



HAL
open science

Local structure of random quadrangulations

Maxim Krikun

► **To cite this version:**

| Maxim Krikun. Local structure of random quadrangulations. 2005. hal-00015811v1

HAL Id: hal-00015811

<https://hal.science/hal-00015811v1>

Preprint submitted on 13 Dec 2005 (v1), last revised 10 Oct 2006 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Local structure of random quadrangulations

Maxim Krikun

December 14, 2005

Abstract

This paper is an adaptation of a method used in [1] to the model of random quadrangulations. We prove local weak convergence of uniform measures on quadrangulations and show that local growth of quadrangulation is governed by certain critical time-reversed branching process. As an intermediate result we calculate a biparametric generating function for certain class of quadrangulations with boundary.

1 Introduction

We consider the set of all finite rooted quadrangulations as a metric space with distance between two quadrangulation defined by

$$d(Q_1, Q_2) = \inf \left\{ \frac{1}{1+R} \mid R : B_R(Q_1) = B_R(Q_2) \right\},$$

where $B_R(Q)$ denotes the ball of radius R around the root, and denote the completion of this space by \mathcal{Q} . Elements of \mathcal{Q} other than finite quadrangulations are, by definition, *infinite quadrangulations*.

Theorem 1 *The sequence μ_N of probability measures uniform on quadrangulations with N faces converges weakly to a probability measure μ with support on infinite quadrangulations.*

The measure μ defines certain random object – a *uniform infinite quadrangulation*, and we are interested in local properties of this object. We show that under distribution μ for each R there exists a cycle γ_R , consisting of vertices at distance R from the root and square diagonals between them, such that γ_R separates the root from the infinite part of quadrangulation. Denote by $|\gamma_R|$ the length of this cycle.

Theorem 2 *$|\gamma_R|$ is a Markov chain with transition probabilities given by*

$$\mathbb{P} \left\{ |\gamma_{r+n}| = k \mid |\gamma_r| = l \right\} = \frac{[t^k]F(t)}{[t^l]F(t)} \cdot \mathbb{P}\{\xi_n = l \mid \xi_0 = k\},$$

where ξ is a critical branching process with offspring generating function

$$\varphi(t) = \frac{1}{2t} \left(\sqrt{(t-9)(t-1)^3} - 3 + 6t - t^2 \right),$$

and $F(t)$ is a generating function of its stationary measure.

Corollary 1 $2|\gamma_R|/R^2$ converges in distribution to $\Gamma(3/2)$ law.

2 Some facts on quadrangulations

2.1 Definitions

Consider a finite planar graph embedded into the sphere, such that each component of the complement to the graph is homeomorphic to a disk. A *planar map* is an equivalence class of such embedded graphs with respect to orientation-preserving homeomorphisms of the sphere.

A planar map is *rooted* if a directed edge, called the *root*, is specified. A rooted planar map has no nontrivial automorphisms. We will refer to the tail vertex of the root as *root vertex*, and to distance from any vertex of a map to this vertex as *distance to the root*.

Quadrangulation is a rooted planar map such that all its faces are squares, i.e. its dual graph is four-valent. Note that every quadrangulation is bipartite (this follows from the fact that any subset of faces in a quadrangulation is necessarily bounded by an even number of edges).

In the following we will distinguish two type of faces based on the distances from the vertices around the face to the root: these distances are either $(R-1, R, R+1, R)$ or $(R-1, R, R-1, R)$ for some R . Since every quadrangulation is bipartite, there are no (R, R) edges.

Quadrangulation with a boundary is rather self-evident notion; formally this is a planar map with all faces being squares except one distinguished face which can be an arbitrary even-sided polygon. When drawing the quadrangulation it is convenient to represent this distinguished face by the infinite face. This face is then excluded from "faces" of quadrangulation and is referred to as "boundary".

We say that a quadrangulation has *simple boundary*, if all vertices of the boundary are distinct (i.e. no vertex is met twice when walking around the boundary), and every second vertex has degree two (see fig. 1).

2.2 Some enumeration results

Let $C(N)$ be the number of rooted quadrangulations with N faces, and let $C(N, m)$ be the number of rooted quadrangulations with N faces and with simple boundary of length $2m$, such that the root is located on the boundary and root vertex has degree two.

