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INFINITELY DIVISIBLE DISTRIBUTIONS FOR RECTANGULAR FREE CONVOLUTION: CLASSIFICATION AND MATRICIAL INTERPRETATION

FLORENT BENAYCH-GEORGES

ABSTRACT. In a previous paper ([B-G1]), we defined the rectangular free convolution \boxplus_λ . Here, we investigate the related notion of infinite divisibility, which happens to be closely related to the classical infinite divisibility: there exists a bijection between the set of classical symmetric infinitely divisible distributions and the set of \boxplus_λ -infinitely divisible distributions, which preserves limit theorems. We give an interpretation of this correspondance in term of random matrices: we construct distributions on sets of complex rectangular matrices which give rise to random matrices with singular laws going from the symmetric classical infinitely divisible distributions to their \boxplus_λ -infinitely divisible correspondants when the dimensions go from one to infinity in a ratio λ .

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INTRODUCTION

In a previous paper ([B-G1]), we modeled the asymptotic behaviour of rectangular random matrices with freeness with amalgamation. Therefore we defined, for each $\lambda \in [0, 1]$, the rectangular free convolution with ratio λ , denoted by \boxplus_λ . It is a binary operation on the set of symmetric probability measures on the real line defined in the following way. Let us call the singular law of a matrix M the uniform law on its singular values, i.e. on the spectrum of its

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absolute value $|M| = (MM^*)^{1/2}$. Consider μ, ν symmetric probability measures on the real line, consider two sequences $q_1(n), q_2(n)$ of integers tending to $+\infty$ such that

$$\frac{q_1(n)}{q_2(n)} \xrightarrow{n \rightarrow \infty} \lambda,$$

and consider, for each n , $M(n), N(n)$ independent $q_1(n) \times q_2(n)$ random matrices, one of them being biunitarily invariant (i.e. having a distribution invariant under the left and right actions of the unitary groups) such that the symmetrization of the singular law of $M(n)$ (resp. of $N(n)$) converges weakly in probability to μ (resp. ν). Then the symmetrization of the singular law of $M(n) + N(n)$ converges weakly in probability to a probability measure which depends only on μ, ν , and λ , denoted by $\mu \boxplus_\lambda \nu$, and called the *rectangular free convolution with ratio λ* of μ and ν .

In the present paper, we study the notion of infinite divisibility for \boxplus_λ , which leads to a Lévy-Kinchine formula for the rectangular R -transform (whose definition we shall recall in section 1): a symmetric probability measure μ is \boxplus_λ -infinitely divisible if and only if there exists a positive finite symmetric measure G (called its *Lévy measure*) such that the rectangular R -transform with ratio λ of μ is given by the formula:

$$C_\mu(z) = z \int_{\mathbb{R}} \frac{1+t^2}{1-zt^2} dG(t).$$

Therefore we can define a bijection Λ_λ between the set of classical symmetric infinitely divisible distributions and the set of \boxplus_λ -infinitely divisible distributions: Λ_λ maps a symmetric $*$ -infinitely divisible distribution to the \boxplus_λ -infinitely divisible distribution with the same Lévy measure. This bijection happens, like the one of Bercovici and Pata ([BPB99]) between $*$ - and \boxplus -infinitely divisible distributions, to have deep properties. It is a semi-groups morphism:

$$\Lambda_\lambda(\mu * \nu) = \Lambda_\lambda(\mu) \boxplus_\lambda \Lambda_\lambda(\nu),$$

and it preserves limit theorems: for all sequences (μ_n) of symmetric distributions and (k_n) of positive integers tending to infinity, we have, for all probability measure μ ,

$$\mu_n^{*k_n} \xrightarrow{n \rightarrow \infty} \mu \iff \mu_n^{\boxplus_\lambda k_n} \xrightarrow{n \rightarrow \infty} \Lambda_\lambda(\mu).$$

Λ_λ will be called the rectangular Bercovici-Pata bijection with ratio λ .

