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# Lower central series for surface braid groups

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## Abstract

We determine the lower central series and corresponding residual properties for braid groups and pure braid groups of orientable surfaces.

## 1 Introduction

Surface braid groups are a natural generalisation of the classical braid groups (corresponding to the case where  $\Sigma$  is a disc) and of fundamental groups of surfaces (corresponding to the case  $n = 1$ ). They were first defined by Zariski during the 1930's (braid groups on the sphere had been considered earlier by Hurwitz), were

re-discovered by Fox during the 1960's, and were used subsequently in the study of mapping class groups.

We recall two definitions of surface braid groups. In Section 5.3, we shall give a third equivalent definition using mapping class groups.

**Surface braid groups via configuration space.** Let  $\Sigma$  be a connected, orientable surface, and let  $\mathcal{P} = \{p_1, \dots, p_n\}$  be a set of  $n$  distinct points (*punctures*) in the interior of  $\Sigma$ . Let  $F_n(\Sigma) = \Sigma^n \setminus \Delta$ , where  $\Delta$  is the fat diagonal, *i.e.* the set of  $n$ -tuples  $x = (x_1, \dots, x_n)$  for which  $x_i = x_j$  for some  $i \neq j$ . The fundamental group  $\pi_1(F_n(\Sigma))$  is called the *pure braid group* on  $n$  strands of the surface  $\Sigma$ ; it shall be denoted by  $P_n(\Sigma)$ . There is a natural action of the symmetric group  $S_n$  on  $F_n(\Sigma)$  by permutation of coordinates. We denote by  $\widehat{F_n(\Sigma)}$  the quotient space  $F_n(\Sigma)/S_n$ . The fundamental group  $\pi_1(\widehat{F_n(\Sigma)})$  is called the *braid group* on  $n$  strands of the surface  $\Sigma$ ; it shall be denoted by  $B_n(\Sigma)$ .

**Surface braid groups as equivalence classes of geometric braids.** A *geometric braid* on  $\Sigma$  based at  $\mathcal{P}$  is a collection  $(\psi_1, \dots, \psi_n)$  of  $n$  disjoint paths (called *strands*) on  $\Sigma \times [0, 1]$  which run monotonically with  $t \in [0, 1]$  and such that  $\psi_i(0) = (p_i, 0)$  and  $\psi_i(1) \subset \mathcal{P} \times \{1\}$ . Two braids are considered to be equivalent if they are isotopic. The usual product of paths defines a group structure on the equivalence classes of braids. This group, which is isomorphic to  $B_n(\Sigma)$ , does not depend on the choice of  $\mathcal{P}$ . A braid is said to be *pure* if  $\psi_i(1) = (p_i, 1)$  for  $i = 1, \dots, n$ . The set of pure braids form a group isomorphic to  $P_n(\Sigma)$ .

Given a group  $G$ , we define the *lower central series* of  $G$  inductively as follows: set  $\Gamma_1(G) = G$ , and for  $i \geq 2$ , define  $\Gamma_i(G) = [G, \Gamma_{i-1}(G)]$ . The group  $G$  is said to be *perfect* if  $G = \Gamma_2(G)$ . From the lower central series of  $G$  one can define another filtration  $D_1(G) \supseteq D_2(G) \supseteq \dots$  setting  $D_1(G) = G$ , and for  $i \geq 2$ , defining  $D_i(G) = \{x \in G \mid x^n \in \Gamma_i(G) \text{ for some } n \in \mathbb{N}^*\}$ . After Garoufalidis and Levine [GLe], this filtration is called *rational lower central series* of  $G$ . Following P. Hall, for any group-theoretic property  $\mathcal{P}$ , a group  $G$  is said to be *residually  $\mathcal{P}$*  if for any (non-trivial) element  $x \in G$ , there exists a group  $H$  with the property  $\mathcal{P}$  and a homomorphism  $\varphi: G \rightarrow H$  such that  $\varphi(x) \neq 1$ . It is well known that a group  $G$  is residually nilpotent if and only if  $\bigcap_{i \geq 1} \Gamma_i(G) = \{1\}$ . On the other hand, a group  $G$  is residually torsion-free nilpotent if and only if  $\bigcap_{i \geq 1} D_i(G) = \{1\}$ .

This paper deals with combinatorial properties of surface braid groups, in particular, their lower central series, and their related residual properties. In the case of the disc  $\mathbb{D}^2$  we have that  $B_n(\mathbb{D}^2)$  is residually nilpotent if and only if  $n = 2$ , and if  $n \geq 3$  then  $\Gamma_3(B_n(\mathbb{D}^2)) = \Gamma_2(B_n(\mathbb{D}^2))$  (see Proposition 4). Moreover, Gorin and Lin [GL] showed that  $\Gamma_2(B_n(\mathbb{D}^2))$  is perfect for  $n \geq 5$ .

The case of the sphere  $\mathbb{S}^2$  and the punctured sphere is currently the subject of investigation by one of the authors [GG2]: in particular  $B_n(\mathbb{S}^2)$  is residually nilpotent if and only if  $n = 2$  and for all  $n \geq 3$ ,  $\Gamma_3(B_n(\mathbb{S}^2)) = \Gamma_2(B_n(\mathbb{S}^2))$ .

Our main results, which concern orientable surfaces of genus at least one, are as follows.

**THEOREM 1** *Let  $\Sigma_g$  be a compact, connected orientable surface without boundary, of genus  $g \geq 1$ , and let  $n \geq 3$ . Then:*

- (a)  $\Gamma_1(B_n(\Sigma_g))/\Gamma_2(B_n(\Sigma_g)) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}_2$ .
- (b)  $\Gamma_2(B_n(\Sigma_g))/\Gamma_3(B_n(\Sigma_g)) \cong \mathbb{Z}_{n-1+g}$ .
- (c)  $\Gamma_3(B_n(\Sigma_g)) = \Gamma_4(B_n(\Sigma_g))$ . Moreover  $\Gamma_3(B_n(\Sigma_g))$  is perfect for  $n \geq 5$ .
- (d)  $B_n(\Sigma_g)$  is not residually nilpotent.

This implies that braid groups of compact, connected orientable surfaces without boundary may be distinguished by their lower central series (indeed by the first two lower central quotients).

**THEOREM 2** *Let  $g \geq 1$ ,  $m \geq 1$  and  $n \geq 3$ . Let  $\Sigma_{g,m}$  be a compact, connected orientable surface of genus  $g$  with  $m$  boundary components. Then:*

- (a)  $\Gamma_1(B_n(\Sigma_{g,m}))/\Gamma_2(B_n(\Sigma_{g,m})) = \mathbb{Z}^{2g+m-1} \oplus \mathbb{Z}_2$ .
- (b)  $\Gamma_2(B_n(\Sigma_{g,m}))/\Gamma_3(B_n(\Sigma_{g,m})) = \mathbb{Z}$ .
- (c)  $\Gamma_3(B_n(\Sigma_{g,m})) = \Gamma_4(B_n(\Sigma_{g,m}))$ . Moreover  $\Gamma_3(B_n(\Sigma_{g,m}))$  is perfect for  $n \geq 5$ .
- (d)  $B_n(\Sigma_{g,m})$  is not residually nilpotent.

Braid groups on 2 strands represent a very difficult and interesting case. In the case of the torus, we are able to prove that its 2-strand braid group is residually nilpotent. Further, using ideas from [GG2] and results of [Ga], we may show that apart from the first term, the lower central series of  $B_2(\mathbb{T}^2)$  and  $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$  coincide, and we may also determine all of their lower central quotients. More precisely:

**THEOREM 3**

- (a)  $B_2(\mathbb{T}^2)$  is residually nilpotent.
- (b) For all  $i \geq 2$ :
  - (i)  $\Gamma_i(B_2(\mathbb{T}^2)) \cong \Gamma_i(\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2)$ ;
  - (ii)  $\Gamma_i(B_2(\mathbb{T}^2))/\Gamma_{i+1}(B_2(\mathbb{T}^2))$  is isomorphic to the direct sum of  $R_i$  copies of  $\mathbb{Z}_2$ , where:

$$R_i = \sum_{j=1}^{i-2} \left( \sum_{\substack{k|i-j \\ k>1}} \mu \left( \frac{i-j}{k} \right) \frac{k\alpha_k}{i-j} \right) \quad \text{and} \quad k\alpha_k = 2^k + 2(-1)^k.$$

(c)  $B_2(\mathbb{T}^2)$  is not residually torsion-free nilpotent.

Finally, as we shall see in Proposition 13,  $B_2(\mathbb{T}^2)$  is not bi-orderable (see Section 2 for a definition).

In Section 5, we recall the relations between mapping class groups and surface braid groups, and prove that pure braid groups of the torus and of surfaces with boundary components are residually torsion-free nilpotent. This is achieved by showing that they may be realised as subgroups of the Torelli group of a surface of higher genus (Lemma 19), which is known to be residually torsion-free nilpotent. In the Appendix, we provide a short proof of this latter fact (Theorem 25).

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## 2 Lower central series for Artin-Tits groups

Let us start by recalling some standard results on combinatorial properties of braid groups. The following result is well known (see [GL] for instance).

**PROPOSITION 4** *Let  $B_n$  be the Artin braid group on  $n \geq 3$  strands. Then  $\Gamma_1(B_n)/\Gamma_2(B_n) \cong \mathbb{Z}$  and  $\Gamma_2(B_n) = \Gamma_3(B_n)$ .*

*Proof.* Let us give an easy proof of the second statement (we use an argument of [GG2]). Let  $\{\sigma_1, \dots, \sigma_{n-1}\}$  be the usual set of generators of  $B_n$ ; the classical relations of  $B_n$ , referred to hereafter as *braid relations*, are as follows:

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \text{ for all } 1 \leq i, j \leq n-1 \text{ and } |i-j| \geq 2, \quad (1)$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for all } 1 \leq i \leq n-2. \quad (2)$$

From this, we see that  $B_n/\Gamma_2(B_n)$  is isomorphic to  $\mathbb{Z}$ .