We will need the following enumeration results (see section 5 for details)

$$q(x) = \sum_{N=0}^{\infty} C(N)x^N = \frac{4}{3} \frac{2\sqrt{1-12x} + 1}{(\sqrt{1-12x} + 1)^2}, \quad (1)$$

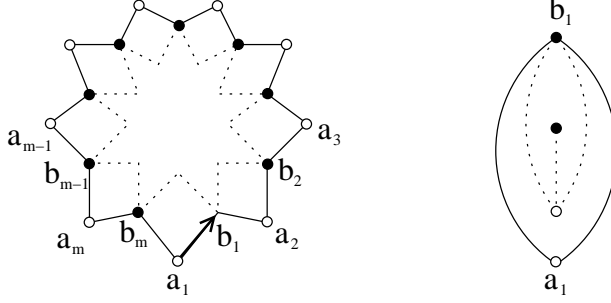


Figure 1: Typical quadrangulation with simple boundary and an example with minimal boundary length.

$$\begin{aligned}
U(x, y) &= \sum_{N, m} C(N, m) x^N y^m \\
&= \frac{1}{2} \left(y - xy^2 - 1 + \sqrt{y^2 - 2xy^3 - 2y + 4xyq(x) + (xy^2 - 1)^2} \right) (2)
\end{aligned}$$

The function $U(x, y)$ is analytic around $(0, 0)$ and its first singularity in x for small y coincides with the singularity of $q(x)$, i.e. $x_0 = 1/12$. From the expansion near this point

$$U(x, y) \Big|_{x=x_0-t} = A(y) + A_1(y)t + B(y)t^{3/2} + O(t^2), \quad (3)$$

where

$$\begin{aligned}
A(y) &= \frac{1}{24} \sqrt{(y-18)(y-2)^3} - \frac{1}{2} + \frac{y}{2} - \frac{y^2}{24}, \\
A_1(y) &= \frac{y^2}{2} + \frac{y}{2} \frac{y^2 - 10y - 32}{\sqrt{(18-y)(2-y)}}, \quad B(y) = \frac{64\sqrt{3} \cdot y}{\sqrt{(y-18)(y-2)^3}},
\end{aligned}$$

one finds the asymptotic of $C(N, m)$ as $N \rightarrow \infty$:

$$C(N, m) = \frac{b(m)}{\Gamma(\frac{3}{2})} N^{-5/2} x_0^{-N} \left(1 + O(N^{-1/2}) \right), \quad b(m) = [y^m] B(y). \quad (4)$$

Note also that $[y]U(x, y) = xq(x)$, thus

$$C(N) = C(N+1, 1) = \frac{b(1)}{x_0 \Gamma(\frac{3}{2})} N^{-5/2} x_0^{-N} \left(1 + O(N^{-1/2}) \right). \quad (5)$$

2.3 Basic probabilities

First let us specify more exactly the definition of ball $B_R(Q)$. Given a rooted quadrangulation Q , $B_R(Q)$ consists of all faces that have at least one vertex at distance strictly less than R from the root. With this definition there are only faces of type $(R-1, R, R+1, R)$ at the boundary of $B_R(Q)$.

Say we want to compute the probability for a uniformly distributed N -faced quadrangulation S_N to have a particular root neighbourhood K . Suppose that K has n faces and, for simplicity, a connected boundary of length $2m$, so that K is a quadrangulation with simple boundary. Take any other quadrangulation S with simple boundary of the same length. We can glue K and S as follows:

- cut m half-squares around the boundary of K , so that the resulting map K' is bounded by m diagonals;
- repeat the same for S , obtaining a map S' bounded by m diagonals;
- identify the boundaries¹ of K' and S' (see fig. 2).

The resulting map $(K' + S')$ has m faces less than K and S had together, i.e. if S has $N - n + m$ faces, $(K' + S')$ would have N faces.

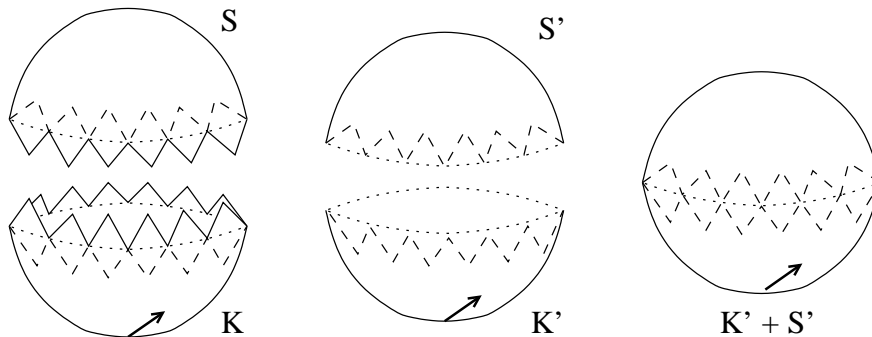


Figure 2: Glueing two quadrangulations with simple boundary.

It's easy to see that the process on fig. 2 is reversible. Indeed, take a quadrangulation Q with root neighbourhood K , cut it in two along the boundary of K' , and add m half-squares to each part. This will give K and some quadrangulation with simple boundary, which then can be used to reconstruct Q .

Thus for each of $C(N - n + m, m)$ maps S with $N - n + m$ faces we get a different N -faced quadrangulation, and every N -faced quadrangulation with root neighbourhood K is obtained this way. In other words

$$P\{B_R(S_N) = K\} = \frac{C(N - n + m, m)}{C(N)}.$$

Combining this with asymptotics (4), (5) we get

¹There are m different ways to do this, but since K is rooted we can choose (in some deterministic way) one of diagonals on the boundary of K' and require that when glueing it is identified with the diagonal of the rooted face of S ; this way the ambiguity is eliminated.

