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UNIFYING THE BGM AND SABR MODELS: A SHORT RIDE IN HYPERBOLIC GEOMETRY

PIERRE HENRY-LABORDÈRE

ABSTRACT. In this short note, using our geometric method introduced in a previous paper [13] and initiated by [4], we derive an asymptotic swaption implied volatility at the first-order for a general stochastic volatility Libor Market Model. This formula is useful to quickly calibrate a model to a full swaption matrix. We apply this formula to a specific model where the forward rates are assumed to follow a multi-dimensional CEV process correlated to a SABR process. For a caplet, this model degenerates to the classical SABR model and our asymptotic swaption implied volatility reduces naturally to the Hagan-al formula [12]. The geometry underlying this model is the hyperbolic manifold \mathbb{H}^{n+1} with n the number of Libor forward rates.

1. INTRODUCTION

The BGM model [6] has recently been the focus of much attention as it gives a theoretical justification for pricing caps-floors using the classical Black-Scholes formula. The basic (physical) random variables are given by the Libor forward rates which are assumed to follow a correlated log-normal process. As the forward swap rate model implied by the BGM is quite complicated (the swap forward rate is not log-normally distributed), the calibration to the swaption matrix is difficult. Asymptotic swaption implied volatility (at the zero-order) were initially derived by Rebonato [18] and Hull-White [10] for the (log-normal) BGM model. Such formula has been obtained by assuming that the ratio of a forward Libor rate over the swap rate and the derivative of the swap rate according to a forward Libor rate are almost constant (and therefore equal to their values at the spot).

Despite its great success, the BGM model presents the same drawbacks as the classical Black-Scholes theory: As the forward rates follow a correlated log-normal process, the model is not able to calibrate the full swaption matrix in/out-the money (in particular the caplets) and give a good dynamics to the Libor rates. The incorporation of a swaption smile can be obtained by introducing more elaborated models which should be flexible enough to calibrate caplets and a grid of swaption volatilities (not necessary at the money) across all swaption expiries and underlying swap maturities. One property that these models must still share is their ability to quickly calibrate the swaption matrix without using complicated numerical routines such as Monte-Carlo simulation which are usually noisy and time-consuming. In this context, Andersen-Andreasen introduced the CEV Libor Market Model (LMM) [1] which assumes that each forward rate follows a CEV process, and showed how to obtain asymptotic swaption smile. Their method is still based on the Rebonato "freezing" argument which is not completely mathematically justified. Recently, for this specific model, Kawai found a better asymptotic formula using the Wiener chaos expansion [16].

Key words and phrases. Heat Kernel expansion, Hyperbolic Geometry, Asymptotic Smile Formula, Stochastic Libor Market Model.

Although giving more flexibility than the BGM model, the CEV LMM model is still not able to calibrate the swaption matrix for in/out strike and in this context, we are naturally led to use stochastic volatility LMM. The literature on this subject is not particularly large. Andersen-al introduced a LMM where the Libors follow a multi-dimensional correlated CEV process coupled (but uncorrelated) to a Heston model [2, 3] and recently V. Piterbarg modifies this model to incorporate term structure [17]. Using an averaging principle, V. Piterbarg derives an asymptotic volatility. Note that as these models are uncorrelated to the stochastic volatility, the swaption fair value is simply given by the fair price in the case of a local volatility model conditional to the stochastic volatility process as explained by the Hull-White decomposition [11]. An asymptotic expression can then be generated by approximating the moments of the volatility process[2].

For pricing exotic options (such as bermudan swaptions for example), it is simpler or more natural to model directly the forward swap rate with a stochastic volatility process. For example, the SABR model [12] was introduced to fulfill his goal. An asymptotic swaption smile formula (at the first-order) was derived for this specific model and help to calibrate quickly the model to liquid market data. In this context, it is natural to try to reconcile/unify the two benchmark models, the BGM and SABR models. We therefore introduce a LMM where the forward rates follow a multi-dimensional CEV process (with one beta for each forward) correlated to a SABR model. As it is the case for the SABR model, we impose that the libors are correlated to the unique volatility and it is therefore not possible to follow the Andersen-al [3] method (i.e. the Hull-White decomposition) to derive an asymptotic swaption smile.

In this paper, we pursue our previous work on the application of the heat kernel expansion on a Riemannian manifold endowed with an Abelian connection [13] to derive an asymptotic smile formula for a swaption. The plan of this paper is as follows: In the first part, we will recall some definitions and present a list of recent Libor Market Models. In the second part, we apply this heat kernel expansion to derive an asymptotic swaption smile formula at the first-order valid for any LLM. In the third part, we present our stochastic LMM and apply this general formula. We will prove that the geometry underlying this model is the hyperbolic manifold \mathbb{H}^{n+1} . Some important properties of this space are then presented. Furthermore, we show that the "freezing" argument is no longer valid when we try to price a swaption in/out the money: The libors should in fact be frozen to the saddle-point (constrained on a particular hyperplane) which minimizes the geodesic distance on \mathbb{H}^{n+1} .

2. LIBOR MARKET MODEL

A swaption gives the right, but not the obligation, to enter into an interest rate swap at a pre-determined rate on an agreed future date [7]. The maturity date for the swaption is noted T_α and T_β is the expiry for the forward swap rate $s_{\alpha\beta}$ given by

$$(2.1) \quad s_{\alpha\beta}(t) = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{(1+\tau F_j(t))}}{\sum_{i=\alpha+1}^{\beta} \tau \prod_{j=\alpha+1}^i \frac{1}{(1+\tau F_j(t))}}$$

with τ the tenor and $F_k(t) \equiv F(t, T_{k-1}, T_k)$ is the forward rate resetting at T_{k-1} .

