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# BIAS-REDUCED EXTREME QUANTILE ESTIMATORS OF WEIBULL TAIL-DISTRIBUTIONS

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**Abstract.** *In this paper, we consider the problem of estimating an extreme quantile of a Weibull tail-distribution. The new extreme quantile estimator has a reduced bias compared to the more classical ones proposed in the literature. It is based on an exponential regression model that was introduced in Diebolt et al. (2005). Its asymptotic normality is established and a small simulation study is provided in order to illustrate its efficiency.*

**Key words and phrases.** Weibull tail-distribution, extreme quantile, bias-reduction, least-squares approach, asymptotic normality.

**AMS Subject classifications.** 62G05, 62G20, 62G30.

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# 1 Introduction

Let  $X_1, \dots, X_n$  be a sequence of independent and identically distributed (i.i.d.) random variables with distribution function  $F$ .

In the present paper, we assume that  $F$  is a Weibull tail-distribution, which means that

$$1 - F(x) = \exp(-H(x)) \quad \text{with} \quad H^{-1}(x) := \inf\{t : H(t) \geq x\} = x^\theta \ell(x), \quad (1)$$

where  $\theta > 0$  denotes the Weibull tail-coefficient and  $\ell$  is a slowly varying function at infinity satisfying

$$\frac{\ell(\lambda x)}{\ell(x)} \rightarrow 1, \text{ as } x \rightarrow \infty, \text{ for all } \lambda > 0. \quad (2)$$

Based on the limited sample  $X_1, \dots, X_n$ , the question is how to obtain a good estimate for a quantile of order  $1 - p_n$ ,  $p_n \rightarrow 0$  defined by

$$x_{p_n} = \inf\{y : F(y) \geq 1 - p_n\},$$

such that the quantile to be estimated is situated on the border of or beyond the range of the data. Extrapolation outside the sample occurs for instance in reliability (Ditlevsen, 1994), hydrology (Smith, 1991), and finance (Embrechts et al., 1997).

Gardes and Girard (2005) investigated this estimation problem and proposed the following estimator of  $x_{p_n}$ :

$$\tilde{x}_{p_n} = X_{n-k_n+1,n} \left( \frac{\log(1/p_n)}{\log(n/k_n)} \right)^{\tilde{\theta}_n} \quad (3)$$

where  $X_{1,n} \leq \dots \leq X_{n,n}$  denote the order statistics associated to the original sample and  $\tilde{\theta}_n$  is the following estimator of the Weibull tail-coefficient proposed by Girard (2004):

$$\tilde{\theta}_n = \frac{\sum_{i=1}^{k_n} (\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n}))}{\sum_{i=1}^{k_n} (\log \log(n/i) - \log \log(n/k_n))}. \quad (4)$$

In the preceding equations,  $k_n$  denotes an intermediate sequence, i.e. a sequence such that  $k_n \rightarrow \infty$  and  $k_n/n \rightarrow 0$  as  $n \rightarrow \infty$ . We refer to Beirlant et al. (1995) and Broniatowski (1993) for other propositions and to Beirlant et al. (2006) for Local Asymptotic Normality (LAN) results. Denoting  $\tau_n = \log(1/p_n)/\log(n/k_n)$ , the estimator (3) can be rewritten as

$$\tilde{x}_{p_n} = X_{n-k_n+1,n} \tau_n^{\tilde{\theta}_n}.$$

It appears that the extreme quantile of order  $1 - p_n$  is estimated through an ordinary quantile of order  $1 - k_n/n$  with a multiplicative correction  $\tau_n^{\tilde{\theta}_n}$ .

The main asymptotic properties (weak consistency and asymptotic normality) of  $\tilde{x}_{p_n}$  have been established by Gardes and Girard (2005). Also, its finite sample performances have been compared to those of classical estimators. All of them exhibit a bias depending on the rate of convergence to 1 of the ratio of the slowly varying function  $\ell$  in (2). In order to quantify this bias, a second-order condition is required. This assumption can be expressed as follows:

**Assumption** ( $R_\ell(b, \rho)$ ). *There exists a constant  $\rho < 0$  and a rate function  $b$  satisfying  $b(x) \rightarrow 0$  as  $x \rightarrow \infty$ , such that for all  $\varepsilon > 0$  and  $1 < A < \infty$ , we have*

$$\sup_{\lambda \in [1, A]} \left| \frac{\log(\ell(\lambda x)/\ell(x))}{b(x)K_\rho(\lambda)} - 1 \right| \leq \varepsilon, \quad \text{for } x \text{ sufficiently large,}$$

$$\text{with } K_\rho(\lambda) = \int_1^\lambda t^{\rho-1} dt.$$

It can be shown that necessarily  $|b|$  is regularly varying with index  $\rho$  (see e.g. Geluk and de Haan, 1987).

In this paper, we focus on the case where the convergence (2) is slow, and thus when the bias term in  $\tilde{\theta}_n$  and therefore in  $\tilde{x}_{p_n}$  is large. This situation is described by the following assumption:

$$xb(x) \rightarrow \infty \text{ as } x \rightarrow \infty, \quad (5)$$

which is fulfilled by Gamma and Gaussian distributions (see for instance Diebolt et al. (2005), Table 1.) The methodology that we propose in order to reduce the bias of  $\tilde{x}_{p_n}$  is to use the following regression model proposed by Diebolt et al. (2005) for the log-spacings of upper order statistics:

$$\begin{aligned} Z_j &:= j \log\left(\frac{n}{j}\right) (\log(X_{n-j+1,n}) - \log(X_{n-j,n})) \\ &= \left( \theta + b\left(\log\left(\frac{n}{k_n}\right)\right) \left(\frac{\log(n/k_n)}{\log(n/j)}\right) \right) f_j + o_{\mathbb{P}}\left(b\left(\log\left(\frac{n}{k_n}\right)\right)\right), \end{aligned} \quad (6)$$

for  $1 \leq j \leq k_n$ , where  $(f_1, \dots, f_{k_n})$  is a vector of independent and standard exponentially distributed random variables and the  $o_{\mathbb{P}}$ -term is uniform in  $j$ .