Lemma 2.1 *Given a quadrangulation K with n faces and simple boundary of length $2m$, such that $B_R(Q) = K$ for some Q ,*

$$\lim_{N \rightarrow \infty} \mathbb{P}\{B_R(S_N) = K\} = \frac{b(m)}{b(1)} x_0^{n-m+1}, \quad (6)$$

where S_N denotes uniformly distributed random quadrangulation with N faces.

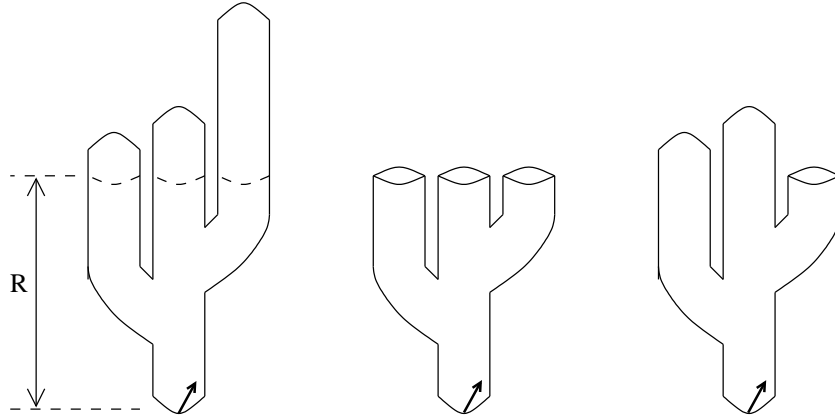


Figure 3: Quadrangulation Q , ball $B_R(Q)$ and hull $\hat{B}_R(Q)$.

In general case, however, the boundary of $B_R(Q)$ may have multiple disjoint components (fig. 3, middle). Following the same reasoning as above and assuming that K has k boundary components ("holes") of length $2m_1, \dots, 2m_k$ and n faces, we'll get the following formula

$$\mathbb{P}\{K = B_R(S_N)\} = \frac{1}{C(N)} \sum_{N_1 + \dots + N_k = N - n} \prod_{j=1}^k C(N_j + m_j, m_j). \quad (7)$$

Here we count all possible ways to "fill" the k holes in K using quadrangulations with appropriate boundary length; N_j is the number of internal faces in quadrangulation used to fill j th hole.

Due to the factor $N^{-5/2}$ in asymptotics (4), for large N the only significant terms in sum (7) are those where one of N_j has order N , while all others are finite. This means that in a large random quadrangulation S_N conditioned to $B_R(S_N) = K$, with high probability only one of the "holes" in K contains the major part of the quadrangulation (we could calculate exact probabilities here, but this is not necessary).

Such observation motivates the following definition: given quadrangulation Q , take the ball $B_R(Q)$ and glue all but the largest components of the complement $Q \setminus B_R(Q)$ back to the ball.² The resulting map is called the *R-hull* of

²if there are multiple components with maximal size, let us chose one of them in some deterministic way; details are not important to us since such situations have small probability for large N .

quadrangulation Q , and is denoted by $\hat{B}_R(Q)$.

Unlike the boundary of the ball, the boundary of $\hat{B}_R(Q)$ is always connected (see fig. 3, right), but similarly to $B_R(Q)$ there are only faces of type $(R, R+1, R, R-1)$ at the boundary of $\hat{B}_R(Q)$, thus the hull is a quadrangulation with simple boundary.

Limiting probabilities for the hull and exactly the same as for the ball:

Lemma 2.2 *Given a quadrangulation K with n faces and boundary of length $2m$, such that $\hat{B}_R(Q) = K$ for some Q ,*

$$\lim_{N \rightarrow \infty} \mathbb{P}\{\hat{B}_R(S_N) = K\} = \frac{b(m)}{b(1)} x_0^{n-m+1} \quad (8)$$

where S_N denotes uniformly distributed random quadrangulation with N faces.

Proof. The proof is essentially the same as for the ball with single hole. Given K , every quadrangulation $(K' + S')$, obtained by glueing K and any quadrangulation S with simple boundary of length $2m$, has the same R -hull $\hat{B}_R(K' + S') = K$ as soon as the number of faces in S is large enough (say larger than n). Thus for $N > 2n$

$$\mathbb{P}\{\hat{B}_R(S_N) = K\} = \frac{C(N - n - m, m)}{C(N)}$$

and the limit (8) follows.

2.4 A note on convergence of measures

The limiting probabilities (8) define a measure μ on \mathcal{Q} , such that for all K and R

$$\mu_N(Q : B_R(Q) = K) \rightarrow \mu(Q : B_R(Q) = K) \quad \text{as } N \rightarrow \infty.$$

However, since \mathcal{Q} is not compact, the existence of this limit does not, by itself, imply weak convergence of μ_N to μ . For the weak convergence to follow one has to show that μ is indeed a probability measure. See [5] for detailed discussion of this question.