As the product of the bond $P(t, T_k)$ with the forward rates $F_k(t)$ is a difference of two bonds with maturity T_{k-1} and T_k , $\frac{1}{\tau}(P(t, T_{k-1}) - P(t, T_k))$, and therefore a traded asset, F_k is a (local) martingale under \mathbb{Q}^k , the (forward) measure associated with the numéraire $P(t, T_k)$. Therefore,

we assume the following driftless dynamics

$$(2.2a) \quad dF_k(t) = \sigma_k(t)\Phi_k(a, F_k)dW_k, \quad \forall t \leq T_{k-1}, \quad k = 1, \dots, n$$

$$(2.2b) \quad dW_k dW_l = \rho_{kl}(t)dt$$

with the initial conditions $a(t=0) = \alpha$ and $F_k(t=0) = F_k^0$.

| Libor market model | SDE |
|--------------------|---|
| BGM | $dF_k = \sigma_k(t)F_k dW_k$ |
| CEV | $dF_k = \sigma_k(t)F_k^\beta dW_k$ |
| Limited CEV | $dF_k = \sigma_k(t)F_k \min(F_k^{\beta-1}, \epsilon^{\beta-1})dW_k$ with ϵ a small positive number |
| Shifted log-normal | $dX_k = \sigma(t)X_k dW_k$ with $F_k = X_k + \alpha$ |
| FL-SV | $dF_k = \sigma_k(t)(\beta F_k + (1-\beta)F_k^0)\sqrt{v}dW_k$ $dv = \lambda(v - \bar{\lambda})dt + \nu\sqrt{v}dZ; dW_k dZ = 0$ |
| FL-TSS | $dF_k = \sigma_k(t)(\beta_k(t)F_k + (1-\beta_k(t))F_k^0)\sqrt{v}dW_k$ $dv = \lambda(v - \bar{\lambda})dt + \nu\sqrt{v}dZ; dW_k dZ = 0$ |

TABLE 1. Examples of stochastic (or local) volatility Libor models.

In order to achieve some flexibility, we assume that the (normal) local volatility $\Phi_k(a, F_k)$ depends on a hidden Markov process a (to be specified later) representing a stochastic volatility. We therefore assume that all the forward rates are coupled with the same stochastic volatility a . (Table 1) presents a list of the different functional forms for ϕ_k used in the literature. The BGM, (limited) CEV and shifted log-normal models correspond to local volatility models ($a = 1$) and the others to stochastic volatility models with a unique stochastic volatility a driven by a Heston process. Note that the stochastic differential equation for the libors L_k has been written in the forward measure \mathbb{Q}^k and the stochastic equation for a remains the same in the forward or forward swap rate measures as a is assumed to be uncorrelated with the Libor rates. This will not be the case in our LLM.

3. ASYMPTOTIC SWAPTION SMILE

The forward swap rate satisfies the following driftless dynamics in the forward-swap measure $\mathbb{Q}^{\alpha\beta}$ (associated to the numéraire $C_{\alpha\beta}(t) = \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)$)

$$(3.1) \quad ds_{\alpha\beta} = \sum_{k=\alpha+1}^{\beta} \frac{\partial s_{\alpha\beta}}{\partial F_k} \sigma_k(t) \phi_k(a, F_k) dZ_k$$

$$\text{with } \frac{\partial s_{\alpha\beta}}{\partial F_j} = \frac{s_{\alpha\beta} \tau_j}{(1 + \tau_j F_j)} \left(\frac{P(t, T_\alpha)}{P(t, T_\beta) - P(t, T_\alpha)} + \frac{\sum_{k=j}^{\alpha-1} \tau_k P(t, T_{k+1})}{C_{\alpha\beta}} \right)$$

The local volatility associated to the forward swap rate ($ds_{\alpha\beta} = \sigma_{loc}^{\alpha\beta}(s_{\alpha\beta}, t) dW_t$) is then by definition

$$(3.2a) \quad (\sigma_{loc}^{\alpha\beta})^2(s, t) \equiv \mathbb{E}^{\alpha\beta} \left[\sum_{i,j=\alpha+1}^{\beta} \rho_{ij}(t) \sigma_i(t) \sigma_j(t) \phi_i(a, F_i) \phi_j(a, F_j) \frac{\partial s_{\alpha\beta}}{\partial F_i} \frac{\partial s_{\alpha\beta}}{\partial F_j} \Big|_{s_{\alpha\beta} = s} \right]$$

$$(3.2b) \quad = \sum_{i,j=\alpha+1}^{\beta} \rho_{ij}(t) \sigma_i(t) \sigma_j(t) \frac{\int_{\mathbb{B}} \phi_i(a, F_i) \phi_j(a, F_j) \frac{\partial s_{\alpha\beta}}{\partial F_i} \frac{\partial s_{\alpha\beta}}{\partial F_j} p da \prod_i dF_i}{\int_{\mathbb{B}} p da \prod_i dF_i}$$

with the submanifold $\mathbb{B} = \{\{F_i\}_i | s_{\alpha\beta} = s\}$ and $p \equiv p(a, F_i, t | \alpha, F_i^0)$ the conditional probability satisfying the (backward) Kolmogorov equation associated to the SDE for the Libors and the volatility a in the forward swap measure $\mathbb{Q}^{\alpha\beta}$. An asymptotic expression in the short time limit for the local volatility $\sigma_{\alpha\beta}(s, t)$ can be found in two steps: find an asymptotic expansion for the conditional probability p (in $\mathbb{Q}^{\alpha\beta}$) and do the integration over \mathbb{B} .