Consider the following short exact sequence:

$$1 \longrightarrow \frac{\Gamma_2(B_n)}{\Gamma_3(B_n)} \longrightarrow \frac{B_n}{\Gamma_3(B_n)} \xrightarrow{p} \frac{B_n}{\Gamma_2(B_n)} \longrightarrow 1,$$

Since all of the  $\sigma_i \in B_n/\Gamma_3(B_n)$  project to the same element of  $B_n/\Gamma_2(B_n)$ , for each  $1 \leq i \leq n-1$ , there exists  $t_i \in \Gamma_2(B_n)/\Gamma_3(B_n)$  (with  $t_1 = 1$ ) such that  $\sigma_i = t_i \sigma_1$ . Projecting the braid relation (2) into  $B_n/\Gamma_3(B_n)$ , we see that  $t_i \sigma_1 t_{i+1} \sigma_1 t_i \sigma_1 = t_{i+1} \sigma_1 t_i \sigma_1 t_{i+1} \sigma_1$ . But the  $t_i$  are central in  $B_n/\Gamma_3(B_n)$ , so  $t_i = t_{i+1}$ , and since  $t_1 = 1$ , we obtain  $\sigma_1 = \dots = \sigma_{n-1}$ . So the surjective homomorphism  $p$  is in fact an isomorphism. ■

REMARKS 5 We recall that classical braid groups are also called Artin-Tits groups of type  $\mathcal{A}$ . Considering the standard group presentations (see for instance [CP]) for Artin-Tits groups and using the same argument as in the above proof, it is easy to show that the lower central series of Artin-Tits groups of type  $\mathcal{B}$  and  $\mathcal{D}$  and of Artin-Tits groups  $E_6$  and  $E_7$  also stabilise at the second term.

REMARKS 6 (a) The following groups are perfect:  $\Gamma_2(B_n)$  for  $n \geq 5$  [GL], and the mapping class group of a compact, connected orientable surface without boundary of genus  $g \geq 3$  [Po].

(b) Given a group  $G$ , the property that the  $i$ th term  $\Gamma_i(G)$  is perfect implies that  $\Gamma_i(G) = \Gamma_{i+1}(G)$ .

It is well known that the pure braid groups  $P_n$  are residually nilpotent. In fact,  $P_n$  is residually torsion-free nilpotent [FR]. Using the faithfulness of the Krammer-Digne representation, Marin has shown recently that the pure Artin-Tits groups of type  $\mathcal{A}$  (the pure braid groups),  $\mathcal{B}$  and  $\mathcal{D}$  are residually torsion-free nilpotent [M].

The fact that a group is residually torsion-free nilpotent has several important consequences, notably that the group is bi-orderable [MR]. We recall that a group  $G$  is said to be *bi-orderable* if there exists a strict total ordering  $<$  on its elements which is invariant under left and right multiplication, in other words,  $g < h$  implies that  $gk < hk$  and  $kg < kh$  for all  $g, h, k \in G$ . We state one interesting property of bi-orderable groups. A group  $G$  is said to have *generalised torsion* if there exist  $g, h_1, \dots, h_k$ , ( $g \neq 1$ ) such that:

$$(h_1gh_1^{-1})(h_2gh_2^{-1}) \cdots (h_kgh_k^{-1}) = 1.$$

PROPOSITION 7 ([GO]) *A bi-orderable group has no generalised torsion.*

The braid group  $B_n$  is not bi-orderable for  $n \geq 3$  since it has generalised torsion [N]. As we shall see in Section 4,  $B_2(\mathbb{T}^2)$  is residually nilpotent, but is not bi-orderable.

### 3 Lower central series for surface braid groups on at least 3 strands

#### 3.1 Surfaces without boundary

This section is devoted to proving Theorem 1. Let  $\Sigma_g$  be a compact, connected orientable surface without boundary, of genus  $g > 0$ . We start by giving a presentation of  $B_n(\Sigma_g)$ .

**THEOREM 8** ([B]) *Let  $n \in \mathbb{N}$ . Then  $B_n(\Sigma_g)$  admits the following group presentation:*

**Generators:**  $a_1, b_1, \dots, a_g, b_g, \sigma_1, \dots, \sigma_{n-1}$ .

**Relations:**

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2 \quad (3)$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for all } 1 \leq i \leq n - 2 \quad (4)$$

$$c_i \sigma_j = \sigma_j c_i \text{ for all } j \geq 2, c_i = a_i \text{ or } b_i \text{ and } i = 1, \dots, g \quad (5)$$

$$c_i \sigma_1 c_i \sigma_1 = \sigma_1 c_i \sigma_1 c_i \text{ for } c_i = a_i \text{ or } b_i \text{ and } i = 1, \dots, g \quad (6)$$

$$a_i \sigma_1 b_i = \sigma_1 b_i \sigma_1 a_i \text{ for } i = 1, \dots, g \quad (7)$$

$$c_i \sigma_1^{-1} c_j \sigma_1 = \sigma_1^{-1} c_j \sigma_1 c_i \text{ for } c_i = a_i \text{ or } b_i, c_j = a_j \text{ or } b_j \text{ and } 1 \leq j < i \leq g \quad (8)$$

$$\prod_{i=1}^g [a_i^{-1}, b_i] = \sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1. \quad (9)$$

*Proof.* Let  $\widetilde{B}_n(\Sigma_g)$  be the group defined by the above presentation, and let  $B_n(\Sigma_g)$  be the group given by the presentation of Theorem 1.2 of [B]. Consider the homomorphism  $\varphi: B_n(\Sigma_g) \rightarrow \widetilde{B}_n(\Sigma_g)$  defined on the generators of  $B_n(\Sigma_g)$  by  $\varphi(\sigma_j) = \sigma_j$  (for  $j = 1, \dots, n-1$ ),  $\varphi(a_i) = a_i^{-1}$  and  $\varphi(b_i) = b_i^{-1}$  (for  $i = 1, \dots, g$ ). It is an easy exercise to check that  $\varphi$  is an isomorphism.  $\blacksquare$

*Proof of Theorem 1.*

(a) Consider the group  $\mathbb{Z}^{2g} \oplus \mathbb{Z}_2$  defined by the presentation  $\langle c_1, \dots, c_{2g}, \sigma \mid \sigma^2 = [c_i, c_j] = [c_i, \sigma] = 1, \text{ for } 1 \leq i, j \leq 2g \rangle$  and  $B_n(\Sigma_g)$  with the group presentation given by Theorem 8. It is easy to check that the homomorphism

$$\varphi: \Gamma_1(B_n(\Sigma_g)) / \Gamma_2(B_n(\Sigma_g)) \rightarrow \mathbb{Z}^{2g} \oplus \mathbb{Z}_2,$$

which sends  $a_k$  to  $c_{2k-1}$ ,  $b_k$  to  $c_{2k}$  and every  $\sigma_j$  to  $\sigma$  is indeed an isomorphism.

(b) Let us start by determining a group presentation for  $B_n(\Sigma_g) / \Gamma_3(B_n(\Sigma_g))$ . Let  $q$  be the canonical projection of  $B_n(\Sigma_g)$  onto  $B_n(\Sigma_g) / \Gamma_3(B_n(\Sigma_g))$ . As in the proof of Proposition 4, the braid relations (4) imply that  $q(\sigma_1) = \cdots = q(\sigma_{n-1})$ ; we denote this element by  $\sigma$ . This implies that the projected relations (3) are trivial. For  $i = 1, \dots, g$ , let us also denote  $q(a_i)$  by  $a_i$  and  $q(b_i)$  by  $b_i$ . Since  $n \geq 3$ , we see from relations (5) that  $\sigma$  is central in  $B_n(\Sigma_g) / \Gamma_3(B_n(\Sigma_g))$  and hence the projected relations (6) become trivial. From relations (8), for all  $1 \leq i, j \leq g, i \neq j$ , one may infer that  $[a_i, b_j] = [a_i, a_j] = [b_i, b_j] = [b_i, a_j] = 1$  in  $B_n(\Sigma_g) / \Gamma_3(B_n(\Sigma_g))$ . Relations (7) and (9) imply that  $[b_i, a_i] = \sigma^{-2}$  for all  $i = 1, \dots, g$ , and  $\prod_{i=1}^g [a_i^{-1}, b_i] = \sigma^{2(n-1)}$  respectively. Conjugating the latter equation by  $a_1 \cdots a_g$  yields  $\prod_{i=1}^g [b_i, a_i] = \sigma^{2(n-1)}$  in  $B_n(\Sigma_g) / \Gamma_3(B_n(\Sigma_g))$  (recall that  $a_i$  commutes with  $a_j$  and  $b_j$  if  $i \neq j$ ), and hence  $\sigma^{-2g} = \prod_{i=1}^g [b_i, a_i] = \sigma^{2(n-1)}$ . Therefore,  $\sigma^{2(g+n-1)} = 1$  in  $B_n(\Sigma_g) / \Gamma_3(B_n(\Sigma_g))$ .