In the next section we will evaluate the sum of limiting probabilities (8) over all possible R -hulls K and show that for each R this sum equals one. This will prove Theorem 1.

3 Quadrangulation and branching process

3.1 Hull decomposition

Consider K such that $K = \hat{B}_R(Q)$ for some quadrangulation Q . If Q is large enough (e.g. if the number of faces in Q is at least twice that of K) then

$$\hat{B}_R(Q) \supseteq \hat{B}_{R-1}(Q) \supseteq \dots \supseteq \hat{B}_1(Q),$$

and this sequence doesn't actually depend on Q .

As noticed earlier, the hull has simple boundary. Let us denote the vertices of the boundary of $\hat{B}_R(Q)$ by $(a_1, b_1, \dots, a_m, b_m)$, as on fig. 1, starting from some arbitrarily chosen vertex and so that all a_i 's are situated at distance $R+1$ from the root, and all b_i 's at distance R from the root.

Let γ_R be the cycle consisting of vertices b_1, \dots, b_m and square diagonals between them. Define cycles $\gamma_{R-1}, \dots, \gamma_1$ similarly. A *layer* L_R is a part of quadrangulation contained between cycles γ_{R-1} and γ_R . It turns out that the layer has very simple structure:

- each edge (b_{i-1}, b_i) of it's *upper boundary* γ_R is a diagonal in a square that touches it's *lower boundary* γ_{R-1} at some point c_i ;
- points c_i are cyclically ordered around γ_{R-1} and there are $l_i \geq 0$ edges of γ_{R-1} between c_i and c_{i+1} ($l_i = 0$ if c_i and c_{i+1} both refer to the same vertex).

Let us call the area $(c_i, b_i, b_{i+1}, c_{i+1})$ a *block*. A layer is uniquely (up to rotation related to choice of vertex a_1) characterized by a sequence of blocks.

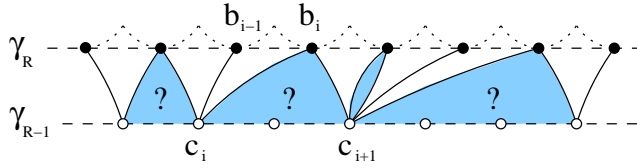


Figure 4: Layer structure. Contents of filled areas is unknown.

The internal structure of a block is not as simple – it can contain arbitrary large subquadrangulation, which can have vertices at distance more than R from the root. This is here where the "reattached" components of $\hat{B}_R(Q) \setminus B_R(Q)$ go. Note that even with $l_i = 0$ the block contents can be non-trivial (fig. 5, right).

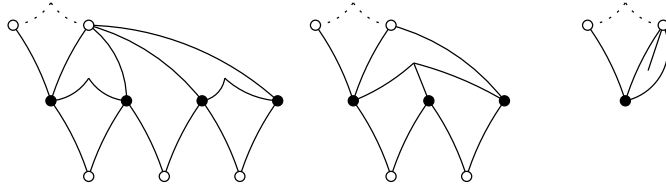


Figure 5: Possible internal structure of a block.

Fortunately there is a bijection between blocks and a class of quadrangulations with simple boundary counted by $C(N, m)$ in section 2.2. The block is converted to quadrangulation via the procedure illustrated on fig. 6. Clearly, this procedure is reversible: one has to choose the topmost vertex on the right-hand side of fig. 6 as the root vertex of the quadrangulation; then the block is

recovered by cutting the quadrangulation along the edge, opposite to the root in the rooted square.

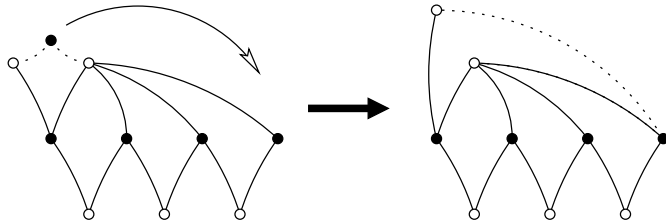


Figure 6: Converting a block to a quadrangulation with simple boundary.

To conclude: the R -hull consists of R layers, each layer consists of one or more blocks, and each block is essentially a quadrangulation with simple boundary.

3.2 Tree structure

The layer/block representation suggests the following tree structure: let the edges of γ_r , $r = 1, \dots, R$ be the nodes of a tree, and connect each edge of γ_r to the edges of γ_{r-1} that belong to the same block.

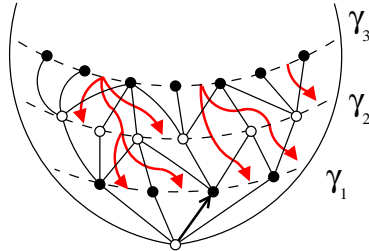


Figure 7: Fragment of tree structure on quadrangulation.