This exponential regression model is similar to the ones proposed by Beirlant et al. (1999, 2002) and Feuerverger and Hall (1999) in the case of Pareto-type distributions.

The model (6) allows us to generate bias-corrected estimates  $\widehat{\theta}_n$  for  $\theta$  through a Least-Square (LS) estimation of  $\theta$  and  $b(\log(n/k_n))$ . The resulting LS estimates are then the

following:

$$\left\{ \begin{array}{l} \widehat{\theta}_n = \overline{Z}_{k_n} - \widehat{b}(\log(n/k_n)) \overline{x}_{k_n} \\ \widehat{b}(\log(n/k_n)) = \frac{\sum_{j=1}^{k_n} (x_j - \overline{x}_{k_n}) Z_j}{\sum_{j=1}^{k_n} (x_j - \overline{x}_{k_n})^2} \end{array} \right.$$

where  $x_j = \frac{\log(n/k_n)}{\log(n/j)}$ ,  $\overline{x}_{k_n} = \frac{1}{k_n} \sum_{j=1}^{k_n} x_j$  and  $\overline{Z}_{k_n} = \frac{1}{k_n} \sum_{j=1}^{k_n} Z_j$ .

The asymptotic normality of the LS-estimator  $\widehat{\theta}_n$  is established in Diebolt et al. (2005). Now, in order to refine  $\widehat{x}_{p_n}$ , we can use the additional information about the slowly varying function  $\ell$  that is provided by the LS-estimates for  $\theta$  and  $b$ . To this aim, condition  $(R_\ell(b, \rho))$  is used to approximate the ratio  $F^{-1}(1 - p_n)/X_{n-k_n+1,n}$ , noting that

$$X_{n-k_n+1,n} \stackrel{d}{=} F^{-1}(U_{n-k_n+1,n}),$$

with  $U_{1,n} \leq \dots \leq U_{n,n}$  the order statistics of a uniform  $(0, 1)$  sample of size  $n$ ,

$$\begin{aligned} \frac{x_{p_n}}{X_{n-k_n+1,n}} &\stackrel{d}{=} \frac{F^{-1}(1 - p_n)}{F^{-1}(U_{n-k_n+1,n})} \\ &= \frac{(-\log(p_n))^\theta}{(-\log(1 - U_{n-k_n+1,n}))^\theta} \frac{\ell(-\log(p_n))}{\ell(-\log(1 - U_{n-k_n+1,n}))} \\ &\stackrel{d}{=} \frac{(-\log(p_n))^\theta}{(-\log(U_{k_n,n}))^\theta} \frac{\ell(-\log(p_n))}{\ell(-\log(U_{k_n,n}))} \\ &\simeq \left( \frac{\log(1/p_n)}{\log(n/k_n)} \right)^\theta \exp \left[ b(\log(n/k_n)) \frac{\left( \frac{\log(1/p_n)}{\log(n/k_n)} \right)^\rho - 1}{\rho} \right]. \end{aligned}$$

The last step follows by replacing  $U_{k_n,n}$  with  $k_n/n$ . Hence, we arrive at the following estimator for extreme quantiles

$$\widehat{x}_{p_n} = X_{n-k_n+1,n} \left( \frac{\log(1/p_n)}{\log(n/k_n)} \right)^{\widehat{\theta}_n} \exp \left[ \widehat{b}(\log(n/k_n)) \frac{\left( \frac{\log(1/p_n)}{\log(n/k_n)} \right)^{\widehat{\rho}_n} - 1}{\widehat{\rho}_n} \right],$$

or equivalently,

$$\widehat{x}_{p_n} = X_{n-k_n+1,n} \tau_n^{\widehat{\theta}_n} \exp \left( \widehat{b}(\log(n/k_n)) K_{\widehat{\rho}_n}(\tau_n) \right).$$

Here,  $\widehat{\rho}_n$  is an arbitrary estimator of  $\rho$ . It will appear in the next section (see Theorem 1(ii)) that, if  $\tau_n$  converges to a constant value  $\tau > 1$ , one can even choose  $\widehat{\rho}_n = \rho^\#$  a constant value, for instance the canonical value  $\rho^\# = -1$ , as suggested by Feuerverger and Hall (1999). Note that the estimator (3) can be seen as a particular case of  $\widehat{x}_{p_n}$  ob-

tained by neglecting the bias-term. In the following, we use the LS-estimators of  $\theta$  and  $b$  defined previously. The study of the asymptotic properties of the extreme quantile estimator is the aim of Section 2, whereas a simulation study is provided in Section 3 to illustrate its usefulness. The proof of our result is postponed to Section 4.

## 2 Bias-reduced extreme quantile estimator

The asymptotic normality of our bias-reduced extreme quantile estimator  $\widehat{x}_{p_n}$  is established in the following theorem.