In the section 4, we characterize the image, by Λ_λ , of the standard Gaussian distribution: when $\lambda > 0$, it has density

$$\frac{[4\lambda - (x^2 - 1 - \lambda)^2]^{1/2}}{2\pi\lambda|x|}$$

on its support $[-1 - \lambda^{1/2}, -1 + \lambda^{1/2}] \cup [1 - \lambda^{1/2}, 1 + \lambda^{1/2}]$. An interesting interpretation of this result is made in a forthcoming paper ([B-G2]) where we construct analogues of Voiculescu's free entropy and free Fisher information for operators between different Hilbert spaces, and where the maximum of entropy and the minimum of Fisher information are realized for operators which absolute value has this (symmetrized) distribution. Another consequence of this result is a new formula for the moments of the Marchenko-Pastur distribution (which is closely related to this distribution): for all $a > 0$, for all $n \geq 1$, the n -th moment of the Marchenko-Pastur distribution with parameter a (see [HP00] p. 65) is equal to $\sum_{\pi} a^{o(\pi)}$, where the sum is taken over all noncrossing pairings of $[2n]$, and where $o(\pi)$ is the number of blocks of a partition π which first element is odd.

In section 5, we shall construct a matricial model for the \boxplus_λ -infinitely divisible laws and present in a maybe more palpable way the Bercovici-Pata bijection with ratio λ (whereas the proofs of the other sections rely on integral transforms and complex analysis): we are going to construct, in the same way as in [B-G04] and in [C-D04], for each $d, d' \geq 1$, for each symmetric $*$ -infinitely divisible distribution μ , an infinitely divisible distribution $\mathbb{P}_{d,d'}^\mu$ on the set of $d \times d'$ complex matrices such that for all μ, ν , $\mathbb{P}_{d,d'}^\mu * \mathbb{P}_{d,d'}^\nu = \mathbb{P}_{d,d'}^{\mu * \nu}$ and such that the symmetrization of the singular law of M (with M random matrix distributed according to $\mathbb{P}_{d,d'}^\mu$) goes from μ to its image by the rectangular Bercovici-Pata bijection with ratio λ when $d, d' \rightarrow \infty$, $\frac{d}{d'} \rightarrow \lambda$.

In the last section, we shall give a representation of the image of the symmetric Poisson distribution by the rectangular Bercovici-Pata bijection with ratio λ as the distribution of the absolute value of sums of rank-one matrices.

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1. PRELIMINARIES

In this section, we shall recall definition and basic results from [B-G1] about the rectangular R -transform C_μ of a symmetric probability measure μ .

Let us denote by $z \mapsto z^{1/2}$ (resp. $z \mapsto \sqrt{z}$) the analytic version of the square root on the complement of the real non positive (resp. non negative) half line such that $1^{1/2} = 1$ (resp. $\sqrt{-1} = i$). Let us define the analytic function U on a neighborhood of zero $U(z) = \frac{-\lambda - 1 + [(\lambda + 1)^2 + 4\lambda z]^{1/2}}{2\lambda}$ (when $\lambda = 0$, $U(z) = z$). Then one can summarize the different steps of the construction of the rectangular R -transform with ratio λ in the following chain

$$\begin{array}{c} \mu \\ \text{sym. prob.} \\ \text{measure} \end{array} \longrightarrow \begin{array}{c} G_\mu(z) = \int \frac{d\mu(t)}{z - t} \\ \text{Cauchy transform} \end{array} \longrightarrow H_\mu(z) = \lambda G_\mu \left(\frac{1}{\sqrt{z}} \right)^2 + (1 - \lambda) \sqrt{z} G_\mu \left(\frac{1}{\sqrt{z}} \right) \longrightarrow \\ C_\mu(z) = U \left(\frac{z}{H_\mu^{-1}(z)} - 1 \right), \\ \text{rect. } R\text{-transf. with ratio } \lambda$$

where H_μ^{-1} is the inverse (for composition) of H_μ . Proposition 1.1 and theorem 1.2 bellow, which have first been established in [B-G1], prove that such an inverse exists, give its domain, and prove that for any tight set A of symmetric probability measures, the properties and the domains of the functions H_μ^{-1} ($\mu \in A$) are “uniform”.

