



**HAL**  
open science

## Propagators for Noncommutative Field Theories

Razvan Gurau, Vincent Rivasseau, Fabien Vignes-Tourneret

► **To cite this version:**

Razvan Gurau, Vincent Rivasseau, Fabien Vignes-Tourneret. Propagators for Noncommutative Field Theories. *Annales Henri Poincaré*, 2006, 7 (7-8), pp.1601-1628. 10.1007/s00023-006-0296-2. hal-00015384

**HAL Id: hal-00015384**

**<https://hal.science/hal-00015384>**

Submitted on 15 Feb 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Propagators for Noncommutative Field Theories

Razvan GURAU, Vincent RIVASSEAU and Fabien VIGNES-TOURNERET

*Laboratoire de Physique Théorique, Bât. 210, Université Paris XI  
F-91405 Orsay Cedex, France*

e-mail: razvan.gurau@th.u-psud.fr, vincent.rivasseau@th.u-psud.fr,  
fabien.vignes@th.u-psud.fr

## Abstract

In this paper we provide exact expressions for propagators of noncommutative Bosonic or Fermionic field theories after adding terms of the Grosse-Wulkenhaar type in order to ensure Langmann-Szabo covariance. We emphasize the new Fermionic case and we give in particular all necessary bounds for the multiscale analysis and renormalization of the noncommutative Gross-Neveu model.

## 1 Introduction

This paper is the first of a series in which we plan to extend the proof of perturbative renormalizability of noncommutative  $\phi_4^4$  field theory [1, 2, 3] to other noncommutative models (see [4] for a general review on noncommutative field theories).

We have in mind in particular Fermionic field theories either of the relativistic type, such as the Gross-Neveu model in Euclidean two dimensional space [5, 6], or of the type used in condensed matter for many body theory, in which there is no symmetry between time and space. In the commutative case, these non-relativistic theories are just renormalizable in any dimension [7, 8, 9]. Their noncommutative version should be relevant for the study of Fermions in 2 dimensions in magnetic fields, hence for the quantum Hall effect. Of course a future goal is also to find the right extension of the Grosse-Wulkenhaar method to gauge theories.

In this paper we generalize the computation of the Bosonic  $\phi_4^4$  propagator of [2] and provide the exact expression of the propagators of Fermionic noncommutative field theories on the Moyal plane. We will restrict our analysis to one pair of noncommuting coordinates, as the generalization to several

pairs are trivial. These propagators are not the ordinary commutative propagators: they have to be modified to obey Langmann-Szabo duality, according to the pioneering papers [10, 2]. We propose to call *vulcanization* this modification of the theory, and to call *vulcanized* the resulting theory and propagators.

Like in [3] we can slice the corresponding noncommutative heat kernels according to the Schwinger parameters in order to derive a multiscale analysis. In this framework the theory with a finite number of slices has a cutoff, and the removal of the cutoff (ultraviolet limit) corresponds to summing over infinitely many slices. However for the Fermionic propagators treated in this paper this multiscale analysis is harder than in the Bosonic case. In  $x$ -space the propagator end-terms oscillate rather than decay as in the Bosonic  $\phi_4^4$  case. In matrix basis the behavior of the propagator is governed by a non-trivial critical point in parameter space and in contrast with the Bosonic case there is no general scaled decay for all indices. The main result of this paper is the detailed analysis of this critical point, leading to Theorem 4.1, namely to the bounds required for the multiscale analysis of the 2 dimensional Euclidean Non-Commutative Gross-Neveu model. These bounds are slightly worse than in the  $\phi_4^4$  case. However they should suffice for a complete proof of renormalizability of the model to all orders (see the discussion after Theorem 4.1). This complete proof is postponed to a future paper [11].

## 2 Conventions

The two dimensional Moyal space  $\mathbb{R}_\theta^2$  is defined by the following associative non-commutative star product

$$(a \star b)(x) = \int \frac{d^2k}{(2\pi)^2} \int d^2y a(x + \frac{1}{2}\theta \cdot k) b(x+y) e^{ik \cdot y} . \quad (2.1)$$

which corresponds to a constant commutator:

$$[x^i, x^j] = i\Theta^{ij} \quad , \text{ where } \Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} . \quad (2.2)$$

In order to perform the second quantization one must first identify the Hilbert space of states of the first quantization. If we deal with a real field theory (for instance the  $\phi^4$  theory which is treated in detail in [12]) one considers a real Hilbert space, whereas for a complex field theory (e.g. any Fermionic field theory) one has to use a complex Hilbert space.

A basis in this Hilbert space is given by the functions  $f_{mn}$  defined in [13, 12].

Any complex-valued function defined on the plane can be decomposed in this basis as:

$$\chi(x) = \sum_{m,n} \chi_{mn} f_{mn}(x). \quad (2.3)$$

A crucial observation is that  $\bar{f}_{mn}(x) = f_{nm}(x)$  (which can be verified on the explicit expression for  $f_{mn}$ ). A real function in this basis obeys:

$$\bar{\chi}(x) = \chi(x) \Rightarrow \overline{\sum \chi_{mn} f_{mn}(x)} = \sum \chi_{pq} f_{pq}(x) \Rightarrow \bar{\chi}_{mn} = \chi_{nm}. \quad (2.4)$$

The scalar product (which must be sesquilinear for a complex Hilbert space) is then defined as:

$$\begin{aligned} \langle \phi, \chi \rangle &= \int \frac{d^2x}{2\pi\theta} \bar{\phi}(x) \chi(x) = \sum \bar{\phi}_{pq} \chi_{rs} \int \frac{d^2x}{2\pi\theta} \bar{f}_{pq} f_{rs} \\ &= \sum \bar{\phi}_{pq} \chi_{rs} \delta_{pr} \delta_{qs} = \sum \bar{\phi}_{pq} \chi_{pq}. \end{aligned} \quad (2.5)$$

Notice that if  $\phi$  is a real field we have:

$$\langle \phi, \chi \rangle = \sum \phi_{qp} \chi_{pq}, \quad (2.6)$$

so that our conventions restrict to those in [12] for the real  $\phi^4$  theory. With this convention  $\langle f_{kl}, \chi \rangle = \chi_{mn} \int \bar{f}_{kl} f_{mn} = \chi_{kl}$ . A linear operator on this space acts like:

$$[A\phi]_{kl} = \langle f_{kl}, A\phi \rangle = \sum_{m,n} \langle f_{kl}, Af_{mn} \rangle \phi_{mn}. \quad (2.7)$$

At this point the convention consistent with that for the real  $\phi^4$  theory found in [12] is to note:

$$\langle f_{kl}, Af_{mn} \rangle = \int \frac{d^2x}{2\pi\theta} \bar{f}_{kl}(x) \int d^2y A(x, y) f_{mn}(y) := A_{l,k;m,n}. \quad (2.8)$$

With this convention the product of operators is

$$[AB]_{p,q;r,s} = \langle f_{qp}, ABf_{rs} \rangle = \sum_{t,u} \langle f_{qp}, Af_{tu} \rangle \langle f_{tu}, Bf_{rs} \rangle = \sum_{t,u} A_{p,q;t,u} B_{u,t;r,s} \quad (2.9)$$

and the identity operator  $I$  has the matrix elements:

$$\phi_{mn} = \sum_{p,q} \langle f_{mn}, If_{pq} \rangle \phi_{pq} \Rightarrow I_{n,m;p,q} = \delta_{pm} \delta_{nq}. \quad (2.10)$$

We pass now to the second quantization. The quadratic part of the action is generically (in  $x$  space):

$$S = \int d^2x \bar{\chi}(x) H(x, y) \chi(y). \quad (2.11)$$

In the matrix basis one has the action:

$$S = 2\pi\theta \sum_{m,n,k,l} \frac{1}{2} \bar{\chi}_{pq} \left( 2 \int \frac{d^2x}{2\pi\theta} \bar{f}_{pq}(x) H(x, y) f_{kl}(y) \right) \chi_{kl}. \quad (2.12)$$

We define the Hamiltonian in the matrix basis as:

$$H_{q,p;k,l} = 2 \int \frac{d^2x}{2\pi\theta} \bar{f}_{pq}(x) H(x, y) f_{kl}(y) \quad (2.13)$$

where the 2 has been included in order to maintain the conventions in [12] for the case of a real field.

### 3 Non-Commutative Schwinger Kernels

In this section we provide the explicit formulas for the Schwinger representation of the non-commutative kernels or propagators of free scalar Bosonic and spin 1/2 Fermionic theories on the Moyal plane. These formulas are essential for a multiscale analysis based on slicing the Schwinger parameter.

The different propagators of interest are expressed via the Schwinger parameter trick as:

$$H^{-1} = \int_0^\infty dt e^{-tH}. \quad (3.1)$$

#### 3.1 Bosonic $x$ Space Kernel

We define  $x \wedge x' = x_0 x'_1 - x_1 x'_0$  and  $x \cdot x' = x_0 x'_0 + x_1 x'_1$ . The following lemma generalizes the Mehler kernel [14]:

**Lemma 3.1** *Let  $H$  be:*

$$H = \frac{1}{2} \left[ -\partial_0^2 - \partial_1^2 + \Omega^2 x^2 - 2iB(x_0 \partial_1 - x_1 \partial_0) \right]. \quad (3.2)$$

*The integral kernel of the operator  $e^{-tH}$  is:*

$$e^{-tH}(x, x') = \frac{\Omega}{2\pi \sinh \Omega t} e^{-A}, \quad (3.3)$$

$$A = \frac{\Omega \cosh \Omega t}{2 \sinh \Omega t} (x^2 + x'^2) - \frac{\Omega \cosh Bt}{\sinh \Omega t} x \cdot x' - i \frac{\Omega \sinh Bt}{\sinh \Omega t} x \wedge x'. \quad (3.4)$$

**Proof** We note that the kernel is correctly normalized: as  $\Omega = B \rightarrow 0$  we have

$$e^{-tH}(x, x') \rightarrow \frac{1}{2\pi t} e^{-\frac{|x-x'|^2}{2t}}, \quad (3.5)$$

which is the normalized heat kernel.

We must then check the equation

$$\frac{d}{dt}e^{-tH} + He^{-tH} = 0. \quad (3.6)$$

In fact

$$\begin{aligned} \frac{d}{dt}e^{-tH} &= \frac{\Omega e^{-A}}{2\pi \sinh \Omega t} \left\{ -\Omega \coth \Omega t + \frac{\Omega^2}{2 \sinh^2 \Omega t} (x^2 + x'^2) \right. \\ &+ \Omega \frac{B \sinh \Omega t \sinh Bt - \Omega \cosh \Omega t \cosh Bt}{\sinh^2 \Omega t} x \cdot x' \\ &\left. + i\Omega \frac{B \cosh Bt \sinh \Omega t - \Omega \sinh Bt \cosh \Omega t}{\sinh^2 \Omega t} x \wedge x' \right\}. \end{aligned} \quad (3.7)$$

Moreover

$$\begin{aligned} \frac{(-\partial_1^2 - \partial_2^2)}{2} e^{-tH} &= \frac{\Omega e^{-A}}{2\pi \sinh \Omega t} \left\{ \Omega \coth \Omega t - \frac{1}{2} \left[ \frac{\Omega \cosh \Omega t}{\sinh \Omega t} x - \frac{\Omega \cosh Bt}{\sinh \Omega t} x' \right]^2 \right. \\ &\left. + \frac{\Omega^2 \sinh^2 Bt}{2 \sinh^2 \Omega t} x'^2 + i \frac{\Omega^2 \cosh \Omega t \sinh Bt}{\sinh^2 \Omega t} x \wedge x' \right\} \end{aligned} \quad (3.8)$$

and

$$iB(x_1 \partial_2 - x_2 \partial_1) = \frac{\Omega}{2\pi \sinh \Omega t} e^{-A} \left\{ (-i \frac{B\Omega \cosh Bt}{\sinh \Omega t} x \wedge x' + \frac{B\Omega \sinh Bt}{\sinh \Omega t} x \cdot x') \right\}. \quad (3.9)$$

