



HAL
open science

Existence of finite test-sets for k -power-freeness of uniform morphisms

Gwenaël Richomme, Francis Wlazinski

► **To cite this version:**

Gwenaël Richomme, Francis Wlazinski. Existence of finite test-sets for k -power-freeness of uniform morphisms. *Discrete Applied Mathematics*, 2007, 155 (15), pp.2001-2016. hal-00015775

HAL Id: hal-00015775

<https://hal.science/hal-00015775>

Submitted on 13 Dec 2005

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

LaRIA : Laboratoire de Recherche en Informatique d'Amiens
Université de Picardie Jules Verne – CNRS FRE 2733
33, rue Saint Leu, 80039 Amiens cedex 01, France
Tel : (+33)[0]3 22 82 88 77
Fax : (+33)[0]3 22 82 54 12
<http://www.laria.u-picardie.fr>

**Existence of finite test-sets
for k -power-freeness of uniform morphisms**

G. Richomme^a, F. Wlazinski^b

**LaRIA RESEARCH REPORT : LRR 2005-10
(December 2005)**

Existence of finite test-sets for k -power-freeness of uniform morphisms

G. Richomme, F. Wlazinski
e-mail: {gwenael.richomme,francis.wlazinski}@u-picardie.fr
LaRIA, Université de Picardie Jules Verne
33 Rue Saint Leu, 80039 Amiens cedex 01, France

13th December 2005

Abstract

A challenging problem is to find an algorithm to decide whether a morphism is k -power-free. We provide such an algorithm when $k \geq 3$ for uniform morphisms showing that in such a case, contrarily to the general case, there exist finite test-sets for k -power-freeness.

Keywords: Formal Languages, Combinatorics on words, k -power-free words, morphisms, test-sets.

1 Introduction

Repetitions in words is a recurrent subject of study in Combinatorics on Words. The reader can consult for instance [7, 13, 14, 15] for surveys of results and applications. The interest for such regularities dates back to the works of A. Thue [23, 24] (see also [3, 4]) who, one century ago, provided examples of some repetition-free words, more precisely some square-free and overlap-free words. The construction of some of these words is simple: they are generated as fixed points of free monoid morphisms. An example is the fixed point (denoted $\Theta^\omega(a)$) of the morphism Θ defined by $\Theta(a) = abc$, $\Theta(b) = ac$ and $\Theta(c) = b$:

$$\Theta^\omega(a) = abcacbabcabcbacbacbacbcb \dots$$

This word is k -power-free [9, 24] for any integer $k \geq 2$, that is, it does not contain any word on the form u^k with u non-empty. May be strangely, for any $k \geq 2$, the morphism Θ is not itself k -power-free: it does not map all k -power-free words on k -power-free words ($\Theta(ab^{k-1}a) = ab(ca)^kbc$). So where'as any k -power-free morphisms generates a k -power-free word, the converse does not hold.

F. Mignosi and P. Séebold [16] have proved that it is decidable whether a morphism generates a k -power-free word: more precisely they proved that, given a word w and a morphism f , it is decidable whether the language $\{f^n(w) \mid n \geq 0\}$ is k -power-free. However, given an integer $k \geq 3$, to decide if a morphism is k -power-free is still an open problem even if some partial results have been achieved especially for morphisms acting on binary alphabets and for 3-power-free morphisms on ternary alphabets [2, 11, 12, 21, 25]. We note that the case $k = 2$ was solved by M. Crochemore [8]. We also observe that properties of k -power-free morphisms are badly known (see for instance [20]) despite of some efforts in the eighties [11, 12] when relations between morphisms and variable-length codes (in the sense of [5]) were studied.

A related problem is the study of overlap-free morphisms: an overlap-free word is a word that does not have any factor of the form $auaua$ with a a letter and u a word; an overlap-free morphism is a morphism preserving overlap-freeness. The study of overlap-free binary morphisms provides ideas of simple tests that can be extended to other classes of morphisms like k -power-free morphisms. For instance, the monoid of overlap-free binary endomorphisms is finitely generated. Unfortunately this is no longer true for both larger alphabets and k -power-free morphisms [8, 18, 21]. Another simple idea is to test overlap-freeness using a finite set of overlap-free words, called test-set for overlap-freeness [6, 19]. Recently [22] we have shown that, in the general case, a finite test-set exists for overlap-freeness of morphisms defined on an alphabet A if and only if A is a binary alphabet. But if we consider only uniform morphisms (the images of the letters have all the same length), such test-sets always exist. Note that the study of uniform overlap-free morphisms is natural since all overlap-free binary endomorphisms are uniform. Another reason to study uniform morphisms is provided by Cobham's theorem stating that a word is automatic if and only if it is the image under a 1-uniform morphism of a fixed point of a uniform morphism (see for instance [1]). Finally let us mention that uniform morphisms are sometimes easier to use to give examples of infinite words with particular properties, as done for instance in [17] where a finite test-set is provided for morphisms mapping α^+ -power-free words onto β^+ -power-free words when α and β are two rational numbers with $1 \leq \alpha \leq \beta \leq 2$.

We started the study of test-sets for k -power-freeness of morphisms in [21] where we obtained a result similar to the case of overlap-freeness: for $k \geq 3$, a finite test-set exists for k -power-freeness of morphisms defined on an alphabet A if and only if A is a binary alphabet. The purpose of this paper is to complete this work showing that, as for overlap-freeness, *there always exist test-sets for k -power-freeness of uniform morphisms* (see Theorem 3.1). Up to now, the existence of such test-sets for uniform morphisms was previously stated only for morphisms defined on two-letter [10, 11, 25] or three-letter alphabets [12].

Despite of the similarities between overlap-freeness and k -power-freeness, we would like to stress many differences between the two studies. Firstly, we mention that the maximal lengths of words involved in the test-sets are different since of course in one case they depend on the parameter k and not just on the size of the alphabet. More important is the fact that we introduce a new way to tackle the decidability of repetition-freeness.

We will only consider test-sets for k -power-freeness when $k \geq 3$. Indeed it is well-known that a uniform morphism is 2-power-free (that is square-free) if and only if the images of 2-power-free words of length 3 are 2-power-free: in our terminology this means that the set of 2-power-free words of length 3 is a test-set for 2-power-freeness of uniform morphisms. The test-sets we obtain are not so simple and depend on both the value of k and the cardinality of A .

We present our test-sets, main tools for the proof and the proof itself in Section 3, Section 4 and Section 5 respectively.

2 Notations and main definitions

We assume the reader is familiar (if not, see for instance [13, 14]) with basic notions on words and morphisms. Let us precise our notations and the main definitions.

Given a finite set X , we denote by $\text{Card}(X)$ its cardinality, that is, the number of its elements. An *alphabet* A is a finite set of symbols called *letters*. A *word* over A is a finite sequence of letters from A . Equipped with the concatenation operation, the set A^* of words over A is a free monoid

with the *empty word* ε as neutral element and A as set of generators. Given a non-empty word $u = a_1 \dots a_n$ with $a_i \in A$, the *length* of u denoted by $|u|$ is the integer n that is the number of letters of u . By convention, we have $|\varepsilon| = 0$.

A word u is a *factor* of a word v if there exist two (possibly empty) words p and s such that $v = pus$. We also say that v *contains* the word u (as a factor). If $p = \varepsilon$, u is a *prefix* of v . If $s = \varepsilon$, u is a *suffix* of v . A word u is a *factor* (resp. a *prefix*, a *suffix*) of a set of words X , if u is a factor (resp. a prefix, a suffix) of a word in X .

Let w be a word and let i, j be two integers such that $0 \leq i - 1 \leq j \leq |w|$. We denote by $w[i..j]$ the factor u of w such that there exist two words p and s with $w = pus$, $|p| = i - 1$, $|pu| = j$. Note that, when $j = i - 1$, we have $w[i..j] = \varepsilon$. When $i = j$, we also denote by $w[i]$ the factor $w[i..i]$ which is the i^{th} letter of w .

Given two words w and u , we denote by $|w|_u$ the number of different words p such that pu is a prefix of w . For instance, if $w = abaababa$, we have $|w|_a = 5$, $|w|_{aba} = 3$.

Powers of a word are defined inductively by $u^0 = \varepsilon$, and for any integer $n \geq 1$, $u^n = uu^{n-1}$: such a word is called a *n-power* when $n \geq 2$ and $u \neq \varepsilon$. A word is *k-power-free* ($k \geq 2$) if it does not contain any *k-power* as factor. A set of *k-power-free* words is said *k-power-free*.

Let us recall two well-known results of combinatorics on words:

Proposition 2.1 [13] *Let A be an alphabet and u, v, w three words over A . If $vu = uw$ and $v \neq \varepsilon$ then there exist two words r and s over A and an integer n such that $u = r(sr)^n$, $v = rs$ and $w = sr$.*

Lemma 2.2 [11, 12] *If a non-empty word v is an internal factor of vv (that is, if there exist two non-empty words x and y such that $vv = xvy$) then there exist a non-empty word t and two integers $i, j \geq 1$ such that $x = t^i$, $y = t^j$ and $v = t^{i+j}$.*

Let A, B be two alphabets. A *morphism* f from A^* to B^* is a mapping from A^* to B^* such that for all words u, v over A , $f(uv) = f(u)f(v)$. When B does not have any importance, we will say that f is a *morphism on A* or that f is defined on A . A morphism on A is entirely known by the images of the letters of A . When $B = A$, f is called an *endomorphism* (on A). Given an integer L , f is *L-uniform* if for each letter a in A we have $|f(a)| = L$. A morphism f is *uniform* if it is *L-uniform* for some integer $L \geq 0$. Given a set X of words over A , and given a morphism f on A , we denote by $f(X)$ the set $\{f(w) \mid w \in X\}$.

A morphism f on A is *k-power-free* if and only if $f(w)$ is *k-power-free* for all *k-power-free* words w over A . For instance, the *empty morphism* ϵ ($\forall a \in A, \epsilon(a) = \varepsilon$) is *k-power-free*.

3 Main result

Let us recall that in all the rest of this paper A is an alphabet containing at least two letters and $k \geq 3$ is an integer.