Summing up, we have obtained the following information:

$$\left. \begin{array}{l} B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g)) \text{ is generated by } a_1, b_1, \dots, a_g, b_g \text{ and } \sigma \\ a_1, b_1, \dots, a_g, b_g \text{ and } \sigma \text{ commute pairwise except for the pairs } (a_i, b_i)_{i=1, \dots, g} \\ [a_1, b_1] = \dots = [a_g, b_g] = \sigma^2; \quad \sigma^{2(n+g-1)} = 1. \end{array} \right\} (10)$$

The remaining relations of  $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$  are those of the form  $[[x, y], z] = 1$  for all  $x, y, z \in B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ . We claim that such relations are implied by those of (10). To see this, recall that  $\Gamma_2(B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g)))$  is the normal subgroup of  $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$  generated by the finite set of commutators  $[a_i, a_j]$ ,  $[b_i, b_j]$ ,  $[a_i, b_j]$ ,  $[a_i, b_i]$ ,  $[a_i, \sigma]$  and  $[b_i, \sigma]$ , for  $1 \leq i \neq j \leq g$ . But the relations of (10) imply that these commutators are all trivial, with the exception of  $[a_i, b_i]$  for  $1 \leq i \leq g$ , which is equal to  $\sigma^2$ . Since  $\sigma$  is central in  $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ , we conclude that  $\Gamma_2(B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))) = \langle \sigma^2 \rangle$ , and that  $[[x, y], z] = 1$  as claimed. Hence (10) is a group presentation for  $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ .

Now consider the following exact sequence:

$$1 \longrightarrow \frac{\Gamma_2(B_n(\Sigma_g))}{\Gamma_3(B_n(\Sigma_g))} \longrightarrow \frac{B_n(\Sigma_g)}{\Gamma_3(B_n(\Sigma_g))} \xrightarrow{p} \frac{B_n(\Sigma_g)}{\Gamma_2(B_n(\Sigma_g))} \longrightarrow 1.$$

From the presentation of  $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$  given by (10), one sees that every element  $x$  of  $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$  may be written in the form  $a_1^{j_1} b_1^{k_1} \dots a_g^{j_g} b_g^{k_g} \sigma^p$ . Since  $B_n(\Sigma_g)/\Gamma_2(B_n(\Sigma_g))$  is isomorphic to  $\mathbb{Z}^{2g} \oplus \mathbb{Z}_2$ , the factors being generated respectively by  $p(a_1), p(b_1), \dots, p(a_g), p(b_g)$  and  $p(\sigma)$ , if  $x \in \text{Ker}(p)$  then  $j_1 = k_1 = \dots = j_g = k_g = 0$  and  $p$  is even, so  $\text{Ker}(p) \subseteq \langle \sigma^2 \rangle$ . The converse is clearly true and so  $\text{Ker}(p) = \langle \sigma^2 \rangle$ .

Let  $d$  denote the order of  $\sigma^2$  in  $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ . From (10), we have that  $d$  divides  $n + g - 1$ . To complete the proof of part (b) of Theorem 1, it suffices to show that  $n + g - 1$  divides  $d$ .

Let  $G$  be the group generated by elements  $a_1, b_1, \dots, a_g, b_g$  and  $\sigma$ , whose relations are  $\sigma^{2(n+g-1)} = 1$ , and the generators commute pairwise except for the pairs  $(a_i, b_i)$  for  $i = 1, \dots, g$ . Then  $G = (\oplus_{i=1}^g \mathbb{F}_2(a_i, b_i)) \oplus \mathbb{Z}_{2(n+g-1)}$ , where  $\mathbb{F}_2(a_i, b_i)$  denotes the free group of rank 2 generated by  $a_i$  and  $b_i$ . Let  $N$  be the subgroup of  $G$  normally generated by the elements  $[a_1, b_1]\sigma^{-2}, \dots, [a_g, b_g]\sigma^{-2}$ , and let  $\rho$  denote the canonical projection  $G \longrightarrow G/N$ . Then  $G/N \cong B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$  by the group presentation given in (10). The cosets modulo  $N$  of the elements  $a_1, b_1, \dots, a_g, b_g$  and  $\sigma$  of  $G$  may be identified respectively with the elements  $a_1, b_1, \dots, a_g, b_g$  and  $\sigma$  of  $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ . Further, by applying the relations of  $G$ , any element of  $N$  may be written in the form  $\prod_{i=1}^g \left( \prod_{k=1}^{m_i} u_{i_k} [a_i, b_i]^{\varepsilon_{i_k}} u_{i_k}^{-1} \right) \sigma^{-2(\sum_{i=1}^g (\sum_{k=1}^{m_i} \varepsilon_{i_k}))}$ , where  $m_i \in \mathbb{N}$  for all  $i = 1, \dots, g$ , and for all  $k = 1, \dots, m_i$ ,  $u_{i_k} \in \mathbb{F}_2(a_i, b_i)$  and  $\varepsilon_{i_k} \in \{1, -1\}$ . Since  $\sigma^{2d} = 1$  in  $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ , and so in  $G/N$ , considered as an element of  $G$ , it follows that  $\sigma^{2d}$  belongs to  $\text{Ker}(\rho)$ . Hence for all  $i = 1, \dots, g$ , there



exists  $m_i \in \mathbb{N}$ , and for  $1 \leq k \leq m_i$ , there exist  $u_{i_k} \in \mathbb{F}_2(a_i, b_i)$  and  $\varepsilon_{i_k} \in \{1, -1\}$  such that  $\sigma^{2d} = \prod_{i=1}^g \left( \prod_{k=1}^{m_i} u_{i_k} [a_i, b_i]^{\varepsilon_{i_k}} u_{i_k}^{-1} \right) \sigma^{-2(\sum_{i=1}^g (\sum_{k=1}^{m_i} \varepsilon_{i_k}))}$ . Thus:

$$\sigma^{2(d + \sum_{i=1}^g (\sum_{k=1}^{m_i} \varepsilon_{i_k}))} = \prod_{i=1}^g \left( \prod_{k=1}^{m_i} u_{i_k} ([a_i, b_i]^{\varepsilon_{i_k}} u_{i_k}^{-1}) \right). \quad (11)$$

From the structure of  $G$ , it follows that both the right- and left-hand sides are equal to 1. Moreover,  $\Gamma_2(G) = \bigoplus_{i=1}^g \Gamma_2(\mathbb{F}_2(a_i, b_i))$ . Let  $1 \leq i \leq g$ . Projecting the right-hand side of equation (11), which belongs to  $\Gamma_2(G)$ , into  $\Gamma_2(\mathbb{F}_2(a_i, b_i))$ , and then into  $\Gamma_2(\mathbb{F}_2(a_i, b_i))/\Gamma_3(\mathbb{F}_2(a_i, b_i))$ , we observe that  $[a_i, b_i]^{\sum_{k=1}^{m_i} \varepsilon_{i_k}} = 1$ . But this quotient is an infinite cyclic group [MKS], hence  $\sum_{k=1}^{m_i} \varepsilon_{i_k} = 0$  for  $i = 1, \dots, g$  and therefore  $\sum_{i=1}^g (\sum_{k=1}^{m_i} \varepsilon_{i_k}) = 0$ . Thus the left-hand side of equation (11) reduces to  $\sigma^{2d} = 1$  in  $G$ , and so  $n + g - 1$  divides  $d$ . It follows that  $\sigma$  is of order  $2(n + g - 1)$  in  $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$  as claimed.

(c) Let  $H$  denote the normal closure in  $B_n(\Sigma_g)$  of the element  $\sigma_1 \sigma_2^{-1}$ . Using the Artin braid relations, one may check that in  $B_n(\Sigma_g)/H$ , the  $\sigma_i$  are all identified to a single element,  $\sigma$ , say, and then that equation (10) defines a group presentation for  $B_n(\Sigma_g)/H$ . Thus  $B_n(\Sigma_g)/H \cong B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$  via an isomorphism  $\iota$ . Now  $B_n(\Sigma_g)$  contains a copy of the usual Artin braid group  $B_n$  which is generated by the  $\sigma_i$ . From the Artin braid relations, it follows that  $\Gamma_2(B_n)$  is the normal closure in  $B_n$  of the elements  $\sigma_i \sigma_{i+1}^{-1}$ ,  $1 \leq i \leq n - 2$ . Moreover, since  $\sigma_{i+1} \sigma_{i+2}^{-1} = \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_{i+2}^{-1} \cdot \sigma_i \sigma_{i+1}^{-1} \cdot \sigma_{i+2} \sigma_{i+1} \sigma_i$  for all  $1 \leq i \leq n - 3$ , we see that  $\Gamma_2(B_n)$  is the normal closure in  $B_n$  of just  $\sigma_1 \sigma_2^{-1}$ , and thus  $\mathcal{N}_{B_n(\Sigma_g)}(\Gamma_2(B_n)) = H$  (if  $X$  is a subset of a group  $G$ , then we denote its normal closure in  $G$  by  $\mathcal{N}_G(X)$ ).

Since  $\Gamma_3(B_n) = \Gamma_2(B_n)$  (by Proposition 4), we have  $\Gamma_4(B_n(\Sigma_g)) \supseteq \Gamma_4(B_n) = \Gamma_2(B_n)$ . Taking normal closures in  $B_n(\Sigma_g)$ , we deduce that  $H$  is a normal subgroup of  $\Gamma_4(B_n(\Sigma_g))$ , and hence we obtain the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \Gamma_4(B_n(\Sigma_g))/H & \longrightarrow & B_n(\Sigma_g)/H & \longrightarrow & B_n(\Sigma_g)/\Gamma_4(B_n(\Sigma_g)) \longrightarrow 1 \\ & & & & \searrow \cong \iota & & \downarrow \\ & & & & & & B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g)). \end{array}$$

Since  $\iota$  is an isomorphism, so is the vertical arrow, and hence its kernel  $\Gamma_3(B_n(\Sigma_g))/\Gamma_4(B_n(\Sigma_g))$  is trivial. This proves the first part of (c). To prove the second part, we have just seen that the normal closure  $H$  of  $\Gamma_2(B_n)$  in  $B_n(\Sigma_g)$ , is isomorphic to  $\Gamma_3(B_n(\Sigma_g))$  (they coincide in fact). Since  $\Gamma_2(B_n)$  is perfect for all  $n \geq 5$  [GL], so are  $H$  and  $\Gamma_3(B_n(\Sigma_g))$ .