It is now straightforward to verify the differential equation (3.6).  $\square$

**Corollary 3.1** *Let  $H$  be:*

$$H = \frac{1}{2} \left[ -\partial_0^2 - \partial_1^2 + \Omega^2 x^2 - 2i\Omega(x_0 \partial_1 - x_1 \partial_0) \right]. \quad (3.10)$$

*The integral kernel of the operator  $e^{-tH}$  is:*

$$e^{-tH}(x, x') = \frac{\Omega}{2\pi \sinh \Omega t} e^{-A}, \quad (3.11)$$

$$A = \frac{\Omega \cosh \Omega t}{2 \sinh \Omega t} (x^2 + x'^2) - \frac{\Omega \cosh \Omega t}{\sinh \Omega t} x \cdot x' - i\Omega x \wedge x'. \quad (3.12)$$

### 3.2 Fermionic $x$ Space Kernel

The two-dimensional free commutative Fermionic field theory is defined by the Lagrangian

$$\mathcal{L} = \bar{\psi}(x) (\not{p} + \mu) \psi(x). \quad (3.13)$$

The propagator of the theory  $(\not{p} + \mu)^{-1}(x, y)$  can be calculated thanks to the usual heat kernel method as

$$\begin{aligned} (\not{p} + \mu)^{-1}(x, y) &= (-\not{p} + \mu) ((\not{p} + \mu) (-\not{p} + \mu))^{-1}(x, y) \\ &= (-\not{p} + \mu) (p^2 + \mu^2)^{-1}(x, y) \end{aligned} \quad (3.14)$$

$$= (-\not{p} + \mu) \int_0^\infty \frac{dt}{4\pi t} e^{-\frac{(x-y)^2}{4t} - \mu^2 t} \quad (3.15)$$

$$= \int_0^\infty \frac{dt}{4\pi t} \left( \frac{-i}{2t} (\not{x} - \not{y}) + \mu \right) e^{-\frac{(x-y)^2}{4t} - \mu^2 t}. \quad (3.16)$$

In the noncommutative case we have to modify the free action, adding a Grosse-Wulkenhaar term to implement Langmann-Szabo duality. This shall prevent ultra-violet infrared mixing in theories with generic interaction of the Gross-Neveu type and allow consistent renormalization to all orders of perturbation.

The free action becomes after vulcanization

$$S_{free} = \int d^2x \bar{\psi}^a(x) (\not{p} + \mu + \Omega \not{\tilde{x}}) \psi^a(x) \quad (3.17)$$

where  $\tilde{x} = 2\Theta^{-1}x$  and  $\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$  and  $a$  is a color index which takes values 1, ...  $N$ . The corresponding propagator  $G$  is diagonal in this color index, so we omit it in this section<sup>a</sup>. To compute this propagator we write as in the commutative case:

$$\begin{aligned} G &= (\not{p} + \mu + \Omega \not{\tilde{x}})^{-1} = (-\not{p} + \mu - \Omega \not{\tilde{x}}) \cdot Q^{-1}, \\ Q &= (\not{p} + \mu + \Omega \not{\tilde{x}}) (-\not{p} + \mu - \Omega \not{\tilde{x}}) \\ &= \mathbb{1}_2 \otimes \left( p^2 + \mu^2 + \frac{4\Omega^2}{\theta^2} x^2 \right) + \frac{4i\Omega}{\theta} \gamma^0 \gamma^1 \otimes \text{Id} + \frac{4\Omega}{\theta} \mathbb{1}_2 \otimes L_2, \end{aligned} \quad (3.18)$$

where  $L_2 = x^0 p_1 - x^1 p_0$ .

To invert  $Q$  we use again the Schwinger trick and obtain:

---

<sup>a</sup>There is no star product in these formulas, since we reduce quadratic expressions in the fields in non-commutative theory to usual integrals with ordinary products.

**Lemma 3.2** *We have:*

$$G(x, y) = -\frac{\Omega}{\theta\pi} \int_0^\infty \frac{dt}{\sinh(2\tilde{\Omega}t)} e^{-\frac{\tilde{\Omega}}{2} \coth(2\tilde{\Omega}t)(x-y)^2 + i\tilde{\Omega}x\wedge y} \quad (3.19)$$

$$\left\{ i\tilde{\Omega} \coth(2\tilde{\Omega}t)(\not{x} - \not{y}) + \Omega(\not{x} - \not{y}) - \mu \right\} e^{-2i\tilde{\Omega}t\gamma^0\gamma^1} e^{-t\mu^2}$$

*It is also convenient to write  $G$  in terms of commutators:*

$$G(x, y) = -\frac{\Omega}{\theta\pi} \int_0^\infty dt \left\{ i\tilde{\Omega} \coth(2\tilde{\Omega}t) [\not{x}, \Gamma^t](x, y) \right. \\ \left. + \Omega [\not{x}, \Gamma^t](x, y) - \mu\Gamma^t(x, y) \right\} e^{-2i\tilde{\Omega}t\gamma^0\gamma^1} e^{-t\mu^2}, \quad (3.20)$$

where

$$\Gamma^t(x, y) = \frac{1}{\sinh(2\tilde{\Omega}t)} e^{-\frac{\tilde{\Omega}}{2} \coth(2\tilde{\Omega}t)(x-y)^2 + i\tilde{\Omega}x\wedge y} \quad (3.21)$$

with  $\tilde{\Omega} = \frac{2\Omega}{\theta}$  and  $x \wedge y = x^0y^1 - x^1y^0$ .

**Proof.** The proof follows along the same lines than for Lemma 3.1 and is given in detail in Appendix B. Note that the constant term  $e^{-2i\tilde{\Omega}t\gamma^0\gamma^1}$  is developed in (B.6).  $\square$

### 3.3 Bosonic Kernel in the Matrix Basis

Let  $H$  be as in Lemma 3.1, with  $\Omega \rightarrow \frac{2\Omega}{\theta}$  and  $B \rightarrow \frac{2B}{\theta}$ . A straightforward computation shows that in the matrix basis we have:

$$H_{m,m+h;l+h,l} = \frac{2}{\theta}(1 + \Omega^2)(2m + h + 1)\delta_{m,l} - \frac{4Bh}{\theta} \delta_{m,l} \quad (3.22)$$

$$- \frac{2}{\theta}(1 - \Omega^2)[\sqrt{(m+h+1)(m+1)} \delta_{m+1,l} + \sqrt{(m+h)m} \delta_{m-1,l}].$$

Notice that in the limiting case  $\Omega = B = 1$  the operator becomes diagonal.

The corresponding propagator in the matrix basis for  $B = 0$  can be found in [2] and [12]. The result is that the only non zero matrix elements of the exponential are:

$$[e^{-\frac{t\theta}{8\tilde{\Omega}}H}]_{m,m+h;l+h,l} = \quad (3.23)$$

$$\sum_{u=\max(0,-h)}^{\min(m,l)} \left( \frac{4\Omega}{(1+\Omega)^2} \right)^{h+2u+1} \left( \frac{1-\Omega}{1+\Omega} \right)^{m+l-2u} \mathcal{E}(m, l, h, u) \mathcal{A}(m, l, h, u)$$



with

$$\mathcal{A}(m, l, h, u) = \frac{\sqrt{m!(m+h)!l!(l+h)!}}{(m-u)!(l-u)!(h+u)!u!}, \quad (3.24)$$

$$\mathcal{E}(m, l, h, u) = \frac{e^{-t(\frac{h+1}{2}+u)}(1-e^{-t})^{m+l-2u}}{\left(1 - \left(\frac{1-\Omega}{1+\Omega}\right)^2 e^{-t}\right)^{m+l+h+1}}. \quad (3.25)$$

Having in mind the slicing of the propagators needed to carry out the renormalization (see [3] for the  $\phi^4$  case), we will use a slightly different representation of the propagator:

$$H^{-1} = \frac{\theta}{8\Omega} \int_0^1 \frac{d\alpha}{1-\alpha} (1-\alpha)^{\frac{\theta}{8\Omega}H}. \quad (3.26)$$

One has then the lemma:

**Lemma 3.3** *Let  $H$  be given by equation (3.22) with  $B = 0$ . We have:*

$$[(1-\alpha)^{\frac{\theta}{8\Omega}H}]_{m,m+h;l+h,l} = \sum_{u=\max(0,-h)}^{\min(m,l)} \mathcal{A}(m, l, h, u) \left(C \frac{1+\Omega}{1-\Omega}\right)^{m+l-2u} \mathcal{E}(m, l, h, u) \quad (3.27)$$

with  $\mathcal{A}(m, l, h, u)$  as before,  $C = \frac{(1-\Omega)^2}{4\Omega}$ , and

$$\mathcal{E}(m, l, h, u) = \frac{(1-\alpha)^{\frac{h+2u+1}{2}} \alpha^{m+l-2u}}{(1+C\alpha)^{m+l+h+1}}. \quad (3.28)$$

The proof is given in Appendix C below. Extending to the  $B \neq 0$  case, we get easily the following corollary, useful for studying the Gross-Neveu model:

**Corollary 3.2** *Let  $B \neq 0$ . Denote  $H_0 = H|_{B=0}$ . We have:*

$$[(1-\alpha)^{\frac{\theta}{8\Omega}H}]_{m,m+h;l+h,l} = [(1-\alpha)^{\frac{\theta}{8\Omega}H_0}]_{m,m+h;l+h,l} (1-\alpha)^{-\frac{4B}{8\Omega}h}. \quad (3.29)$$

### 3.4 Fermionic Kernel in the Matrix Basis

Let  $L_2 = -\imath(x^0\partial_1 - x^1\partial_0)$ . The inverse of the quadratic form

$$\Delta = Q - \frac{4\imath\Omega}{\theta} \gamma^0 \gamma^1 = p^2 + \mu^2 + \frac{4\Omega^2}{\theta^2} x^2 + \frac{4B}{\theta} L_2 \quad (3.30)$$

is given by the previous section:

$$\Gamma_{m,m+h;l+h,l} = \frac{\theta}{8\Omega} \int_0^1 d\alpha \frac{(1-\alpha)^{\frac{\mu^2\theta}{8\Omega}-\frac{1}{2}}}{(1+C\alpha)} \Gamma_{m,m+h;l+h,l}^\alpha,$$

$$\Gamma_{m,m+h;l+h,l}^{(\alpha)} = \left( \frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{m+l+h} (1-\alpha)^{-\frac{Bh}{2\Omega}} \quad (3.31)$$

$$\sum_{u=0}^{\min(m,l)} \mathcal{A}(m, l, h, u) \left( \frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}(1-\Omega)} \right)^{m+l-2u}, \quad (3.32)$$

where  $\mathcal{A}(m, l, h, u)$  is given by (3.24) and  $C$  is defined in Lemma 3.3.