The whole hull K then can be represented by a planar forest T_K of height³ R , where with each vertex one associates a quadrangulation with simple boundary, so that for a vertex v_i of outdegree l_i the associated quadrangulation has boundary length $2(l_i + 1)$. Unfortunately this representation is not unique: given T_K and its associated quadrangulations, we can reconstruct K and the root vertex of K , but not the root edge. In order to include the information on root edge position into the tree structure, we apply to K the following modification (see fig. 8):

- cut K along the root edge, obtaining a hole of length two;

³note that this forest is "reversed" with respect to K : it starts at γ_R and grows down to the root of K . In the following we will keep using such reversed notation and will refer to nodes corresponding to γ_r as the r -th level of T_K .

- attach a new square to the boundary of this hole;
- identify two remaining edges of this square and make the resulting edge a new root.

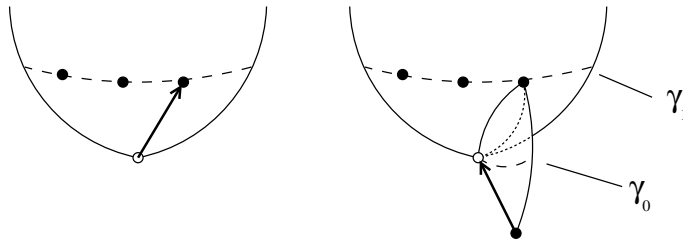


Figure 8: Adding an extra square at the root.

One diagonal of a new square has its ends identified; this gives an extra cycle γ_0 , which always has length one. In terms of tree structure this means that we add one child to some γ_1 -vertex of T_K ; this new vertex has no associated quadrangulation. Call this extended forest a *skeleton* of hull K .

Note that since there is no natural "first" edge in γ_R , the tree structure implies only cyclic order on the trees of T_K . However for convenience we will consider T_K as linearly ordered, and will keep in mind that the same tree structure can be represented by several forests, which differ by cyclic permutation of trees.

Apart from this ambiguity the hull K is uniquely characterized by its skeleton and associated quadrangulations.

3.3 Analogy with branching process

Branching process is a random process with discrete time. It starts with one or more *particles*, and at each step every particle independently of the others is replaced by zero, one, or more *child particles* according to the *offspring distribution* $\{p_i\}_{i=0}^{\infty}$, which remains fixed throughout the whole process.

It is convenient to represent the trajectory of a branching process by a planar tree (or forest, if starting from multiple particles). The probability to see certain trajectory tree T is then a product over all vertices of probability for a particle to have an offspring of size equal to the outdegree of the vertex:

$$P\{T\} = \prod_{v \in T} p_{i|_{i=\text{outdeg}(v)}} \quad (9)$$

We will attempt to do the reverse: apply the theory of branching processes to the analysis of the tree structure described above.

Say we want to compute the probability for an R -hull of uniformly distributed quadrangulation S_N to have a particular skeleton T_K . As explained above, every such R -hull is obtained from T_K by choosing an appropriate set

of associated quadrangulations. On the other hand, taking for every vertex $v_i \in T_K$ with outdegree l_i a quadrangulation with simple boundary of length $2(l_i + 1)$ will give a valid R -hull, which has the required skeleton. A simple calculation shows that if i 'th associated quadrangulation has n_i faces, the hull will have $m - 1 + \sum_i (n_i - 1)$ faces, where m is half the length of hull boundary (or equivalently the number of trees in the skeleton).

Combining this with Lemma 2.2, we'll get the following formula

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \mathbb{P}\{\text{skel}_R(S_N) = T_K\} \\
&= \sum_{n_1, n_2, \dots = 0}^{\infty} \left(\prod_{v_i \in T_K} C(n_i, l_i + 1) \right) \frac{b(m)}{b(1)} x_0^{\sum_i (n_i - 1)} \\
&= \frac{b(m)}{b(1)} \prod_{v_i \in T_K} \sum_{n_i=0}^{\infty} C(n_i, l_i + 1) x_0^{n_i - 1} \\
&= \frac{b(m)}{b(1)} \prod_{v_i \in T_K} \frac{1}{x_0} [y^{l_i+1}] A(y), \tag{10}
\end{aligned}$$

where $A(y)$ is the first coefficient of expansion (3).

The last product in (10) looks similar to the product (9). There is however one important difference – product terms of (10) do not define a probability distribution.