Our main result (Theorem 3.1) is the existence of test-sets for *k-power-freeness* of uniform morphisms whatever is A and k : A *test-set for k-power-freeness of uniform morphisms* on A is a set $T \subseteq A^*$ such that, for any uniform morphism f on A , f is *k-power-free* if and only if $f(T)$ is *k-power-free*.

This existence is provided by the set

$$T_{A,k} = U_{k,A} \cup (k\text{PF}(A) \cap V_{k,A})$$

where $U_{k,A}$, $k\text{PF}(A)$ and $V_{k,A}$ are defined as follows:

- $U_{k,A}$ is the set of k -power-free words over A of length at most $k + 1$,
- $k\text{PF}(A)$ is the set of all k -power-free words over A , and
- $V_{k,A}$ is the set of words over A that can be written $a_0w_1a_1w_2 \dots a_{k-1}w_k a_k$ where a_0, a_1, \dots, a_k are letters of A and w_1, w_2, \dots, w_k are words over A verifying $||w_i| - |w_j|| \leq 1$ and $|w_i|_a \leq 1$; $\forall 1 \leq i, j \leq k$ and $\forall a \in A$.

In the previous definition, the inequality $|w_i|_a \leq 1$ means that any letter of A appears at most once in w_i . In particular, it follows that $\max\{|w| \mid w \in T_{A,k}\} \leq b_{k,A}$ where $b_{k,A} = k \times \text{Card}(A) + k + 1$.

Theorem 3.1 $T_{A,k}$ is a test-set for k -power-freeness of uniform morphisms on A .

An immediate consequence is the following corollary that gives a simple bound for the length of the words whose images we have to check to verify the k -power-freeness of a morphism:

Corollary 3.2 A uniform morphism on A is k -power-free for an integer $k \geq 3$ if and only if the images by f of all k -power-free words of length at most $k \times \text{Card}(A) + k + 1$ are k -power-free.

4 Tools

In this section we recall or introduce some useful tools. May be the reader will read them when needed in the proof of Theorem 3.1, but we would like to present the novelties of our approach (from Section 4.2).

4.1 ps-morphisms

A morphism f is a *ps-morphism* (Keränen [11] called it *ps-code*) if

$$f(a) = ps, \text{ and } f(b) = ps', f(c) = p's$$

with $a, b, c \in A$ (possibly $c = b$), p, s, s', p' in B^* then necessarily $b = a$ or $c = a$. Any any ps-morphism is injective. A basic result about these morphisms is:

Lemma 4.1 [11, 12] *If all the k -power-free words of length at most $k + 1$ have a k -power-free image by a morphism f , then f is a ps-morphism.*

4.2 Decomposition of k -powers

One situation that we will quickly meet in the proof of Theorem 3.1 is: f is a L -uniform ps-morphism ($L \geq 0$), w is a k -power-free word such that $f(w)$ contains a k -power u^k and $|w| \geq k + 1$. In this case, Lemma 4.2 below will enable us to decompose u^k using factors of $f(w)$ (see also Figure 1).

We observe that (possibly by replacing w by one of its factors) we can consider that u^k is *directly covered* by $f(w)$. This means that u^k is not a factor of the image of a proper factor of w . More precisely, if p_0 and s_k are the words such that $f(w) = p_0 u^k s_k$ then $|p_0| < L$ and $|s_k| < L$. The present situation verifies:

Lemma 4.2 *Let f be a uniform morphism and let $k \geq 3$ be an integer. A k -power u^k ($u \neq \varepsilon$) is directly covered by the image of a word w of length at least $k + 1$ if and only if there exist words $(p_i)_{i=0,\dots,k}$, $(s_i)_{i=0,\dots,k}$, $(w_i)_{i=1,\dots,k}$ and letters $(a_i)_{i=0,\dots,k}$ such that:*

- (1) $w = a_0 w_1 a_1 \dots a_{k-1} w_k a_k$,
- (2) $f(a_i) = p_i s_i$ ($0 \leq i \leq k$),
- (3) $s_0 \neq \varepsilon$,
- (4) $p_i \neq \varepsilon$ ($1 \leq i \leq k$),
- (5) $u = s_{i-1} f(w_i) p_i$ ($1 \leq i \leq k$).

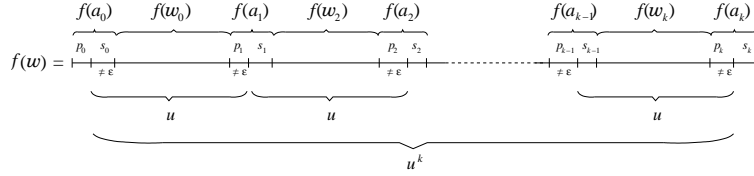


Figure 1: $(p_i, s_i, x_i, w_i)_{i=0,\dots,k}$ -decomposition of u^k in $f(w)$

Proof of Lemma 4.2. By definition f is uniform: Let L be the integer such that $|f(b)| = L$ for each letter b . Clearly if words $(p_i)_{i=0,\dots,k}$, $(s_i)_{i=0,\dots,k}$, $(w_i)_{i=1,\dots,k}$ and letters $(a_i)_{i=0,\dots,k}$ verify Conditions (1) to (5), then $|w| \geq k + 1$ and u^k is directly covered by $f(w)$.

Assume now that u^k is covered by $f(w)$ with $|w| \geq k + 1$. Let p_0 and s_k be the words such that $f(w) = p_0 u^k s_k$. For each integer ℓ between 0 and k , let i_ℓ be the least non-zero integer such that $p u^\ell$ is a prefix of $f(w[1..i_\ell])$. Since u^k is covered by $f(w)$, $i_0 = 1$, $i_k = |w|$ and $i_0 \leq i_1 \leq i_2 \leq \dots \leq i_k$. If $i_\ell = i_{\ell+1}$ for (at least) one integer ℓ between 0 and $k - 1$, then $|u| \leq |f(a_\ell)| = L$. For any integer m between 0 and $k - 1$, since f is L -uniform and $|u| \leq L = |f(a_m)|$, $i_m + 1 \geq i_{m+1}$ ($i_m = i_{m+1}$ or $i_m + 1 = i_{m+1}$). Hence $|w| = i_k \leq i_{\ell+1} + (k - \ell - 1) = i_\ell + (k - \ell - 1) \leq (i_0 + \ell) + (k - \ell - 1) = k$: a contradiction.

So $i_0 < i_1 < i_2 < \dots < i_k$. We define for each integer ℓ between 1 and k the words $w_\ell = w[i_{\ell-1} + 1 \dots i_\ell - 1]$ and p_ℓ such that $f(w[1..i_{\ell+1} - 1])p_\ell = p_0 u^\ell$. Moreover let $a_\ell = w[i_\ell]$ for $0 \leq \ell \leq k$. By construction for $0 \leq \ell \leq k - 1$, the word p_ℓ is a non-empty prefix of $f(a_\ell)$ and so we can consider the word s_ℓ such that $f(a_\ell) = p_\ell s_\ell$. Up to now by construction, we have Conditions (1), (2), (4) and (5). Since u^k is covered by $f(w)$, Condition (3) is also verified. \square

Definition 4.3 When a k -power u^k is directly covered by the image (by a uniform morphism f) of a word of length at least $k + 1$, if $(p_i, s_i, a_i, w_i)_{i=0,\dots,k}$ is a $(4k + 4)$ -uple such that $w_0 = \varepsilon$ and the other $4k + 3$ words verify Conditions (1) to (5) of Lemma 4.2, we will say that u^k has a $(p_i, s_i, a_i, w_i)_{i=0,\dots,k}$ -decomposition in $f(w)$, or that $(p_i, s_i, a_i, w_i)_{i=0,\dots,k}$ is a decomposition of u^k in $f(w)$.

4.3 Non-synchronized decompositions of k -powers

Between all decompositions that a k -power can have in the image of a word by a L -uniform morphism f , Lemma 4.5 will allow us to eliminate the following possibility:

Definition 4.4 Let $(p_i, s_i, a_i, w_i)_{i=0, \dots, k}$ be as in Definition 4.3. When $|s_i| = |s_{i+1}|$ for an integer i between 1 and $k - 2$, the decomposition is said synchronized (with respect to images of factor of w), or shortly that the k -power u^k is synchronized in $f(w)$.

Let us make several remarks about this definition.

First it is immediate that a decomposition $(p_i, s_i, a_i, w_i)_{i=0, \dots, k}$ of a k -power is synchronized if and only if for all integers i, j with $1 \leq i < j \leq k - 1$, we have $|s_i| = |s_j|$. Since f is uniform, and since $f(a_\ell) = p_\ell s_\ell$ (for all ℓ , $1 \leq \ell \leq k - 1$), it is also equivalent that $|p_i| = |p_j|$ for all $1 \leq i < j \leq k - 1$, or that $|p_i| = |p_{i+1}|$ for all $1 \leq i \leq k - 2$.

One aspect may appear strange: why do not we allow $i = 0$ in the definition of a synchronized decomposition? This is due to the dissymmetry brought by Conditions (3) and (4) in the definition of a decomposition. Assume that $(p_i, s_i, a_i, w_i)_{i=0, \dots, k}$ is a synchronized decomposition of a k -power u^k in $f(w)$ with f L -uniform. Since $|s_1| = |s_2|$, we have $|p_1| = |p_2|$. Moreover $u = s_0 f(w_1) p_1 = s_1 f(w_2) p_2$. Thus $p_1 = p_2$ and $s_0 f(w_1) = s_1 f(w_2)$. When $s_1 \neq \varepsilon$, since also $p_1 \neq \varepsilon$, we have $0 < |s_1| < L$. In this case $s_0 = s_1$. But when $s_1 = \varepsilon$, since $s_0 \neq \varepsilon$, we have $s_0 \neq s_1$, $p_0 = \varepsilon$ and $s_0 = f(a_0) \neq s_1$.

Of course we do not consider $i = k - 1$ in the definition of a synchronized decomposition simply because s_k is not a factor of u^k .

Lemma 4.5 Let f be a uniform ps-morphism defined on an alphabet A , and let $k \geq 3$ be an integer. Any k -power directly covered by the image by f of a k -power-free word of length at least $k + 1$ is not synchronized.