The Fermionic propagator  $G$  (3.20) in matrix space can be deduced from this kernel. One should simply take  $B = \Omega$ , add the missing  $\gamma^0\gamma^1$  term, and compute the action of  $-\not{p} - \Omega\not{x} + \mu$  on  $\Gamma$ . Hence we have to compute  $[x^\nu, \Gamma]$  in the matrix basis. It is easy to express the multiplicative operator  $x^\nu$  in this matrix basis. Its commutator with  $\Gamma$  follows from

$$\begin{aligned} [x^0, \Gamma]_{m,n;k,l} = & 2\pi\theta \sqrt{\frac{\theta}{8}} \left\{ \sqrt{m+1}\Gamma_{m+1,n;k,l} - \sqrt{l}\Gamma_{m,n;k,l-1} + \sqrt{m}\Gamma_{m-1,n;k,l} \right. \\ & - \sqrt{l+1}\Gamma_{m,n;k,l+1} + \sqrt{n+1}\Gamma_{m,n+1;k,l} - \sqrt{k}\Gamma_{m,n;k-1,l} \\ & \left. + \sqrt{n}\Gamma_{m,n-1;k,l} - \sqrt{k+1}\Gamma_{m,n;k+1,l} \right\}, \end{aligned} \quad (3.33)$$

$$\begin{aligned} [x^1, \Gamma]_{m,n;k,l} = & 2i\pi\theta \sqrt{\frac{\theta}{8}} \left\{ \sqrt{m+1}\Gamma_{m+1,n;k,l} - \sqrt{l}\Gamma_{m,n;k,l-1} - \sqrt{m}\Gamma_{m-1,n;k,l} \right. \\ & + \sqrt{l+1}\Gamma_{m,n;k,l+1} - \sqrt{n+1}\Gamma_{m,n+1;k,l} + \sqrt{k}\Gamma_{m,n;k-1,l} \\ & \left. + \sqrt{n}\Gamma_{m,n-1;k,l} - \sqrt{k+1}\Gamma_{m,n;k+1,l} \right\}. \end{aligned} \quad (3.34)$$

This leads to the formula for  $G$  in matrix space:

**Lemma 3.4** *Let  $G_{m,n;k,l}$  be the matrix basis kernel of the operator  $(\not{p} + \Omega\not{x} + \mu)^{-1}$ . We have:*

$$\begin{aligned} G_{m,n;k,l} &= -\frac{2\Omega}{\theta^2\pi^2} \int_0^1 d\alpha G_{m,n;k,l}^\alpha \\ G_{m,n;k,l}^\alpha &= \left( i\tilde{\Omega} \frac{2-\alpha}{\alpha} [\not{x}, \Gamma^\alpha]_{m,n;k,l} + \Omega [\not{x}, \Gamma^\alpha]_{m,n;k,l} - \mu \Gamma_{m,n;k,l}^\alpha \right) \\ &\quad \times \left( \frac{2-\alpha}{2\sqrt{1-\alpha}} \mathbb{1}_2 - i \frac{\alpha}{2\sqrt{1-\alpha}} \gamma^0\gamma^1 \right). \end{aligned} \quad (3.35)$$

where  $\Gamma^\alpha$  is given by (3.32) and the commutators by formulae (3.33) and (3.34).

The first two terms in (3.35) contain commutators and are grouped together under the name  $G_{m,n;k,l}^{\alpha,\text{comm}}$ . The last term is called  $G_{m,n;k,l}^{\alpha,\text{mass}}$ . Hence

$$G_{m,n;k,l}^{\alpha,\text{comm}} = \left( i\tilde{\Omega} \frac{2-\alpha}{\alpha} [\not{x}, \Gamma^\alpha]_{m,n;k,l} + \Omega [\not{x}, \Gamma^\alpha]_{m,n;k,l} \right) \times \left( \frac{2-\alpha}{2\sqrt{1-\alpha}} \mathbb{1}_2 - i \frac{\alpha}{2\sqrt{1-\alpha}} \gamma^0 \gamma^1 \right). \quad (3.36)$$

$$G_{m,n;k,l}^{\alpha,\text{mass}} = -\mu \Gamma_{m,n;k,l}^\alpha \times \left( \frac{2-\alpha}{2\sqrt{1-\alpha}} \mathbb{1}_2 - i \frac{\alpha}{2\sqrt{1-\alpha}} \gamma^0 \gamma^1 \right). \quad (3.37)$$

## 4 Non-Commutative Gross-Neveu Model in the Matrix Basis

The Euclidean two-dimensional Gross-Neveu model is written in terms of  $N$  pairs of conjugate Grassmann fields  $\bar{\psi}^a, \psi^a$ ,  $a = 1, \dots, N$ . In the commutative case the interaction has the following "N-vector" form:

$$\lambda \left( \sum_a \bar{\psi}^a \psi^a \right)^2(x). \quad (4.1)$$

There are several non-commutative generalizations of this interaction, depending on how to put the star product with respect to color and conjugation. The most general action involving a vertex with two colors, each present on one  $\bar{\psi}$  and one  $\psi$  takes the form (using cyclicity of the integral trace of star products):

$$S = S_{free} + \int \text{Tr} V(\bar{\psi}, \psi), \quad V(\bar{\psi}, \psi) = \sum_{a,b} \lambda_1 \bar{\psi}^a \star \psi^a \star \bar{\psi}^b \star \psi^b + \lambda_2 \bar{\psi}^a \star \psi^b \star \bar{\psi}^b \star \psi^a + \lambda_3 \bar{\psi}^a \star \bar{\psi}^b \star \psi^a \star \psi^b + \lambda_4 \bar{\psi}^a \star \bar{\psi}^b \star \psi^b \star \psi^a \quad (4.2)$$

where  $S_{free}$  is defined in (3.17). Recall that in the matrix basis the Grassmann fields  $\bar{\psi}^a, \psi^a$ ,  $a = 1, \dots, N$  are Grassmann matrices  $\bar{\psi}_{mn}^a, \psi_{mn}^a$ ,  $a = 1, \dots, N$ ,  $m, n \in \mathbb{N}$ .

In the cases  $\lambda_3 = \lambda_4 = 0$ , there are no non planar tadpole of the type of figure 1 which lead to infrared-ultraviolet mixing in  $\phi_4^4$ . Hence one may superficially conclude that there is no need to vulcanize the free action, hence

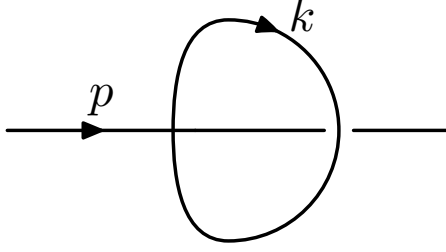


Figure 1: The non-planar tadpole

to use (3.17). However even for  $\lambda_3 = \lambda_4 = 0$ , four point functions lead to "logarithmic" IR/UV mixing, hence to "renormalons" effects (large graphs with amplitudes of size  $n!$  at order  $n$ ). Since we want anyway to renormalize generic Gross-Neveu actions in which  $\lambda_3 \neq 0$  or  $\lambda_4 \neq 0$ , we shall always use the vulcanized free action and propagator.

A proof of the BPHZ renormalization theorem according to the multiscale analysis [3] decomposes into two main steps. First one has to prove bounds on the sliced propagator; then using these bounds one has to prove that irrelevant operators in the multiscale analysis give rise to convergent sums, and that this is also the case for marginal and relevant operators *after* subtracting a singular part of the same form than the original action.

In this paper we provide the first part of this proof, namely the appropriate bounds. The rest of the proof of renormalizability to all orders of the generic model (4.2) is postponed to a future paper [11]. However we illustrate on a particular example below why the bounds of this paper (which are optimal in a certain sense) should be sufficient for this task, which is however significantly more difficult than the corresponding task for  $\phi_4^4$ .

The multiscale slice decomposition is performed as in [3]

$$\int_0^1 d\alpha = \sum_{i=1}^{\infty} \int_{M^{-i}}^{M^{-i+1}} d\alpha \quad (4.3)$$

and leads to the following propagator for the  $i^{\text{th}}$  slice:

$$\Gamma_{m,m+h,l+h,l}^i = \frac{\theta}{8\Omega} \int_{M^{-i}}^{M^{-i+1}} d\alpha \frac{(1-\alpha)^{\frac{\mu_0^2 \theta}{8\Omega} - \frac{1}{2}}}{(1+C\alpha)} \Gamma_{m,m+h;l+h,l}^{(\alpha)} \quad (4.4)$$

$$G_{m,n;k,l} = \sum_{i=1}^{\infty} G_{m,n;k,l}^i ; G_{m,n;k,l}^i = -\frac{2\Omega}{\theta^2 \pi^2} \int_{M^{-i}}^{M^{-i+1}} d\alpha G_{m,n;k,l}^\alpha \quad (4.5)$$

We split  $G$  according to (3.36) and (3.37), and we now bound  $|G_{m,n;k,l}^i|$ . We define  $h = n - m$  and  $p = l - m$ . By obvious symmetry of the integer indices we can assume  $h \geq 0$ , and  $p \geq 0$ , so that the smallest of the four integers  $m, n, k, l$  is  $m$  and the largest is  $k = m + h + p$ . The main result of this paper is the following bound:

**Theorem 4.1** *Under the assumed conditions  $h = n - m \geq 0$  and  $p = l - m \geq 0$  the Gross-Neveu propagator in a slice  $i$  obeys the bound*

$$|G_{m,n;k,l}^{i,\text{comm}}| \leq KM^{-i/2} \left( \frac{\exp\left\{-\frac{cp^2}{1+kM^{-i}} - \frac{cM^{-i}}{1+k} \left(h - \frac{k}{1+C}\right)^2\right\}}{(1 + \sqrt{kM^{-i}})} + e^{-ckM^{-i}-cp} \right). \quad (4.6)$$

for some (large) constant  $K$  and (small) constant  $c$  which depend only on  $\Omega$ . Furthermore the part with the mass term has a slightly different bound:

$$|G_{m,n;k,l}^{i,\text{mass}}| \leq KM^{-i} \left( \frac{\exp\left\{-\frac{cp^2}{1+kM^{-i}} - \frac{cM^{-i}}{1+k} \left(h - \frac{k}{1+C}\right)^2\right\}}{1 + \sqrt{kM^{-i}}} + e^{-ckM^{-i}-cp} \right). \quad (4.7)$$

The rest of the paper is devoted to the proof of this theorem. We give this proof only for  $i \gg 1$ , the “first slices” being unimportant for renormalization.

In the rest of this section we indicate how this bound leads to efficient power counting estimates for renormalization.

Recall first that for any non-commutative Feynman graph  $G$  we can define the genus of the graph, called  $g$  and the number of faces “broken by external legs”, called  $B$  as in [2]-[3]. We have  $g \geq 0$  and  $B \geq 1$ . The power counting established for  $\phi_4^4$  in [2]-[3] involves the superficial degree of divergence of a graph

$$\omega(G) = (2 - N/2) - 4g - 2(B - 1), \quad (4.8)$$

and it is positive only for  $N = 2$  and  $N = 4$  subgraphs with  $g = 0$  and  $B = 1$ . These are the only non-vacuum graphs that have to be renormalized. We expect the same conclusion for the two-dimensional Gross-Neveu model, since this holds for the commutative counterpart of these models.

Let us sketch now why the multislice analysis based on bound (4.1) proves that a graph with internal propagators in slice  $i \gg 1$  and external legs in slice 1 with  $N \geq 6$  or  $N = 4$  and  $g \geq 1$  does not require renormalization. First remark that the second term in bound (4.1) gives exactly the same decay proved in [3]. The  $O(1)$  decay in  $p$  means that the model is quasi-local in the sense of [12]. Hence all indices except those of independent faces cost  $O(1)$  to sum. Each main “face index” is summed with the scale decay

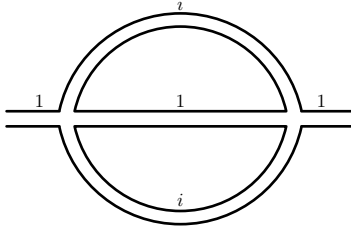


Figure 2: The Sunset Graph

$kM^{-i}$ , hence each face sum costs  $M^i$  in two dimensions. Combined with the  $M^{-i/2}$  scaling factors of the propagators in (4.1) one recovers the usual power counting in  $\omega$ .