3.1. First step: Heat Kernel expansion. As explained previously, the first step can be achieved using the heat kernel expansion. In that purpose, the Kolmogorov equation is rewritten as the heat kernel equation on a $(n + 1)$ -dimensional Riemannian manifold \mathcal{M}^{n+1} endowed with an Abelian connection as explained in [13, 14]. Let's assume that our multi-dimensional stochastic equations (in $\mathbb{Q}^{\alpha\beta}$) are written as

$$(3.3) \quad dx^\mu = b^\mu(x, t)dt + \sigma^\mu(x, t)dW^\mu$$

with $dW^\mu dW^\nu = \rho_{\mu\nu}dt$ (note that the indices $1, \dots, n$ (resp. $n + 1$) correspond(s) to the forward F^i (resp. a)). Then, the metric $g_{\mu\nu}$ depends only on the diffusion terms σ_μ and the connection \mathcal{A}_μ on the drift terms b^μ as well

$$(3.4a) \quad g_{\mu\nu} = 2 \frac{\rho^{\mu\nu}}{\sigma_\mu \sigma_\nu}, \quad \mu, \nu = 1 \dots n + 1, \quad \rho^{\mu\nu} \equiv [\rho^{-1}]_{\mu\nu}$$

$$(3.4b) \quad \mathcal{A}^\mu = \frac{1}{2} \left(b^\mu - \sum_{\nu=1}^{n+1} g^{-\frac{1}{2}} \partial_\nu (g^{1/2} g^{\mu\nu}) \right), \quad \mu = 1 \dots n + 1$$

In terms of these functions, the asymptotic solution to the Kolmogorov equation in the short-time limit is given by $(x = (a, F_i), x^0 = (\alpha, F_i^0))$

$$(3.5) \quad p(x, t | x^0) = \frac{\sqrt{g(x)}}{(4\pi t)^{\frac{n}{2}}} \sqrt{\Delta(x, x^0)} \mathcal{P}(x, x^0) e^{-\frac{\sigma(x, x^0)}{2\tau}} \sum_{n=1}^{\infty} a_n(x, x^0) t^n, \quad \tau \rightarrow 0$$

- Here, $\sigma(x, x^0)$ is the Synge world function equal to one half of the square of geodesic distance $d(x, x^0)$ between x and x^0 for the metric g . This distance is defined as the minimizer of

$$(3.6) \quad d(x, x^0)^2 = \min_C \int_0^T g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} dt$$

and t parameterizes the curve C joining $x(t=0) \equiv x^0$ and $x(T) \equiv x$.

- $\Delta(x, x^0)$ is the so-called Van Vleck-Morette determinant

$$(3.7) \quad \Delta(x, x^0) = g(x)^{-\frac{1}{2}} \det \left(-\frac{\partial^2 \sigma(x, x^0)}{\partial x \partial x^0} \right) g(x^0)^{-\frac{1}{2}}$$

with $g(x) = \det[g_{\mu\nu}(x, x)]$

- $\mathcal{P}(x, x^0)$ is the parallel transport of the Abelian connection along the geodesic from the point x^0 to x

$$(3.8) \quad \mathcal{P}(x, x^0) = e^{\int_{C(x, x^0)} \mathcal{A}_\mu dx^\mu}$$

- The $a_i(x, x^0)$ ($a_0(x, x^0) = 1$) are smooth functions on M and depend on geometric invariants such as the scalar curvature R . More details can be found in [13].

3.2. Second step: Saddle-point method. The integration over \mathbb{B} is obtained by using a saddle-point method which consists in approximating at the first order the integral $\int f(x)e^{\epsilon\phi(x)}dx$ in the limit ϵ small by [8]

$$(3.9) \quad \int f(x)e^{\epsilon\phi(x)}dx \sim_{\epsilon \ll 1} f(x^*)e^{\epsilon\phi(x^*)} \left(1 + \frac{1}{\epsilon} \left(-\frac{\partial_{\alpha\beta}f}{2f} A_{\alpha\beta} + \left(\frac{\partial_{\alpha}f}{2f} \partial_{\beta\gamma\delta}\phi + \frac{1}{8} \partial_{\alpha\beta\gamma\delta}\phi \right) A_{\alpha\beta} A_{\gamma\delta} - \frac{\partial_{\alpha\beta\gamma}\phi \partial_{\delta\mu\nu}\phi}{72} A_{\alpha\beta} A_{\gamma\delta} A_{\mu\nu} \right) \right)$$

with $A^{\alpha\beta} = [\partial_{\alpha\beta}\phi]^{-1}$, $dx \equiv \prod_{i=1}^n dx_i$ and x^* the saddle-point (which minimizes $\phi(x)$). This expression can be obtained by developing $\phi(x)$ and $f(x)$ in series around x^* . The quadratic part in $\phi(x)$ leads to a Gaussian integration over x which can be performed.

3.2.1. Saddle-point. As the conditional probability at the zero-order is proportional to $e^{-\frac{d(x,x^0)^2}{4t}}$, the saddle-point corresponds to the point $x = (a, F_{i=1, \dots, n})$ on the submanifold $s_{\alpha\beta} = s$ which minimizes the geodesic distance $d(x, x^0)$ [5, 4]

$$(3.10) \quad (a^*, \{F_i^*\}) \equiv (a, \{F_i\}) \text{ such as } \min_{a, \{F_i\} | s_{\alpha\beta}=s} [d(x, x^0)^2]$$

Introducing a Lagrange multiplier, λ , this is equivalent to

$$(3.11) \quad (a^*, \{F_i^*\}) \equiv (a, \{F_i\}) \text{ such as } \min_{a, \{F_i\}, \lambda} [d^2(x, x^0) + \lambda(s_{\alpha\beta}(F) - s)]$$