Proof of Lemma 4.5. By definition f is uniform: Let L be the integer such that $|f(b)| = L$ for each letter b . Assume there exists a k -power u^k that has a synchronized decomposition $(p_i, s_i, a_i, w_i)_{i=0, \dots, k}$ in $f(w)$, where w is a k -power-free word. By hypothesis $s_i = s_j$ and $p_i = p_j$ for all $0 < i < j < k$. We denote $s = s_1$ and $p = p_1$. From $u = s f(w_2) p = s f(w_k) p_k$, we deduce that $|p| = |p_k| \pmod L$. Since $0 < |p|, |p_k| \leq L$, we get $p_k = p$. Hence $u = s_0 f(w_1) p$ and $u = s f(w_i) p$ for all $2 \leq i \leq k$. We have seen before the lemma's statement that $s_0 = s$ when $s \neq \varepsilon$ and $s_0 = f(a_0)$ when $s = \varepsilon$.

Assume first $s = \varepsilon$ and $s_0 = f(a_0)$. Since f is injective, we get $a_0 w_1 a_1 = w_i a_i$ for all $2 \leq i \leq k$. Thus $w = (a_0 w_1 a_1)^k$. This contradicts the fact that w is k -power-free.

So $s \neq \varepsilon$ and $s_0 = s$. Since f is injective, $w_i = w_1$ for all $1 \leq i \leq k$ and $a_i = a_1$ for all $1 \leq i \leq k - 1$. Hence $w = a_0 (w_1 a_1)^{k-1} w_1 a_k$. Since w is k -power-free, $a_0 \neq a_1$ and $a_k \neq a_1$. Let $a = a_1$, $b = a_k$, $c = a_0$, $p' = p_0$ and $s' = s_k$: $f(c) = p's$, $f(b) = ps'$. From $f(a) = ps$, we deduce that f is not a ps-morphism. \square

We end this section with some examples of non-synchronized k -powers.

Example 4.6 $f(a) = baaba$, $f(b) = bcdab$, $f(c) = cdabc$, $f(d) = dbaab$:

$$f(abcd) = baab(abcd)^3 baab.$$

The decomposition of $(abcd)^3$ in $f(abcd)$ is given by $a_0 = a$, $a_1 = b$, $a_2 = c$, $a_3 = d$, $w_1 = w_2 = w_3 = \varepsilon$, $p_0 = baab = s_3$, $s_0 = a$, $p_1 = bcd$, $s_1 = ab$, $p_2 = cd$, $s_2 = abc$, $p_3 = d$.

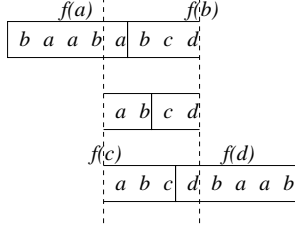


Figure 2: Example 4.6

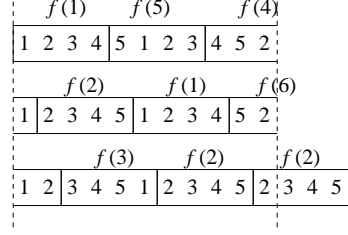


Figure 3: Example 4.7

Example 4.7 $f(1) = 1234; f(2) = 2345, f(3) = 3451, f(4) = 4521, f(5) = 5123, f(6) = 5212 :$

$$f(154216322) = (12345123452)^3 345$$

The decomposition of $(12345123452)^3$ in $f(154216322)$ is given by $a_0 = 1, a_1 = 4, a_2 = 6, a_3 = 2, w_1 = 5, w_2 = 21, w_3 = 32, p_0 = \varepsilon, p_1 = 452, p_2 = 52, p_3 = 2, s_0 = f(a_0), s_1 = 1, s_2 = 12, s_3 = 345$.

4.4 Reduction of a k -power

In this section, we introduce the key technic of the proof of Theorem 3.1. It consists in the possibility to reduce the length of k -powers in order to consider only k -powers covered by the image of a word in $V_{k,A}$.

Proposition 4.8 *Let f be an injective uniform morphism on A . If there exists a k -power-free word W of length at least $k + 1$ such that U^k is directly covered by $f(W)$ then there exists a word w of length at least $k + 1$ such that $w \in V_{k,A}, |w| \leq |W|$ and $f(w)$ covers a k -power u^k . Moreover the k -powers u^k and U^k are both synchronized or both non-synchronized.*

This proposition is a direct corollary of Lemma 4.9 (to be used inductively) whose idea is illustrated by Figure 4.

We denote by $\text{Reduced}(U^k, W)$ the set of pairs (u^k, w) that can be obtained in conclusion of Proposition 4.8.

Lemma 4.9 (Reduction lemma) *Let f be an injective uniform morphism on A and let w be a word over A . We assume that there exists a non-empty word u such that the k -power u^k has a $(p_i, s_i, a_i, w_i)_{i=0, \dots, k}$ -decomposition in $f(w)$. We also assume that there exist an integer $1 \leq \ell \leq k$ and a letter a in A such that $w_\ell = x_\ell y_\ell z_\ell$ and both x_ℓ and y_ℓ end with a . Then:*

1. For all integers i such that $1 \leq i \leq k$, there exist three words x_i, y_i, z_i such that $w_i = x_i y_i z_i, |s_{\ell-1} f(x_\ell)| - |f(a)| < |s_{i-1} f(x_i)| \leq |s_{\ell-1} f(x_\ell)|$ and $|y_i| = |y_\ell|$.
2. Let $u' = s_{\ell-1} f(x_\ell z_\ell) p_\ell$ and $w' = a_0 \prod_{i=1}^k (x_i z_i a_i)$. The k -power $(u')^k$ has a $(p_i, s_i, a_i, x_i z_i)_{i=0, \dots, k}$ -decomposition in $f(w')$.
3. $|w'| < |w|$.

$$u = s_{q-1} f(x_q y_q z_q) p_q$$

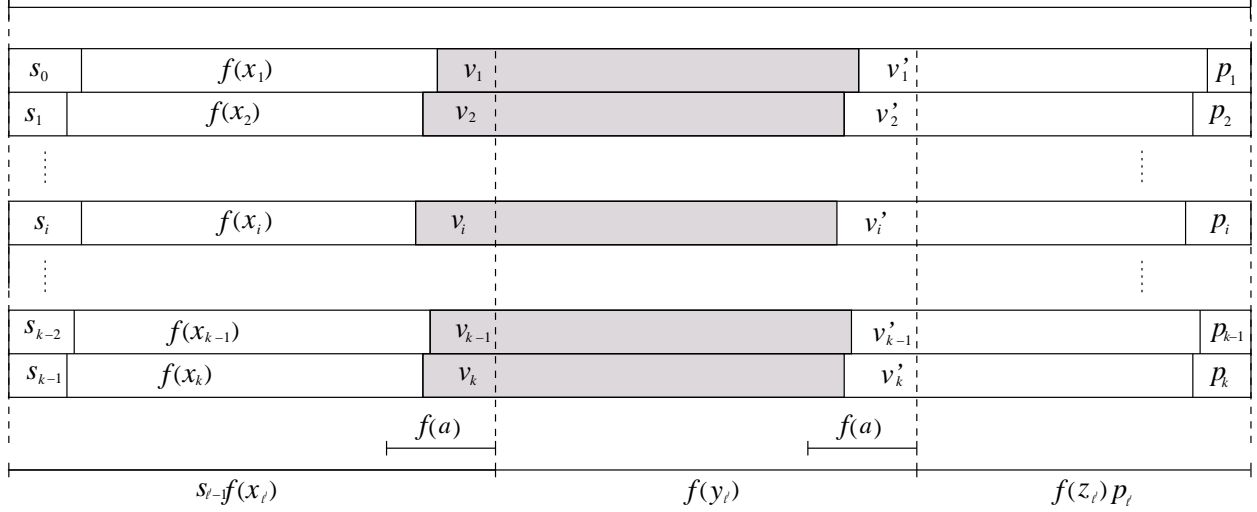


Figure 4:

To explain Figure 4, let us say that the grey parts are deleted and that the two occurrences of $f(a)$ allow to merge the left and right non-grey parts in order to have the new k -power $(u')^k$ directly covered by the image of the new word w' .

Proof of lemma 4.9.

1. By definition f is uniform: Let L be the integer such that $|f(b)| = L$ for each letter b .

Let i be an integer such that $1 \leq i \leq k$. We have $u = s_{i-1}f(w_i)p_i = s_{\ell-1}f(x_\ell y_\ell z_\ell)p_\ell$. Let us observe that:

$$|s_{i-1}| \leq |s_{\ell-1}f(x_\ell)| \leq |s_{i-1}f(w_i)|$$

Indeed, since $|f(x_\ell)| \neq 0$ (x_ℓ ends with a), we have $|s_{i-1}| \leq |f(a_{i-1})| = L = |f(x_\ell)| \leq |s_{\ell-1}f(x_\ell)|$. Moreover $|s_{\ell-1}f(x_\ell)| \leq |s_{\ell-1}f(x_\ell)| + |f(y_\ell)| - |f(a)| \leq |s_{\ell-1}f(w_\ell)| - |f(a)| \leq |s_{\ell-1}f(w_\ell)| + |p_\ell| - |p_i| = |s_{i-1}f(w_i)|$.

Thus we can define x_i as the greatest prefix (maybe empty) of w_i such that $s_{i-1}f(x_i)$ is a prefix of $s_{\ell-1}f(x_\ell)$. Since f is uniform, we have:

$$|s_{\ell-1}f(x_\ell)| - |f(a)| < |s_{i-1}f(x_i)| \leq |s_{\ell-1}f(x_\ell)|$$

It follows that $|s_{i-1}f(x_i)| \leq |s_{\ell-1}f(x_\ell)| < |s_{\ell-1}f(x_\ell y_\ell)|$. Let y_i be the greatest word such that $x_i y_i$ is a prefix of w_i and $s_{i-1}f(x_i y_i)$ is a prefix of $s_{\ell-1}f(x_\ell y_\ell)$. Let z_i be the word such that $w_i = x_i y_i z_i$

Let v'_i be the word such that $s_{i-1}f(x_i y_i) v'_i = s_{\ell-1}f(x_\ell y_\ell)$. We have $v'_i f(z_i) p_\ell = f(z_i) p_i$. Assume $|v'_i| \geq L$. The definition of y_i implies that $z_i = \varepsilon$. The equality $|v'_i f(z_i) p_\ell| = |p_i|$ with $|p_\ell| \neq 0$ is incompatible with $|p_i| \leq L$. Thus $|v'_i| < L$. It follows:

$$|s_{\ell-1}f(x_\ell y_\ell)| - |f(a)| < |s_{i-1}f(x_i y_i)| \leq |s_{\ell-1}f(x_\ell y_\ell)|$$

From this double inequality and the previous one concerning $|s_{i-1}f(x_i)|$, we deduce that $|f(y_\ell)| - |f(a)| < |f(y_i)| < |f(y_\ell)| + |f(a)|$.