Hence let us concentrate on the more difficult case of the first part of bound (4.1), and for instance consider the “sunset” graph  $G$  of Figure 2

When the scales of the three internal lines are roughly identical, this graph should be renormalized as a two-point subgraph and nothing particularly new happens. But something new occurs when the two *exterior* lines have scales  $i \gg 1$  and the interior line has scale 1 (like the external legs), as shown on the figure. In the usual multiscale analysis we do not have a divergent two point subgraph in the traditional sense. The subgraph made of the two external lines is “dangerous”, i.e. has all internal scales above all external ones. But this graph has  $N = 4$  and  $g = 0$ ,  $B = 2$ , and it should *not* and in fact *cannot* be renormalized. However applying bound (4.6) the sum over  $i$  diverges logarithmically! Indeed the sums over the  $p$  indices cost only  $O(1)$  as usual for a quasi-matrix model, but the two internal faces sums, together with their lines prefactor give

$$\sum_{k, \delta h=0}^{\infty} M^{-i/2} M^{-i/2} e^{-M^{-i}k} \frac{e^{-(\delta h)^2/k}}{1 + \sqrt{k}} = O(1). \quad (4.9)$$

What is the solution to this riddle? In this case it is the full *two-point* subgraph  $G$  which has to be renormalized. This works because the renormalization improvement brings modified “composite propagators” *solely* on the exterior face of the graph [2]. These improved propagators have scale  $i$ , hence they bring a factor  $M^{-i}$ , and the sum over  $i$  converges.

One has to generalize this argument, and show that all the counterterms are of the right form to complete the BPHZ theorem for this kind of models [11]. In short all dangerous subgraphs with  $N = 2, 4$ ,  $g = 0$ ,  $B = 1$ , and  $N = 4$ ,  $g = 0$ ,  $B = 2$  should be renormalized, the last ones being renormalized by the corresponding *two-point* function counterterms. This subtlety makes

the multislice formulation of renormalization in the non-commutative Gross-Neveu model more complicated than in the non-commutative  $\phi_4^4$ .

## 5 Proof of Theorem 4.1

We cast the propagator for  $B = \Omega$  in the following form:

$$\Gamma = \int_0^1 d\alpha \frac{(1-\alpha)^{-1/2}}{1+C\alpha} \Gamma^\alpha \quad (5.1)$$

with:

$$\Gamma^\alpha = \left( \frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{2m+p} \frac{1}{(1+C\alpha)^h} \sum_{u=0}^m \left( \frac{\alpha \sqrt{C(1+C)}}{\sqrt{1-\alpha}} \right)^{2m+p-2u} \mathcal{A}(m, m+p, h, u). \quad (5.2)$$

We have:

$$\Gamma^\alpha = e^{(2m+p) \ln \frac{\sqrt{1-\alpha}}{1+C\alpha} - h \ln(1+C\alpha)} \sum_{0 \leq v=m-u \leq m} e^{(2v+p) \ln \frac{\alpha \sqrt{C(1+C)}}{\sqrt{1-\alpha}}} \mathcal{A}(m, m+p, h, u). \quad (5.3)$$

We consider the regime  $\alpha \ll C \ll 1$  hence we limit ourselves as usually to a parameter  $\Omega$  close to 1. We use Stirling's formula to write:

$$\Gamma^\alpha \leq K \sum_{0 \leq v \leq k-h-p} e^{f(m,l,v)} \frac{\left( (2\pi)^4 k(k-h)(k-p)(k-h-p) \right)^{1/4}}{\sqrt{(2\pi)^4 v(v+p)(k-v-h-p)(k-v-p)}} \quad (5.4)$$

with:

$$\begin{aligned} f &= (2k-2h-p) \ln \frac{\sqrt{1-\alpha}}{1+C\alpha} - h \ln(1+C\alpha) + (2v+p) \ln \frac{\alpha \sqrt{C(1+C)}}{\sqrt{1-\alpha}} \\ &+ \frac{k-h-p}{2} \ln(k-h-p) + \frac{k-h}{2} \ln(k-h) + \frac{k-p}{2} \ln(k-p) \\ &+ \frac{k}{2} \ln k - v \ln v - (v+p) \ln(v+p) - (k-v-p) \ln(k-v-p) \\ &- (k-v-h-p) \ln(k-v-h-p). \end{aligned} \quad (5.5)$$

We define the reduced variables  $x = v/k$ ,  $y = h/k$ ,  $z = p/k$ . These parameters live in the compact simplex  $0 \leq x, y, z \leq 1$ ,  $0 \leq x + y + z \leq 1$ . In our propagator bound we can replace the Riemann sum over  $v$  with the integral. Since we have a bounded function on a compact interval, this is a rigorous

upper bound (up to some inessential overall constant) for  $k$  large, which is the case of interest:

$$\Gamma^\alpha \leq \int_0^{1-y-z} dx \frac{[(1-y)(1-z)(1-z-y)]^{1/4}}{[x(x+z)(1-x-z)(1-x-y-z)]^{1/2}} e^{kg(x,y,z)} \quad (5.6)$$

with the function  $g$  defined by:

$$\begin{aligned} g = & (2-2y-z) \ln \frac{\sqrt{1-\alpha}}{1+C\alpha} + (2x+z) \ln \frac{\alpha\sqrt{C(1+C)}}{\sqrt{1-\alpha}} - y \ln(1+C\alpha) \\ & + \frac{1-y}{2} \ln(1-y) + \frac{1-z}{2} \ln(1-z) + \frac{1-y-z}{2} \ln(1-y-z) \\ & - x \ln x - (x+z) \ln(x+z) - (1-x-z) \ln(1-x-z) \\ & - (1-x-y-z) \ln(1-x-y-z). \end{aligned} \quad (5.7)$$

The differential is

$$\begin{aligned} dg = & dx \left\{ \ln \frac{\alpha^2 C(1+C)(1-x-z)(1-x-y-z)}{(1-\alpha)x(x+z)} \right\} \\ & + dy \left\{ \ln \frac{(1+C\alpha)(1-x-y-z)}{(1-\alpha)\sqrt{(1-y)(1-y-z)}} \right\} \\ & + dz \left\{ \ln \frac{\alpha\sqrt{C(1+C)}(1+C\alpha)(1-x-z)(1-x-y-z)}{(1-\alpha)(x+z)\sqrt{(1-z)(1-y-z)}} \right\}. \end{aligned} \quad (5.8)$$

The second derivatives are

$$\begin{aligned} \frac{\partial^2 g}{\partial x^2} &= -\frac{1}{x} - \frac{1}{x+z} - \frac{1}{1-x-z} - \frac{1}{1-x-y-z}, \\ \frac{\partial^2 g}{\partial y^2} &= \frac{1}{2(1-y)} + \frac{1}{2(1-y-z)} - \frac{1}{1-x-y-z}, \\ \frac{\partial^2 g}{\partial z^2} &= \frac{1}{2(1-z)} + \frac{1}{2(1-y-z)} - \frac{1}{x+z} - \frac{1}{1-x-z} - \frac{1}{1-x-y-z}, \\ \frac{\partial^2 g}{\partial x \partial y} &= -\frac{1}{1-x-y-z}, \\ \frac{\partial^2 g}{\partial x \partial z} &= -\frac{1}{x+z} - \frac{1}{1-x-z} - \frac{1}{1-x-y-z}, \\ \frac{\partial^2 g}{\partial y \partial z} &= \frac{1}{2(1-y-z)} - \frac{1}{1-x-y-z}. \end{aligned} \quad (5.9)$$

**Lemma 5.1** *The function  $g$  is concave in the simplex.*



**Proof** We have to prove that the quadratic form  $-Q$  defined by the 3 by 3 symmetric matrix of the second derivatives is negative in the whole simplex. In others words we should prove that the oppposite quadratic form

$$\begin{aligned}
Q &= \left[ \frac{1}{x} + \frac{1}{x+z} + \frac{1}{1-x-z} + \frac{1}{1-x-y-z} \right] u^2 \\
&+ \left[ \frac{1}{1-x-y-z} - \frac{1}{2(1-y)} - \frac{1}{2(1-y-z)} \right] v^2 \\
&+ \left[ \frac{1}{x+z} + \frac{1}{1-x-z} + \frac{1}{1-x-y-z} - \frac{1}{2(1-z)} - \frac{1}{2(1-y-z)} \right] w^2 \\
&+ \frac{2}{1-x-y-z} uv + \left[ \frac{2}{x+z} + \frac{2}{1-x-z} + \frac{2}{1-x-y-z} \right] uw \\
&+ \left[ \frac{2}{1-x-y-z} - \frac{1}{1-y-z} \right] vw
\end{aligned} \tag{5.10}$$

is positive. But we have

$$\begin{aligned}
Q &= \frac{(u+v+w)^2}{1-x-y-z} + \frac{(u+w)^2}{(x+z)(1-x-z)} + \frac{u^2}{x} - \frac{v^2}{2(1-y)} - \frac{(v+w)^2}{2(1-y-z)} \\
&\geq \frac{(u+v+w)^2}{1-x-y-z} + \frac{(u+w)^2}{2(x+z)} + \frac{u^2}{2x} - \frac{v^2}{2(1-y)} - \frac{(v+w)^2}{2(1-y-z)} \\
&= \frac{1}{2} \left( \left[ \frac{(u+v+w)^2}{1-x-y-z} - \frac{v^2}{1-y} + \frac{(u+w)^2}{x+z} \right] \right. \\
&\quad \left. + \left[ \frac{(u+v+w)^2}{1-x-y-z} - \frac{(v+w)^2}{1-y-z} + \frac{u^2}{x} \right] \right) \\
&= \frac{1}{2} \left( \frac{1}{1-y} \left[ \sqrt{\frac{x+z}{1-x-y-z}} (u+v+w) + \sqrt{\frac{1-x-y-z}{x+z}} (u+w) \right]^2 \right. \\
&\quad \left. + \frac{1}{1-y-z} \left[ \sqrt{\frac{x}{1-x-y-z}} (u+v+w) + \sqrt{\frac{1-x-y-z}{x}} u \right]^2 \right) \\
&\geq 0.
\end{aligned} \tag{5.11}$$

□

**Lemma 5.2** *The only critical point of the function in the closed simplex is at  $x_0 = \frac{C\alpha}{1+C\alpha}$ ,  $y_0 = \frac{1}{1+C}$ ,  $z = 0$ , where the function  $g = 0$ .*

**Proof** One can easily check that:

$$g(x_0, y_0, 0) = \partial_x g(x_0, y_0, 0) = \partial_y g(x_0, y_0, 0) = \partial_z g(x_0, y_0, 0) = 0. \tag{5.12}$$

The unicity follows from the concavity of the function  $g$ .  $\square$

Our bound on  $g$  will be inspired by the steepest descent method around the critical point. We divide now the simplex into:

- the neighborhood of the maximum. We call this region the “mountain top”. It corresponds to  $\delta x = |x - x_0| \ll \alpha$ ,  $\delta y = |y - y_0| \ll O(1)$ ,  $z \ll \alpha$ . For aesthetic reasons we prefer to use a reference quadratic form  $Q_0$  to define a smooth border of this region. Hence putting  $X = (\delta x, \delta y, z)$  we define the mountain top by the condition

$$XQ_0{}^tX = \frac{(\delta x)^2 + z^2}{\alpha^2} + (\delta y)^2 \leq \eta \quad (5.13)$$

where  $\eta$  is a small constant,

- the rest of the simplex. This region is defined by  $XQ_0{}^tX \geq \eta$ .

## 5.1 The “Mountain Top”

In this region we use the Hessian approximation and check that the cubic correction terms are small with respect to this leading order.