**Theorem 1.** *Suppose (1) holds together with  $(R_\ell(b, \rho))$  and (5). We assume that*

$$k_n \rightarrow \infty, \frac{\sqrt{k_n}}{\log(n/k_n)} b(\log(n/k_n)) \rightarrow \lambda \in \mathbb{R}, \quad (7)$$

and if  $\lambda = 0$ ,

$$\frac{\sqrt{k_n}}{\log(n/k_n)} \rightarrow \infty \text{ and } \frac{\log^2(k_n)}{\log(n/k_n)} \rightarrow 0. \quad (8)$$

Under the additional condition that

$$|\widehat{\rho}_n - \rho| \log(\tau_n) = O_{\mathbb{P}}(1), \quad (9)$$

we have

(i) if  $\tau_n \rightarrow \infty$

$$\frac{\sqrt{k_n}}{\log(n/k_n) \log(\tau_n)} \left( \log(\widehat{x}_{p_n}) - \log(x_{p_n}) \right) \xrightarrow{d} \mathcal{N}(0, \theta^2),$$

(ii) if  $\tau_n \rightarrow \tau, \tau > 1$ , and if we replace  $\widehat{\rho}_n$  by a canonical choice  $\rho^\# < 0$ , then

$$\frac{\sqrt{k_n}}{\log(n/k_n)} \left( \log(\widehat{x}_{p_n}) - \log(x_{p_n}) \right) \xrightarrow{d} \mathcal{N}(\lambda \mu(\tau), \theta^2 \sigma^2(\tau)),$$

with

$$\sigma^2(\tau) = \left( K_{\rho^\#}(\tau) - \log(\tau) \right)^2,$$

and

$$\mu(\tau) = \left( K_{\rho^\#}(\tau) - K_\rho(\tau) \right).$$

The proof of this theorem is postponed to Section 4. In the following remark we provide some possible choices for the sequences  $(k_n)$  and  $(p_n)$ .

**Remark 1.** Suppose (1) holds together with  $(R_\ell(b, \rho))$  and (5). Then, choosing

$$k_n = \left( \lambda \frac{\log(n)}{b(\log(n))} \right)^2, \quad \lambda > 0, \quad p_n = n^{-\tau}, \quad \tau > 1, \quad \text{and} \quad \widehat{\rho}_n = \rho^\# < 0,$$

Theorem 1(ii) applies and thus

$$\frac{1}{b(\log(n))} \left( \log(\widehat{x}_{p_n}) - \log(x_{p_n}) \right) \xrightarrow{d} \mathcal{N} \left( \mu(\tau), \left( \frac{\theta}{\lambda} \right)^2 \sigma^2(\tau) \right).$$

Clearly, the faster  $b$  converges to 0, the faster  $\widehat{x}_{p_n}$  converges to  $x_{p_n}$ .

In order to illustrate that our estimator has a reduced bias, we will provide a small simulation study in the next section.

### 3 A small simulation study

The finite sample performance of the estimators  $\widetilde{x}_{p_n}$  and  $\widehat{x}_{p_n}$  are investigated on 4 different distributions:  $|\mathcal{N}(0, 1)|$ ,  $\Gamma(0.25, 0.25)$ ,  $\Gamma(4, 1)$  and  $\mathcal{W}(0.25, 0.25)$ . It is shown in Gardes and Girard (2005) that  $\widetilde{x}_{p_n}$  gives better results than the other approaches (Hosking and Wallis, 1987; Breiman et al., 1990; Beirlant et al., 1995). This explains why  $\widehat{x}_{p_n}$  is only compared to the estimator  $\widetilde{x}_{p_n}$ . In the following, we take  $p_n := p_n(\tau) = n^{-\tau}$  with  $\tau = 1, 2$  and  $4$  and we choose  $\widehat{\rho}_n = -1$ . We simulate  $N = 500$  samples  $(\mathcal{X}_{n,i})_{i=1, \dots, N}$  of size  $n = 500$ . On each sample  $(\mathcal{X}_{n,i})$ , the estimates  $\widetilde{x}_{p_n(\tau), i}(k_n)$ ,  $\tau = 1, 2$  and  $4$  are computed for  $k_n = 2, \dots, 360$ . We present the plots obtained by drawing the points

$$(k_n, RE(\widetilde{x}_{p_n(\tau)}(k_n))) \text{ for } \tau = 1, 2 \text{ and } 4,$$

where  $RE(\widetilde{x}_{p_n(\tau)}(k_n))$  is the median value of the relative errors

$$\left| \frac{\widetilde{x}_{p_n(\tau), i}(k_n) - x_{p_n(\tau)}}{x_{p_n(\tau)}} \right|_{i=1, \dots, N}.$$

The same procedure is achieved for the estimator  $\widehat{x}_{p_n}$ . Finally, in order to make a better comparison of the two estimators, we draw the points

$$\left( k_n, \frac{RE(\widetilde{x}_{p_n(\tau)}(k_n))}{RE(\widehat{x}_{p_n(\tau)}(k_n))} \right) \text{ for } \tau = 1, 2 \text{ and } 4.$$

Results are presented on figures 1–4. For the  $|\mathcal{N}(0, 1)|$ ,  $\Gamma(0.25, 0.25)$  and  $\Gamma(4, 1)$  distributions, the relative error of  $\widehat{x}_{p_n}$  is smaller than the one of  $\widetilde{x}_{p_n}$  at least for large values of  $k_n$  ( $k_n > 100$ ). Note that for the  $\mathcal{W}(0.25, 0.25)$  distribution, Theorem 1 does not apply since  $xb(x) = 0$ . In this situation  $\widetilde{x}_{p_n}$  performs slightly better.