3.3. Asymptotic local volatility. Plugging our asymptotic expression for the conditional probability (3.5) into (3.9), we finally obtain the local volatility at the first-order

$$(3.12) \quad (\sigma_{loc}^{\alpha\beta})^2(s, t) = \sum_{i,j=1}^n \rho_{ij}(t) \sigma_i(t) \sigma_j(t) f_{ij}(F^*, a^*) \left(1 + 2t \sum_{\mu,\nu=1}^{n+1} A^{\mu\nu} \left(\frac{\partial_{\mu\nu} f_{ij}(F^*, a^*)}{f_{ij}(F^*, a^*)} + 2 \frac{\partial_{\mu} f_{ij}(F^*, a^*)}{f_{ij}(F^*, a^*)} \frac{\partial_{\nu} \psi(F^*, a^*)}{\psi(F^*, a^*)} - \sum_{\gamma,\delta=1}^{n+1} A^{\gamma\delta} \frac{\partial_{\mu} f_{ij}(F^*, a^*)}{f_{ij}(F^*, a^*)} \partial_{\nu\gamma\delta}\phi \right) \right)$$

with $f_{ij}(F, a) = a^2 C_i(F_i) C_j(F_j) \frac{\partial s_{\alpha\beta}}{\partial F_i} \frac{\partial s_{\alpha\beta}}{\partial F_j}$, $\psi(F, a) = \sqrt{g\Delta\mathcal{P}}$ and $A^{\mu\nu} = [\partial_{\mu\nu} d^2]^{-1}$.

Note that as opposed to other asymptotic methods presented in the literature, this formula is exact at $t = 0$. The zero-order formula (independent of the time t for $\sigma_i(t)$, $\rho_{ij}(t)$ constant) was derived for a general multi-dimensional local volatility model by [4, 5]. Moreover, in the expansion, we assumed that time is small but we made no assumption that F_k is close to the spot libor or that the volatility of volatility is small.

3.4. Asymptotic Smile. The asymptotic smile can be derived in two steps from the asymptotic local volatility: first, we derive a time-homogeneous local volatility from the average local volatility over time [17]

$$(3.13) \quad ds = \frac{\sigma_{loc}^{\alpha\beta}(s, t)}{\sigma_{loc}^{\alpha\beta}(s_0, t)} \sigma_{loc}^{\alpha\beta}(s_0, t) dW_t$$

Doing a change of local time $t' = \int_0^t \sigma_{loc}^{\alpha\beta}(s_0, u)^2 du$, we now have the associated local volatility model for the swap rate

$$(3.14) \quad ds_{\alpha\beta} = \bar{\sigma}_{loc}^{\alpha\beta}(s, t) dW_t$$

$$\text{with } \bar{\sigma}_{loc}^{\alpha\beta}(s, t) = \frac{\sigma_{loc}^{\alpha\beta}(s, t)}{\sigma_{loc}^{\alpha\beta}(s_0, t)}.$$

Secondly, we know that there is a one-to-one correspondence between this local volatility and the smile [13] given at the first-order by $(C(f) \equiv \bar{\sigma}_{loc}^{\alpha\beta}(s, t = 0))$

$$(3.15) \quad \begin{aligned} \sigma_{BS}(K, T) = & \sqrt{\frac{\int_0^T \sigma_{loc}^{\alpha\beta}(s_0, u)^2 du}{T}} \frac{\ln(\frac{K}{s_0})}{\int_{s_0}^K \frac{df'}{C(f')}} \left(1 + \frac{C^2(f_{av}) \int_0^T \sigma_{loc}^{\alpha\beta}(s_0, u)^2 du}{24} \left(2 \frac{C''(f)}{C(f_{av})} - \left(\frac{C'(f_{av})}{C(f_{av})}\right)^2 + \frac{1}{f_{av}^2}\right.\right. \\ & \left. \left.+ 12 \frac{\partial_t \bar{\sigma}_{loc}^{\alpha\beta}(f_{av}, t = 0)}{C^3(f_{av})}\right) \Big|_{f_{av} = \frac{s_0 + K}{2}} \right) \end{aligned}$$

4. SABR-LMM MODEL

We have seen that the asymptotic local and implied volatilities can be computed if we know the geodesic distance and a parametrization of geodesic curves on \mathcal{M}^{n+1} . This is the case for the hyperbolic space \mathbb{H}^n for all n . This manifold has a lot of important properties. As such, it appears to be the perfect toy model (usually its Lorentzian version AdS/dS) in a number of domain: chaos, cosmology, string theory, In the first part, we present our BGM-LLM-SABR model and show that the underlying geometry is \mathbb{H}^{n+1} (with n the number of forward Libor rates). Using this connection, we will find an asymptotic local volatility and an asymptotic swaption implied volatility.

4.1. Dynamics. We introduce the model SABR-LMM, given by the following SDE under the spot Libor measure \mathbb{Q} (associated to the numéraire $B_d(t) = \prod_{j=1}^{\beta(t)-1} (1 + \tau_j F_j(T_j - 1)) P(t, T_{\beta(t)-1})$ where $\beta(t) = m$ if $T_{m-2} < t < T_{m-1}$)

$$(4.1a) \quad dF_k = a^2 B^k(F, t) dt + \sigma_k(t) a C_k(F_k) dZ_k$$

$$(4.1b) \quad da = \nu a dZ_{n+1}; \quad , \quad dZ_i dZ_j = \rho_{ij} dt \quad i, j = 1, \dots, n+1$$

with

$$(4.2) \quad C_k(F_k) = F_k^{\beta_k}$$

$$(4.3) \quad B^k(F, t) = \sum_{j=\beta(t)}^k \frac{\tau_j \rho_{jk} \sigma_k(t) \sigma_j(t) C_k(F_i) C_i(F_k)}{(1 + \tau_j F_j)}$$