**Lemma 5.3** *In the mountain top region, for some small constant  $c$  (which may depend on  $C$ , hence on  $\Omega$ ):*

$$g(X) \leq -c\alpha XQ_0{}^tX = -c\alpha \left[ \frac{(\delta x)^2 + z^2}{\alpha^2} + (\delta y)^2 \right] \quad (5.14)$$

**Proof** From (5.15) we evaluate the second order derivatives of  $g$  at leading order in  $\alpha$  at the maximum:

$$\begin{aligned} \frac{\partial^2 g}{\partial x^2} &\approx -\frac{2}{\alpha C}; & \frac{\partial^2 g}{\partial x \partial z} &\approx -\frac{1}{\alpha C}, \\ \frac{\partial^2 g}{\partial z^2} &\approx -\frac{1}{\alpha C}; & \frac{\partial^2 g}{\partial x \partial y} &\approx -\frac{1+C}{C}, \\ \frac{\partial^2 g}{\partial y^2} &\approx -\alpha \frac{1+C}{C}; & \frac{\partial^2 g}{\partial y \partial z} &\approx -\frac{1+C}{2C}. \end{aligned} \quad (5.15)$$

It is easy to diagonalize the corresponding 3 by 3 quadratic form, and to check that in the neighborhood of the maximum it is smaller than  $-c\alpha Q_0$  for some small constant  $c$ :

$$g_{Hessian}(\delta x, \delta y, z) = XQ_{Hessian}{}^tX \leq -\alpha c XQ_0{}^tX = -\alpha c \left( \frac{(\delta x)^2 + z^2}{\alpha^2} + (\delta y)^2 \right), \quad (5.16)$$

where  $g_{Hessian}$  is the Hessian approximation to the function  $g$ . It is easy to check from the expression (5.8) of the differential  $dg$  that the third order derivatives scale in the appropriate way so that choosing the constant  $\eta$  small enough in (5.13), the function  $g$  obeys the same bound than (5.16) with a slightly different constant  $c$ .  $\square$

## 5.2 The Rest of the Simplex

To bound the function  $g$  in the whole simplex we use the previous notation  $X = (\delta x, \delta y, z)$ . Drawing the segment from point  $X$  to the origin (i.e. the mountain top), we cross the border of the mountain top at  $X_0 = \lambda X$  with  $X_0 Q_0 {}^t X_0 = \eta$ . We define  $X_1 = X_0/2 = (\lambda/2)X$ .  $X$  out of the mountain top means that  $\lambda = \sqrt{\eta/XQ_0 {}^t X} \leq 1$ .

**Lemma 5.4** *Out of the mountain top region the function  $g(X) = g(\delta x, \delta y, z)$  obeys the bound, for some small enough constant  $c$ :*

$$g(\delta x, \delta y, z) \leq -c(\alpha + |\delta x| + z). \quad (5.17)$$

**Proof** We use concavity of the function on the segment considered, which means that the function  $g$  is below its first order Taylor approximation at  $X_1$ :

$$g(X) \leq g(X_1) + \langle dg(X_1), X - X_1 \rangle. \quad (5.18)$$

At  $X_1$  the Hessian approximation of  $g$  is valid, say up to a factor 2. Hence

$$g(X) \leq (1/2)[X_1 Q_{Hessian} {}^t X_1 + 2(X - X_1) Q_{Hessian} {}^t X_1]. \quad (5.19)$$

Using (5.16) we can relate  $Q_{Hessian}$  to our reference quadratic form  $Q_0$  up to a constant and get for some small  $c$ :

$$\begin{aligned} g(X) &\leq -c\alpha[X_1 Q_0 {}^t X_1 + 2(X - X_1) Q_0 {}^t X_1] \\ &= -c\alpha[(\eta/4) + \lambda(1 - \lambda/2)XQ_0 {}^t X] \\ &= -c'\alpha[1 + \sqrt{XQ_0 {}^t X}] \end{aligned} \quad (5.20)$$

for some constant  $c'$  smaller than  $c$ . In the last line we used  $1 - \lambda/2 \geq 1/2$  and  $\lambda = \sqrt{\eta/XQ_0 {}^t X}$ . Finally

$$\alpha\sqrt{XQ_0 {}^t X} \geq \sqrt{(\delta x)^2 + z^2} \geq (|\delta x| + z)/\sqrt{2} \quad (5.21)$$

completes the proof of (5.17).  $\square$

### 5.3 Integration on $x$

It remains now to prove some explicit decay of the function  $G$  in the variable  $z$  after integration in  $x$  in (5.6). The decay in  $z$  is necessary to prove that the model is a quasi-local matrix model in the sense of [1]. For the mountain top region, we do not have any decay in  $k$  so we want also to exhibit the decay in  $y$ .

**Lemma 5.5** *For some large constant  $K$  and small constant  $c$ , under the condition  $\alpha k \geq 1$  we have*

$$\Gamma^\alpha \leq K \left( \frac{\exp\left\{-\frac{c}{\alpha k} p^2 - \frac{c\alpha}{k} \left(h - \frac{k}{1+C}\right)^2\right\}}{\sqrt{\alpha k}} + e^{-c\alpha k - cp} \right). \quad (5.22)$$

**Proof** In the integration on  $x$  in (5.6) we can insert  $1 = \chi + (1 - \chi)$  where  $\chi$  is the characteristic function of the mountain top. In the first term we apply the bound (5.14) and in the second the bound (5.17). In this second case we use a better estimation of the prefactors in front of  $e^{kg}$  in (5.6). Actually their expression in (5.6) leads to a spurious logarithmic divergence due to the bad behaviour of the Stirling approximation close to 0. We will use

$$K\sqrt{n+1}n^n e^{-n} \leq n! \leq K'\sqrt{n}n^n e^{-n}, \quad (5.23)$$

which leads to an integral of the type

$$\int_0^1 \frac{dx}{x+1/k} e^{-ckx} \leq K \int_{1/k}^\infty \frac{dx}{x} e^{-ckx}, \quad (5.24)$$

which is bounded by a constant. Scaling back to the original variables completes the proof of Lemma 5.5.  $\square$

### 5.4 The Region $\alpha k \leq 1$

In this region we do not need Stirling's formula at all. It is easier to derive a direct simple bound on  $\Gamma_{m,m+h;k,m+p}^\alpha$  (recall that  $k = m + h + p$ ):

**Lemma 5.6** *For  $M$  large enough and  $\alpha k < 1$ , there exists constants  $K$  and  $c$  such that*

$$\Gamma_{m,m+h;k,m+p}^\alpha \leq K e^{-c(\alpha k + p)}. \quad (5.25)$$

**Proof.** We assume  $h \geq 0$  and  $p \geq 0$ . For  $\alpha k \leq 1$ , the following crude bound on  $\Gamma^\alpha$  follows easily from (5.2):

$$\Gamma_{m,m+h;k,m+p}^\alpha \leq e^{-\alpha(C+1/2)(2m+p) - (C/2)\alpha h} \sum_{u=0}^m \frac{X^{m-u}}{(m-u)!} \frac{Y^{m+p-u}}{(m+p-u)!} \quad (5.26)$$

where  $X = \frac{\alpha\sqrt{C(C+1)}}{\sqrt{1-\alpha}}\sqrt{m(m+h)}$  and  $Y = \frac{\alpha\sqrt{C(C+1)}}{\sqrt{1-\alpha}}\sqrt{(m+p)(m+p+h)}$ .

$$\Gamma_{m,m+h,l+h,l}^\alpha \leq e^{-\alpha(C+1/2)(2m+p)-(C/2)\alpha h} \frac{\left(\frac{\alpha k\sqrt{C(C+1)}}{\sqrt{1-\alpha}}\right)^p}{p!} \sum_{u=0}^m \frac{\left(\frac{\alpha k\sqrt{C(C+1)}}{\sqrt{1-\alpha}}\right)^{2(m-u)}}{(m-u)!^2} \quad (5.27)$$

The sum over  $u$  is bounded by a constant. For  $C$  small (i.e.  $\Omega$  close to 1) we have certainly  $\frac{\sqrt{C(C+1)}}{\sqrt{1-\alpha}} \leq 1/2$  hence we get the desired result.  $\square$

Combining with (5.22) we conclude that Lemma 5.5 always holds:

**Lemma 5.7** *For some large constant  $K$  and small constant  $c$  we have*

$$\Gamma^\alpha \leq K \left( \frac{\exp\left\{-\frac{c}{1+\alpha k}p^2 - \frac{c\alpha}{1+k}\left(h - \frac{k}{1+C}\right)^2\right\}}{1 + \sqrt{\alpha k}} + e^{-c\alpha k - cp} \right). \quad (5.28)$$

## 5.5 Numerator Terms

In this section we check that the numerators in the Gross-Neveu propagator bring the missing power counting factors, hence we complete the proof of Theorem 4.1.

The bound (4.7) is nothing but the direct consequence of multiplying the bound of Lemma 5.7 by the width  $M^{-i}$  of the integration interval over  $\alpha$ . Hence we now prove (4.6).

In the commutator terms (3.36) the  $\Omega$  and the  $i\frac{\alpha}{2\sqrt{1-\alpha}}\gamma^0\gamma^1$  are smaller by at least an  $\alpha$  factor. Therefore the largest piece is the  $O(1/\alpha)[\tilde{x}, \Gamma]$  term. The  $1/\alpha$  factor compensates the width  $M^{-i}$  of the integration interval over  $\alpha$ . Hence we need to prove that a  $[\tilde{x}, \Gamma]$  numerator adds a factor  $\sqrt{\alpha}$  to the bound of Lemma 5.7:

(The  $\Omega$  and the  $i\frac{\alpha}{2\sqrt{1-\alpha}}\gamma^0\gamma^1$  terms in (3.36) have an additional factor  $\alpha$ , hence are much easier to bound and left to the reader. The bound for the mass term  $\mu\Gamma$  just involves multiplying the bound of Lemma 5.7 by the appropriate weight  $M^{-i}$  coming from the  $\alpha$  integration, which leads easily to (4.7).

The commutator  $[\not{x}, \Gamma]$  involves terms like

$$\begin{aligned} & \sqrt{m+1}\Gamma_{m+1,n;k,l} - \sqrt{l}\Gamma_{m,n;k,l-1} \\ &= \left(\sqrt{m+1} - \sqrt{l}\right)\Gamma_{m,n;k,l-1} + \sqrt{m+1}(\Gamma_{m+1,n;k,l} - \Gamma_{m,n;k,l-1}). \end{aligned} \quad (5.29)$$

### 5.5.1 The first term

The first term is the easiest to bound. It is zero unless  $p = l - m - 1 \geq 1$ . In this case, we have

$$\sqrt{m+1} - \sqrt{l} \leq \frac{2p}{1 + \sqrt{l}}. \quad (5.30)$$

Using Lemma 5.8:

- On the mountain top we have  $l = k - h \simeq \frac{C}{1+C}k$ , hence  $\frac{2p}{1+\sqrt{l}} \leq O(1)\frac{p}{1+\sqrt{k}}$ . An additional factor  $\alpha$  comes from (5.31), hence we have a bound in  $O(1)\frac{\alpha p}{1+\sqrt{k}}$ . Using a fraction of the decay  $e^{-\alpha p^2/k}$  from Lemma 5.7 bounds  $O(1)\frac{\alpha p}{1+\sqrt{k}}$  by  $\sqrt{\alpha}$ .
- Out of the mountain top we have a factor  $\alpha\sqrt{kl}$  which comes from (5.32), and we can use  $pe^{-cp} \leq e^{-c'p}$  and  $\sqrt{\alpha k}e^{-\alpha k} \leq e^{-c'\alpha k}$ . Hence the desired factor  $\sqrt{\alpha}$  is obtained.