In order to make the analogy with branching process complete, we apply the following normalization procedure: for each square crossed by one of the cycles $\gamma_0, \dots, \gamma_R$ write y_0 on the upper half-square and y_0^{-1} on the lower half-square. Each block then gets an extra term $y_0^{l_i-1}$; plus there is y_0^m on the upper boundary of the hull, due to m half-squares above γ_R , and y_0^{-1} due to one half-square below γ_0 . So we can continue (10) with

$$\begin{aligned}
\dots &= \frac{b(m)}{b(1)} y_0^{m-1} \prod_{v_i \in T_K} \frac{1}{x_0} y_0^{l_i-1} [y^{l_i+1}] A(y) \\
&= \frac{b(m)}{b(1)} y_0^{m-1} \prod_{v_i \in T_K} [t_i^{l_i}] \varphi(t), \tag{11}
\end{aligned}$$

where

$$\varphi(t) = \frac{1}{t x_0 y_0^2} A(y_0 t).$$

For the Taylor coefficients of $\varphi(t)$ to define a probability distribution it has to satisfy the equation $\varphi(1) = 1$, which is equivalent to $A(y_0) = x_0 y_0$. Solving this last equation we find $y_0 = 2$ and

$$\varphi(t) = \frac{1}{2t} \left(\sqrt{(t-9)(t-1)^3} - 3 + 6t - t^2 \right).$$

3.4 Remaining proofs

Proof of Theorem 1. Using (10), (11) let us compute the probability $|\gamma_R| = m$ with respect to μ .

$$\mathbb{P}_\mu\{|\gamma_R| = m\} = \frac{1}{m} \frac{b(m)}{b(1)} y_0^{m-1} \sum_T \prod_{v_i \in T} [t_i^l] \varphi(t), \quad (12)$$

where the sum is taken over all forests T of height $R + 1$ that have m vertices on level R and exactly one vertex at level 0 (in "reversed" notation). The term $1/m$ appears in (12) because for each R -hull with $|\gamma_R| = m$ there are exactly m linearly ordered forests describing the tree structure of this hull.

Let ξ be a branching process with offspring generating function $\varphi(t)$. The sum in (12) can be interpreted as the probability for ξ starting from state m at time 0, to reach state 1 at time R . Let

$$F(t) = \sum_{m=0}^{\infty} \frac{b(m)/m}{b(1)} y_0^{m-1} t^m.$$

Then (12) can be rewritten as

$$\mathbb{P}_\mu\{|\gamma_R| = m\} = [t^m] F(t) \cdot \mathbb{P}\{\xi_{R+1} = 1 | \xi_0 = m\}. \quad (13)$$

The R -step transition probabilities of a branching process are expressed via it's offspring generating function as

$$\mathbb{P}\{\xi_R = n | \xi_0 = m\} = [t^n] \varphi_R^m(t),$$

where $\varphi_R(t)$ stands for the R 'th iteration of $\varphi(t)$. Thus

$$\sum_{m=1}^{\infty} \mathbb{P}_\mu\{|\gamma_R| = m\} = [t] F(\varphi_R(t)).$$

Since $b(m) = [y] B(y)$, where $B(y)$ is a coefficient in (3), we find that

$$F(t) = \frac{1}{y_0 b(1)} \int_0^{ty_0} \frac{B(y)}{y} dy = \frac{3}{4} \left(\sqrt{\frac{9-t}{1-t}} - 3 \right)$$

and a direct computation shows that $F(t)$ satisfies the Abel equation

$$F(\varphi(t)) - F(\varphi(0)) = F(t). \quad (14)$$

In particular this means that $[t] F(\varphi_r(t)) = [t] F(t)$ for all r , and since $[t] F(t) = 1$

$$\sum_{m=1}^{\infty} \mathbb{P}_\mu\{|\gamma_R| = m\} = 1.$$

But the last sum is also the sum of limiting probabilities (8) over all possible R -hulls. This completes the proof of Theorem 1.

Proof of Theorem 2. To prove theorem 2 first note that

$$\mathbb{P}_\mu\{|\gamma_r| = l, |\gamma_{r+n}| = k\} = [t^k]F(t) \cdot \mathbb{P}\{\xi_n = l, \xi_{r+n} = 1 | \xi_0 = k\}.$$

This formula is obtained by taking sum of probabilities (11) over all skeletons with k vertices at level $(r+n)$ and l vertices at level r . Now since ξ is Markovian

$$\mathbb{P}\{\xi_n = l, \xi_{r+n} = 1 | \xi_0 = k\} = \mathbb{P}\{\xi_n = l | \xi_0 = k\} \mathbb{P}\{\xi_{r+n} = 1 | \xi_n = l\},$$

and combining this with (13) we get

$$\mathbb{P}\{|\gamma_{r+n}| = k | |\gamma_r| = l\} = \frac{[t^k]F(t)}{[t^l]F(t)} \cdot \mathbb{P}\{\xi_n = l | \xi_0 = k\}. \quad (15)$$

The Abel equation (14) means that $F(t)$ is a generating function of stationary measure for process ξ (see [3], section 1.4), so the right-hand side of (15) is indeed the transition probability for a reversed branching process.