Since f is uniform, it follows that $|f(y_\ell)| = |f(y_i)|$ and $|y_i| = |y_\ell|$ (see Figure 4).

2. For all integers $1 \leq i \leq k$, let v_i be the word such that $s_{i-1}f(x_i)v_i = s_{\ell-1}f(x_\ell)$. By definition of x_i , we have $0 \leq |v_i| < |f(a)|$. Moreover $f(y_i z_i)p_i = v_i f(y_\ell z_\ell)p_\ell$. Since $|y_i| = |y_\ell|$, we get $|s_{\ell-1}f(x_\ell)| = |s_{\ell-1}f(x_\ell y_\ell)| - |f(y_\ell)| = |s_{i-1}f(x_i y_i)v'_i| - |f(y_i)| = |s_{i-1}f(x_i)v'_i|$. It follows that $|v_i| = |v'_i|$. Since x_ℓ and y_ℓ both end with a and since $|v_i| = |v'_i| < |f(a)|$, it follows that v_i and v'_i are both suffixes of $f(a)$ and so $v_i = v'_i$.

Let $w' = a_0 \prod_{i=1}^k (x_i z_i a_i)$. For all integers i such that $1 \leq i \leq k$, we have $u' = s_{\ell-1}f(x_\ell z_\ell)p_\ell = s_{i-1}f(x_i)v_i f(z_\ell)p_\ell = s_{i-1}f(x_i z_i)p_i$. Thus, $f(w') = p_0 s_0 \prod_{i=1}^k f(x_i z_i)p_i s_i = p_0 \left(\prod_{i=1}^k s_{i-1}f(x_i z_i)p_i \right) s_k = p_0 u'^k s_k$.

3. Since $y_\ell \neq \varepsilon$ (y_ℓ ends with a), we have $|w'| < |w|$.

□

Let us give an example of reduction:

Example 4.10 Let us consider the morphism defined by $f(1) = 1234$; $f(2) = 2345$, $f(3) = 3451$, $f(4) = 4521$, $f(5) = 5123$, $f(6) = 5212$, $f(7) = 5178$, $f(8) = 6234$, $f(9) = 1781$, $f(a) = 2346$, $f(b) = 7812$, $f(c) = 3462$. This morphism is not 3-power-free (it is not a ps-morphism). We observe (see Figure 5) that $f(17185429a2163bc322)$ contains the cube $(12345178123462345123452)^3$. This 3-power can be reduced on two ways. First, using the fact that $f(1)$ appears twice in the first occurrence of u , we can obtain the cube $(123462345123452)^3$ in the image of $f(1854a216c322)$ as shown by Figure 6. Second, using the fact that $f(3)$ appears twice in the first occurrence of u , we can obtain the cube $(12345123452)^3$ in the image of $f(154216322)$ as shown by Figure 7.

$f(1)$				$f(7)$				$f(1)$				$f(8)$				$f(5)$				$f(4)$									
1	2	3	4	5	1	7	8	1	2	3	4	6	2	3	4	5	1	2	3	4	5	2							
$f(2)$					$f(9)$					$f(a)$					$f(2)$					$f(1)$					$f(6)$				
1	2	3	4	5	1	7	8	1	2	3	4	6	2	3	4	5	1	2	3	4	5	2							
$f(3)$				$f(b)$				$f(c)$				$f(3)$				$f(2)$				$f(2)$									
1	2	3	4	5	1	7	8	1	2	3	4	6	2	3	4	5	1	2	3	4	5	2	3	4	5				

Figure 5: Example 4.10

$f(1)$				$f(8)$				$f(5)$				$f(4)$							
1	2	3	4	6	2	3	4	5	1	2	3	4	5	2					
$f(a)$					$f(2)$					$f(1)$					$f(6)$				
1	2	3	4	6	2	3	4	5	1	2	3	4	5	2					
$f(c)$				$f(3)$				$f(2)$				$f(2)$							
1	2	3	4	6	2	3	4	5	1	2	3	4	5	2	3	4	5		

Figure 6: first possible reduction

$f(1)$				$f(5)$				$f(4)$						
1	2	3	4	5	1	2	3	4	5	2				
$f(2)$					$f(1)$					$f(6)$				
1	2	3	4	5	1	2	3	4	5	2				
$f(3)$				$f(2)$				$f(2)$						
1	2	3	4	5	1	2	3	4	5	2	3	4	5	

Figure 7: second possible reduction

We observe in Example 4.10 that the two possible reductions verify the first Reduction Rule, and the different words obtained are both in $V_{k,A}$. The one chosen will be the first reduction according to the second rule.

We end with two remarks (using notations from Lemma 4.9) that will be useful in the end of Theorem 3.1. The first remark is a direct consequence of the first part of Lemma 4.9. To understand the second remark, we observe that since $|u| = |s_{\ell-1}f(x_\ell y_\ell z_\ell)p_\ell| = |s_{j-1}f(x_j y_j z_j)p_j|$ for all integers j such that $1 \leq j \leq k$, we also have $|f(z_j)p_j| - |f(a)| < |f(z_\ell)p_\ell| \leq |f(z_j)p_j|$.

Remark 4.11

1. *If there exists an integer q such that $x_q = \varepsilon$ then $x_\ell = a$ and $|s_{\ell-1}| < |s_{q-1}|$.*
2. *If there exists an integer q such that $z_q = \varepsilon$ then $z_\ell = \varepsilon$ and $|p_\ell| \leq |p_q|$.*

4.5 More precisions on the reduction

Proposition 4.8 will enable us to prove Theorem 3.1 when $k \geq 4$.

More precisely given two words W and U with W k -power-free, $|W| \geq k + 1$ and U^k directly covered by $f(W)$, we will construct (using this proposition) some words w and u such that (u^k, w) belongs to $\text{Reduced}(U^k, W)$, $|w| \leq |W|$ and $w \in V_{k,A}$. Moreover the decomposition of u^k in $f(w)$ will be non-synchronized. Since the word w belongs to $V_{k,A}$ and since $f(T_{A,k})$ is k -power-free, we can see that w is not k -power-free, and so there exists a non-empty word v such that v^k is a factor of w . We will be able to prove that this situation will be possible only if $k = 3$ and $|v| = 1$.

But when $k = 3$, the following example shows that there can exist words w and u such that u^k has a non-synchronized decomposition in $f(w)$: so we will need to be more precise in our use of the reductions.

Example 4.12 Let f be the morphism from $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}^*$ to $\{a, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, b\}^*$ defined by $f(1) = a0123$, $f(2) = 40125$, $f(3) = 67892$, $f(4) = 34012$, $f(5) = 56789$, $f(6) = 23401$, $f(7) = 25678$, $f(8) = 92340$, $f(9) = 1234b$. We have (see Figure 8):

$$f(1234445666789) = a(012340125678923401234)^3b$$

Thus this 5-uniform morphism f is a ps-morphism for which there exists a non-synchronized k -power. We let the reader verify that f is a 3-power-free morphism and so $f(T_{3,\{1,2,3,4,5,6,7,8,9\}})$ is 3-power-free.

We now explain how we tackle the situation $k = 3$ and $|v| = 1$. As we have just seen by Example 8, there can exist words $w \in V_{k,A}$ and u such that u^k has a non-synchronized decomposition in $f(w)$. We will show that, under all current hypotheses, w and u cannot be obtained by successive reductions from the words W and U defined in the previous section. For this purpose, we will be more precise on the way the reductions are made to obtain a couple (u^k, w) in $\text{Reduced}(U^k, W)$. Actually one can observe that if a word does not belong to $V_{k,A}$, there can exist many different ways to reduce it using Lemma 4.9. We will apply the two following additional rules (with the notations of Lemma 4.9):

	$f(1)$	$f(2)$	$f(3)$	$f(4)$	
a	0 1 2 3	4 0 1 2 5	6 7 8 9 2	3 4 0 1 2	3 4
	$f(4)$	$f(4)$	$f(5)$	$f(6)$	$f(6)$
	0 1 2	3 4 0 1 2	5 6 7 8 9	2 3 4 0 1	2 3 4
		$f(6)$	$f(7)$	$f(8)$	$f(9)$
	0 1	2 3 4 0 1	2 5 6 7 8	9 2 3 4 0	1 2 3 4
					b

Figure 8: An example of non-synchronized 3-power

Reduction rules:

1. $|x_\ell|_a = 1$ and $|y_\ell|_a = 1$
2. if there exist an integer $1 \leq \ell' \leq k$ and a letter a' in A such that $w_{\ell'} = x_{\ell'}y_{\ell'}z_{\ell'}$ and both $x_{\ell'}$ and $y_{\ell'}$ end with a' and such that $(\ell, a) \neq (\ell', a')$, then $|s_{\ell-1}f(x_\ell)| < |s_{\ell'-1}f(x_{\ell'})|$.

These rules mean that we always made the leftmost reduction possible. The determinism introduced by these rules will be a key element of the proof.

5 Proof of Theorem 3.1

In this section, we prove Theorem 3.1 which means: given any L -uniform morphism f on A (with $L \geq 0$ an integer), f is k -power-free if and only if $f(T_{A,k})$ is k -power-free.

Let f is a uniform morphism from A^* to B^* where B is an alphabet not necessarily equals to A , and let L be the integer such that $|f(b)| = L$ for each letter b . The “only if” part of the theorem follows immediately from the definition and the “if” part is also immediate when $L = 0$. Thus from now on $L \geq 1$. We assume that $f(T_{A,k})$ is k -power-free and we show (by contradiction) that f is k -power-free.

Since $U_{k,A} \subseteq T_{A,k}$, by Lemma 4.1, we have

Fact 1 f is a *ps-morphism*.

Let us recall that this implies that f is injective

Assume by contradiction that f is not k -power-free.