(d) We first remark that  $\Gamma_3(B_n(\Sigma_g)) \neq \{1\}$ . For if  $\Gamma_3(B_n(\Sigma_g))$  were trivial, by (c), we would have  $\Gamma_2(B_n(\Sigma_g)) \cong \mathbb{Z}_{n-1+g}$ . But by [VB],  $B_n(\Sigma_g)$  is torsion free, which yields a contradiction. From this it follows that  $\bigcap_{i \in \mathbb{N}} \Gamma_i(B_n(\Sigma_g)) \neq \{1\}$ . This completes the proof of the theorem.

■

### 3.2 Surfaces with non-empty boundary

In this section, we study the case of orientable surfaces with boundary, and prove Theorem 2. We identify  $\Sigma_{g,0}$  with  $\Sigma_g$ . As in Theorem 8, from Theorem 1.1 of [B], one obtains the following presentation of  $B_n(\Sigma_{g,m})$ .

**THEOREM 9** *Let  $n \in \mathbb{N}$ . Then  $B_n(\Sigma_{g,m})$  admits the following group presentation:*

**Generators:**  $a_1, b_1, \dots, a_g, b_g, z_1, \dots, z_{m-1}, \sigma_1, \dots, \sigma_{n-1}$ .

**Relations:**

*Relations (3) – (8) of Theorem 8*

$$z_i \sigma_j = \sigma_j z_i \text{ for all } j \geq 2 \text{ and } i = 1, \dots, m-1 \quad (12)$$

$$z_i \sigma_1 z_i \sigma_1 = \sigma_1 z_i \sigma_1 z_i \text{ for } i = 1, \dots, m-1 \quad (13)$$

$$z_i \sigma_1^{-1} z_j \sigma_1 = \sigma_1^{-1} z_j \sigma_1 z_i \text{ for } 1 \leq j < i \leq m-1 \quad (14)$$

$$c_i \sigma_1^{-1} z_j \sigma_1 = \sigma_1^{-1} z_j \sigma_1 c_i \text{ for } c_i = a_i \text{ or } b_i, i = 1, \dots, g \text{ and } j = 1, \dots, m-1. \quad (15)$$

*Proof of Theorem 2.* Statement (a) may be proved in the same way as (a) of Theorem 1.

We now prove part (b). As in the proof of part (b) of Theorem 1, one may check that  $\Gamma_2(B_n(\Sigma_{g,m}))/\Gamma_3(B_n(\Sigma_{g,m})) = \langle \sigma^2 \rangle$ , where for all  $i = 1, \dots, n-1$ ,  $\sigma$  is the projection of  $\sigma_i$  in  $B_n(\Sigma_{g,m})/\Gamma_3(B_n(\Sigma_{g,m}))$ . It thus suffices to show that  $\sigma^2$  is of infinite order.

Instead of repeating the arguments used in Theorem 1, we propose a different proof, based on geometric relations between surface braid groups. Suppose that  $\sigma^{2d} = 1$  for some  $d \in \mathbb{N}$ . This is equivalent to saying that  $\sigma_i^{2d}$  belongs to  $\Gamma_3(B_n(\Sigma_{g,m}))$  for all  $i = 1, \dots, n-1$ .

Let  $1 \leq i \leq m$ . To each boundary component  $\partial_i$  of  $\Sigma_{g,m}$  let us associate a surface  $\Sigma_{g_i,1}$  of positive genus  $g_i$ . We choose the  $g_i$  so that  $h = g + \sum_{i=1}^m g_i > d - (n-1)$ . Let  $\Sigma_h$  denote the compact, orientable surface without boundary and of genus  $h$  obtained by glueing  $\partial \Sigma_{g_i,1}$  to  $\partial_i$  for all  $i = 1, \dots, m$ . The embedding of  $\Sigma_{g,m}$  into  $\Sigma_h$  induces a natural homomorphism  $\varphi$  between  $B_n(\Sigma_{g,m})$  and  $B_n(\Sigma_h)$ , sending geometric generators of  $B_n(\Sigma_{g,m})$  to the corresponding elements of  $B_n(\Sigma_h)$ . In particular,  $\varphi(\sigma_i) = \sigma_i$  for all  $i = 1, \dots, n-1$ .

Since  $\sigma_i^{2d}$  belongs to  $\Gamma_3(B_n(\Sigma_{g,m}))$ , it follows that  $\varphi(\sigma_i^{2d}) = \sigma_i^{2d}$  belongs to  $\Gamma_3(B_n(\Sigma_h))$ , and hence  $\sigma^{2d} = 1$  in  $\Gamma_2(B_n(\Sigma_h))/\Gamma_3(B_n(\Sigma_h))$  (recall that, by Theorem 1,  $\Gamma_2(B_n(\Sigma_h))/\Gamma_3(B_n(\Sigma_h)) = \langle \sigma^2 \rangle \cong \mathbb{Z}_{h+n-1}$ ). But this would imply that  $h+n-1 \leq d$  – a contradiction. This proves part (b).

Part (c) may be proved in the same way as (c) of Theorem 1; indeed, the quotient  $B_n(\Sigma_{g,m})/\Gamma_3(B_n(\Sigma_{g,m}))$  has a presentation similar to that of (10) (with the  $z_i$  central, but without the last relation), and is isomorphic to  $B_n(\Sigma_{g,m})/H$ , where  $H$  is the normal closure of  $\sigma_1\sigma_2^{-1}$  in  $B_n(\Sigma_{g,m})$ , and thus is equal to the normal closure of  $\Gamma_2(B_n)$  in  $B_n(\Sigma_{g,m})$ . As in Theorem 1, one may show that  $H = \Gamma_3(B_n(\Sigma_{g,m}))$  is perfect for  $n \geq 5$ .

Finally, to prove part (d), as in Theorem 1 it suffices to prove that  $\Gamma_3(B_n(\Sigma_{g,m})) \neq \{1\}$ . Suppose that  $\Gamma_3(B_n(\Sigma_{g,m})) = \{1\}$ . Then  $\Gamma_2(B_n(\Sigma_{g,m})) \cong \mathbb{Z}$  by (b), and since  $B_n(\Sigma_{g,m}) \supset B_n$ , it follows that  $\Gamma_2(B_n)$  is cyclic; but since  $n \geq 3$ , this contradicts the results of [GL]. ■

## 4 Braid groups on 2 strands: properties and open questions

The aim of this section is to prove Theorem 3. Consider first the group presentation given by Theorem 8, and take  $n = 2$  and  $g = 1$ . Setting  $\alpha = a\sigma_1$ ,  $\beta = b\sigma_1$  and  $\gamma = a\sigma_1b$ , one obtains the following presentation of  $B_2(\mathbb{T}^2)$ :

**THEOREM 10** ([BG])  *$B_2(\mathbb{T}^2)$  is generated by  $\alpha$ ,  $\beta$  and  $\gamma$ , subject to the relations:*

$$\begin{aligned} \alpha^2 \text{ and } \beta^2 \text{ are central} \\ \alpha^2\beta^2 = \gamma^2. \end{aligned}$$

*Further,  $\alpha^2$  and  $\beta^2$  generate the centre of  $B_2(\mathbb{T}^2)$ .*

Let  $p: B_2(\mathbb{T}^2) \rightarrow B_2(\mathbb{T}^2)/Z(B_2(\mathbb{T}^2))$  denote the canonical projection. From this presentation, it follows that  $B_2(\mathbb{T}^2)/Z(B_2(\mathbb{T}^2))$  is generated by  $\bar{\alpha} = p(\alpha)$ ,  $\bar{\beta} = p(\beta)$  and  $\bar{\gamma} = p(\gamma)$ , subject to the relations  $\bar{\alpha}^2 = \bar{\beta}^2 = \bar{\gamma}^2 = 1$ . So  $B_2(\mathbb{T}^2)/Z(B_2(\mathbb{T}^2))$ , which we identify with  $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ , is the Coxeter group  $W(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$  associated to the free group  $\mathbb{F}_3(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ , and  $B_2(\mathbb{T}^2)$  is a central extension of  $W(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ :

$$1 \longrightarrow Z(B_2(\mathbb{T}^2)) \longrightarrow B_2(\mathbb{T}^2) \xrightarrow{p} \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \longrightarrow 1.$$

This presentation of  $B_2(\mathbb{T}^2)$  was considered in [BG], where the following length functions  $\ell_{\hat{\alpha}}, \ell_{\hat{\beta}}$  were defined. If  $x \in \{\alpha, \beta, \gamma\}$ , set:

$$\ell_{\hat{\alpha}}(x) = \begin{cases} 1 & \text{if } x \neq \alpha \\ 0 & \text{if } x = \alpha, \end{cases}$$

and similarly for  $\ell_{\hat{\beta}}$ . From Theorem 10, it follows that each of  $\ell_{\hat{\alpha}}$  and  $\ell_{\hat{\beta}}$  extends to a homomorphism of  $B_2(\mathbb{T}^2)$  onto  $\mathbb{Z}$ .

The following observation will be used in the proof of Theorem 3.

PROPOSITION 11 *The intersection of  $\Gamma_2(B_2(\mathbb{T}^2))$  and  $Z(B_2(\mathbb{T}^2))$  is trivial.*

*Proof.* Let  $x \in Z(B_2(\mathbb{T}^2))$ . By Theorem 10, there exist  $m, n \in \mathbb{Z}$  such that  $x = a^{2m}b^{2n}$ , and thus  $\ell_{\hat{\alpha}}(x) = 2n$  and  $\ell_{\hat{\beta}}(x) = 2m$ . But  $x \in \Gamma_2(B_2(\mathbb{T}^2))$ , so  $\ell_{\hat{\alpha}}(x) = \ell_{\hat{\beta}}(x) = 0$ . We conclude that  $m = n = 0$ , and hence  $x = 1$ .  $\blacksquare$

We are now able to prove Theorem 3.

*Proof of Theorem 3.* Set  $G = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ .

(a) Suppose that  $x \in \bigcap_{i \in \mathbb{N}} \Gamma_i(B_2(\mathbb{T}^2))$ . Then  $p(x) \in \bigcap_{i \in \mathbb{N}} \Gamma_i(G)$ , but since  $G$  is residually nilpotent [G], it follows that  $x \in \text{Ker}(p) = Z(B_2(\mathbb{T}^2))$ . So  $x = 1$  by Proposition 11, and hence  $B_2(\mathbb{T}^2)$  is residually nilpotent.

(b) (i) Let us consider the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \Gamma_2(B_2(\mathbb{T}^2)) & \longrightarrow & B_2(\mathbb{T}^2) & \longrightarrow & B_n(\mathbb{T}^2)/\Gamma_2(B_n(\mathbb{T}^2)) \longrightarrow 1 \\ & & \downarrow p_2 & & \downarrow p & & \downarrow \\ 1 & \longrightarrow & \Gamma_2(G) & \longrightarrow & G & \longrightarrow & G/\Gamma_2(G) \longrightarrow 1 \end{array}$$

The first and third vertical arrows are those induced by  $p$ . More generally, for  $i \geq 2$ , let  $p_i: \Gamma_i(B_2(\mathbb{T}^2)) \rightarrow \Gamma_i(G)$  denote the epimorphism induced by  $p$ . But it follows from Proposition 11 that  $p_2$  is also injective, so is an isomorphism. Since for  $i \geq 3$ ,  $p_i$  is the restriction of  $p_2$  to  $\Gamma_i(B_2(\mathbb{T}^2))$ ,  $p_i$  is an isomorphism too.

(ii) From (b)(i), it follows that  $\Gamma_i(B_2(\mathbb{T}^2))/\Gamma_{i+1}(B_2(\mathbb{T}^2)) \cong \Gamma_i(G)/\Gamma_{i+1}(G)$ , so it suffices to prove the result for  $G$ . We break the proof down into two parts as follows:

(1). Recall that the elements  $\bar{\alpha}, \bar{\beta}$  and  $\bar{\gamma}$  are each of order 2, and generate  $G$ . We claim that every non-trivial element of  $\Gamma_i(G)/\Gamma_{i+1}(G)$  is of order 2. Since  $\Gamma_i(G)/\Gamma_{i+1}(G)$  is a finitely-generated Abelian group by [MKS], this will imply that it is isomorphic to a finite number,  $R_i$  say, of copies of  $\mathbb{Z}_2$ . To prove the claim, recall from [MKS] that  $\Gamma_i(G)/\Gamma_{i+1}(G)$  is generated by the cosets modulo  $\Gamma_{i+1}(G)$  of the  $i$ -fold simple commutators  $[[\cdot \cdot \cdot [[\rho_1, \rho_2], \rho_3] \cdot \cdot \cdot, \rho_{i-1}], \rho_i]$ , where  $\rho_j \in \{\bar{\alpha}, \bar{\beta}, \bar{\gamma}\}$  for all  $1 \leq j \leq i$ . We argue by induction on  $i \geq 2$ . Firstly, let  $i = 2$ . Then  $\Gamma_2(G)/\Gamma_3(G)$  is generated by the cosets of the  $[\rho_1, \rho_2]$ . But modulo  $\Gamma_3(G)$ ,  $[\rho_1, \rho_2]^2 \equiv [\rho_1^2, \rho_2] \equiv 1$ , and since  $\Gamma_2(G)/\Gamma_3(G)$  is Abelian, this implies that all of its non-trivial elements are of order 2. Now suppose that  $i \geq 3$ , and suppose by induction that the result holds for  $i - 1$ , so that  $x^2 \equiv 1$  modulo  $\Gamma_i(G)$  for all  $x \in \Gamma_{i-1}(G)$ . Every  $i$ -fold simple commutator may be written in the form  $[x, \rho_i]$ , where  $x$  is a  $(i - 1)$ -fold simple commutator, so belongs to  $\Gamma_{i-1}(G)$ , and  $\rho_i \in \{\bar{\alpha}, \bar{\beta}, \bar{\gamma}\}$ , so belongs to  $G$ . By the induction hypothesis,  $x^2 \in \Gamma_i(G)$ , so  $[x, \rho_i]^2 \equiv [x^2, \rho_i] \equiv 1$  modulo  $\Gamma_{i+1}(G)$ , and once more, since  $\Gamma_i(G)/\Gamma_{i+1}(G)$  is Abelian, all of its non-trivial elements are of order 2. This proves the claim.

(2). The number  $R_i$  of summands of  $\mathbb{Z}_2$  is given by Theorem 3.4 of [Ga]. We refer to Gaglione's notation in what follows. Since  $U_\infty(x) = 0$ , the  $R_\infty^j$  are all zero ( $R_\infty^j$  represents the rank of the free abelian factor of  $\Gamma_j(G)/\Gamma_{j+1}(G)$ ), and so  $R_i$  is as given in the statement of the theorem. It just remains to determine  $k\alpha_k$  for all  $k \geq 2$ . A simple calculation shows that  $1 - U(x) = (1+x)^2(1-2x)$ , hence:

$$\frac{d}{dx} \ln(1 - U(x)) = \frac{2}{x+1} + \frac{2}{2x-1},$$

and that for  $k \geq 2$ ,

$$\frac{d^k}{dx^k} \ln(1 - U(x)) = (-1)^{k+1} (k-1)! \left( \frac{2}{(x+1)^k} + \frac{2^k}{(2x-1)^k} \right).$$

So

$$k\alpha_k = -\frac{1}{(k-1)!} \left( \frac{d^k}{dx^k} \ln(1 - U(x)) \right) \Big|_{x=0} = 2^k + 2(-1)^k,$$

as required.

(c) Given a group  $G$ , the quotient group  $D_i(G)/D_{i+1}(G)$  is torsion free and it is isomorphic to  $\Gamma_i(G)/\Gamma_{i+1}(G)$  modulo torsion, for  $i \geq 1$  [P]. Therefore, from part (b) one deduces that  $D_2(B_2(\mathbb{T}^2)) = D_3(B_2(\mathbb{T}^2))$ . On the other hand one can easily verify that  $B_2(\mathbb{T}^2)/\Gamma_2(B_2(\mathbb{T}^2)) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2$  and therefore  $B_2(\mathbb{T}^2)/D_2(B_2(\mathbb{T}^2)) \cong \mathbb{Z}^2$ . Since  $B_2(\mathbb{T}^2)$  is not abelian, it follows that  $D_2(B_2(\mathbb{T}^2))$  is not trivial and then  $\bigcap_{i \in \mathbb{N}} D_i(B_2(\mathbb{T}^2)) \neq \{1\}$ . ■

REMARKS 12 From Theorem 3 one concludes that  $\Gamma_i(B_2(\Sigma_{1,p})) \neq \Gamma_{i+1}(B_2(\Sigma_{1,p}))$ . On the other hand, the group  $\Gamma_2(B_2(\Sigma_{g,p}))$  is generated by the set of conjugates of the commutators of the form  $[g, g']$ , where  $g, g'$  are generators of  $B_2(\Sigma_{g,p})$ . Therefore  $\Gamma_2(B_2(\Sigma_{g,p})) \subset P_2(\Sigma_{g,p})$ . Since  $P_2(\Sigma_{g,p})$  is residually nilpotent for  $p \geq 1$  (see Section 5), one deduces that  $B_2(\Sigma_{g,p})$  is residually soluble for  $p \geq 1$ . The question of whether  $B_2(\Sigma)$  is in fact residually nilpotent, when  $\Sigma$  is a surface of positive genus possibly with boundary different from the torus, is open.

To finish this section, we prove the following result:

PROPOSITION 13 *The group  $B_2(\mathbb{T}^2)$  is not bi-orderable.*

*Proof.* Consider  $B_2(\mathbb{T}^2)$  with the group presentation in Theorem 10. Set  $g = \alpha\beta\gamma^{-1}$ . The following equality holds in  $B_2(\mathbb{T}^2)$ :

$$((\alpha\gamma)^{-1}g(\alpha\gamma))(\gamma^{-1}g\gamma)(\alpha^{-1}g\alpha)(g) = 1.$$

Since  $g \neq 1$ , the group  $B_2(\mathbb{T}^2)$  is not bi-orderable by Proposition 7. ■

Let  $\Sigma$  be an orientable surface, possibly with boundary. If  $n \geq 3$ ,  $B_n(\Sigma)$  is not bi-orderable since it contains a copy of  $B_n$  which is not bi-orderable [Go]. If  $n = 1$ , the group  $B_1(\Sigma)$  is isomorphic to  $\pi_1(\Sigma)$  which is known to be residually free. Therefore it is also residually torsion-free nilpotent and hence bi-orderable.

REMARK 14 If  $\Sigma$  is an orientable surface, possibly with boundary, different from the torus, the sphere and the disc, the question of whether  $B_2(\Sigma)$  is in fact bi-orderable is open.

## 5 Residual torsion free nilpotence of surface pure braid groups

In this section we give a short survey on relations between surface braids and mapping classes, and we show that pure braid groups of surfaces with non-empty boundary may be realised as subgroups of Torelli groups of surfaces with one boundary component.

### 5.1 Surface pure braid groups

We start by recalling a group presentation for pure braid groups of surfaces with one boundary component [B].