Proposition 1.1. *Let A be a set of symmetric probability measures on the real line. Then the following assertions are equivalent*

- (i) A is tight,
- (ii) for every $0 < \theta < \pi$, $\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \theta}} \frac{1}{z} H_\mu(z) = 1$ uniformly in $\mu \in A$,
- (iii) $\lim_{\substack{x \rightarrow 0 \\ x \in (-\infty, 0)}} \frac{1}{x} H_\mu(x) = 1$ uniformly in $\mu \in A$.

Define, for $\alpha \in (0, \pi)$, $\beta > 0$, $\Delta_{\alpha, \beta}$ to be the set of complex numbers z such that $|\arg z - \pi| < \alpha$ and $|z| < \beta$.

Let \mathcal{H} be the set of functions f which are analytic in a domain \mathcal{D}_f such that for all $\alpha \in (0, \pi)$, there exists β positive such that

$$\Delta_{\alpha, \beta} \subset \mathcal{D}_f.$$

A family $(f_a)_{a \in A}$ of functions of \mathcal{H} is said to be *uniform* if for all $\alpha \in (0, \pi)$, there exists β positive such that

$$\forall a \in A, \quad \Delta_{\alpha, \beta} \subset \mathcal{D}_{f_a}.$$

Theorem 1.2. *Let $(H_a)_{a \in A}$ be a uniform family of functions of \mathcal{H} such that for every $\alpha \in (0, \pi)$,*

$$\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} \frac{H_a(z)}{z} = 1 \text{ uniformly in } a \in A.$$

Then there exists a uniform family $(F_a)_{a \in A}$ of functions of \mathcal{H} such that for every $\alpha \in (0, \pi)$,

$$\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} \frac{F_a(z)}{z} = 1 \text{ uniformly in } a \in A,$$

and there exists β positive such that

$$\forall a \in A, \quad H_a \circ F_a = F_a \circ H_a = I_d \text{ on } \Delta_{\alpha, \beta}.$$

Moreover, the family $(F_a)_{a \in A}$ is unique in the following sense: if a family $(\tilde{F}_a)_{a \in A}$ of functions of \mathcal{H} satisfies the same conditions, then for all $\alpha \in (0, \pi)$, there exists β positive such that

$$\forall a \in A, \quad F_a = \tilde{F}_a \text{ on } \Delta_{\alpha, \beta}.$$

Using the theory of cumulants in operator-valued free probability theory, we prove (equation (42) of [B-G1]) the *additivity of rectangular R -transform*: for all μ, ν , symmetric probability measures, we have

$$C_{\mu \boxplus \nu} = C_\mu + C_\nu.$$

We shall mention here two other results, proved in [B-G1]. The second of them allows us to claim that \boxplus_λ is continuous with respect to weak convergence.

Lemma 1.3 (Tightness and rectangular R -transform). *Let A be a set of symmetric probability measures. Then we have equivalence between :*

- (i) *A is tight,*
- (ii) *for any $0 < \alpha < \pi$, $\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} C_\mu(z) = 0$ uniformly in $\mu \in A$,*
- (iii) *$\lim_{\substack{x \rightarrow 0 \\ x \in (-\infty, 0)}} C_\mu(x) = 0$ uniformly in $\mu \in A$.*

Theorem 1.4 (Paul Lévy's theorem for rectangular R -transform). *Let (μ_n) be a sequence of symmetric probability measures. Then we have equivalence between:*

- (i) *(μ_n) converges weakly to a symmetric probability measure;*
- (ii) *there exists α, β such that*
 - (a) *$\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} C_{\mu_n}(z) = 0$ uniformly in n ,*
 - (b) *the sequence (C_{μ_n}) converges uniformly on every compact set of $\Delta_{\alpha, \beta}$ when $n \rightarrow \infty$;*
- (iii) (a) *$\lim_{\substack{x \rightarrow 0 \\ x \in (-\infty, 0)}} C_{\mu_n}(x) = 0$ uniformly in n ,*

- (b) *there exists $\beta > 0$ such that the sequence (C_{μ_n}) converges pointwise on $[-\beta, 0)$ when $n \rightarrow \infty$.*

Moreover, in this case, denoting by μ the weak limit of (μ_n) , for every α , there exists β such that the sequence (C_{μ_n}) converges uniformly to C_μ on every compact set of $\Delta_{\alpha, \beta}$ when $n \rightarrow \infty$.