The stochastic equation for a was written in the spot Libor measure in order to get a drift independent of a specific underlying swap $s_{\alpha\beta}$ or a forward bond. Under the forward swap measure $\mathbb{Q}^{\alpha\beta}$, we have

$$(4.4a) \quad dF_k = a^2 b^k(F, t) dt + \sigma_k(t) a C_k(F_k) dZ_k$$

$$(4.4b) \quad da = -\nu^2 a^2 b^a(F, t) + \nu a dZ_{n+1}; \quad , \quad dZ_i dZ_j = \rho_{ij}(t) dt \quad i, j = 1, \dots, n+1$$

| | |
|-----------------|--|
| BGM parameters | $a, b, c, d, \phi_i, \rho_L, \delta_A, \delta_B$ |
| Cev parameters | $\beta_i, i = 1, \dots, n$ |
| SABR parameters | $\alpha, \nu, \rho_{ia} \ i = 1, \dots, n,$ |

TABLE 2. SABR-LMM: $9 + 3n$ parameters

with

$$(4.5) \quad b^k(F, t) = \sum_{j=\alpha+1}^{\beta} (21_{j \leq k} - 1) \tau_j \frac{P(t, T_j)}{C_{\alpha\beta}(t)} \sum_{i=\min(k+1, j+1)}^{\max(k, j)} \frac{\tau_i \rho_{ki} \sigma_i(t) \sigma_k(t) C_i(F_i) C_k(F_k)}{(1 + \tau_i F_i)}$$

$$(4.6) \quad C_{\alpha\beta}(t) = \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)$$

$$(4.7) \quad b^a(F, t) = \sum_{i=\alpha+1}^{\beta} \tau_i \omega_i(t) \sum_{k=1}^i \frac{\tau_k C_k(F_k) \rho_{ka} \sigma_k(t)}{1 + \tau_k F_k(t)}$$

$$\text{and with } \omega_i(t) = \frac{\prod_{k=1}^i \frac{1}{(1 + \tau_k F_k)}}{\sum_{j=\alpha+1}^{\beta} \prod_{k=1}^j \frac{1}{(1 + \tau_k F_k)}}$$

Note that the forward-rate dynamics under the forward measure \mathbb{Q}^k is much simpler and given by the following stochastic differential equations (SDE)

$$(4.8) \quad dF_k(t) = \sigma_k(t) a C_k(F_k) dW_k, \quad dW_k dW_p = \rho_{kp}(t) dt$$

As it is the case for the BGM model, we can use a piecewise parametric form or a functional form for the serial volatilities $\sigma_i(t)$ and the correlation $\rho_{ij}(t)$ (here full rank) as

$$(4.9) \quad \sigma_i(t) = \phi_i [(a(T_{i-1} - t) + d) e^{-b(T_{i-1} - t)} + c] \forall t \leq T_{i-1}$$

$$(4.10) \quad \rho_{ij}(t = 0) = \rho_L + (1 - \rho_L) e^{-(\delta_A - \delta_B \min[T_{i-1}, T_j - 1]) |T_{i-1} - T_j - 1|}$$

The model depends on $9 + 3n$ parameters (see Tab. 2) which are calibrated on the swaption matrix. In the next subsection, we derive the metric, the geodesic distance and the Abelian connection underlying this model.

4.2. Hyperbolic geometry. By definition, the infinitesimal distance between the point x^α and $x^\alpha + dx^\alpha$ (3.4a) ($ds^2 = \sum_{\alpha, \beta=1}^{n+1} g_{\alpha\beta} dx^\alpha dx^\beta$) is given by ($\rho^{ij} \equiv [\rho^{-1}]_{ij}$, $(i, j) = (1, \dots, n)$ and $\rho^{ia} \equiv [\rho^{-1}]_{ia}$ are the components of the inverse of the correlation matrix ρ)

$$(4.11) \quad ds^2 = \frac{2}{\nu^2 a^2} \left(\sum_{i, j=1}^n \rho^{ij} \frac{\nu dF_i}{C_i(F_i)} \frac{\nu dF_j}{C_j(F_j)} + 2 \sum_{i=1}^n \rho^{ia} \frac{\nu dF_i}{C_i(F_i)} da + \rho^{aa} da^2 \right)$$

After some algebraic manipulations, we show that in the new coordinates $[x_k]_{k=1 \dots n+1}$ (L is the Cholesky decomposition of the (reduced) correlation matrix: $[\rho]_{i, j=1 \dots n} = [\hat{L} \hat{L}^\dagger]_{i, j=1 \dots n}$)

$$(4.12) \quad x_k = \sum_{i=1}^n \nu \hat{L}^{ki} \int_{F_i^0}^{F_i} \frac{dF'_i}{C_i(F'_i)} + \sum_{i=1}^n \rho^{ia} \hat{L}_{ik} a, \quad k = 1, \dots, n$$

$$(4.13) \quad x_{n+1} = \left(1 - \sum_{i, j} \rho^{ia} \rho^{ja} \rho_{ij} \right)^{\frac{1}{2}} a$$

the metric becomes

$$(4.14) \quad ds^2 = \frac{2(1 - \sum_{i,j}^n \rho^{ia} \rho^{ja} \rho_{ij})}{\nu^2} \frac{\sum_{i=1}^n dx_i^2 + dx_{n+1}^2}{x_{n+1}^2}$$