Proof of Corollary 1. The R th iteration of $\varphi(t)$ is

$$\varphi_R(t) = 1 - \frac{8}{\left(\sqrt{\frac{9-t}{1-t}} + 2R\right)^2 - 1},$$

this can be verified by induction. The distribution of $|\gamma_R|$ is given by

$$\mathbb{P}\{|\gamma_R| = m\} = [t^m]F(t) \cdot [t]\varphi_R^m(t).$$

Calculating explicitly

$$[t]\varphi_R^m(t) = \frac{4}{3}m(2R+3) \frac{(R^2+3R)^{m-1}}{(R^2+3R+2)^{m+1}},$$

$$[t^m]F(t) = \frac{3}{\sqrt{2\pi}}m^{-1/2} + O(1),$$

and putting $m = xR^2$ we find for large R

$$\mathbb{P}\{|\gamma_R| = xR^2\} = \frac{8}{\sqrt{2\pi}} \frac{x^{1/2}e^{-2x}}{R^2} + O\left(\frac{1}{R^3}\right).$$

This implies convergence of $2|\gamma_R|/R^2$ to $\Gamma(3/2)$ law.

3.5 Linear cycle

The cycle γ_R is a natural analog of circle in Euclidean geometry: this is a closed curve, and it's points are situated *exactly* at distance R from the center (root). The relation $|\gamma_R| = O(R^2)$ is, however, quite different from the usual $L = 2\pi R$.

A natural question to ask is what happens, if we weaken restrictions, for example, by allowing the separating cycle to contain any points at distance *at least* R from the root?

It turns out that there exists such cycle with length *linear* in R . This cycle is built as follows:

- consider all vertices of γ_R , and group together the vertices that have common parent in γ_{2R} ;
- in each group there is a leftmost element and a rightmost element. Take a path from the leftmost up to the common parent and then down to the rightmost (approx. $2R$ steps);
- join these paths together to form a separating cycle ℓ_R . γ_{2R} that have non-empty offspring at γ_R .

The length of ℓ_R is $2R \cdot \theta_R$, where θ_R is the number of vertices at γ_{2R} that have nonempty offspring at γ_R . It remains to show that θ_R has finite distribution.

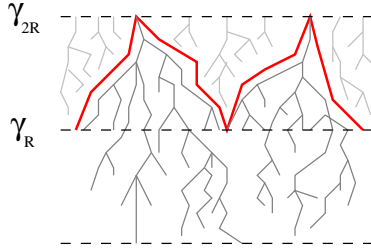


Figure 9: fragment of linear cycle

From (15) we have

$$\mathbb{P}\{\theta_R = k | \gamma_R = m\} = \sum_{n=1}^{\infty} \frac{[t^n]F(t)}{[t^m]F(t)} \binom{n}{k} \varphi_R^{n-k}(0) \cdot [t^m](\varphi_R(t) - \varphi_R(0))^k,$$

$$\mathbb{P}\{\gamma_R = m\} = [t^m]F(t) \cdot [t]\varphi_R^m(t),$$

thus

$$\mathbb{E}(y^{\theta_R} | \gamma_R = m) = \frac{[t^m]F(\varphi_R(0) + y(\varphi_R(t) - \varphi_R(0)))}{[t^m]F(t)},$$

$$\mathbb{E}y^{\theta_R} = [t]F(\varphi_R(0) + y(\varphi_{2R}(t) - \varphi_R(0))).$$

This is a generating function of θ_R . A direct calculation of it's derivatives at $y = 1$ shows that as $R \rightarrow \infty$ the first and second moments of θ_R converge to $11/2$ and $171/2$ respectively.

4 Discussion

Random planar maps are considered a natural model of space with fluctuating geometry in 2-dimensional quantum gravity. Ambjorn and Watabiki [4] suggested that the internal Hausdorff dimension such random space is 4, and this relation doesn't depend on the choice of triangulations, quadrangulations, or some other reasonable distribution of polygons as the underlying model.

Theorem 1 for triangulations was proved by Angel and Schramm [5]. They also provided estimates for the growth of hull boundary, which was shown to be quadratic in radius with polylog corrections [6], but no exact limit was found.

A similar theorem was proved by Chassaing and Durhuus [8], who showed the local weak convergence of well-labeled trees, which are known to be bijective to quadrangulations [9]. This bijection, however, is continuous only in one direction: from quadrangulations to trees (with respect topology of \mathcal{Q} and the natural local topology on the space of trees), so this result is not equivalent to Theorem 1.

Related model of random triangulations with free boundary was considered by Malyshev and Krikun [7]. It was shown that at the critical boundary parameter value the boundary of a random quadrangulation with N faces has about \sqrt{N} edges and the ratio converges in distribution to the square of $\Gamma(3/4)$ law. Nothing is known about the diameter of the triangulation, but this is natural to suggest that the diameter has order $N^{1/4}$.

The skeleton construction used in this paper was proposed in [1], where it was applied to random triangulations. The branching process obtained for triangulations differs from ξ , but it has non-extinction probabilities of the same order $1/R^2$ and it's generating function has the main singularity of the same order $3/2$.

It seems natural to look for convergence of profile $|\gamma_R|$ to a continuous time process

$$\frac{|\gamma_{tR}|}{R^2} \rightarrow \zeta_t$$

Given what we know about b.p. ξ , a natural candidate for ζ_t is a time-reversed continuous state branching process of order $1/2$. If true, this means that ζ_t has a.s. discontinuous trajectories.