THEOREM 15 *Let  $\Sigma_{g,1}$  be a compact, connected orientable surface of genus  $g \geq 1$  with one boundary component. The group  $P_n(\Sigma_{g,1})$  admits the following presentation:*

**Generators:**  $\{A_{i,j} \mid 1 \leq i \leq 2g + n - 1, 2g + 1 \leq j \leq 2g + n, i < j\}$ .

**Relations:**

- (PR1)  $A_{i,j}^{-1} A_{r,s} A_{i,j} = A_{r,s}$  if  $(i < j < r < s)$  or  $(r + 1 < i < j < s)$ ,  
or  $(i = r + 1 < j < s)$  for even  $r < 2g$  or  $r > 2g$ ;
- (PR2)  $A_{i,j}^{-1} A_{j,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s}^{-1}$  if  $(i < j < s)$ ;
- (PR3)  $A_{i,j}^{-1} A_{i,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s} A_{j,s}^{-1} A_{i,s}^{-1}$  if  $(i < j < s)$ ;
- (PR4)  $A_{i,j}^{-1} A_{r,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s}^{-1} A_{j,s}^{-1} A_{r,s} A_{j,s} A_{i,s} A_{j,s}^{-1} A_{i,s}^{-1}$   
if  $(i + 1 < r < j < s)$  or  
 $(i + 1 = r < j < s)$  for odd  $r < 2g$  or  $r > 2g$ ;

- (ER1)  $A_{r+1,j}^{-1}A_{r,s}A_{r+1,j} = A_{r,s}A_{r+1,s}A_{j,s}^{-1}A_{r+1,s}^{-1}$   
if  $r$  odd and  $r < 2g$ ;
- (ER2)  $A_{r-1,j}^{-1}A_{r,s}A_{r-1,j} = A_{r-1,s}A_{j,s}A_{r-1,s}^{-1}A_{r,s}A_{j,s}A_{r-1,s}A_{j,s}^{-1}A_{r-1,s}^{-1}$   
if  $r$  even and  $r < 2g$ .

As a representative of the generator  $A_{i,j}$ , we may take a geometric braid whose only non-trivial (non-vertical) strand is the  $(j-2g)$ th one. In Figure 1, we illustrate the projection of such braids on the surface  $\Sigma_{g,1}$  (see also Figure 8 of [B]). Some misprints in Relations (ER1) and (ER2) of Theorem 5.1 of [B] have been corrected.

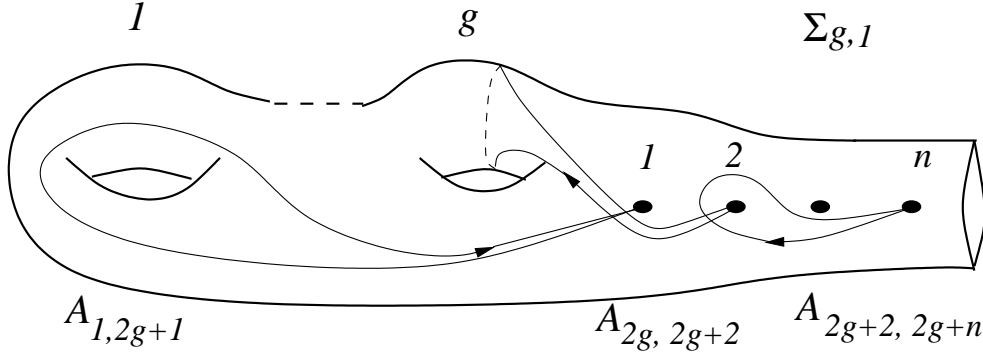


Figure 1: Projection of representatives of the generators  $A_{i,j}$ . We represent  $A_{i,j}$  by its only non-trivial strand.

With respect to the presentation of  $B_n(\Sigma_g)$  given in Theorem 8, the elements  $A_{i,j}$  are the following braids:

- $A_{i,j} = \sigma_{j-2g} \cdots \sigma_{i+1-2g} \sigma_{i-2g}^2 \sigma_{i+1-2g}^{-1} \cdots \sigma_{j-2g}^{-1}$ , for  $i \geq 2g$ ;
- $A_{2i,j} = \sigma_{j-2g} \cdots \sigma_1 a_{g-i+1} \sigma_1^{-1} \cdots \sigma_{j-2g}^{-1}$ , for  $1 \leq i \leq g$ ;
- $A_{2i-1,j} = \sigma_{j-2g} \cdots \sigma_1 b_{g-i+1} \sigma_1^{-1} \cdots \sigma_{j-2g}^{-1}$ , for  $1 \leq i \leq g$ .

Relations (PR1), ..., (PR4) correspond to the classical relations for the pure braid group  $P_n$  [Bir]. New relations arise when we consider two generators  $A_{2i,j}$ ,  $A_{2i-1,k}$ , for  $1 \leq i \leq g$  and  $j \neq k$ . They correspond to loops based at two different points which go around the same handle.

## 5.2 Mapping class groups

The *mapping class group* of a surface  $\Sigma_{g,p}$ , denoted by  $\mathcal{M}_{g,p}$ , is the group of isotopy classes of orientation-preserving self-homeomorphisms which fix the boundary components pointwise. If the surface has empty boundary then we shall just write  $\mathcal{M}_g$ . Note that we will denote the composition in the mapping class groups from left to right<sup>1</sup>.

Let  $\mathcal{P} = \{x_1, \dots, x_n\}$  be a set of  $n$  distinct points in the interior of the surface  $\Sigma_{g,p}$ . The *punctured mapping class group* of  $\Sigma_g$  relative to  $\mathcal{P}$  is defined to be the group of isotopy classes of orientation-preserving self-homeomorphisms which fix the boundary components pointwise, and which fix  $\mathcal{P}$  setwise. This group, denoted by  $\mathcal{M}_{g,p}^{(n)}$ , does not depend on the choice of  $\mathcal{P}$ , but just on its cardinal. We define the *pure punctured mapping class group*, denoted by  $\mathcal{PM}_{g,p}^{(n)}$ , to be the subgroup of (isotopy classes of) homeomorphisms which fix the set  $\mathcal{P}$  pointwise. We recall that a Dehn twist  $T_C$  along a simple closed curve  $C$  is the isotopy class of a positive twist along  $C$ .

Let  $C$  and  $D$  be two simple closed curves bounding an annulus containing the single puncture  $x_j$ . We shall say that the multitwist  $T_C T_D^{-1}$  is a *j-bounding pair braid*.

REMARK 16 According to the definition given in [IIM], a *j-bounding pair braid* is a 1-string bounding pair braid with string based at the puncture  $x_j$ .

## 5.3 Pure braids, bounding pair braids and Torelli groups

Surface braid groups are related to mapping class groups as follows:

THEOREM 17 (**Birman [Bir]**) *Let  $g \geq 1$  and  $p \geq 0$ . Let  $\psi : \mathcal{M}_{g,p}^{(n)} \longrightarrow \mathcal{M}_{g,p}$  be the homomorphism induced by the map which forgets the set  $\mathcal{P}$ . If  $\Sigma_{g,p}$  is different from the torus then  $\text{Ker}(\psi)$  is isomorphic to  $B_n(\Sigma_{g,p})$ .*

REMARKS 18 In particular, if  $\Sigma$  is an orientable surface (possibly with boundary) of positive genus and different from the torus, the surface pure braid group  $P_n(\Sigma)$  may be identified with the group generated by bounding pair braids. Indeed, consider two simple closed curves  $C$  and  $D$  bounding an annulus containing a single puncture, say the  $j$ th one. We suppose that at least one curve between  $C$  and  $D$  either does not bound a disc, or it bounds a disc containing at least two punctures (otherwise the corresponding bounding pair braid is trivial). Let  $h$  be a homeomorphism of  $\Sigma$  which is a representative of  $T_C T_D^{-1}$ . Then  $h$  is isotopic to the identity, but the points  $\{x_1, \dots, x_n\}$  may move during the isotopy. The track of these points in  $\Sigma \times [0, 1]$

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<sup>1</sup>We do this in order to have the same group-composition in braid groups and mapping class groups.



yields the geometric braid  $\beta_h$  corresponding to (the class of) the homeomorphism  $h$ . Let  $L : S^1 \rightarrow \Sigma$  be an embedding such that  $L \cap \mathcal{P} = \{x_j\}$  and such that  $L$  is contained in the annulus bounded by  $C$  and  $D$ . Up to braid isotopy, we may suppose that the  $j$ th strand  $\psi_j$  of the braid  $\beta_h$  is such that  $\psi_j(t) = (L(e^{2\pi it}), t)$  (or  $\psi_j(t) = (L(e^{2\pi i(1-t)}), t)$ ) for every  $t$  in  $[0, 1]$ , and the other  $n - 1$  strands are trivial (vertical). Since this set of braids generates  $P_n(\Sigma)$  (see Theorems 5.1 and 5.2 of [B] for instance), we deduce that bounding pair braids generate a group isomorphic to  $P_n(\Sigma)$ .

## 5.4 Torelli groups

We recall that the Torelli group  $\mathcal{T}_{g,1}$  is the subgroup of the mapping class group  $\mathcal{M}_{g,1}$  which acts trivially on the first homology group of the surface  $\Sigma_{g,1}$ .

Before stating the main theorem of this section, we recall the following exact sequence:

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \mathcal{M}_{g,n+p} \xrightarrow{q} \mathcal{PM}_{g,p}^{(n)} \longrightarrow 1, \quad (16)$$

where  $\mathbb{Z}^n$  is generated by Dehn twists along the first  $n$  boundary components of  $\Sigma_{g,n+p}$ . Geometrically, the projection  $q$  may be obtained by glueing one-punctured discs  $\mathbb{D}_1, \dots, \mathbb{D}_n$ , say, onto the first  $n$  boundary components.