Written in the coordinates $[x_i]$, the metric is therefore the standard hyperbolic metric on \mathbb{H}^{n+1} (modulo a constant factor $\frac{2(1 - \sum_{i,j,k=1}^n \rho^{ia} \rho^{ja} \rho_{ij})}{\nu^2}$). In order to compute our saddle-point (3.11), we need the geodesic distance which is given by [19]

Proposition 4.2.1. *The geodesic distance $d(x, x')$ on \mathbb{H}^{n+1} is given by*

$$(4.15) \quad d(x, x^0) = \cosh^{-1} \left[1 + \frac{\sum_{i=1}^{n+1} (x_i - x_i^0)^2}{2x_{n+1}x_{n+1}^0} \right]$$

Using the geodesic distance on \mathbb{H}^{n+1} between the points $x = (\{F\}_k, a)$ and the initial point $x^0 = (\{F^0\}_k, \alpha)$ ($q_i = \int_{F_i^0}^{F_i} \frac{dF_i'}{C_i(F_i')}$) given by

$$d(x, x^0) = \frac{\sqrt{2}(1 - \sum_{i,j}^n \rho^{ia} \rho^{ja} \rho_{ij})^{\frac{1}{2}}}{\nu} \cosh^{-1} \left[1 + \frac{\nu^2 \sum_{i,j=1}^n \rho^{ij} q_i q_j + 2\nu \sum_{j=1}^n \rho^{ja} q_j + (a - \alpha)^2}{2(1 - \sum_{i,j=1}^n \rho^{ia} \rho^{ja} \rho_{ij}) a \alpha} \right]$$

we derive the following non-linear equations (3.11) satisfied by the saddle-point $a^*(s), q_i^*(s)$ which implicitly depends on s , the swaption strike:

$$(4.16a) \quad a^*(s) = \sqrt{\alpha^2 + 2\nu \sum_{i=1}^n \rho^{ia} q_i^* + \nu^2 \sum_{i,j=1}^n \rho^{ij} q_i^* q_j^*}$$

$$(4.16b) \quad (2\nu \rho^{ia} + 2 \sum_{j=1}^n \nu^2 \rho^{ij} q_j^*) \frac{d(a^*, \{q_i^*\})}{a^*(s) (\cosh(d(a^*, \{q_i^*\}))^2 - 1)^{\frac{1}{2}}} = -\lambda \alpha (1 - \sum_{i,j=1}^n \rho^{ia} \rho^{ja} \rho_{ij}) \frac{\partial s_{\alpha\beta}}{\partial q_i} \Big|_*$$

with

$$(4.17) \quad \cosh(d(a^*, \{q_i^*\})) = 1 + \frac{a^*(s) - \alpha}{(1 - \sum_{i,j=1}^n \rho^{ia} \rho^{ja} \rho_{ij}) \alpha}$$

and with

$$(4.18) \quad q_i^* = \frac{F_i^{*1-\beta_i} - F_i^{01-\beta_i}}{(1 - \beta_i)}, \quad \beta_i \neq 1$$

$$(4.19) \quad q_i^* = \ln\left(\frac{F_i^*}{F_i^0}\right), \quad \beta_i = 1$$

The saddle-point is determined by solving these non-linear equations (4.16) and an approximation (which could be used as a guess solution in a numerical optimization routine) is found by linearizing these equations around the spot Libor rates (i.e. $q_i = 0$)

$$(4.20a) \quad \lambda^* \alpha^2 = \frac{-2\nu^2 (s - s_0) - 2\nu \sum_{j=1}^n \rho_{ij} \rho^{ja} \omega_j}{(1 - \sum_{p,q=1}^n \rho^{pa} \rho^{qa} \rho_{pq}) \sum_{i,j=1}^n \rho_{ij} \omega_i \omega_j} + o((s - s_0)^2)$$

$$(4.20b) \quad q_i^*(s) = \frac{\sum_{j=1}^n \rho_{ij} \omega_j (s - s_0)}{\sum_{p,q=1}^n \omega_p \omega_q \rho_{pq}} + o((s - s_0)^2)$$

with $\omega_i \equiv \frac{\partial s_{\alpha\beta}}{\partial q_i}(q_i = 0)$. Note that when the strike is close to at-the-money, the saddle-points are close to the spot Libors and $a^* = \alpha$.

Moreover, by using the explicit expression for the hyperbolic distance, the Van-Vleck-Morette determinant is

$$(4.21) \quad \Delta(F, a, \alpha) = \frac{d(a, F|\alpha)}{\sqrt{\cosh^2(d(a, F|\alpha)) - 1}}$$

4.3. Connection. The Abelian connection is given by (3.4b) ¹

$$(4.24) \quad \mathcal{A}_i = \frac{1}{C_i(F_i)} \left[\sum_{j=1}^n \rho^{ij} \left(\frac{b^j(F, t)}{C_j(F_j)} - \frac{\partial_j C_j(F_j)}{2} \right) - \nu \rho^{ia} b^a(F, t) \right]$$

$$(4.25) \quad \mathcal{A}_a = \frac{1}{\nu} \left[\sum_{j=1}^n \rho^{aj} \left(\frac{b^j(F, t)}{C_j(F_j)} - \frac{\partial_j C_j(F_j)}{2} \right) - \nu \rho^{aa} b^a(F, t) \right]$$

where we have used that

$$(4.26) \quad \sqrt{g} = \frac{2^{\frac{n+1}{2}} \det[\rho]^{-\frac{1}{2}}}{\nu a^2 \prod_{i=1}^n C_i(F_i)}$$