We first make a crucial choice.

Choice 1: let W be a k -power-free word of *smallest* length such that $f(W)$ directly covers a k -power.

Let U be a word such that U^k is directly covered by $f(W)$.

Since $U_{k,A} \subseteq T_{A,k}$, $|W| \geq k + 1$. Let $(p_i, s_i, a_i, W_i)_{i=0,\dots,k}$ be a decomposition of U^k in $f(W)$. By Lemma 4.5, this decomposition is not synchronized, that is, $s_i \neq s_j$ and $p_i \neq p_j$ for all integers i, j with $0 < i < j < k$.

Applying iteratively the Reduction Lemma 4.9 with the deterministic rules chosen in Section 4.5, we construct some words w and u such that (u^k, w) belongs to $\text{Reduced}(U^k, W)$, $|w| \leq |W|$ and

$w \in V_{k,A}$. We know that the decomposition of u^k in $f(w)$ is $(p_i, s_i, a_i, w_i)_{i=0,\dots,k}$ for some words $(w_i)_{i=0,\dots,k}$.

Let us observe that since the decomposition $(p_i, s_i, a_i, W_i)_{i=0,\dots,k}$ is not synchronized, it follows the definition that $(p_i, s_i, a_i, w_i)_{i=0,\dots,k}$ is also not synchronized.

Since $w \in V_{k,A}$ and $f(k\text{PF}(A) \cap V_{k,A})$ is k -power-free, we deduce that:

Fact 2 w is not k -power-free.

Choice 2: let v^k be a smallest k -power factor of w ($v \neq \varepsilon$).

We denote v_1, v_2 words such that $w = v_1 v^k v_2$.

Fact 3 No powers respectively of $f(v)$ and of u have a common factor of length greater than or equals to $|f(v)| + |u| - \gcd(|f(v)|, |u|)$.

This fact is a consequence of the following proposition which is a corollary of the well-known Fine and Wilf's theorem (see [13, 14] for instance).

Proposition 5.1 [11] *Let x and y be two words. If a power of x and a power of y have a common factor of length at least equal to $|x| + |y| - \gcd(|x|, |y|)$ then there exist two words t_1 and t_2 such that x is a power of $t_1 t_2$ and y is a power of $t_2 t_1$ with $t_1 t_2$ and $t_2 t_1$ primitive words. Furthermore, if $|x| > |y|$ then x is not primitive.*

Proof of Fact 3. Assume the opposite. By Proposition 5.1, there exist two words t_1, t_2 and two integers n_1, n_2 such that $f(v) = (t_1 t_2)^{n_1}$ and $u = (t_2 t_1)^{n_2}$. Since $u \neq \varepsilon$ and $v \neq \varepsilon$, we have $t_1 t_2 \neq \varepsilon$, $n_1 \geq 1$ and $n_2 \geq 1$. If $n_1 \geq 2$, $f(v^{\lceil k/2 \rceil}) = (t_1 t_2)^{n_1 \lceil k/2 \rceil}$ contains the k -power $(t_1 t_2)^k$. Since $k \geq 3$, $\lceil k/2 \rceil < k$, and so $|v^{\lceil k/2 \rceil}| < |v^k| \leq |w| \leq |W|$. By choice of v , $v^{\lceil k/2 \rceil}$ is k -power-free: this contradicts Choice 1 on W . So $n_1 = 1$. We get $|u| = |f(v)^{n_2}| = n_2 |f(v)|$ and so $|u| = 0 \pmod L$. For all integers j between 1 and k , $|u s_j| = |s_{j-1} f(w_j) p_j s_j| = |s_{j-1} f(w_j a_j)|$, and so $|s_j| = |s_{j-1}| \pmod L$. But for $j \geq 1$, $p_j \neq \varepsilon$, so that $|s_j| < L$. It follows that $|s_j| = |s_{j-1}|$ for all $j \geq 2$. This contradicts the fact that the decomposition of u^k is not synchronized. \square

Fact 4 $|v| = 1$ and $k = 3$

The proof of this fact is made of three steps.

Step 4.1 If $|f(v)| \geq |u|$ then $|v| = 1$ and $k = 3$.

Proof. Since $v \neq \varepsilon$, we can write $v = xv' = v''y$ for two letters x, y and two words v', v'' . Since $f(w) = f(v_1) f(v)^k f(v_2) = pu^k s$ with $|p| < L$ and $|s| < L$, the word $C = f(v'v^{k-2}v'')$ is a common factor of $f(v^k)$ and u^k . We have $|C| = |f(v)^k| - |f(xy)| = kL|v| - 2L$. When $|v| \geq 2$ or when $k \geq 4$, $|C| \geq 2L|v| = 2|f(v)| \geq |f(v)| + |u|$. So by Fact 3, we cannot have $|v| \geq 2$ or $k \geq 4$, that is (since $v \neq \varepsilon$), we must have $|v| = 1$ and $k = 3$. \square

Step 4.2 If $|f(v)| < |u|$ then $|v| = 1$.

Proof. Let us assume by contradiction that $|v| \geq 2$. There exist two letters x, y and a word v' such that $v = xv'y$. Since $|f(v_1) f(v)^k f(v_2)| = |f(w)| = |pu^k s| \geq |u^k| > |f(v^k)|$, we have $|v_1 v_2| \geq 1$.

If $v_1 = \varepsilon$, we get $x = a_0$ and $f(x) = p_0 s_0$. Since $|f(xv'yv^{k-1})| = |f(v)^k| < |u^k| \leq |pu^k|$, the word $C = s_0 f(v'yv^{k-1})$ is a prefix of u^k and so a common factor of $f(v)^k$ and u^k . Let us recall that $w \in V_{k,A}$. This means in particular that $|w_1|_y \leq 1$ and so, since $w = xw_1 a_1 \prod_{i=2}^k w_i a_i$

starts with $xv'yxv'y$, w_1 is a prefix of $v'yxv'$. Consequently $|u| \leq |s_0f(v'yxv'y)|$. It follows that $|C| = |s_0f(v'yxv'y)| + (k-2)|f(v)| \geq |u| + |f(v)|$. This contradicts Fact 3. So $v_1 \neq \varepsilon$.

Similarly we can prove that $v_2 \neq \varepsilon$ and so $v^k = (xv'y)^k$ is a factor of $w_1 \prod_{i=2}^k a_i w_i$. Thus $f(v)^k$ is a common factor of $f(v)^k$ and u^k . Since $w \in V_{k,A}$, we have $|w_i|_x \leq 1$ and $|w_i|_y \leq 1$ for all $1 \leq i \leq k$. This implies that $|xv'yxv'y| \geq |w_i| + 2$ for all $1 \leq i \leq k$ and thus $|f(v)^2| = |f(xv'yxv'y)| \geq |u|$. Consequently $|f(v^3)| \geq |f(v)| + |u|$, and once again we have a contradiction with Fact 3. \square

Step 4.3 If $|f(v)| < |u|$ then $k = 3$.

Proof. By the previous step, we know that $|v| = 1$. So $v = x$ for a letter x . Since $|f(v_1)f(x)^k f(v_2)| = |f(w)| = |pu^k s| \geq k|u| > k|f(x)|$, we have $|v_1 v_2| \geq 1$.

If $v_1 = \varepsilon$, x^k is a prefix of w , $x = a_0$ and $f(x) = p_0 s_0$. Since $|s_0 f(x)^{k-1}| \leq |f(x)^k| < |u^k|$ and since $s_0 f(\prod_{i=1}^k w_i a_i) = u^k s_k$, the word $s_0 f(x)^{k-1}$ is a prefix of u^k and so a common factor of $f(v)^k$ and u^k . Since $w \in V_{k,A}$, $|w_1|_x \leq 1$. This implies $w_1 = \varepsilon$ or $w_1 = x$ and so $|u| = |s_0 f(w_1) p_1| \leq |s_0 f(xx)|$. If $k \geq 4$, $|s_0 f(x)^{k-1}| \geq |s_0 f(xx)| + |f(x)| \geq |f(v)| + |u|$: this contradicts Fact 3. So $k = 3$.

When $v_2 = \varepsilon$, symmetrically we can prove $k = 3$.

Now we consider the case where $v_1 \neq \varepsilon$ and $v_2 \neq \varepsilon$. The word $f(x)^k$ is a common factor of $f(v)^k$ and u^k . Let us recall that $w = a_0 \prod_{i=1}^k w_i a_i$, and $|w_i|_x \leq 1$ for each integer i with $1 \leq i \leq k$. Since here x^k is a factor of $w_1 \prod_{i=2}^{k-1} a_i w_{i+1}$, there must exist an integer i , $1 \leq i \leq k$ such that $w_i = x$. Thus $|u| + |f(x)| = |s_{i-1} f(w_i) p_i| + |f(x)| \leq |f(x^4)|$. This contradicts Fact 3 when $k \geq 4$. So $k = 3$.

\square

We now make a break in the proof of the theorem to explain the situation. Up to now, we have proved this theorem when $k \geq 4$ showing that, when $f(T_{A,k})$ is k -power-free, there cannot exist words like w and u such that $w \in V_{k,A}$ and u^k has a non-synchronized decomposition in $f(w)$. Example 4.12 shows that this is possible when $k = 3$. Consequently when dealing with Case $k = 3$ (and $|v| = 1$), we have to consider the sequence of reductions of the couple (U^k, W) into the couple (u^k, w) . This will occur only in Cases 3 and 7 below. Actually Example 4.12 belongs to Case 3.

We now continue and end the proof of Theorem 3.1 treating Case $k = 3$ and $|v| = 1$.

Let us recall that $w = a_0 w_1 a_1 w_2 a_2 w_3 a_3 = v_1 v^3 v_2$. Since $|v| = 1$, from now on, we replace the notation v by x . Since $w \in V_{k,A}$, $|w_1|_x \leq 1$, $|w_2|_x \leq 1$, $|w_3|_x \leq 1$. Thus for at least one integer i , $a_i = x$. More precisely, we distinguish nine cases depending on the relative position of x^3 with respect to the w_i 's and to the a_i 's (see Figure 9: note that Case 2 and Case 8 are split into two subcases).