## 4 Proofs

**Proof of Theorem 1.** We decompose our quantile estimator as follows:

$$\begin{aligned}
\log(\widehat{x}_{p_n}) - \log(x_{p_n}) &= \log(X_{n-k_n+1,n}) + \widehat{\theta}_n \log(\tau_n) \\
&+ \widehat{b}(\log(n/k_n)) K_{\widehat{\rho}_n}(\tau_n) - \log\left((- \log(p_n))^\theta \ell(- \log(p_n))\right) \\
&\stackrel{d}{=} \theta \{\log(- \log(U_{k_n,n})) - \log \log(n/k_n)\} \\
&+ (\widehat{\theta}_n - \theta) \log(\tau_n) \\
&+ \{\log \ell(- \log(U_{k_n,n})) - \log \ell(\log(n/k_n))\} \\
&+ \{\log \ell(\log(n/k_n)) - \log \ell(- \log(p_n)) + b(\log(n/k_n)) K_\rho(\tau_n)\} \\
&+ (\widehat{b}(\log(n/k_n)) - b(\log(n/k_n))) K_{\widehat{\rho}_n}(\tau_n) \\
&+ b(\log(n/k_n)) \{K_{\widehat{\rho}_n}(\tau_n) - K_\rho(\tau_n)\} \\
&=: \sum_{j=1}^6 B_{j,k_n}.
\end{aligned}$$

We successively discuss each of the terms  $B_{j,k_n}$ ,  $j = 1, \dots, 6$ . First concerning  $B_{1,k_n}$ , remark that

$$\log(- \log(U_{k_n,n})) - \log \log(n/k_n) \stackrel{d}{=} \log\left(\frac{T_{n-k_n+1,n}}{\log(n/k_n)}\right),$$

where  $T_{j,n}$  denotes the order statistics from an i.i.d. standard exponential sample of size  $n$ . Since it is well known that

$$\sqrt{k_n} (T_{n-k_n+1,n} - \log(n/k_n)) \xrightarrow{d} \mathcal{N}(0, 1), \quad (10)$$

we clearly have

$$\frac{\sqrt{k_n}}{\log(n/k_n) \log(\tau_n)} B_{1,k_n} = O_{\mathbb{P}}\left(\frac{1}{\log^2(n/k_n) \log \tau_n}\right) = o_{\mathbb{P}}(1). \quad (11)$$

Remark now that

$$\frac{\sqrt{k_n}}{\log(n/k_n) \log(\tau_n)} B_{2,k_n} = \frac{\sqrt{k_n}}{\log(n/k_n)} (\widehat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}(0, \theta^2), \quad (12)$$

by Theorem 3 in Diebolt et al. (2005).

Next, using  $(R_\ell(b, \rho))$  and (10), we get

$$\begin{aligned}
B_{3,k_n} &\stackrel{d}{=} \log\left(\frac{\ell(T_{n-k_n+1,n})}{\ell(\log(n/k_n))}\right) \\
&= K_\rho\left(\frac{T_{n-k_n+1,n}}{\log(n/k_n)}\right) b(\log(n/k_n)) (1 + o_{\mathbb{P}}(1))
\end{aligned}$$



$$\begin{aligned}
&= \left( \frac{T_{n-k_n+1,n}}{\log(n/k_n)} - 1 \right) b(\log(n/k_n))(1 + o_{\mathbb{P}}(1)) \\
&= O_{\mathbb{P}} \left( \frac{b(\log(n/k_n))}{\sqrt{k_n} \log(n/k_n)} \right),
\end{aligned}$$

so that under (7),

$$\frac{\sqrt{k_n}}{\log(n/k_n) \log(\tau_n)} B_{3,k_n} = O_{\mathbb{P}} \left( \frac{1}{\sqrt{k_n} \log(n/k_n) \log(\tau_n)} \right) = o_{\mathbb{P}}(1). \quad (13)$$

Next, under  $(R_{\ell}(b, \rho))$  one has (for a suitably chosen  $b$ ) that for all  $\epsilon > 0$  (see Drees, 1998, Lemma 2.1)

$$\sup_{t>1} t^{-(\epsilon+\rho)} \left| \frac{\log \ell(tx) - \log \ell(x)}{b(x)} - K_{\rho}(t) \right| \rightarrow 0.$$

Hence, we conclude, choosing  $\epsilon < -\rho$ , that

$$\left| \frac{B_{4,k_n}}{b(\log(n/k_n))} \right| = \left| \frac{\log \ell(\log(n/k_n)) - \log \ell(\tau_n \log(n/k_n))}{b(\log(n/k_n))} + K_{\rho}(\tau_n) \right| \rightarrow 0,$$

which implies that, under (7), we have

$$\frac{\sqrt{k_n}}{\log(n/k_n) \log(\tau_n)} B_{4,k_n} = o \left( \frac{1}{\log(\tau_n)} \right) = o(1). \quad (14)$$

Next, we can check that, according to (6),

$$B_{5,k_n} = K_{\widehat{\rho}_n}(\tau_n) \frac{1}{k_n} \sum_{j=1}^{k_n} \beta_{j,n} (f_j - 1),$$

where

$$\beta_{j,n} := \frac{(x_j - \bar{x}_{k_n})(\theta + b(\log(n/k_n))x_j)}{\frac{1}{k_n} \sum_{i=1}^{k_n} (x_i - \bar{x}_{k_n})^2}.$$

A direct application of Lyapounov's theorem, combined with Lemma 5 in Diebolt et al. (2005) yields

$$\frac{\sqrt{k_n}}{\log(n/k_n) k_n} \sum_{j=1}^{k_n} \beta_{j,n} (f_j - 1) \xrightarrow{d} \mathcal{N}(0, \theta^2).$$

Therefore

$$\frac{\sqrt{k_n}}{\log(n/k_n) \log(\tau_n)} B_{5,k_n} = \frac{K_{\widehat{\rho}_n}(\tau_n)}{\log(\tau_n)} \xi_{1,n} \quad \text{with} \quad \xi_{1,n} \xrightarrow{d} \mathcal{N}(0, \theta^2). \quad (15)$$