**LEMMA 19** *Let  $\Sigma_{g,1}$  be a surface of genus greater than or equal to one with one boundary component. Then the group  $P_n(\Sigma_{g,1})$  embeds in  $\mathcal{T}_{g+n,1}$ .*

*Proof.* Applying Theorem 17 and Remark 18, we identify  $P_n(\Sigma_{g,1})$  with the subgroup of  $\mathcal{PM}_{g,1}^{(n)}$  generated by bounding pair braids. Let us first embed  $P_n(\Sigma_{g,1})$  in  $\mathcal{M}_{g,n+1}$ . To achieve this, we construct a section  $s$  on  $P_n(\Sigma_{g,1})$  of the sequence (16). For each generator  $A_{i,j}$  of  $P_n(\Sigma_{g,1})$ , we define  $s(A_{i,j})$  as follows. Consider two simple closed curves  $a$  and  $a'$  lying in  $\Sigma_{g,1}$  such that  $A_{i,j}$  is equal to the boundary pair braids  $T_a T_{a'}^{-1}$ . These two curves may be chosen so as to avoid the discs  $\mathbb{D}_1, \dots, \mathbb{D}_n$ , and thus may be seen as lying in  $\Sigma_{g,n+1}$ . If  $d_j$  is a simple closed curve parallel to the  $j$ th-boundary component, we set  $s(A_{i,j}) = T_a T_{a'}^{-1} T_{d_j}$ , which we denote by  $A'_{i,j}$ . Since the Dehn twists  $T_{d_1}, \dots, T_{d_n}$  belong to the kernel of  $q$ , one has  $q \circ s = \text{Id}$ , and hence  $s$  is injective. We claim that  $s$  is a homomorphism. To prove this, we have to show that relations (PR1-4) and (ER1-2) are satisfied in  $\mathcal{M}_{g,n+1}$  via  $s$ .

The four first relations may be written in the form  $h A_{r,s} h^{-1} = A_{r,s}$ , where  $h$  is a word in the  $A_{i,j}$ 's. These relations are compatible with  $s$ , since for all simple closed curves  $a$  in  $\Sigma_{g,n+1}$ , and all  $h$  in  $\mathcal{M}_{g,n+1}$ , one has:

$$T_{h(a)} = h^{-1} T_a h. \quad (17)$$

For example, relation (PR1) is compatible with  $s$  because the curves occurring in  $A'_{i,j}$  are disjoint from those occurring in  $A'_{r,s}$ . For (PR2), the bounding pair

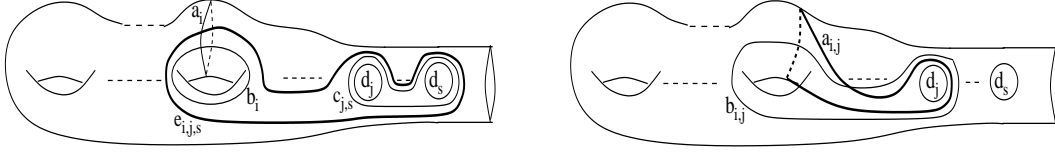


Figure 2: curves for relations (PR2) and (ER2)

braids  $A_{j,s}$  (resp.  $A_{i,j}$ ,  $A_{i,s}$ ) is equal to  $T_{d_j}T_{c_{j,s}}^{-1}$  (resp.  $T_aT_{a'}^{-1}$ ,  $T_bT_{b'}^{-1}$  for  $(a, a') \in \{(a_i, a_{i,j}), (b_i, b_{i,j}), (d_i, c_{i,j})\}$  and  $(b, b') \in \{(a_i, a_{i,s}), (b_i, b_{i,s}), (d_i, c_{i,s})\}$ , where curves are those described by Figure 2). Thus we have:

$$\begin{aligned} A'_{i,j}{}^{-1}A'_{j,s}A'_{i,j} &= [T_a^{-1}T_{a'}T_{d_j}^{-1}]T_{d_j}T_{c_{j,s}}^{-1}T_{d_s}[T_{d_j}T_{a'}^{-1}T_a] \\ &= [T_a^{-1}T_{a'}]T_{c_{j,s}}^{-1}[T_{a'}^{-1}T_a]T_{d_j}T_{d_s} \quad \text{since the } T_{d_k}'\text{'s are central} \\ &= T_{T_aT_{a'}^{-1}(c_{j,s})}^{-1}T_{d_j}T_{d_s} \quad \text{by (17),} \end{aligned}$$

and similarly,  $A'_{i,s}A'_{j,s}A'_{i,s}{}^{-1} = T_{T_b^{-1}T_{b'}(c_{j,s})}^{-1}T_{d_j}T_{d_s}$ . Now, it is easy to see that

$$T_aT_{a'}^{-1}(c_{j,s}) = T_{a'}^{-1}(c_{j,s}) = T_b^{-1}T_{b'}(c_{j,s}),$$

which yields the required relation. The compatibility of relations (PR3-4) with  $s$  may be proved in the same way; we leave this as an exercise for the reader.

Relation (ER1) is a consequence of the lantern relation and relation (17). Indeed, if we consider the seven curves  $b_i$ ,  $d_j$ ,  $d_s$ ,  $e_{i,j,s}$ ,  $b_{i,s}$ ,  $b_{j,s}$  and  $c_{j,s}$  shown in Figure 2 (where  $r = 2i - 1$ ), the lantern relation may be written as:

$$T_{e_{i,j,s}}T_{b_i}T_{d_j}T_{d_s} = T_{b_{i,s}}T_{b_{i,j}}T_{c_{j,s}},$$

which implies that

$$A'_{r,s} = T_{b_i}T_{b_{i,s}}^{-1}T_{d_s} = T_{b_{i,j}}T_{e_{i,j,s}}^{-1}T_{c_{j,s}}T_{d_j}^{-1} = T_{b_{i,j}}T_{e_{i,j,s}}^{-1}T_{d_s}A'_{j,s}{}^{-1}.$$

Since  $A'_{r+1,j} = T_{a_i}T_{a_{i,j}}^{-1}T_{d_j}$ , we obtain

$$\begin{aligned} A'_{r+1,j}{}^{-1}A'_{r,j}A'_{r+1,j} &= \left[ T_{d_j}^{-1}T_{a_{i,j}}T_{a_i}^{-1}T_{b_{i,j}}T_{e_{i,j,s}}^{-1}T_{d_s}T_{a_i}T_{a_{i,j}}^{-1}T_{d_j} \right] \left[ A'_{r+1,j}{}^{-1}A'_{j,s}{}^{-1}A'_{r+1,j} \right] \\ &= \left[ T_{a_{i,j}}T_{a_i}^{-1}T_{b_{i,j}}T_{e_{i,j,s}}^{-1}T_{a_i}T_{a_{i,j}}^{-1} \right] T_{d_s} \left[ A'_{r+1,s}A'_{j,s}{}^{-1}A'_{r+1,s} \right] \quad \text{by (PR2)} \\ &= T_{T_{a_i}T_{a_{i,j}}^{-1}(b_{i,j})}T_{T_{a_i}T_{a_{i,j}}^{-1}(e_{i,j,s})}^{-1}T_{d_s} \left[ A'_{r+1,s}A'_{j,s}{}^{-1}A'_{r+1,s} \right] \quad \text{by (17)}. \end{aligned}$$

But  $T_{a_i}T_{a_{i,j}}^{-1}(b_{i,j}) = b_i$  and  $T_{a_i}T_{a_{i,j}}^{-1}(e_{i,j,s}) = b_{i,s}$ , so

$$\begin{aligned} A'_{r+1,j}{}^{-1}A'_{r,j}A'_{r+1,j} &= T_{b_i}T_{b_{i,s}}^{-1}T_{d_s} \left[ A'_{r+1,s}A'_{j,s}{}^{-1}A'_{r+1,s} \right] \\ &= A'_{r,s}A'_{r+1,s}A'_{j,s}{}^{-1}A'_{r+1,s}, \end{aligned}$$

which is relation (ER1). Relation (ER2) is also a consequence of a lantern: again, we leave the details to the reader.

Hence  $s : P_n(\Sigma_{g,1}) \longrightarrow \mathcal{M}_{g,n+1}$  is an embedding. Glueing a one-holed torus onto each of the first  $n$  boundary components of  $\Sigma_{g,n+1}$ , we obtain a homomorphism  $\varphi : \mathcal{M}_{g,n+1} \longrightarrow \mathcal{M}_{g+n,1}$  which is injective (see [PR2]). Clearly, the image under  $\varphi$  of each  $s(A_{i,j})$  acts trivially on the homology group  $H_1(\Sigma_{g+n,1}; \mathbb{Z})$ . Thus  $\varphi s(P_n(\Sigma_{g,1}))$  lies in the Torelli group of  $\Sigma_{g+n,1}$ . ■

REMARK 20 The sequence (16) does not split, since the first homology group of  $\mathcal{M}_{g,n}$  is trivial.

REMARK 21 This embedding of  $P_n(\Sigma_{g,1})$  into  $\mathcal{T}_{g+n,1}$  does not hold for surfaces with empty boundary. Indeed, the group  $P_n(\Sigma_g)$  has an extra relation (TR) (see [B]) which is not satisfied by the section  $s$ .

THEOREM 22 *Let  $\Sigma$  be the sphere, the torus, or a surface of positive genus with non-empty boundary. Then the group  $P_n(\Sigma)$  is residually torsion-free nilpotent.*

*Proof.* Let  $\Sigma_{g,1}$  be a surface of genus greater than or equal to one, and with one boundary component. First, we remark that Lemma 19 and Theorem 25 imply that  $P_n(\Sigma_{g,1})$  is residually torsion-free nilpotent. Now let  $\Sigma_{g,p}$  be a surface with  $p > 1$  boundary components. The group  $P_n(\Sigma_{g,p})$  may be realised as the subgroup of  $P_{n+p-1}(\Sigma_{g,1})$  formed by the braids whose first  $p-1$  strands are vertical. Therefore  $P_n(\Sigma_{g,p})$  is residually torsion-free nilpotent.