- Case 1: $a_0 = x = w_1 = a_1$;
- Case 2: $a_0 = x$, $w_1 = \varepsilon$, $a_1 = x$, $w_2 a_2$ starts with x ;
- Case 3: w_1 ends with x , $a_1 = x$, $w_2 \neq \varepsilon$, w_2 starts with x ;
- Case 4: w_1 ends with x , $a_1 = x$, $w_2 = \varepsilon$, $a_2 = x$;
- Case 5: $a_1 = x = w_2 = a_2$;
- Case 6: $a_1 = x$, $w_2 = \varepsilon$, $a_2 = x$, w_3 starts with x ;
- Case 7: w_2 ends with x , $a_2 = x$, w_3 starts with x ;
- Case 8: $a_1 w_2$ ends with x , $a_2 = x = a_3$, $w_3 = \varepsilon$;
- Case 9: $a_2 = x = w_3 = a_3$.

Case	a_0	w_1	a_1	w_2	a_2	w_3	a_3
1	x	x	x				
2	x	ε	x	$x\dots$			
	x	ε	x	ε	x		
3		$\dots x$	x	$x\dots$			
4		$\dots x$	x	ε	x		
5			x	x	x		
6			x	ε	x	$x\dots$	
7				$\dots x$	x	$x\dots$	
8			x	ε	x	ε	x
				$\dots x$	x	ε	x
9					x	x	x

Figure 9: Nine cases

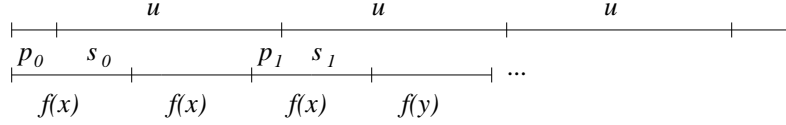


Figure 10: Case 1

Of course some cases are symmetric: Cases 1 and 9 (and Case 5 is very close), Cases 2 and 8, Cases 3 and 7, Cases 4 and 6. In what follows we prove that all cases are impossible since they contradict previous facts or hypotheses. Firstly:

Fact 5 *Cases 2, 4, 6 and 8 are not possible.*

Indeed in these cases, we can see that u^3 and $f(x)^3$ have a common factor of length $|u| + |f(x)|$ (for instance in Case 2, the common factor is $s_0 p_1 s_1 x = u s_1 f(x)$): this contradicts Fact 3.

Fact 6 *Case 1 is not possible.*

Proof. In this case, we have $f(x) = p_0 s_0 = p_1 s_1$ (see Figure 10).

Let y be the first letter of $w_2 a_2$ (that is, $w_2 \neq \varepsilon$ and y is the first letter of w_2 , or, $w_2 = \varepsilon$ and $a_2 = y$). Assume $y = x$. The word $u = s_0 f(x) p_1$ is a factor of $f(x^3)$. Since u^3 is not synchronized in $f(w)$, $|s_0| \neq |s_1|$. If $|s_0| < |s_1|$, $s_0 f(x)$ is a prefix of $s_1 f(x)$. If $|s_1| < |s_0|$, $s_1 f(x)$ is a prefix of $s_0 f(x)$. In both cases $f(x)$ is an internal factor of $f(xx)$: By Lemma 2.2, $f(x)$ is not primitive. This implies that $f(xx)$ contains a 3-power, a contradiction with the 3-power-freeness of $f(T_{A,k})$.

So $y \neq x$. We now consider two subcases.

Case 1.a: $|s_0| > |s_1|$

Since f is uniform, $|f(x)| = |f(y)|$ and so $|s_0 f(x)| > |s_1 f(y)|$. In this case $u = s_0 f(x) p_1$ starts with $s_1 f(y)$. Let v'_3 be the word such that $s_0 f(x) = s_1 f(y) v'_3$. We have $|v'_3| = |s_0| - |s_1| \leq |s_0| \leq |f(a_0)| = |f(x)|$. Thus v'_3 is a suffix of $f(x)$. Let v_3 be the word such that $f(x) = v_3 v'_3$. Since s_0 is a suffix of $f(x)$, v'_3 is a suffix of s_0 . For length reason, it follows that $f(y) = v'_3 v_3$. If $|p_1| \geq |v_3|$

then, since $f(x) = p_1s_1 = v_3v'_3$, v_3 is a prefix of p_1 and, since f is injective, yy is a factor of w_2 : this contradicts the fact that $w \in V_{k,A}$.

So $|p_1| < |v_3|$ and consequently $|s_1| > |v'_3|$ (since $f(x) = p_1s_1 = v_3v'_3$). We observe that $s_0 = s_1v'_3$ and s_1 is a suffix of s_0 (remember $f(x) = p_0s_0$), that is $s_0 = s_1v'_3 = v_4s_1$ for a word v_4 . Lemma 2.1 implies the existence of words α, β and of an integer r such that $s_1 = (\alpha\beta)^r\alpha$, $v'_3 = \beta\alpha$ and $s_0 = (\alpha\beta)^{r+1}\alpha$ (and $v_4 = \alpha\beta$). Note that $r \geq 1$ and $\alpha\beta \neq \varepsilon$ since $|s_1| > |v'_3| \neq 0$. Thus the words $f(xy)$ contains the factor $s_0v'_3 = (\alpha\beta)^{r+2}\alpha$ which contains the 3-power $(\alpha\beta)^3$. Since $xy \in T_{A,k}$, this contradicts the 3-power-freeness of $f(T_{A,k})$.

Case 1.b: $|s_0| < |s_1|$

We have $f(x) = p_0s_0 = p_1s_1$ and u starts with both s_0 and s_1 . Let v_3 and v_4 be the words such that $v_3s_0 = s_1 = s_0v_4$. Lemma 2.1 implies the existence of words α, β and of an integer r such that $v_3 = \alpha\beta$, $s_0 = (\alpha\beta)^r\alpha$, $v_4 = \beta\alpha$. Since u starts both with $s_0f(x)$ and with s_1 , the word v_4 is a prefix of $f(x)$. Thus $f(xx)$ contains the factor $s_1v_4 = (\alpha\beta)^{r+2}\alpha$. Since $xx \in T_{A,k}$ and $f(T_{A,k})$ is 3-power-free, we have $r = 0$, that is $s_1 = s_0\beta s_0$.

If $|f(x)| \geq |\beta s_0| + |s_1|$, then $f(x) = v_4ts_1 = \beta s_0ts_0\beta s_0$ for a word t and $p_1 = \beta s_0t$: $u = s_0f(x)p_1 = s_0\beta s_0ts_0\beta s_0\beta s_0t$. Since the word uu starts with $s_1f(y) = s_0\beta s_0f(y)$ and since $|f(y)| = |f(x)|$, we have $f(y) = ts_0\beta s_0\beta s_0$. It follows that $f(yx)$ contains the 3-power $(s_0\beta)^3$: this contradicts the 3-power-freeness of $f(T_{A,k})$.

So $|f(x)| < |\beta s_0| + |s_1|$. Since u starts with $s_0f(x)$ and with $s_1 = s_0\beta s_0$, there exists a word v_5 such that $f(x) = (\beta s_0)v_5$. Let us recall that $f(x) = p_1s_0(\beta s_0)$. Lemma 2.1 implies the existence of some words γ, δ and of an integer s such that $p_1s_0 = \gamma\delta$, $\beta s_0 = (\gamma\delta)^s\gamma$ and $v_5 = \delta\gamma$. The word uu starts both with $s_0f(x) = s_0\beta s_0v_5$ and with $s_1f(y) = s_0\beta s_0f(y)$. Consequently v_5 is a prefix of $f(y)$ and $f(xy)$ starts with $(\gamma\delta)^{s+2}\gamma$. Since $xy \in T_{A,k}$ and $f(T_{A,k})$ is 3-power-free, $s = 0$: $\gamma = \beta s_0$, $p_1s_0 = \beta s_0\delta$, $v_5 = \delta\beta s_0$ and $f(x) = \beta s_0\delta\beta s_0$. Let us recall that $|p_1| + |s_1| = |f(x)| < |\beta s_0| + |s_1|$, that is $|p_1| < |\beta s_0|$. Consequently from $p_1s_0 = \beta s_0\delta$, we deduce that δ is a suffix of s_0 . Thus $f(xxy)$ contains the factor $s_0f(x)v_5$ that ends with $(\delta\beta s_0)^3$. Since $x \neq y$, $xy \in T_{A,k}$. We have a contradiction with the 3-power-freeness of $f(T_{A,k})$. This ends the proof of impossibility of Case 1.

□

As already said, Case 9 is symmetric to Case 1. Moreover Case 5 can be treated as previously. Hence

Fact 7 *Cases 5 and 9 are not possible*

We end the proof of Theorem 3.1 with the proof of the final following case:

Fact 8 *Cases 3 and 7 are not possible.*

Proof. Since Cases 3 and 7 are symmetric, from now on we only consider Case 3. Let us consider the sequence of words obtained by successive reductions of W leading to w . More precisely, let $(\nu_i, \sigma_i)_{1 \leq i \leq m}$ be the couple of words such that $(\nu_1, \sigma_1) = (U, W)$, $(\nu_m, \sigma_m) = (u, w)$ and for each i , $1 \leq i < m$, $(\nu_{i+1}^k, \sigma_{i+1})$ is the word in $\text{Reduced}(\nu_i^k, \sigma_i)$ obtained by applying Lemma 4.9 with the additional Reduction Rules chosen in Section 4.5. By the reduction process, we know that each one of the k -powers ν_j^k ($1 \leq j \leq m$) has a $(p_i, s_i, a_i, w_{i,j})_{0 \leq i \leq 3}$ -decomposition in $f(\sigma_i)$ for some words $(w_{i,j})_{1 \leq i \leq 3}$.

By hypotheses of Case 3, $\sigma_m = w$ contains the 3-power x centered in a_1 . We mean more precisely that $w_1 = w_{m,1}$ ends with x (and so is not the empty word), $a_1 = a_{m,1} = x$ and $w_2 = w_{m,2}$ starts

with x (and is also not the empty word). On other part, $\sigma_1 = W$ is 3-power-free. Thus there exists an integer q with $1 \leq q < m$ such that σ_q does not contains xxx centered in a_1 where'as σ_j contains xxx centered in a_1 for all j such that $q + 1 \leq j \leq m$. To simplify temporarily the notation, we set $W_1 = \sigma_q$, $W_2 = \sigma_{q+1}$, $U_1 = \nu_q$ and $U_2 = \nu_{q+1}$.