Finally, following the method of proof of Lemma 1 in de Haan and Rootzén (1993), we

find that

$$\begin{aligned}
& \left| \int_1^{\tau_n} s^{\rho-1} (s^{\widehat{\rho}_n - \rho} - 1) ds - (\widehat{\rho}_n - \rho) \int_1^{\tau_n} s^{\rho-1} \log(s) ds \right| \\
& \leq |\widehat{\rho}_n - \rho| \int_1^{\tau_n} s^{\rho-1} \log(s) (s^{|\widehat{\rho}_n - \rho|} - 1) ds \\
& \leq |\widehat{\rho}_n - \rho| \left( \int_1^{\tau_n} s^{\rho-1} \log(s) ds \right) (\tau_n^{|\widehat{\rho}_n - \rho|} - 1),
\end{aligned}$$

where the first inequality comes from the fact that  $\left| \frac{e^x - 1}{x} - 1 \right| \leq e^{|x|} - 1$ . Hence

$$\begin{aligned}
\frac{\sqrt{k_n}}{\log(n/k_n) \log(\tau_n)} B_{6,k_n} &= \frac{\sqrt{k_n}}{\log(n/k_n)} b(\log(n/k_n)) \\
&\times (\widehat{\rho}_n - \rho) \frac{\int_1^{\tau_n} x^{\rho-1} \log(x) dx}{\log(\tau_n)} \left\{ 1 + O\left(\tau_n^{|\widehat{\rho}_n - \rho|} - 1\right) \right\}.
\end{aligned}$$

If  $\tau_n \rightarrow \infty$ , this implies that  $\widehat{\rho}_n \xrightarrow{\mathbb{P}} \rho$  by the assumption (9) and therefore

$$\frac{\sqrt{k_n}}{\log(n/k_n) \log(\tau_n)} B_{6,k_n} = o_{\mathbb{P}}(1). \tag{16}$$

Combining (11)-(16), Theorem 1 (i) follows.

If  $\tau_n \rightarrow \tau, \tau > 1$ , then the normalization factor  $\log(\tau_n) \rightarrow \log(\tau) \neq 0$  can be omitted in (11), (13) and (14) while preserving the negligibility of these terms. Besides, we can replace  $\widehat{\rho}_n$  with any canonical choice, for instance  $\rho^\# < 0$ , and therefore

$$\begin{aligned}
\frac{\sqrt{k_n}}{\log(n/k_n)} B_{6,k_n} &= \frac{\sqrt{k_n}}{\log(n/k_n)} b(\log(n/k_n)) (K_{\rho^\#}(\tau_n) - K_\rho(\tau_n)) \\
&\longrightarrow \lambda \mu(\tau) := \lambda (K_{\rho^\#}(\tau) - K_\rho(\tau)).
\end{aligned} \tag{17}$$

The limiting distribution is then given by (12) and (15) with a bias term due to (17). To conclude with the second part of our Theorem 1, we have to establish the limiting distribution of

$$U_n := \frac{\sqrt{k_n} K_{\rho^\#}(\tau_n)}{\log(n/k_n)} \left( \widehat{b}(\log(n/k_n)) - b(\log(n/k_n)) \right) + \frac{\sqrt{k_n} \log(\tau_n)}{\log(n/k_n)} (\widehat{\theta}_n - \theta).$$

To this aim, remark that

$$U_n = \frac{k_n^{-\frac{1}{2}}}{\log(n/k_n)} \sum_{j=1}^{k_n} \omega_{j,n}(f_j - 1) + o_{\mathbb{P}}(1),$$

where

$$\omega_{j,n} = \beta_{j,n} K_{\rho^*}(\tau_n) + \alpha_{j,n} \log(\tau_n)$$

and

$$\alpha_{j,n} = \left( \theta + b(\log(n/k_n)) x_j \right) \left( 1 - \frac{x_j - \bar{x}_{k_n}}{\frac{1}{k_n} \sum_{i=1}^{k_n} (x_i - \bar{x}_{k_n})^2} \bar{x}_{k_n} \right).$$

Using Lemma 5 in Diebolt et al. (2005), direct computations lead to

$$\sum_{j=1}^{k_n} \text{Var}(\omega_{j,n}(f_j - 1)) = \sum_{j=1}^{k_n} \omega_{j,n}^2 \sim \theta^2 (\log(n/k_n))^2 k_n \sigma^2(\tau)$$

and

$$\sum_{j=1}^{k_n} \mathbb{E}(\omega_{j,n}(f_j - 1))^4 = 9 \sum_{j=1}^{k_n} \omega_{j,n}^4 \sim C k_n (\log(n/k_n))^4,$$

where  $C$  is a suitable constant. Therefore a direct application of Lyapounov's theorem yields

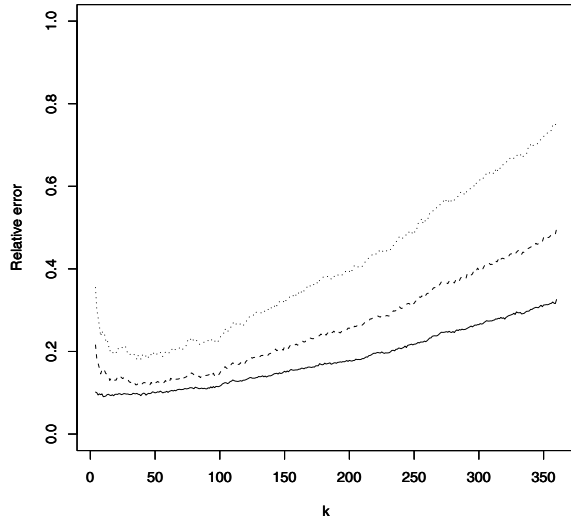
$$U_n \xrightarrow{d} \mathcal{N}(0, \theta^2 \sigma^2(\tau)),$$