Finally, the Abelian 1-form connection is

$$\mathcal{A} = \frac{1}{\nu} \sum_{j=1}^n \left[\left(\frac{b^j(F, t)}{C_j(F_j)} - \frac{\partial_j C_j(F_j)}{2} \right) (\nu \sum_{i=1}^n \rho^{ij} dq_i + \rho^{aj} da) \right] - b^a(F, t) (\nu \sum_{i=1}^n \rho^{ia} dq_i + \rho^{aa} da)$$

In order to compute the log of the parallel gauge transport $\ln(\mathcal{P})(a, q|\alpha) = \int_{\mathcal{C}} \mathcal{A}$, we need to know a parametrization of the geodesic curve on \mathbb{H}^{n+1} . However, we can directly find $\ln(\mathcal{P})(a, q|\alpha)$ if we approximate the drifts $b^k(F, t)$ by their values at the Libor spots (and $t = 0$). A similar approximation was done in the Hagan-al formula [12] as was shown in [13]. Modulo this approximation,

$$\ln(\mathcal{P})(a, q|\alpha) \sim \frac{1}{\nu} \sum_{j=1}^n \left[\left(\frac{b^j(F^0, 0)}{C_j(F_j^0)} - \frac{\partial_j C_j(F_j^0)}{2} \right) (\nu \sum_{i=1}^n \rho^{ij} q_i + \rho^{aj} (a - \alpha)) \right] - b^a(F^0, 0) (\nu \sum_{i=1}^n \rho^{ia} q_i + \rho^{aa} (a - \alpha))$$

4.4. Asymptotic Smile-Summary. The asymptotic local volatility is given by (3.12)

$$\begin{aligned} (\sigma_{loc}^{\alpha\beta})^2(s, \tau) &= \sum_{i,j=1}^n \rho_{ij} \sigma_i(t) \sigma_j(t) f_{ij}(a, F) \left(1 + 2\tau \sum_{\mu,\nu=1}^{n+1} A^{\mu\nu} \left(\frac{\partial_{\mu\nu} f_{ij}(a, F)}{f_{ij}(a, F)} + 2 \frac{\partial_{\mu} f_{ij}(a, F)}{f_{ij}(a, F)} \frac{\partial_{\nu} \psi(a, F)}{\psi(a, F)} \right) - \right. \\ &\quad \left. \sum_{\gamma,\delta=1}^{n+1} A^{\gamma\delta} \frac{\partial_{\mu} f_{ij}(a, F)}{f_{ij}(a, F)} \partial_{\nu\gamma\delta} \phi(a, F) \right) \Big|_{(a,F)=(a^*(s), F^*(s))} \end{aligned}$$

with $(a^{*2}(s), \{F_i^*\}_i(s))$ the saddle-point satisfying the equations (4.16) and approximated by (4.20) and

1

$$(4.22) \quad \mathcal{A}^i = \frac{a^2}{2} (b^i - \frac{1}{2} C_i \partial_i C_i)$$

$$(4.23) \quad \mathcal{A}^a = -\frac{\nu^2 a^2 b^a(F, t)}{2}$$

$$\begin{aligned}
f_{ij}(a, F) &= a^2 C_i(F_i) C_j(F_j) \frac{\partial s_{\alpha\beta}}{\partial F_i} \frac{\partial s_{\alpha\beta}}{\partial F_j}, \quad \psi(a, F) = \sqrt{g \Delta \mathcal{P}}, \quad A^{\alpha\beta} = [\partial_{\alpha\beta} d^2]^{-1} \\
d(a, F) &= \frac{\sqrt{2}(1 - \sum_{i,j=1}^n \rho^{ia} \rho^{ja} \rho_{ij})^{\frac{1}{2}}}{\nu} \cosh^{-1} \left[1 + \frac{\nu^2 \sum_{i,j=1}^n \rho^{ij} q_i q_j + 2\nu \sum_{j=1}^n \rho^{ja} q_j + (a - \alpha)^2}{2(1 - \sum_{i,j=1}^n \rho^{ia} \rho^{ja} \rho_{ij}) a \alpha} \right] \\
\ln(\mathcal{P})(a, q|\alpha) &\sim \frac{1}{\nu} \sum_{j=1}^n \left[\left(\frac{b^j(F^0, 0)}{C_j(F_j^0)} - \frac{\partial_j C_j(F_j^0)}{2} \right) (\nu \sum_{i=1}^n \rho^{ij} q_i + \rho^{aj} (a - \alpha)) \right] - b^a(F^0, 0) (\nu \sum_{i=1}^n \rho^{ia} q_i + \rho^{aa} (a - \alpha)) \\
\Delta(F, a, \alpha) &= \frac{d(a, F|\alpha)}{\sqrt{\cosh^2(d(a, F|\alpha)) - 1}} \\
\sqrt{g} &= \frac{2^{\frac{n+1}{2}} \det[\rho]^{-\frac{1}{2}}}{\nu a^2 \prod_{i=1}^n C_i(F_i)}
\end{aligned}$$

Note that this expression is *exact* when τ goes to zero. The smile at the first-order is then obtained by plugging the above expression into (3.15).