By the reduction process, there exist words $(x_i, y_i, z_i)_{1 \leq i \leq 3}$ (set also $x_0 = y_0 = z_0 = \varepsilon$) such that U_1^3 has a $(p_i, s_i, a_i, x_i y_i z_i)_{i=0, \dots, 3}$ -decomposition in $f(W_1)$ and U_2^3 has a $(p_i, s_i, a_i, x_i z_i)_{i=0, \dots, 3}$ -decomposition in $f(W_2)$. Since W_2 is obtained from W_1 by the Reduction Lemma 4.9 there exist an integer $1 \leq \ell \leq 3$ and a letter a in A such that both x_ℓ and y_ℓ end with a and $|s_{\ell-1}f(x_\ell)| - |f(a)| < |s_{i-1}f(x_i)| \leq |s_{\ell-1}f(x_\ell)|$ and $|y_i| = |y_\ell|$. By the Reduction Rule 1, $|x_\ell|_a = |y_\ell|_a = 1$.

Finally let us stress that by definition of W_1 and W_2 , we assume that $x_1 z_1$ ends with x , $x_2 z_2$ starts with x and that either $x_1 y_1 z_1$ does not end with x or $x_2 y_2 z_2$ does not start with x . We end in two steps showing first that $x_1 y_1 z_1$ must end with x , and second that $x_2 y_2 z_2$ must start with x : This contradicts the previous sentence.

Step 1: $x_1 y_1 z_1$ must end with x

Assume by contradiction that $x_1 y_1 z_1$ does not end with x . Since $x_1 z_1$ ends with x , we have $z_1 = \varepsilon$ and y_1 ends with $b \neq x$ (since x_ℓ and y_ℓ ends with the same letter, it also means that $l \neq 1$). By Remark 4.11(2), $z_\ell = \varepsilon$ and $|p_\ell| \leq |p_1|$. Thus $U_1 = s_0 f(x_1 y_1) p_1 = s_{\ell-1} f(x_\ell y_\ell) p_\ell$ with $|y_1| = |y_\ell|$, $x_1 (= x_1 z_1)$ ends with x and both x_ℓ and y_ℓ end with a . Let c be the first letter of y_1 (see figure 11).

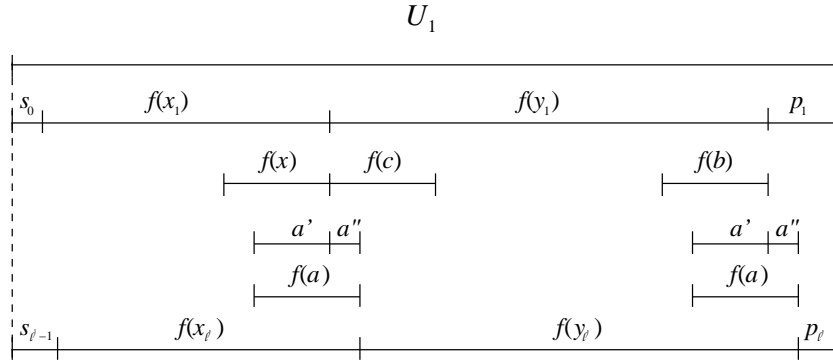


Figure 11:

Let a'' be the suffix of $f(a)$ such that $p_1 = a'' p_\ell$ and let a' be the prefix of $f(a)$ such that $f(a) = a' a''$. Since $f(b) p_1$ and $f(a) p_\ell$ are both suffixes of U_1 , we get that $f(b)$ ends with a' . Since $|f(y_1) p_1| = |f(y_\ell) a'' p_\ell|$, we get that $|s_0 f(x_1) a''| = |s_0 f(x_1 y_1) p_1| + |a''| - |f(y_1) p_1| = |s_{\ell-1} f(x_\ell)|$. So $f(x)$ ends with a' and $f(c)$ starts with a'' . Since p_1 and $s_0 a''$ are prefixes of $f(x)$, by a length criterion, it follows that $f(x) = a'' a'$.

If $c \neq x$, $b x^2 c$ is 3-power-free and $f(b x^2 c)$ contains the 3-power $(a' a'')^3$: this contradicts the 3-power-freeness of $f(T_{A,k})$.

Thus $c = x$. If $|x_\ell| \geq 2$, let e be the letter such that x_ℓ ends with ea . Since y_1 contains b and c with $b \neq x = c$, we have $|y_\ell| = |y_1| \geq 2$. Let d be the first letter of y_ℓ . We have $d \neq a$ and $e \neq a$ since $|x_\ell|_a = |y_\ell|_a = 1$. Since $f(y_1) p_1 = a'' f(y_\ell) p_\ell$ and since $f(y_1)$ starts with $f(x) = a'' a'$, we get that $f(d)$ starts with a' . Since $s_0 f(x_1) a'' = s_{\ell-1} f(x_\ell)$ and since $f(x_1)$ ends with $f(x)$, we get that

$f(e)$ ends with a'' . It follows that $f(ea^2d)$ contains $(a''a')^3$ although ea^2d is a 3-power free word: this contradicts the 3-power-freeness of $f(T_{A,k})$.

Thus $c = x$ and $|x_\ell| = 1$. Consequently $x_\ell = a$. Since $|s_{\ell-1}f(x_\ell)| - |f(a)| < |s_0f(x_1)| \leq |s_{\ell-1}f(x_\ell)|$ and since x_1 ends with x , we have $x_1 = x$. Thus $U_2 = s_0f(x)p_1$. Since $|u| \geq |s_0f(x)p_1|$, we deduce that $U_2 = u$ and $W_2 = w$. It follows that $u = s_0f(x)p_1$.

Let us recall that moreover $f(x) = p_1s_1$ and $s_1f(x)$ is a prefix of u . If $|s_0| < |s_1|$ then $f(x)$ is an internal factor of $f(xx)$ and (by Lemma 2.2) $f(x^2)$ contains a 3-power: this contradicts the 3-power-freeness of $f(T_{A,k})$. Thus $|s_0| \geq |s_1|$. Let s_0'' be the suffix of s_0 and p_2' be the word such that $s_0''f(x) = f(x)p_2'$ and $s_1f(x)p_2'$ is a prefix of u . By Lemma 2.1, there exist two words α and β such that $s_0'' = \alpha\beta (\neq \varepsilon)$, $p_2' = \beta\alpha$ and $f(x) = (\alpha\beta)^r\alpha$ for an integer r . We have $|s_0''| = |s_0| - |s_1| \leq |f(a_0)|$. If $|s_0''| = |f(a_0)|$, then $s_0 = f(a_0)$ and $s_1 = \varepsilon$: this contradicts the fact that u^k is not synchronized in $f(w)$. Thus $|s_0''| < |f(a_0)| = |f(x)|$. Consequently $r \geq 1$. Let γ be the letter such that $x\gamma$ is a prefix of w_2a_2 : p_2' is a prefix of $f(\gamma)$. By Fact 3, no powers respectively of $f(x)$ and of u^3 have a common factor of length greater than $|f(x)| + |u|$. Hence $a_0 \neq x$. But then $a_0x\gamma$ is 3-power-free where'as $f(a_0x\gamma)$ contains $(\alpha\beta)^{r+2}$: this contradicts the 3-power-freeness of $f(T_{A,k})$, and so $x_1y_1z_1$ must end with x .

Step 2: $x_2y_2z_2$ must start with x

From what precedes, we know now that it remains to consider the case where $x_2y_2z_2$ does not start with x . We will show that this assumption leads to a final contradiction.

Since x_2z_2 starts with x , we have $x_2 = \varepsilon$ and y_2 starts with $b \neq x$. By Remark 4.11(1), $x_\ell = a$ and $|s_{\ell-1}| < |s_1|$ (and so $l \neq 2$). Thus $U_1 = s_1f(y_2z_2)p_2 = s_{\ell-1}f(ay_\ell z_\ell)p_\ell$ with $|y_2| = |y_\ell|$, $z_2 = x_2z_2$ starts with x and y_ℓ ends with a . Let c be the last letter of y_2 (see figure 12).

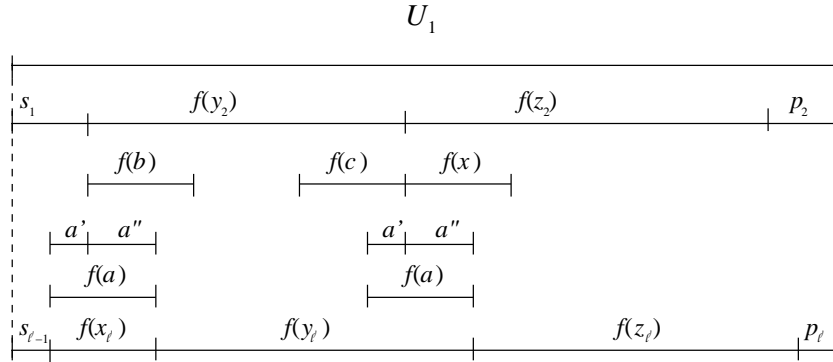


Figure 12:

Let a' be the prefix of $f(a)$ such that $s_1 = s_{\ell-1}a'$ and let a'' be the suffix of $f(a)$ such that $f(a) = a'a''$. Since $s_{\ell-1}f(a)$ and $s_1f(b)$ are both prefixes of U_1 , the word $f(b)$ starts with a'' . Since $|s_1f(y_2)| = |s_{\ell-1}a'f(y_\ell)| = |s_{\ell-1}f(ay_\ell)| - |a''| = |U_1| - |f(z_\ell)p_\ell a''|$, we have $|f(z_2)p_2| = |U_1| - |s_1f(y_2)| = |a''f(z_\ell)p_\ell|$. Since $f(z_2)p_2$ and $a''f(z_\ell p_\ell)$ are both suffixes of U_1 , it follows that $f(z_2)p_2 = a''f(z_\ell)p_\ell$ and we get that $f(x)$ starts with a'' and $f(c)$ ends with a' . Since a' is a suffix of s_1 and so of $f(x)$, by a length criterion, we get $f(x) = a''a'$.