It remains to prove

**Lemma 5.8** *Let  $(m, l, h) \in \mathbb{N}^3$  with  $p = l - m - 1 \geq 1$ . We have*

- *On the mountain top*

$$\Gamma_{m,m+h;k,m+p} \leq K\alpha \Gamma_{m,m+h;m+p-1+h,m+p-1}, \quad (5.31)$$

- *if  $\Gamma$  is out the mountain top region*

$$\Gamma_{m,m+h;k,m+p} \leq K\alpha\sqrt{kl} \Gamma_{m,m+h;m+p-1+h,m+p-1}. \quad (5.32)$$

**Proof.** Let  $p \geq 1$ . The kernel  $\Gamma_{m,m+h;m+p+h,m+p}$  is given by

$$\begin{aligned} \Gamma_{m,m+h;m+p+h,m+p} &= \left( \frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{2m+p} (1+C\alpha)^{-h} \\ &\quad \sum_{u=0}^m (\alpha D)^{2(m-u)+p} \mathcal{A}(m, m+p, h, u) \end{aligned} \quad (5.33)$$

with  $\mathcal{A}$  defined by (3.24) and  $D(\alpha) = \sqrt{\frac{C(C+1)}{1-\alpha}}$ . The scaling factor  $(\alpha D)^{2(m-u)+p}$  reaches its maximum at  $u = m$ . There we have a factor  $(\alpha D)^p$ . For  $p \geq 1$ , we factorize  $\alpha D$  to get

$$\begin{aligned} \Gamma_{m,m+h;m+p+h,m+p} &= \alpha D \left( \frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{2m+p} (1+C\alpha)^{-h} \\ &\quad \sum_{u=0}^m (\alpha D)^{2(m-u)+p-1} \mathcal{A}(m, m+p, h, u). \end{aligned} \quad (5.34)$$

Using  $\binom{m+p}{m+p-u} = \frac{m+p}{m+p-u} \binom{m+p-1}{m+p-1-u}$  and  $\binom{m+p+h}{m+p-u} = \frac{m+p+h}{m+p-u} \binom{m+p-1+h}{m+p-1-u}$ , we have

$$\begin{aligned} \Gamma_{m,m+h;m+p+h,m+p} &\leq K\alpha \left( \frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{2m+p-1} (1+C\alpha)^{-h} \sum_{u=0}^m (\alpha D)^{2(m-u)+p-1} \\ &\quad \times \mathcal{A}(m, m+p-1, h, u) \frac{\sqrt{(m+p)(m+p+h)}}{m+p-u}. \end{aligned} \quad (5.35)$$

On the ‘‘mountain top’’ we express  $m+p-u$  in term of  $k$  and get

$$\frac{1}{m+p-u} = \frac{1}{k-u} \frac{1}{1-\frac{h}{k-u}}. \quad (5.36)$$

Moreover  $u \simeq \alpha k$  and  $(k-u)^{-1} \simeq k^{-1}$ , which leads to (5.31).

Out of the mountain top we use simply  $m+p-u \geq 1$  and  $\sqrt{(m+p)(m+p+h)} = \sqrt{kl}$ , and (5.32) follows.  $\square$

### 5.5.2 The second term

We now focus on the terms involving differences of  $\Gamma$ 's. For this we need some identities on the combinatorial factor  $\mathcal{A}$ :

$$\mathcal{A}(m, l, h, u) = \frac{\sqrt{ml}}{u} \mathcal{A}(m-1, l-1, h+1, u-1), \text{ for } u \geq 1, \quad (5.37)$$

$$\mathcal{A}(m, l, h, u) = \frac{\sqrt{m(m+h)}}{m-u} \mathcal{A}(m-1, l, h, u), \quad (5.38)$$

$$\mathcal{A}(m, l, h, u) = \frac{\sqrt{m(m+h)l(l+h)}}{(m-u)(l-u)} \mathcal{A}(m-1, l-1, h, u).. \quad (5.39)$$

Let us recall  $h = n - (m+1)$  and  $p = l - (m+1)$ . Then  $\Gamma_{m+1,n;k,l} = \Gamma_{m+1,m+1+h;l+h,l}$  and  $\Gamma_{m,n;k,l-1} = \Gamma_{m,m+h+1;l+h,l-1}$ .

Using Lemma (5.9) we get:

$$\sqrt{m+1} (\Gamma_{m+1,n;k,l} - \Gamma_{m,n;k,l-1}) \leq K\sqrt{\alpha}\Gamma^\alpha, \quad (5.40)$$

on the mountain top by (5.48) and the sequel, as well as out of the critical region by (5.50). The factors thus obtained together with Lemma (5.7) complete the proof of (4.1).

**Lemma 5.9** *For  $M$  large enough and  $\Omega$  close enough to 1, there exist  $K$  such that*

$$G_{m,n;k,l}^\alpha \leq K\sqrt{\alpha}\Gamma_{m-1,n;k-1,l-2}^\alpha. \quad (5.41)$$

**Proof.** It remains to bound the second term in (5.29), namely  $\sqrt{m+1}(\Gamma_{m+1,n;k,l} - \Gamma_{m,n;k,l-1})$ .

$$\begin{aligned} & \Gamma_{m+1,m+1+h;l+h,l} - \Gamma_{m,m+h+1;l+h,l-1} \tag{5.42} \\ &= \left( \frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{m+1+l} (1+C\alpha)^{-h} \sum_{u_1=0}^{m+1} (\alpha D)^{m+1+l-2u_1} \mathcal{A}(u_1; m+1, l, h) \\ & \quad - \left( \frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{m+l-1} (1+C\alpha)^{-h-1} \sum_{u_2=0}^m (\alpha D)^{m+l-1-2u_2} \mathcal{A}(u_2; m, l, h+1). \end{aligned}$$

Thus we conclude, again up to boundary terms (treated in Subsection 5.5.3 below)

$$\begin{aligned} & \Gamma_{m+1,m+1+h;l+h,l} - \Gamma_{m,m+h+1;l+h,l-1} \tag{5.43} \\ & \leq \left( \frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{m+l+1} (1+C\alpha)^{-h} \sum_{u_1=1}^{m+1} (\alpha D)^{m+1+l-2u_1} \mathcal{A}(u_1; m+1, l, h) \\ & \quad - \left( \frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{m+l-1} (1+C\alpha)^{-h-1} \sum_{u_2=0}^m (\alpha D)^{m+l-1-2u_2} \mathcal{A}(u_2; m, l-1, h+1) \end{aligned}$$

Let  $u_1 = u + 1$ . Thanks to (5.37) we write the sum over  $u_1$  as

$$\begin{aligned} & \sum_{u_1=1}^{m+1} (\alpha D)^{m+1+l-2u_1} \mathcal{A}(u_1; m+1, l, h) \\ &= \sum_{u=0}^m (\alpha D)^{m+l-1-2u} \mathcal{A}(u; m, l-1, h+1) \frac{\sqrt{l(m+1)}}{u+1}. \tag{5.44} \end{aligned}$$

The difference in (5.43) is now written as

$$\begin{aligned} \mathcal{D} & \stackrel{\text{def}}{=} \left( \frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{m+l-1} (1+C\alpha)^{-h-1} \sum_{u=0}^m (\alpha D)^{m+l-1-2u} \mathcal{A}(u; m, l-1, h+1) \\ & \quad \times \left\{ \left( \frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^2 (1+C\alpha) \frac{\sqrt{l(m+1)}}{u+1} - 1 \right\}. \tag{5.45} \end{aligned}$$

The factor between braces is expressed as  $\left\{ \frac{\sqrt{l(m+1)}}{u+1} - 1 - \alpha \frac{1+C}{1+C\alpha} \frac{\sqrt{l(m+1)}}{u+1} \right\}$ .

The scaling factor  $(\alpha D)^{m+l-1-2u}$  is maximum for  $u = m$  where it reaches  $(\alpha D)^p$ . We can consider up to boundary terms that the  $u$ -sum goes only up to



$m - 1$ . We can factorize  $\alpha^2$  and prove that the remaining terms are smaller than  $1/\alpha$  on the “mountain top” or  $k$  outside this critical region.

For the  $u$ -sum, with  $u$  running only to  $m - 1$  we have:

$$\mathcal{D} = i \left( \frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{m+l-1} (1+C\alpha)^{-h-1} \sum_{u=0}^{m-1} (\alpha D)^{m+l-1-2u} \mathcal{A}(u; m, l-1, h+1) \times \left\{ \frac{\sqrt{l(m+1)}}{u+1} - 1 - \alpha \frac{1+C}{1+C\alpha} \frac{\sqrt{l(m+1)}}{u+1} \right\}, \quad (5.46)$$

$$\begin{aligned} \mathcal{D} &\leq K \alpha^2 \left( \frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{m+l-3} (1+C\alpha)^{-h-1} \\ &\times \sum_{u=0}^{m-1} (\alpha D)^{m+l-3-2u} \mathcal{A}(u; m-1, l-2, h+1) \\ &\times \frac{\sqrt{m(m+h+1)(l-1)(l+h)}}{(m-u)(l-1-u)} \left\{ \frac{\sqrt{l(m+1)}}{u+1} - 1 - c_1 \alpha \frac{\sqrt{l(m+1)}}{u+1} \right\} \\ &= K \sum_{u=0}^{m-1} \Gamma(u, m-1, l-2, h+1) E(u, m, l, h) \end{aligned} \quad (5.47)$$

where  $c_1 = \frac{1+C}{1+C\alpha}$ , and

$$E(u, m, l, h) = \alpha^2 \frac{\sqrt{m(m+h+1)(l-1)(l+h)}}{(m-u)(l-1-u)(u+1)(1+C\alpha)} \times \left\{ (1-\alpha)\sqrt{l(m+1)} - (1+C\alpha)(u+1) \right\}.$$

Once more the subsequent procedure depends on the value of the indices in the configuration space  $\mathbb{N}^3$ . If they stand on the mountain top, we pass to the variables  $x = v/k = C\alpha/(1+C\alpha) + \delta x$ ,  $y = h+1/k = 1/(1+C) + \delta y$ ,  $z = p/k = \delta z$ . We can assume  $\alpha k \geq 1$  and we can evaluate  $E$  as:

$$\begin{aligned} E &\leq O(1)[(1-\alpha)\sqrt{(1-y)(1-y-z)} - (1+C\alpha)(1-x-y-z)] + O(1/k) \\ &\leq O(|\alpha\delta y| + |\delta x| + |\delta z|) + O(1/k) \end{aligned} \quad (5.48)$$

The rest of the sum reconstructs the usual mountain bound of lemma 5.7. Using the Hessian estimates  $\delta y \simeq 1/\sqrt{\alpha k}$ ,  $\delta x \simeq \delta z \simeq \sqrt{\alpha k}$ , the  $E$  correction simply provides an additional factor  $\sqrt{\alpha/k}$  to the estimate of Lemma 5.7. Adding the  $\sqrt{m} \leq \sqrt{k}$  factor we recover the desired  $\sqrt{\alpha}$  factor.

If the indices are outside the critical region,

$$\mathcal{D} \leq K\alpha^2 \Gamma_{m-1, m+h; l+h-1, l-2} \max_{u \leq m-1} \left( \frac{\sqrt{m(m+h+1)(l-1)(l+h)}}{(m-u)(l-1-u)} \right) \quad (5.49)$$

$$\begin{aligned} & \times \left\{ \frac{l-1-u}{u+1} - c_1 \alpha \frac{\sqrt{l(m+1)}}{u+1} \right\} \leq K\alpha^2 \Gamma_{m-1, m+h; l+h-1, l-2} k \sqrt{1 + \frac{p}{m}} \\ \sqrt{m+1} \mathcal{D} & \leq K\sqrt{\alpha} e^{-c(p+\alpha k)} \end{aligned} \quad (5.50)$$

This is a better bound.  $\square$

### 5.5.3 Boundary Terms

For purists we add the treatment of the boounary terms, and show that they do not perturb the previous computations. We start with the  $u_1 = 0$  term in (5.43). We have:

$$\begin{aligned} \mathcal{O} & \stackrel{\text{def}}{=} \left( \frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{m+1+l} (1+C\alpha)^{-h} (\alpha D)^{m+1+l} \mathcal{A}(0; m+1, l, h) \\ & = \left( \frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{m+1+l} (1+C\alpha)^{-h} (\alpha D)^{m+1+l} \sqrt{\binom{m+1+h}{m+1} \binom{l+h}{l}} \\ & \leq (\alpha D)^{m+1+l} e^{-\alpha(C+1/2)(m+1+l) - \frac{C'}{2}\alpha h} \max_h e^{-\frac{C'}{4}\alpha h} \frac{(m+1+h)^{(m+1)/2}}{\sqrt{(m+1)!}} \\ & \quad \times \max_h e^{-\frac{C'}{4}\alpha h} \frac{(l+h)^{l/2}}{\sqrt{l!}} \end{aligned} \quad (5.51)$$

$$\begin{aligned} & \leq \alpha^{(m+1+l)/2} \left( \frac{D\sqrt{2}}{\sqrt{C'}} \right)^{m+1+l} e^{-\alpha(C+1/2 - \frac{C'}{4})(m+1+l) - \frac{C'}{2}\alpha h} \\ & \leq (9\alpha)^{(m+1+l)/2} e^{-\frac{1}{2}\alpha(m+1+l) - \frac{C'}{2}\alpha h} \leq (9\alpha)^{p/2} e^{-cak} \end{aligned} \quad (5.52)$$

Then  $\sqrt{m+1}\mathcal{O} \leq K\sqrt{\alpha} e^{-c(\alpha k+p)}$  which is the desired result.