Remark 4.4.1 (Libor CEV model). Note that our model for ν goes to zero (and $\alpha \equiv 1$) gives the Andersen-Andreasen CEV libor model (with different CEV parameters for each libors) and the above expressions degenerates into

$$(4.27) \quad f_{ij}(F) = C_i(F_i) C_j(F_j) \frac{\partial s_{\alpha\beta}}{\partial F_i} \frac{\partial s_{\alpha\beta}}{\partial F_j}$$

$$(4.28) \quad d(F) = \sqrt{2 \sum_{i,j=1}^n \rho^{ij} q_i q_j}$$

$$(4.29) \quad \ln(\mathcal{P})(q) = \sum_{j=1}^n \left(\frac{b^j(F^0, 0)}{C_j(F_j^0)} - \frac{\partial_j C_j(F_j^0)}{2} \right) \sum_{i=1}^n \rho^{ij} q_i$$

$$(4.30) \quad \Delta(F, F^0) = 1$$

$$(4.31) \quad \sqrt{g} = \frac{2^{\frac{n}{2}} \det[\rho]^{-\frac{1}{2}}}{\prod_{i=1}^n C_i(F_i)}$$

with the saddle-points (4.16) satisfying the non-linear equations

$$(4.32) \quad \rho^{ij} q_j^* = -\frac{\lambda}{4} \frac{\partial s_{\alpha\beta}}{\partial q_i} \Big|_*$$

4.5. Comments. It is interesting to note that for $n = 1$, i.e. for a caplet, the caplet asymptotic smile reduces to the classical SABR formula by construction.

Moreover, the asymptotic local volatility is given at the zero-order by

$$\sigma_{\alpha\beta}^2(s, t) \tau = \sum_{i,j=1}^n \rho_{ij}(t) \sigma_i(t) \sigma_j(t) a^{*2}(F^*) C_i(F_i^*) C_i(F_i^*) \frac{\partial s_{\alpha\beta}}{\partial F_i}(F^*) \frac{\partial s_{\alpha\beta}}{\partial F_j}(F^*)$$

with F^* depending implicitly on s via (4.16). At this stage, it is useful to recall how a similar asymptotic local volatility is derived using the "freezing" argument. The forward swap rate satisfies the following SDE in the forward swap numéraire $\mathbb{Q}^{\alpha\beta}$

$$(4.33) \quad ds_{\alpha\beta} = \sum_{k=1}^n \frac{\partial s_{\alpha\beta}}{\partial F_k} \sigma_k(t) a C_k(F_k) dZ_k$$

The "freezing" argument consists in assuming that the terms $\frac{\partial s_{\alpha\beta}}{\partial F_k}$ and $\frac{C(s)}{C(F_i)}$ are almost constant. Therefore, the SDE (4.33) can be approximated by

$$(4.34) \quad ds_{\alpha\beta} = \sum_{k=1}^n \frac{\partial s_{\alpha\beta}}{\partial F_k}(F^0) \sigma_k(t) a \frac{C_k(F_k^0)}{C_k(s^0)} C_k(s) dZ_k$$

and the local volatility is

$$(\sigma_{loc}^{\alpha\beta})^2(s, t) = \sum_{i,j=1}^n \rho_{ij} \sigma_i(t) \sigma_j(t) a^{*2}(s) \frac{C_i(F_i^0)}{C_i(s^0)} \frac{C_j(F_j^0)}{C_j(s^0)} \frac{\partial s_{\alpha\beta}}{\partial F_i}(F^0) \frac{\partial s_{\alpha\beta}}{\partial F_j}(F^0) C_i(s) C_j(s)$$

For the swaption smile ATM, we reproduce this formula as the saddle-point Libor rates coincides with the spot rates. This is not the case in/out the money and therefore our expression (exact at the zero-order) shows that the freezing argument is no longer correct when we try to fit a swaption implied smile in/out the money.

In [13], we explained how to derive a general asymptotic smile for any stochastic volatility model using this geometric framework. As an application, we derived an asymptotic smile for a SABR model with a mean-reversion term. In the following, we try to consider some natural extensions of our SABR-BGM model where we add a non trivial drift to the volatility process. The only modification comes from the expression of the Abelian connection.

4.6. Extensions. Under the spot Libor measure, we assumed that the volatility follows the process

$$(4.35) \quad da = -\nu a^2 \psi^a(a) dt + \nu a dZ_{n+1}$$

with $\psi^a(a)$ a general analytical function of a (the scaling a^2 in front of $\psi^a(a)$ has been put for convenience). After some algebraic computations, we derive the new Abelian 1-form connection

$$\mathcal{A} = \frac{1}{\nu} \sum_{j=1}^n \left[\left(\frac{b^j(F, t)}{C_j(F_j)} - \frac{\partial_j C_j(F_j)}{2} \right) (\nu \sum_{i=1}^n \rho^{ij} dq_i + \rho^{aj} da) \right] - (b^a(F, t) + \psi(a)) (\nu \sum_{i=1}^n \rho^{ia} dq_i + \rho^{aa} da)$$

Using a similar approximation as before, i.e. $C_j(F_j) \sim C_j(F_j^0)$ and $\psi^a(a) \sim \psi^a(\alpha)$, we obtain for the parallel gauge transport

$$(4.36) \quad \begin{aligned} \ln(\mathcal{P})(a, q|\alpha) \sim & \frac{1}{\nu} \sum_{j=1}^n \left[\left(\frac{b^j(F^0, 0)}{C_j(F_j^0)} - \frac{\partial_j C_j(F_j^0)}{2} \right) (\nu \sum_{i=1}^n \rho^{ij} q_i + \rho^{aj}(a - \alpha)) \right] - (b^a(F^0, 0) \\ & + \psi(\alpha)) (\nu \sum_{i=1}^n \rho^{ia} q_i + \rho^{aa}(a - \alpha)) \end{aligned}$$

Finally, the smile is obtained using our general formula (3.15). Note that the metric and the geodesic equations remain unchanged when we only modify the drift terms.

5. CONCLUSION

In this short note, we have introduced a LMM model coupled to a SABR stochastic volatility process. By using the heat kernel expansion technique in the short time limit, we have obtained an asymptotic swaption implied volatility at the first-order, compatible with the Hagan-al classical formula for caplets. Moreover, we have seen that this exact expression (when the expiry is very short) is incompatible with the analog expression obtained using the freezing argument.

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