which achieves the proof of the second part of Theorem 1.  $\square$

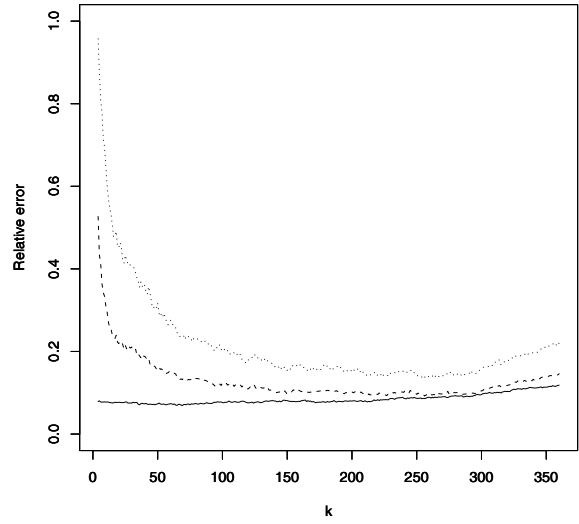
## References

- [1] Beirlant, J., Bouquiaux, C., Werker, B., (2006), Semiparametric lower bounds for tail index estimation, *Journal of Statistical Planning and Inference*, **136**, 705-729.
- [2] Beirlant, J., Broniatowski, M., Teugels, J.L., Vynckier, P., (1995), The mean residual life function at great age: Applications to tail estimation, *Journal of Statistical Planning and Inference*, **45**, 21-48.
- [3] Beirlant, J., Dierckx, G., Goegebeur, Y., Matthys, G., (1999), Tail index estimation and an exponential regression model, *Extremes*, **2**, 177-200.
- [4] Beirlant, J., Dierckx, G., Guillou, A., Starica, C., (2002), On exponential representations of log-spacings of extreme order statistics, *Extremes*, **5** (2), 157-180.
- [5] Breiman, L., Stone, C. J., Kooperberg, C. (1990), Robust confidence bounds for extreme upper quantiles, *Journal of Computational Statistics and Simulation*, **37**, 127-149.

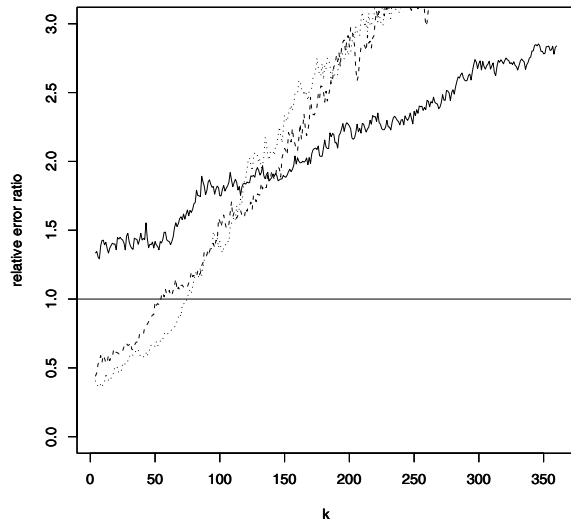
- [6] Broniatowski, M., (1993), On the estimation of the Weibull tail coefficient, *Journal of Statistical Planning and Inference*, **35**, 349–366.
- [7] Diebolt, J., Gardes, L., Girard, S., Guillou, A., (2005), Bias-reduced estimators of the Weibull-tail coefficient, Technical Report LMC RR-1078, <https://hal.ccsd.cnrs.fr/ccsd-00008881>, submitted.
- [8] Ditlevsen, O., (1994), Distribution Arbitrariness in Structural Reliability, *Structural Safety and Reliability*, 1241–1247, Balkema, Rotterdam.
- [9] Drees, H., (1998), On smooth statistical tail functionals, *Scandinavian Journal of Statistics*, **25**, 187-210.
- [10] Embrechts, P., Klüppelberg, C., Mikosch, T., (1997), *Modelling extremal events*, Springer.
- [11] Feuerverger, A., Hall, P., (1999), Estimating a Tail Exponent by Modelling Departure from a Pareto Distribution, *Annals of Statistics*, **27**, 760–781.
- [12] Gardes, L., Girard, S., (2005), Estimating extreme quantiles of Weibull tail-distributions, *Communication in Statistics - Theory and Methods*, **34**, 1065-1080.
- [13] Geluk, J.L., de Haan, L., (1987), Regular Variation, Extensions and Tauberian Theorems, *Math Centre Tracts*, **40**, Centre for Mathematics and Computer Science, Amsterdam.
- [14] Girard, S., (2004), A Hill type estimate of the Weibull tail-coefficient, *Communication in Statistics - Theory and Methods*, **33**(2), 205–234.
- [15] de Haan, L., Rootzén, H., (1993), On the estimation of high quantiles, *Journal of Statistical Planning and Inference*, **35**, 1-13.
- [16] Hosking, J., Wallis, J., (1987), Parameter and quantile estimation for the generalized Pareto distribution, *Technometrics*, **29**, 339-349.
- [17] Smith, J., (1991), Estimating the upper tail of flood frequency distributions, *Water Resources Research*, **23**(8), 1657–1666.



(a) Relative error of  $\tilde{x}_{p_n}$  for  $\tau = 1$  (straight line),  $\tau = 2$  (dashed line) and  $\tau = 4$  (dotted line).

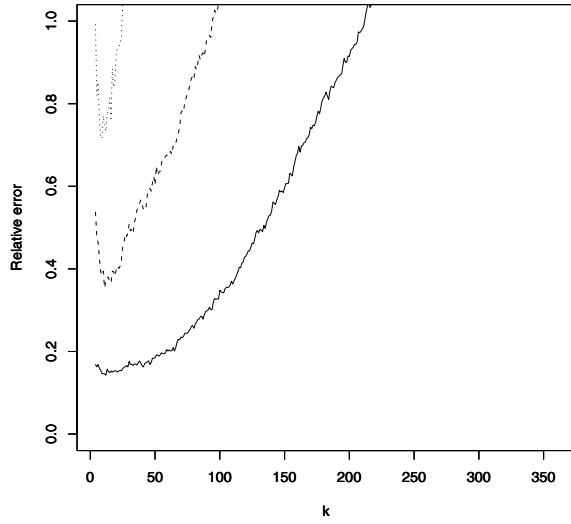


(b) Relative error of  $\hat{x}_{p_n}$  for  $\tau = 1$  (straight line),  $\tau = 2$  (dashed line) and  $\tau = 4$  (dotted line).

