the crossover parameter  $g$ , (or the Ginzburg number), as mentioned in §.4.2]. As an essential new consequence, we note that  $t_X \propto \frac{1}{Y_c}$  along the critical isochore, for  $T > T_c$ . Therefore, our two-term asymptotic hyperscaling seems also compatible with (at least) the first-order contribution to the critical crossover. However this observed supplementary constraint is not a necessity from the field theory framework [24]. Consequently, our next work [46] is to provide thermodynamic fundamentals for the asymptotic master behavior of thermodynamic and correlation functions which was inferred from the above minimal information. In addition to the derivation of such thermodynamic fundamentals, we also propose a convenient mean form [41, 42] of the max and min forms for each complete crossover function recently derived by Bagnuls and Bervillier [24]. Such means functions can be appropriately modified to account for the results obtained by the scale dilatation method [41], extending thus the analysis of the crossover behavior of the one-component fluids outside their Ising-like preasymptotic domain.

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- [11] The subscript  $a_{\bar{p}}$  in  $Q_{c,a_{\bar{p}}}^{min}$  notation, recalls for the Helmholtz free energy per particle  $a_{\bar{p}}(T, v_{\bar{p}}) = \frac{A(T, \frac{V}{N}, 1)}{N}$  where  $A(T, V, N)$  is the total Helmholtz free energy of the fluid and  $T$  (temperature),  $V$  (total volume),  $N$  (total temperature), its associated three natural variables.  $v_{\bar{p}} = \frac{V}{N}$  is the particle volume. From the general point of view of the thermodynamics, the equilibrium states of a one-component system at constant amount of matter  $N$ , are entirely defined by the knowledge of the normalized potential  $a_{\bar{p}}(T, v_{\bar{p}})$  or, alternatively but equivalently, by the knowledge of the two equations of state  $s_{\bar{p}}(T, v_{\bar{p}}) = \left(\frac{\partial a_{\bar{p}}}{\partial T}\right)_{v_{\bar{p}}}$  and  $p(T, v_{\bar{p}}) = \left(\frac{\partial a_{\bar{p}}}{\partial v_{\bar{p}}}\right)_T$ .  $s_{\bar{p}}$  is the entropy of the particle and  $p$  is the pressure. These equilibrium states are represented by one 3D characteristic surface of equation  $\Phi(a_{\bar{p}}, T, v_{\bar{p}}) = 0$ , or alternatively but equivalently, by two 3D phase surfaces of equations  $\Phi(s_{\bar{p}}, T, v_{\bar{p}}) = 0$  and  $\Phi(p, v_{\bar{p}}, T) = 0$ , respectively. Only the latter phase surface can be constructed from  $p, V, T, M$  (total mass) measurements, when the mass  $m_{\bar{p}}$  of the particle is known [with  $M = Nm_{\bar{p}}$ ].
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