The remaining cases are the pure braid groups of  $\mathbb{S}^2$  and  $\mathbb{T}^2$ . In the case of the sphere, the group  $P_n(\mathbb{S}^2)$  is isomorphic to  $\mathbb{Z}_2 \times P_{n-2}(\Sigma_{0,3})$  (see [GG1]). Therefore, for  $i > 1$ ,  $\Gamma_i(P_n(\mathbb{S}^2))$  and  $\Gamma_i(P_{n-2}(\Sigma_{0,3}))$  are isomorphic. Since  $P_{n-2}(\Sigma_{0,3})$  is a subgroup of  $P_n$  (which may be realised geometrically as the subgroup of braids whose last strand is vertical), from [FR] it follows that  $P_n(\mathbb{S}^2)$  is residually torsion-free nilpotent. In a similar way, using Lemma 23, one sees that the group  $P_n(\mathbb{T}^2)$  is residually torsion-free nilpotent. ■

LEMMA 23 *The group  $P_n(\mathbb{T}^2)$  is isomorphic to  $P_{n-1}(\Sigma_{1,1}) \times \mathbb{Z}^2$ .*

*Proof.* Consider the pure braid group exact sequence for an orientable surface  $\Sigma$ :

$$(PBS) \quad 1 \longrightarrow P_{n-1}(\Sigma \setminus \{x_1\}) \longrightarrow P_n(\Sigma) \xrightarrow{\theta} \pi_1(\Sigma) \longrightarrow 1,$$

where geometrically,  $\theta$  is the map that forgets the paths pointed at  $x_2, \dots, x_n$ . Since  $ZP_n(\Sigma_{1,1})$  is trivial [PR1], we deduce that the restriction of  $\theta$  to  $ZP_n(\mathbb{T}^2)$  is injective. Since  $ZP_n(\mathbb{T}^2) = \mathbb{Z}^2$  [PR1], the restriction of  $\theta$  to  $ZP_n(\mathbb{T}^2)$  is in fact an isomorphism, and we conclude that  $P_n(\mathbb{T}^2)$  is isomorphic to the direct product  $P_{n-1}(\Sigma_{1,1}) \times \mathbb{Z}^2$ . ■

REMARK 24 Pure braid groups of surfaces of genus  $g \geq 2$  with empty boundary are bi-orderable ([Go]). To the best of our knowledge, it is not known whether they are residually torsion-free nilpotent.

## Appendix: Residually torsion-free nilpotence of Torelli groups

Let  $\Sigma_{g,1}$  be the surface of genus  $g \geq 1$  with one boundary component. In this section, we give a short proof of the fact that the Torelli group  $\mathcal{T}_{g,1}$  is residually torsion-free nilpotent. This result is folklore (see Section 14 of [H]) but we did not find a detailed proof of it in the literature. In what follows, the fundamental group of  $\Sigma_{g,1}$ , which is free, will be denoted by  $\pi$ , and its lower central series by  $\pi_1 = \pi$  and  $\pi_{k+1} = [\pi, \pi_k]$  for  $k \geq 1$ .

THEOREM 25 *The Torelli group of a surface of genus  $g \geq 1$  with one boundary component is residually torsion-free nilpotent.*

In order to prove the theorem, let us recall the "machinery" of Johnson's homomorphism. For any positive integer  $k$ , we consider the subgroup  $\mathcal{M}[k]$  of  $\mathcal{M}_{g,1}$  consisting of the elements which act trivially on  $\pi/\pi_k$ . Thus  $\mathcal{M}[1] = \mathcal{M}_{g,1}$ ,  $\mathcal{M}[2] = \mathcal{T}_{g,1}$  and  $\mathcal{M}[k+1] \subset \mathcal{M}[k]$  for all  $k$ . We also recall that the  $k$ th Johnson's homomorphism is a map  $J_k : \mathcal{M}[k] \longrightarrow \text{Hom}(\pi/\pi_2, \pi_k/\pi_{k+1})$  such that  $\text{Ker}(J_k) = \mathcal{M}[k+1]$  (see [J] or [Mo]).

PROPOSITION 26 *The filtration  $\mathcal{M}[1] \supset \mathcal{M}[2] \supset \dots$  has the following properties:*

- (a)  $\bigcap_{k \in \mathbb{N}^*} \mathcal{M}[k] = \{\text{Id}\}$ ;
- (b) *all quotients  $\mathcal{M}[k]/\mathcal{M}[k+1]$  are torsion free.*

*Proof.*

(a) Let  $f$  be an element of  $\bigcap_{k \in \mathbb{N}^*} \mathcal{M}[k]$ . Then for all  $x$  in  $\pi$  and all positive integers  $k$ ,  $f(x)x^{-1}$  is an element of  $\pi_k$ . Since  $\pi$  is residually nilpotent (see [MKS]), we see that  $f(x)x^{-1} = 1$ . Thus  $f$  acts trivially on the fundamental group of  $\Sigma_{g,1}$ . By Nielsen's theorem, this means that  $f$  is the identity map.

(b) The kernel of the  $k$ th Johnson's morphism is  $\mathcal{M}[k+1]$ , and thus  $\mathcal{M}[k]/\mathcal{M}[k+1]$  is isomorphic to a subgroup of  $\text{Hom}(\pi/\pi_2, \pi_k/\pi_{k+1})$ , which is torsion free since  $\pi_j/\pi_{j+1}$  is a free group for all  $j$ .

■

PROPOSITION 27 For all positive integers  $k$ , one has  $D_k(\mathcal{T}_{g,1}) \subset \mathcal{M}[k+1]$ .

Clearly, Propositions 26 and 27 give a proof of Theorem 25.

Before proving Proposition 27, let us state a lemma about commutator calculus.

LEMMA 28 If  $f$  is an element of  $\mathcal{M}[k]$  and  $x$  an element of  $\pi_j$ , then  $x^{-1}f(x)$  belongs to  $\pi_{k+j-1}$ .

*Proof.* We proceed by induction on  $j$ . For  $j = 1$ , this is the definition of  $\mathcal{M}[k]$ . Let  $j$  be a positive integer, and suppose that  $x^{-1}f(x) \in \pi_{k+i-1}$  for all  $x \in \pi_i$  and all  $i$ ,  $1 \leq i \leq j$ . If  $(x, y)$  is an element of  $\pi_j \times \pi$  then there exists  $(a, a') \in \pi_{k+j-1} \times \pi_k$  such that  $f(x) = xa$  and  $f(y) = ya'$ . So:

$$\begin{aligned} f([x, y]) &= [f(x), f(y)] \\ &= [xa, ya'] \\ &= xay a' a^{-1} x^{-1} a'^{-1} y^{-1} \\ &= x[a, ya'] ya' x^{-1} a'^{-1} y^{-1} \\ &= x[a, ya'] x^{-1} [x, y] ya' x^{-1} a'^{-1} y^{-1} \\ &= x[a, ya'] x^{-1} [x, y] y [x, a'] y^{-1}. \end{aligned}$$

Since  $a \in \pi_{k+j-1}$ ,  $x \in \pi_j$  and  $a' \in \pi_k$ , one has  $[a, ya'] \in \pi_{k+j}$  and  $[x, a'] \in \pi_{k+j}$ . Thus  $[x, y]^{-1}f([x, y])$  belongs to  $\pi_{k+j}$ . ■

*Proof of Proposition 27.* First we prove that  $\Gamma_k(\mathcal{T}_{g,1}) \subset \mathcal{M}[k+1]$ . We proceed by induction on  $k$ . If  $k = 1$ , then  $\mathcal{T}_{g,1} = \mathcal{M}[2]$ . Suppose that  $\Gamma_k(\mathcal{T}_{g,1}) \subset \mathcal{M}[k+1]$ , and consider an element  $f$  (resp.  $g$ ) of  $\Gamma_k(\mathcal{T}_{g,1})$  (resp.  $\mathcal{T}_{g,1}$ ). We want to prove that the commutator  $[f, g] \in \mathcal{M}[k+2]$ , in other words:

$$([f, g])(x)x^{-1} \in \pi_{k+2} \text{ for all } x \text{ in } \pi.$$

So let  $x$  be an element of  $\pi$ . There exists  $(d_1, d_2) \in \pi_{k+1} \times \pi_2$  such that

$$f^{-1}(g^{-1}(x)) = g^{-1}(x)d_1 \text{ and } g^{-1}(x) = d_2x.$$

Then:

$$([f, g])(x) = fgf^{-1}(g^{-1}(x)) = fg(g^{-1}(x)d_1) = f(xg(d_1)).$$

By Lemma 28, there is an element  $d_3$  in  $\pi_{k+2}$  such that  $g(d_1) = d_1d_3$ . From this, we obtain

$$\begin{aligned} ([f, g])(x) &= f(xd_1d_3) = f\left(xg^{-1}(x)^{-1}f^{-1}(g^{-1}(x))\right)f(d_3) \\ &= f(d_2^{-1})g^{-1}(x)f(d_3). \end{aligned}$$

Since  $f \in \mathcal{M}[k+1]$  and  $d_2^{-1} \in \pi_2$ , applying Lemma 28 yields an element  $d_4$  of  $\pi_{k+2}$  such that  $f(d_2^{-1}) = d_4 d_2^{-1}$ . Thus we obtain

$$([f, g])(x) = d_4 d_2^{-1} g^{-1}(x) f(d_3) = d_4 x f(d_3) = d_4 [x, f(d_3)] f(d_3) x.$$

Since  $d_3$  and  $d_4$  are elements of  $\pi_{k+2}$ , we see that  $([f, g])(x) x^{-1} \in \pi_{k+2}$ .

Let  $g \in D_k(\mathcal{T}_{g,1})$ . Then for some  $n$ ,  $g^n$  belongs to  $\Gamma_k(\mathcal{T}_{g,1})$  and so to  $\mathcal{M}[k+1]$ . By Proposition 26 it follows that  $g$  belongs to  $\mathcal{M}[k+1]$ . ■

REMARK 29 The above commutator calculus may be found in [HP].

REMARK 30 In [H] a sketch of a proof of Proposition 26 is outlined, and Proposition 27 is stated without proof (Proposition 14.9 and the following remarks).

## References

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