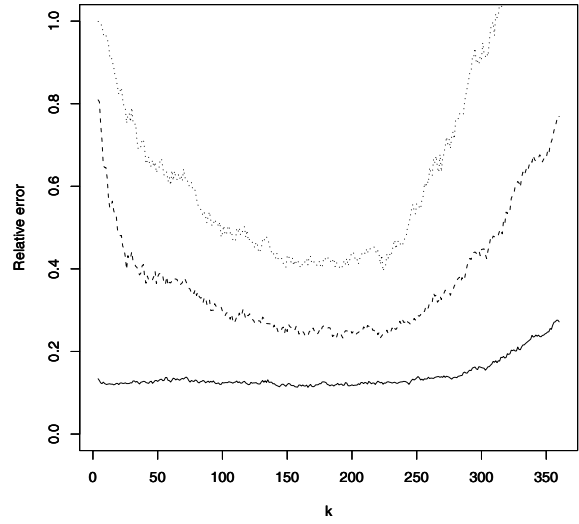


(c) Relative error of  $\tilde{x}_{p_n}$  divided by the relative error of  $\hat{x}_{p_n}$  for  $\tau = 1$  (straight line),  $\tau = 2$  (dashed line) and  $\tau = 4$  (dotted line).

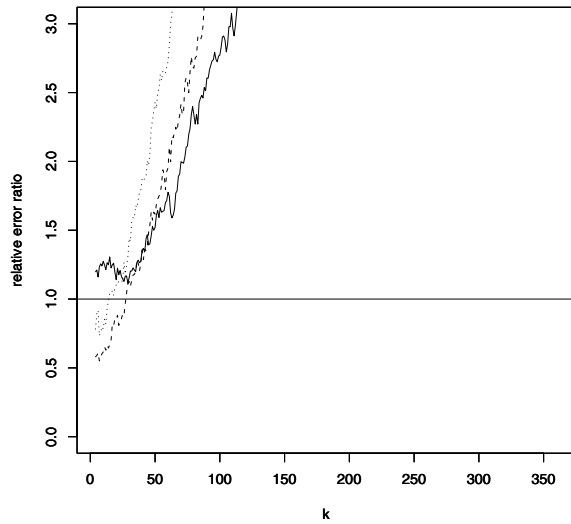
Figure 1: Comparison of the estimators  $\tilde{x}_{p_n}$  and  $\hat{x}_{p_n}$  for the  $|\mathcal{N}(0, 1)|$  distribution.



(a) Relative error of  $\tilde{x}_{p_n}$  for  $\tau = 1$  (straight line),  $\tau = 2$  (dashed line) and  $\tau = 4$  (dotted line).

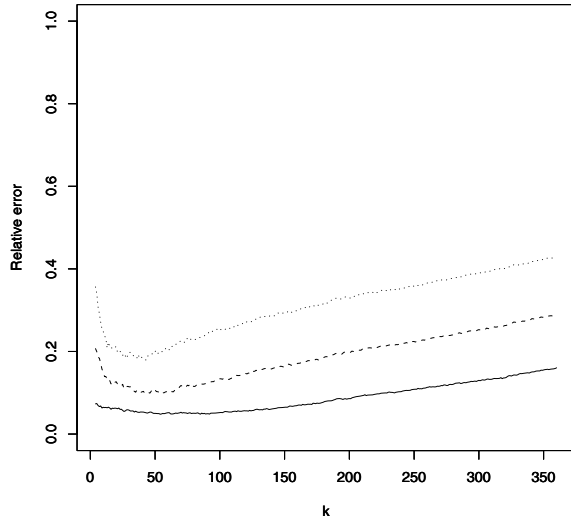


(b) Relative error of  $\widehat{x}_{p_n}$  for  $\tau = 1$  (straight line),  $\tau = 2$  (dashed line) and  $\tau = 4$  (dotted line).

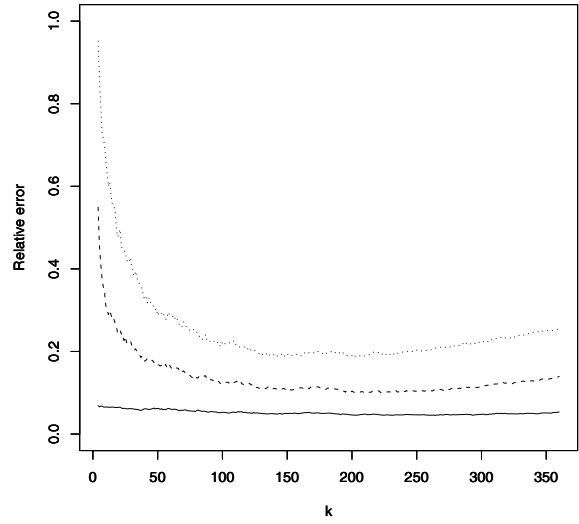


(c) Relative error of  $\tilde{x}_{p_n}$  divided by the relative error of  $\widehat{x}_{p_n}$  for  $\tau = 1$  (straight line),  $\tau = 2$  (dashed line) and  $\tau = 4$  (dotted line).