We now study the  $u = m$  term in (5.45) and prove that it obeys (5.41):

$$\begin{aligned}
\mathcal{M} &\stackrel{\text{def}}{=} \left( \frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{m+l-1} (1+C\alpha)^{-h-1} (\alpha D)^p \mathcal{A}(m; m, l-1, h+1) \\
&\times \left\{ \left( \frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^2 (1+C\alpha) \frac{\sqrt{l(m+1)}}{m+1} - 1 \right\} \\
&\leq \Gamma_{m, m+h+1; l+h, l-1} \times \left\{ \frac{\sqrt{l(m+1)}}{m+1} - 1 - \alpha \frac{1+C}{1+C\alpha} \frac{\sqrt{l(m+1)}}{m+1} \right\} \\
&\leq \Gamma_{m, m+h+1; l+h, l-1} \times \left\{ \frac{l}{m+1} - 1 - \alpha \frac{1+C}{1+C\alpha} \sqrt{1 + \frac{p}{m+1}} \right\} \\
&\leq K\Gamma_{m, m+h+1; l+h, l-1} \times \left\{ \frac{p}{m+1} + \alpha \right\}. \tag{5.53}
\end{aligned}$$

Remark that the  $u = m$  term is not on the mountain top so that for the function  $\Gamma_{m, m+h+1; l+h, l-1}$  appearing in the equations (5.53) we use the bounds outside the mountain top region. Then we have to treat two different cases according to the value of  $\alpha k$ . If  $\alpha k \leq 1$ , thanks to (5.26)

$$\begin{aligned}
\mathcal{M} &\leq e^{-c\alpha k} \left( \frac{\alpha \sqrt{(m+p)(m+1+p+h)}}{2} \right)^p \times \left\{ \frac{p}{m+1} + \alpha \right\}, \\
\sqrt{m+1}\mathcal{M} &\leq \alpha p \sqrt{\frac{(m+p)(m+1+p+h)}{m+1}} \left( \frac{\alpha k}{2} \right)^{p-1} e^{-c\alpha k} \\
&\quad + \alpha \sqrt{m+1} e^{-c\alpha k - c'p} \\
&\leq \left( \alpha p \sqrt{k} \sqrt{1 + \frac{p}{m+1}} + \sqrt{\alpha} \right) e^{-c(p+\alpha k)} \leq K\sqrt{\alpha} e^{-c(p+\alpha k)}. \tag{5.54}
\end{aligned}$$

If  $\alpha k \geq 1$ ,

$$\sqrt{m+1}\mathcal{M} \leq \left( \frac{p}{\sqrt{m+1}} + \alpha \sqrt{m+1} \right) e^{-c(p+\alpha k)} \leq K\sqrt{\alpha} e^{-\frac{c}{2}(p+\alpha k)}. \tag{5.55}$$

## A The ordinary $B=0$ bound

In this section we use the above analysis to revisit the bounds on the  $\phi_4^4$  propagator given in [3]. By convention the indices of the  $\phi^4$  propagator are

two-dimensionnal indices. Using obvious notations

$$G_{m,m+h;l+h,l}^{\phi^4} = \int_0^1 d\alpha \frac{(1-\alpha)^{\frac{\mu^2\theta}{8\Omega}}}{(1+C\alpha)^2} G_{m,m+h;l+h,l}^{\alpha,\phi^4}, \quad (\text{A.1})$$

$$G_{m,m+h;l+h,l}^{\alpha,\phi^4} = \Gamma_{m,m+h;l+h,l}^\alpha (1-\alpha)^{h/2} \leq \Gamma_{m,m+h;l+h,l}^\alpha e^{-\frac{\alpha h}{2}}. \quad (\text{A.2})$$

We have then the following result (recalling  $p = l - m$ )

**Theorem A.1** *The  $\phi_4^4$  propagator in a slice obeys the bound*

$$G_{m,n;k,l}^i \leq KM^{-i} \min(1, (\alpha k)^p) e^{-c(\alpha k + p)} \quad (\text{A.3})$$

for some (large) constant  $K$  and (small) constant  $c$  which depend only on  $\Omega$ .

**Proof.** In this context the analysis of Lemmas 5.6 and 5.7 leads to the modified bound

$$G_{m,n;k,l}^\alpha \leq Ke^{-\frac{\alpha h}{2}} \left( \frac{\exp\left\{-\frac{c}{\alpha k}p^2 - \frac{c\alpha}{k}\left(h - \frac{k}{1+C}\right)^2\right\}}{\sqrt{\alpha k}} + \min(1, (\alpha k)^p) e^{-c(\alpha k + p)} \right). \quad (\text{A.4})$$

The second term is already of the desired form in (A.3). Moreover:

$$\exp\left\{-\frac{c\alpha}{k}\left(h - \frac{k}{1+C}\right)^2 - \frac{\alpha h}{2}\right\} \leq \exp\left\{-\frac{c}{1+C}\alpha k - \alpha h\left(\frac{1}{2} - \frac{2c}{1+C}\right)\right\}. \quad (\text{A.5})$$

Then choosing  $c$  small enough, the first term in (A.4) is bounded by

$$K \exp\left\{-c\alpha k - \frac{c}{\alpha k}p^2\right\} \leq K \exp\left\{-\frac{c}{2}\alpha k - c\sqrt{2}p\right\}. \quad (\text{A.6})$$

This completes the proof.  $\square$

Remark that in contrast with the Gross-Neveu case, the  $\phi^4$  propagator has no critical point in index space. The bound is the same for all  $(m, l, h) \in \mathbb{N}^3$  and looks like the bound (5.17). Note also that the bound (A.3) allows to recover the propositions 1, 2, 3 and 4 of [3] in a very direct manner.

## B Proof of Lemma 3.2

Let us compute first

$$\begin{aligned}
Q^{-1} &= \int_0^\infty dt e^{-tQ}, \\
Q^{-1}(x, y) &= \frac{\Omega}{\theta\pi} \int_0^\infty dt \frac{e^{-t\mu^2}}{\sinh(2\tilde{\Omega}t)} e^{-\frac{\tilde{\Omega}}{2} \coth(2\tilde{\Omega}t)(x-y)^2 + \imath\tilde{\Omega}x\wedge y} e^{-2\imath\tilde{\Omega}t\gamma^0\gamma^1} \\
&\stackrel{\text{def}}{=} \frac{\Omega}{\theta\pi} \int_0^\infty dt e^{-t\mu^2} e^{-2\imath\tilde{\Omega}t\gamma^0\gamma^1} \Gamma^t(x, y), \\
\Gamma^t(x, y) &= \frac{1}{\sinh(2\tilde{\Omega}t)} e^{-\frac{\tilde{\Omega}}{2} \coth(2\tilde{\Omega}t)(x-y)^2 + \imath\tilde{\Omega}x\wedge y} \tag{B.1}
\end{aligned}$$

with  $\tilde{\Omega} = \frac{2\Omega}{\theta}$  and  $x \wedge y = x^0y^1 - x^1y^0$ .

We only have to check that  $e^{-tQ}$  is a solution of

$$\frac{dP}{dt} + QP = 0. \tag{B.2}$$

The constant is fixed by the requirement that in the limit  $\Omega \rightarrow 0$ ,  $Q^{-1}$  goes to the usual heat kernel.

$$\begin{aligned}
\frac{d\Gamma^t}{dt} &= \frac{e^{-\frac{\tilde{\Omega}}{2} \coth(2\tilde{\Omega}t)(x-y)^2 + \imath\Omega x\wedge y}}{\sinh(2\tilde{\Omega}t)} \left( -\frac{2\tilde{\Omega} \cosh(2\tilde{\Omega}t)}{\sinh(2\tilde{\Omega}t)} + \frac{\tilde{\Omega}^2}{\sinh^2(2\tilde{\Omega}t)} (x-y)^2 \right), \\
\partial^\nu \Gamma^t &= \frac{e^{-\frac{\tilde{\Omega}}{2} \coth(2\tilde{\Omega}t)(x-y)^2 - \imath\Omega x \cdot \tilde{y}}}{\sinh(2\tilde{\Omega}t)} \left( -\tilde{\Omega} \coth(2\tilde{\Omega}t)(x-y)^\nu - \imath\Omega \tilde{y}^\nu \right), \\
\Delta \Gamma^t &= \frac{e^{-\frac{\tilde{\Omega}}{2} \coth(2\tilde{\Omega}t)(x-y)^2 - \imath\Omega x \cdot \tilde{y}}}{\sinh(2\tilde{\Omega}t)} \times \\
&\quad \left( -2\tilde{\Omega} \coth(2\tilde{\Omega}t) + \left( -\tilde{\Omega} \coth(2\tilde{\Omega}t)(x-y)^\mu - \imath\Omega \tilde{y}^\mu \right)^2 \right) \\
&= \frac{e^{-\frac{\tilde{\Omega}}{2} \coth(2\tilde{\Omega}t)(x-y)^2 - \imath\Omega x \cdot \tilde{y}}}{\sinh(2\tilde{\Omega}t)} \left( -2\tilde{\Omega} \coth(2\tilde{\Omega}t) + \tilde{\Omega}^2 \coth^2(2\tilde{\Omega}t)(x-y)^2 \right. \\
&\quad \left. - \Omega^2 \tilde{y}^2 + 2\imath\Omega \tilde{\Omega} \coth(2\tilde{\Omega}t)x \cdot \tilde{y} \right). \tag{B.3}
\end{aligned}$$

The operator  $L_2 = x^0 p_1 - x^1 p_0 = -ix^0 \partial_1 + ix^1 \partial_0$  acting on  $\Gamma^t$  gives:

$$\begin{aligned}
L_2 \Gamma^t &= -i \frac{x^0}{\sinh(2\tilde{\Omega}t)} \left( -\tilde{\Omega} \coth(2\tilde{\Omega}t)(x-y)^1 - i\Omega \tilde{y}^1 \right) \\
&+ i \frac{x^1}{\sinh(2\tilde{\Omega}t)} \left( -\tilde{\Omega} \coth(2\tilde{\Omega}t)(x-y)^0 - i\Omega \tilde{y}^0 \right) e^{-\frac{\tilde{\Omega}}{2} \coth(2\tilde{\Omega}t)(x-y)^2 - i\Omega x \cdot \tilde{y}} \\
&= \frac{e^{-\frac{\tilde{\Omega}}{2} \coth(2\tilde{\Omega}t)(x-y)^2 - i\Omega x \cdot \tilde{y}}}{\sinh(2\tilde{\Omega}t)} \left( i \coth(2\tilde{\Omega}t) \Omega x \cdot \tilde{y} - \tilde{\Omega} x \cdot y \right). \tag{B.4}
\end{aligned}$$

$$\begin{aligned}
\left( -\Delta + 2\tilde{\Omega} L_2 + \tilde{\Omega}^2 x^2 \right) \Gamma^t &= \frac{e^{-\frac{\tilde{\Omega}}{2} \coth(2\tilde{\Omega}t)(x-y)^2 - i\Omega x \cdot \tilde{y}}}{\sinh(2\tilde{\Omega}t)} \left( 2\tilde{\Omega} \coth(2\tilde{\Omega}t) \right. \\
&\quad \left. - \tilde{\Omega}^2 \coth^2(2\tilde{\Omega}t)(x-y)^2 + \tilde{\Omega}^2 y^2 - 2i\Omega \tilde{\Omega} \coth(2\tilde{\Omega}t) x \cdot \tilde{y} \right. \\
&\quad \left. + 2i\Omega \tilde{\Omega} \coth(2\tilde{\Omega}t) x \cdot \tilde{y} - 2\tilde{\Omega}^2 x \cdot y + \tilde{\Omega}^2 x^2 \right) \\
&= \frac{e^{-\frac{\tilde{\Omega}}{2} \coth(2\tilde{\Omega}t)(x-y)^2 - i\Omega x \cdot \tilde{y}}}{\sinh(2\tilde{\Omega}t)} \times \left( 2\tilde{\Omega} \coth(2\tilde{\Omega}t) - \frac{\tilde{\Omega}^2}{\sinh^2(2\tilde{\Omega}t)} (x-y)^2 \right) \tag{B.5}
\end{aligned}$$

With the  $\mu^2$  and  $\gamma^0 \gamma^1$  terms, eq. (B.2) is satisfied.