If $c \neq x$, cx^2b is 3-power-free and $f(cx^2b)$ contains $(a'a'')^3$: this contradicts the 3-power-freeness of $f(T_{A,k})$.

Thus $c = x$ and y_2 contains two different letters b and x . We get $|y_\ell| = |y_2| \geq 2$. Let d be the letter such that y_ℓ ends with da . Since $|y_\ell|_a = 1$, we have $d \neq a$. Since $f(x)a''$ and $f(da)$ are both suffixes of $f(y_\ell)$, the word $f(d)$ ends with a'' . Since $x_2z_2 = z_2$ starts with x , $|z_2|_x \neq 0$. Let z'_2 and z''_2 be the words such that $z_2 = z'_2x z''_2$ with $|z''_2|_x = 0$. Let z'_ℓ be the word and e be the letter such that $z'_\ell e$ is the prefix of $z_\ell a_\ell$ verifying $|s_{\ell-1}f(ay_\ell z'_\ell)| < |s_1f(y_2 z'_2 x)| \leq |s_{\ell-1}f(ay_\ell z'_\ell e)|$. Let us recall that $s_1 = s_{\ell-1}a'$ and so $|s_1| = |s_{\ell-1}a'|$. Moreover $s_1f(y_2 z'_2 x)$ and $s_{\ell-1}f(ay_\ell z'_\ell)$ are both prefixes of U_1 , and $|s_1a''| = |s_{\ell-1}| + |f(x)| = |s_{\ell-1}| \pmod L$. Thus $s_1f(y_2 z'_2 x) = s_1f(y_2 z'_2)a''a' = s_{\ell-1}f(ay_\ell z'_\ell)a'$. It follows that $f(e)$ starts with a' . If $e \neq a$, da^2e is 3-power-free and $f(da^2e)$ contains $(a''a')^3$: this contradicts the 3-power-freeness of $f(T_{A,k})$.

Thus $e = a$. Assume $|z'_\ell e| \leq |z_\ell|$. Let us recall that the reductions are assumed to be made under two rules. The second Reduction Rule implies that, having made a reduction with, in Lemma 4.9, an integer ℓ and a letter a , then if $|z_\ell|_a \neq 0$, the next $|z_\ell|_a$ reductions are made with the same integer ℓ and the same letter a . Thus here the words $\sigma_{q+2}, \dots, \sigma_{q+1+|z'_\ell a|_a}$ exist and are obtained using, in Lemma 4.9, the same integer ℓ and the same letter a than the ones used to reduce $\sigma_q = W_1$ into $\sigma_{q+1} = W_2$. Moreover $\nu_{q+1+|z'_\ell a|_a} = s_1f(z''_2)p_2$. Since $|\nu_{q+1+|z'_\ell a|_a}| \geq |\nu_m| \geq |s_1f(x)p_2|$, we have $z''_2 \neq \varepsilon$. But since $|z''_2|_x = 0$, we have a contradiction with the fact that w contains xxx centered in a_1 .

Thus $|z'_\ell e| > |z_\ell|$, that is, $z_\ell = z'_\ell$, $e = a_\ell (= a)$. It follows that $z''_2 = \varepsilon$. Since σ_j contains xxx centered in a_1 for all $q+1 \leq j \leq m$, we must have $w_2 = x$ and $u = s_1f(x)p_2$. Let us recall that moreover $f(x) = p_1s_1$ and $f(x)p_1$ is a suffix of u . If $|p_1| > |p_2|$ then $f(x)$ is an internal factor of $f(xx)$ and (by Lemma 2.2) $f(x^2)$ contains a 3-power: this contradicts the 3-power-freeness of $f(T_{A,k})$. Since the decomposition is not synchronized, we have $|p_1| \neq |p_2|$. Thus $|p_1| < |p_2|$. Let p''_2 be the prefix of p_2 and s'_0 be the word such that $p_2 = p''_2p_1$, $f(x)p''_2 = s'_0f(x)$ and $s'_0f(x)p_1$ is a suffix of u . By Lemma 2.1, there exist two words α and β such that $s'_0 = \alpha\beta (\neq \varepsilon)$, $p''_2 = \beta\alpha$ and $f(x) = (\alpha\beta)^r\alpha$ for an integer r . Since $|s'_0| = |p_2| - |p_1| < |f(x)|$ (remember $|p_1| \neq 0$), we have $r \geq 1$. Let γ be the letter such that γx is a suffix of a_0w_1 : s'_0 is a suffix of $f(\gamma)$. By Fact 3, no power respectively of $f(x)$ and of u^3 have a common factor of length greater than $|f(x)| + |u|$. Hence $a_2 \neq x$. But then γxa_2 is 3-power-free whereas $f(\gamma xa_2)$ contains $(\alpha\beta)^{r+2}$: this contradicts the 3-power-freeness of $f(T_{A,k})$. This is a final contradiction proving that Case 3 is not possible. So consequently Theorem 3.1 holds. \square

6 Conclusion

Theorem 3.1 and Corollary 3.2 lead to some natural questions: is $T_{A,k}$ the smaller test-set? Is the bound $b_{k,A} = k \times \text{Card}(A) + k + 1$ optimal? The answer to these questions are negative at least in most of the previously known cases. As already mentioned in the introduction, M. Leconte [12] has previously got a test-set when $\text{Card}(A) = 3$. He proved [12] : a uniform morphism f defined on a three-letter alphabet is k -power-free ($k \geq 3$) if and only if the images of all k -power-free words of length at most $3k + 5$ are k -power-free. We observe that in case $k = 3$, we obtain a better bound than M. Leconte. But in all other cases, the bound of M. Leconte is better than our. Another result shows the non-optimality of our bound $b_{k,A}$. When $\text{Card}(A) = 2$ (and $k \geq 3$), V. Keränen proved: a uniform and primitive morphism defined on a two-letter alphabet is k -power-free if and only if the images of length at most 4 are k -power-free. This bound in this result does not depend on the value of k and is far better than our general bound $b_{k,\{a,b\}} \geq b_{3,\{a,b\}} = 13$.

To end, let us mention further works. In this paper, we propose a new technic to tackle the

decidability of k -power-freeness of uniform morphisms. We are now looking to extension of this technic to the decidability of k -power-freeness of arbitrary morphisms.

References

- [1] J.-P. Allouche and J. Shallit. *Automatic sequences*. Cambridge University Press, 2003.
- [2] D. R. Bean, A. Ehrenfeucht, and G. F. McNulty. Avoidable patterns in strings of symbols. *Pacific J. of Math*, 85(2):261–294, 1979.
- [3] J. Berstel. Axel Thue’s work on repetition in words. Technical Report 92.70, LITP, septembre 1992. 4th Conf. on Formal Power Series and Algebraic Combinatorics, Montréal 1992.
- [4] J. Berstel. Axel Thue’s papers on repetition in words: a translation. Technical Report 20, Laboratoire de Combinatoire et d’Informatique Mathématique, Université du Québec, Montréal, 1995.
- [5] J. Berstel and D. Perrin. *Theory of codes*. Academic Press, 1985. See also J. Berstel’s web page.
- [6] J. Berstel and P. Séébold. A characterization of overlap-free morphisms. *Discrete Applied Mathematics*, 46:275–281, 1993.
- [7] C. Choffrut and J. Karhumäki. *Handbook of Formal Languages*, volume 1, chapter Combinatorics of words. Springer, 1997.
- [8] M. Crochemore. Sharp characterizations of squarefree morphisms. *Theoretical Computer Science*, 18:221–226, 1982.
- [9] S. Istrail. On irreducible alanguages and nonrational numbers. *Bull. Math. de la Soc. Sci. Math. de la R.S. de Roumanie*, 21 (69)(3-4):301–308, 1977.
- [10] V. Keränen. On k -repetition freeness of length uniform morphisms over a binary alphabet. *Discrete Applied Mathematics*, 9:301–305, 1984.
- [11] V. Keränen. On the k -freeness of morphisms on free monoids. *Annales Academiae Scientiarum Fennicae* 61, Series A, 1986.
- [12] M. Leconte. *Codes sans répétition*. PhD thesis, LITP Université Paris 6, october 1985.
- [13] M. Lothaire. *Combinatorics on words*, volume 17 of *Encyclopedia of Mathematics*. Addison-Wesley, 1983. Reprinted in 1997 by Cambridge University Press in the Cambridge Mathematical Library, Cambridge, UK, 1997.
- [14] M. Lothaire. *Algebraic Combinatorics on words*, volume 90 of *Encyclopedia of Mathematics*. Cambridge University Press, Cambridge, UK, 2002.
- [15] M. Lothaire. *Combinatorics on words and applications*, volume 105 of *Encyclopedia of Mathematics*. Cambridge University Press, Cambridge, UK, 2005.

- [16] F. Mignosi and Séébold. If a D0L language is k -power free then it is circular. In *ICALP 93*, volume 700 of *Lect. Notes in Comp. Sci.*, pages 507–518, 1993.
- [17] P. Ochem. *Graph coloring and combinatorics on words*. PhD thesis, Université de Bordeaux I, Nov. 2005.
- [18] G. Richomme. Some non finitely generated monoids of repetition-free endomorphisms. *Information Processing Letters*, 85:61–66, 2003.
- [19] G. Richomme and P. Séébold. Characterization of test-sets for overlap-free morphisms. *Discrete Applied Mathematics*, 98:151–157, 1999.
- [20] G. Richomme and P. Séébold. Conjectures and results on morphisms generating k -power-free words. *Int. J. of Foundations of Computer Science*, 15(2):307–316, 2004.
- [21] G. Richomme and F. Wlazinski. Some results on k -power-free morphisms. *Theoretical Computer Science*, 273:119–142, 2002.
- [22] G. Richomme and F. Wlazinski. Overlap-free morphisms and finite test-sets. *Discrete Applied Mathematics*, 143:92–109, 2004.
- [23] A. Thue. Über unendliche zeichenreihen. *Kristiania Videnskapsselskapets Skrifter Klasse I. Mat.-naturv*, 7:1–22, 1906.
- [24] A. Thue. Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen. *Kristiania Videnskapsselskapets Skrifter Klasse I. Mat.-naturv*, 1:1–67, 1912.
- [25] F. Wlazinski. A test-set for k -power-free binary morphisms. *RAIRO Theoretical Informatics and Applications*, 35:437–452, 2001.