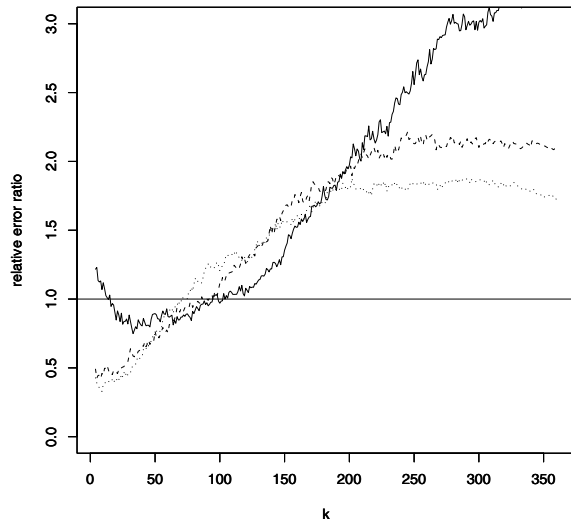
Figure 2: Comparison of the estimators  $\tilde{x}_{p_n}$  and  $\widehat{x}_{p_n}$  for the  $\Gamma(0.25, 0.25)$  distribution.



(a) Relative error of  $\tilde{x}_{p_n}$  for  $\tau = 1$  (straight line),  $\tau = 2$  (dashed line) and  $\tau = 4$  (dotted line).

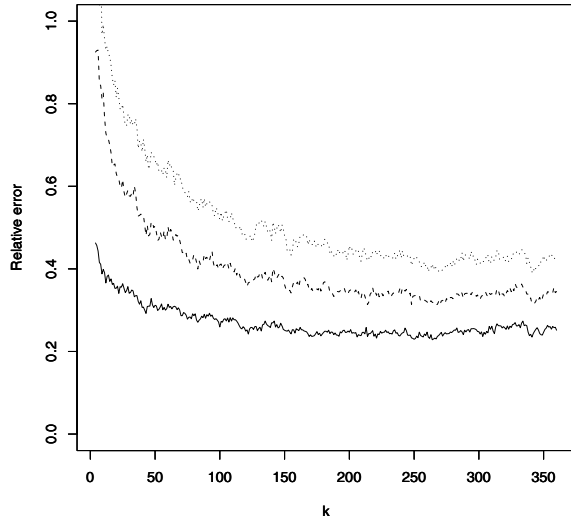


(b) Relative error of  $\widehat{x}_{p_n}$  for  $\tau = 1$  (straight line),  $\tau = 2$  (dashed line) and  $\tau = 4$  (dotted line).

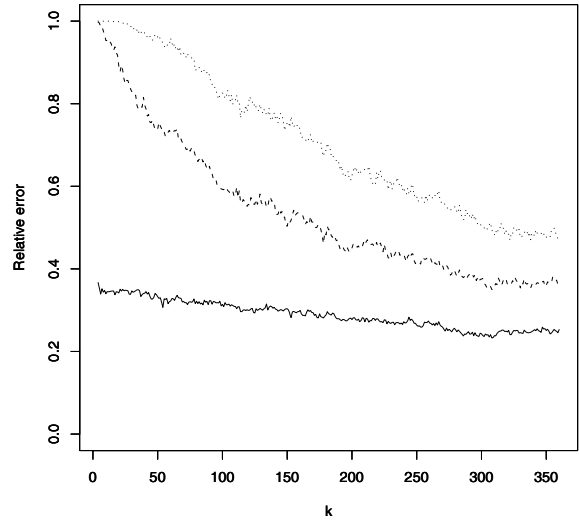


(c) Relative error of  $\tilde{x}_{p_n}$  divided by the relative error of  $\widehat{x}_{p_n}$  for  $\tau = 1$  (straight line),  $\tau = 2$  (dashed line) and  $\tau = 4$  (dotted line).

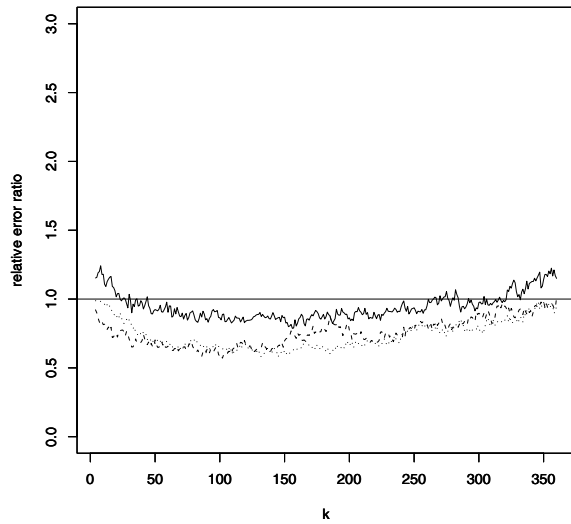
Figure 3: Comparison of the estimators  $\tilde{x}_{p_n}$  and  $\widehat{x}_{p_n}$  for the  $\Gamma(4, 1)$  distribution.



(a) Relative error of  $\tilde{x}_{p_n}$  for  $\tau = 1$  (straight line),  $\tau = 2$  (dashed line) and  $\tau = 4$  (dotted line).



(b) Relative error of  $\widehat{x}_{p_n}$  for  $\tau = 1$  (straight line),  $\tau = 2$  (dashed line) and  $\tau = 4$  (dotted line).



(c) Relative error of  $\tilde{x}_{p_n}$  divided by the relative error of  $\widehat{x}_{p_n}$  for  $\tau = 1$  (straight line),  $\tau = 2$  (dashed line) and  $\tau = 4$  (dotted line).

Figure 4: Comparison of the estimators  $\tilde{x}_{p_n}$  and  $\widehat{x}_{p_n}$  for the  $\mathcal{W}(0.25, 0.25)$  distribution.