For the full propagator it remains to compute  $e^{-2i\tilde{\Omega}t\gamma^0\gamma^1}$  and the action of  $(-\not{p} + \mu - \Omega \not{x})$  on  $e^{-tQ}$ .

$$\begin{aligned}
e^{-2i\tilde{\Omega}t\gamma^0\gamma^1} &= \sum_{n \geq 0} \frac{(-2i\tilde{\Omega}t)^{2n}}{(2n)!} (-\mathbf{1}_2)^n + \sum_{n \geq 1} \frac{(-2i\tilde{\Omega}t)^{2n+1}}{(2n+1)!} (-\gamma^0 \gamma^1)^n \\
&= \cosh(2\tilde{\Omega}t) \mathbf{1}_2 - i \sinh(2\tilde{\Omega}t) \gamma^0 \gamma^1. \tag{B.6}
\end{aligned}$$

With the convention  $\not{p} = -i\not{\partial}$  we have  $-\not{p} = i\gamma^\nu \partial_\nu$  and

$$-\not{p} \Gamma^t = \frac{i\gamma^\nu e^{-\frac{\tilde{\Omega}}{2} \coth(2\tilde{\Omega}t)(x-y)^2 - i\Omega x \cdot \tilde{y}}}{\sinh(2\tilde{\Omega}t)} \left( -\tilde{\Omega} \coth(2\tilde{\Omega}t)(x-y)_\nu - i\Omega \tilde{y}_\nu \right). \tag{B.7}$$

which completes the proof of the Lemma.

## C Proof of Lemma 3.3

We will restrict our attention to the case  $h \geq 0$ . We see that:

$$\lim_{\alpha \rightarrow 0} \mathcal{E} = \delta_{mu} \delta_{tu} \rightarrow (1 - \alpha) \frac{\theta}{8\Omega} H|_{\alpha=0} = I \tag{C.1}$$

so that we have the correct normalization. One must now only check the differential equation:

$$(1 - \alpha) \frac{d}{d\alpha} (1 - \alpha)^{\frac{\theta}{8\Omega} H} + \frac{\theta}{8\Omega} H (1 - \alpha)^{\frac{\theta}{8\Omega} H} = 0. \quad (\text{C.2})$$

A straightforward computation yields the result:

$$\begin{aligned} -(1 - \alpha) \frac{d}{d\alpha} [(1 - \alpha)^{\frac{\theta}{8\Omega} H}]_{m, m+h; l+h, l} &= \sum_{u=0}^{\min(m, l)} \mathcal{A}(m, l, h, u) \left( C \frac{1 + \Omega}{1 - \Omega} \right)^{m+l-2u} \\ &\times \left[ (2m + h + 1) \left( \frac{1}{2} + C \frac{1 - \alpha}{1 + C\alpha} \right) - (m - u) \left( 1 + \frac{1 - \alpha}{\alpha} + C \frac{1 - \alpha}{1 + C\alpha} \right) \right. \\ &\quad \left. - (l - u) \left( \frac{1 - \alpha}{\alpha} - C \frac{1 - \alpha}{1 + C\alpha} \right) \right] \mathcal{E}(m, l, h, u). \end{aligned} \quad (\text{C.3})$$

We will treat the last term in the above sum. Using the equality:

$$\begin{aligned} (l - u) \mathcal{A}(m, l, h, u) &= \mathcal{A}(m, l, h, u + 1) \times \\ &\left[ \frac{(m + 1)(m + h + 1)}{m - u} - (2m + h + 1) + (m - u - 1) \right] \end{aligned} \quad (\text{C.4})$$

and changing the dummy variable from  $u$  to  $v = u + 1$ , the term rewrites as:

$$\begin{aligned} \frac{\alpha C^2 \left( \frac{1 + \Omega}{1 - \Omega} \right)^2}{1 + C\alpha} \sum_v \left[ \frac{(m + 1)(m + h + 1)}{m + 1 - v} - (2m + h + 1) + (m - v) \right] \\ \mathcal{A}(m, l, h, v) \left( C \frac{1 + \Omega}{1 - \Omega} \right)^{m+l-2v} \mathcal{E}(m, l, h, v). \end{aligned} \quad (\text{C.5})$$

Coupling the identical terms in the two sums we get the coefficients of:

$$2m + h + 1 \rightarrow \frac{1}{2} + C \frac{1 - \alpha}{1 + C\alpha} + \frac{\alpha C^2 \left( \frac{1 + \Omega}{1 - \Omega} \right)^2}{1 + C\alpha} = C + \frac{1}{2} = \frac{1 + \Omega^2}{4\Omega} \quad (\text{C.6})$$

$$m - u \rightarrow (-1) \left[ \frac{1}{\alpha} + C \frac{1 - \alpha}{1 + C\alpha} + \frac{\alpha C^2 \left( \frac{1 + \Omega}{1 - \Omega} \right)^2}{1 + C\alpha} \right] = -\frac{1 + C\alpha}{\alpha}. \quad (\text{C.7})$$

The complete sum is then:

$$\begin{aligned} \sum_u \left( C \frac{1 + \Omega}{1 - \Omega} \right)^{m+l-2u} \mathcal{A}(m, l, h, u) \mathcal{E}(m, l, h, u) \times \\ \left[ (2m + h + 1) \frac{1 + \Omega^2}{4\Omega} - (m - u) \frac{1 + C\alpha}{\alpha} - \frac{(m + 1)(m + h + 1)}{m + 1 - u} \frac{\alpha C^2 \left( \frac{1 + \Omega}{1 - \Omega} \right)^2}{1 + C\alpha} \right]. \end{aligned} \quad (\text{C.8})$$

Using

$$\begin{aligned} (m-u)\mathcal{A}(m,l,h,u) &= \sqrt{m(m+h)}\mathcal{A}(m-1,l,h,u), & (C.9) \\ \frac{(m+1)(m+h+1)}{(m+1-v)}\mathcal{A}(m,l,h,u) &= \sqrt{(m+1)(m+h+1)}\mathcal{A}(m+1,l,h,u), \end{aligned}$$

one can cast the result into the form:

$$\begin{aligned} -(1-\alpha)\frac{d}{dt}[(1-\alpha)^{\frac{\theta}{8\Omega}H}]_{m,m+h;l+h,l} &= & (C.10) \\ \frac{1+\Omega^2}{4\Omega}(2m+h+1)[(1-\alpha)^{\frac{\theta}{8\Omega}H}]_{m,m+h;l+h,l} & \\ -C\frac{1+\Omega}{1-\Omega}\sqrt{m(m+h)}[(1-\alpha)^{\frac{\theta}{8\Omega}H}]_{m-1,m-1+h;l+h,l} & \\ -\sqrt{(m+1)(m+h+1)}C\frac{1-\Omega}{1+\Omega}[(1-\alpha)^{\frac{\theta}{8\Omega}H}]_{m+1,m+1+h;l+h,l}. & (C.11) \end{aligned}$$

On the other hand:

$$\begin{aligned} \frac{\theta}{8\Omega}H_{m,m+h;p+h,p} &= \frac{1+\Omega^2}{4\Omega}(2m+h+1)\delta_{mp} \\ -\frac{1-\Omega^2}{4\Omega}[\sqrt{(m+h+1)(m+1)}\delta_{m+1,p} + \sqrt{(m+h)m}\delta_{m-1,p}] & (C.12) \end{aligned}$$

and the differential equation is checked.

**Acknowledgement** We thank S. Al Jaber, V. Gayral, J. Magnen and R. Wulkenhaar for discussions at various stages of this work. We also thank our anonymous referee for interesting comments which led to this improved version.

## References

- [1] H. Grosse and R. Wulkenhaar, “Power-counting theorem for non-local matrix models and renormalisation,” *Commun. Math. Phys.* **254** (2005), no. 1, 91–127, [hep-th/0305066](#).
- [2] H. Grosse and R. Wulkenhaar, “Renormalisation of  $\phi^4$ -theory on noncommutative  $\mathbb{R}^4$  in the matrix base,” *Commun. Math. Phys.* **256** (2005), no. 2, 305–374, [hep-th/0401128](#).
- [3] V. Rivasseau, F. Vignes-Tourneret, and R. Wulkenhaar, “Renormalization of noncommutative  $\phi^4$ -theory by multi-scale analysis,” *Commun. Math. Phys. (Online First)* DOI: [10.1007/s00220-005-1440-4](#) (2005) [hep-th/0501036](#).



- [4] M. R. Douglas and N. A. Nekrasov, “Noncommutative field theory,” *Rev. Mod. Phys.* **73** (2001) 977–1029, [hep-th/0106048](#).
- [5] P. K. Mitter and P. H. Weisz, “Asymptotic scale invariance in a massive thirring model with  $u(n)$  symmetry,” *Phys. Rev.* **D8** (1973) 4410–4429.
- [6] D. J. Gross and A. Neveu, “Dynamical symmetry breaking in asymptotically free field theories,” *Phys. Rev.* **D10** (1974) 3235.
- [7] J. Feldman and E. Trubowitz, “Perturbation theory for many fermion systems,” *Helv. Phys. Acta* **63** (1990) 156–260.
- [8] G. Benfatto and G. Gallavotti, *Renormalization group*, ch. 11 and references therein. Physics notes: 1. Princeton Univ. Pr., Princeton, USA, 1995. 142 p.
- [9] M. Salmhofer, *Renormalization, an introduction*. Texts and monographs in physics. Springer, 1999. 231 p.
- [10] E. Langmann and R. J. Szabo, “Duality in scalar field theory on noncommutative phase spaces,” *Phys. Lett.* **B533** (2002) 168–177, [hep-th/0202039](#).
- [11] F. Vignes-Tourneret, “Perturbative renormalizability of the noncommutative Gross-Neveu model.” Work in progress.
- [12] H. Grosse and R. Wulkenhaar, “Renormalisation of  $\phi^4$ -theory on noncommutative  $\mathbb{R}^2$  in the matrix base,” *JHEP* **12** (2003) 019, [hep-th/0307017](#).
- [13] J. M. Gracia-Bondía and J. C. Várilly, “Algebras of distributions suitable for phase space quantum mechanics. I,” *J. Math. Phys.* **29** (1988) 869–879.
- [14] B. Simon, *Functional integration and quantum physics*, vol. 86 of *Pure and applied mathematics*. Academic Press, New York, 1979.