There should also exist certain natural mapping of branching process structure into the brownian map [10].

Finally, we want to note the following statement in [4]: *"A boundary of l links will have the discrete length l in lattice units, but if we view the boundary from the interior of the surface its true linear extension r will only be \sqrt{l} , since the boundary can be viewed as a random walk from the interior"*.

5 Enumeration

The formula (1) is obtained from a more general formula for the number of bicubic (bipartite, trivalent) planar maps due to Tutte [2]. No doubt, (2) could

also be derived from the same source, but we shall give a slightly more straightforward proof.

Consider first the class \mathcal{Q}' of quadrangulations with simple boundary with no double edges. Every such quadrangulation has at least two faces. Take the vertex opposite to the root vertex in the rooted face and cut the quadrangulation along every edge, incident to this vertex. Forget the rooted face. This operation produces one or more components, each being either a single face, either again a quadrangulation from \mathcal{Q}' , and the boundary of each component consists of two segments – one is a part of original quadrangulation's boundary, another consists of previously internal faces.

This is a bijection – given an ordered collection of quadrangulations, each with boundary separated in two segments, the original quadrangulation can be reconstructed. Thus

$$u(x, y) = xy \cdot \sum_{n=0}^{\infty} w(x, y, 1)^n,$$

where $u(x, y)$ is a generating function of quadrangulations in \mathcal{Q}' with x counting faces and y counting faces at the boundary, and

$$w(x, y, z) = \frac{u(x, y) - u(x, z)}{y - z}$$

is a generating function of quadrangulations with segmented boundary, y and z counting the length (in faces) of each segment.

Express $w(x, y, 1)$ from the first equation and substitute into the second one with $z = 1$. This gives

$$1 - \frac{xy}{u(x, y)} = \frac{u(x, 1) - u(x, y)}{1 - y}.$$

Note that this equation is quadratic in $u(x, y)$.

To pass from quadrangulations without double edges to the general class of quadrangulations \mathcal{Q} we attach at each internal edge of quadrangulation from \mathcal{Q}' a general sphere quadrangulation. More exactly, we cut this edge and identify two sides of obtained hole with two sides of analogous hole, obtained by cutting the root edge of quadrangulation being attached. (This is the *extension* procedure, best explained in [2], section 7).

Expansion is equivalent to a substitution $x \rightarrow q^2x$, $y \rightarrow q^{-2}y$. Under this substitution we get

$$\begin{aligned} u(q^2x, q^{-2}y) &= U(x, y), \\ u(q^2x, 1) &= q - 2xq^2 - 1. \end{aligned}$$

Correction term $2xyq^2$ correspond to two quadrangulations with boundary of length 2 (as on fig. 1, right), which cannot be obtained from quadrangulation without double edges by expansion, and 1 corresponds to a degenerate quadrangulation which consists of a single edge. When this degenerate quadrangulation is attached to and edge during expansion it doesn't change anything.

Combining two last equations with quadratic equation on $u(x, y)$ we get (2).

References

- [1] M. Krikun. Uniform infinite planar triangulation and related time-reversed branching process. *Journal of mathematical sciences*, vol 131, 2, pp. 5520–5537, 2005; arXiv:math.PR/0311127
- [2] W. Tutte. A census of planar maps. *Canad. J. Math*, 15:249–271, 1963.
- [3] F.M. Hoppe, E. Seneta. Analytical methods for discrete branching processes. in *Advances in probability and related topics*, vol. 5, A. Joffe, P.E. Ney eds., 1978, pp. 219-261.
- [4] J. Ambjorn, Y. Watabiki. Scaling in quantum gravity. *Nucl. Phys. B* 445, 1995, 1, 129–142.
- [5] O. Angel, O. Schramm. Uniform Infinite Planar Triangulations. *Comm. Math. Phys.* vol 241, no. 2-3, pp. 191-213, 2003, arXiv:math.PR/0207153
- [6] O. Angel. Growth and Percolation on the Uniform Infinite Planar Triangulation. arXiv:math.PR/0208123
- [7] V.A. Malyshev, M. Krikun. Random Boundary of a Planar Map in "Trends in Mathematics. Mathematics and Computer Science II. Algorithms, Trees, Combinatorics and Probabilities", BirkHauser, 2002, pp.83–94
- [8] P. Chassaing, B. Durhuus. Local limit of labelled trees and expected volume growth in a random quadrangulation. arXiv:math.PR/0311532
- [9] G. Schaeffer. Conjugation d'arbres et cartes combinatoires aleatoires. Ph.D. thesis, Universite Bordeaux I, Bordeaux, 1998.
- [10] J.F. Marcert, A. Mokkadem. Limit of normalized quadrangulations: the brownian map. arXiv:math.PR/0403398

Maxim Krikun
Institut Elie Cartan,
Universite Henri Poincare,
Nancy, France
krikun@iecn.u-nancy.fr