2. LÉVY-KINCHINE FORMULA FOR \boxplus_λ -INFINITELY DIVISIBLE DISTRIBUTIONS

\boxplus_λ -infinitely divisible distributions are defined in the same way as $*$ - and \boxplus -infinitely divisible distributions:

Definition 2.1. *A symmetric probability measure ν is said to be \boxplus_λ -infinitely divisible if for each $n \in \mathbb{N}^*$, there exists a symmetric distribution ν_n such that $\nu_n^{\boxplus_\lambda n} = \nu$.*

As for $*$ - and \boxplus -, we have the following characterization of \boxplus_λ -infinite divisibility.

Theorem 2.2. *Let ν be a symmetric distribution. Then ν is \boxplus_λ -infinitely divisible if and only if there exists a sequence (ν_n) such that $\nu_n^{\boxplus_\lambda n}$ converges weakly to ν .*

Proof. If ν is \boxplus_λ -infinitely divisible, it is clear. Assume the existence of a sequence (ν_n) such that $\nu_n^{\boxplus_\lambda n}$ converges weakly to ν . Consider $k \geq 1$. Let us show that there exists a symmetric probability measure σ such that $\sigma^{\boxplus_\lambda k} = \nu$. We have $\lim_{\substack{y \rightarrow 0 \\ y < 0}} nC_{\nu_n}(y) = 0$ uniformly in n , so

$\lim_{\substack{y \rightarrow 0 \\ y < 0}} nC_{\nu_{kn}}(y) = 0$ uniformly in n . So by lemma 1.3, the sequence $(\nu_{kn}^{\boxplus_\lambda n})$ is tight. If the symmetric distribution σ is the limit of one of its subsequences, we have

$$\sigma^{\boxplus_\lambda k} = \lim \left(\nu_{kn}^{\boxplus_\lambda n} \right)^{\boxplus_\lambda k} = \lim \nu_{kn}^{\boxplus_\lambda nk} = \nu.$$

□

Corollary 2.3. *The set of \boxplus_λ -infinitely divisible distributions is closed under weak convergence.*

Proof. If a sequence (μ_n) of \boxplus_λ -infinitely divisible distributions converges weakly to a distribution μ , then if for every n , $\nu_n^{\boxplus_\lambda n} = \mu_n$, the sequence $(\nu_n^{\boxplus_\lambda n})$ converges weakly to μ . □

To prove the Lévy-Kinchine formula for \boxplus_λ -infinitely divisible distributions, we need the following lemma, which is the analogue of propositions 2.6 and 2.7 of [BPB99].

Lemma 2.4. *Let (ν_n) be a sequence of measures that converges weakly to δ_0 . Consider $\alpha \in (0, \pi)$. Then there exists $\beta > 0$ such that on $\Delta_{\alpha, \beta}$,*

$$C_{\nu_n}(z) = \left(\frac{1}{\sqrt{z}} G_{\nu_n} \left(\frac{1}{\sqrt{z}} \right) - 1 \right) (1 + v_n(z))$$

where the functions v_n , defined on $\Delta_{\alpha, \beta}$, are such that

- (i) $\forall n, \forall z, |v_n(z)| \leq 1/2$ and $\lim_{z \rightarrow 0} v_n(z) = 0$ uniformly in n ,
- (ii) $\forall z, \lim_{n \rightarrow \infty} v_n(z) = 0$.

Proof. First, note that unless $\nu_n = \delta_0$ (in which case $v_n = 0$ is suitable), for all z , $\frac{1}{\sqrt{z}}G_{\nu_n}\left(\frac{1}{\sqrt{z}}\right) - 1 \neq 0$, so there is a function v_n on the domain of C_{ν_n} such that

$$C_{\nu_n}(z) = \left(\frac{1}{\sqrt{z}}G_{\nu_n}\left(\frac{1}{\sqrt{z}}\right) - 1 \right) (1 + v_n(z)).$$

To prove (i), we will only use the tightness of $\{\nu_n; n \in \mathbb{N}\}$. It suffices to show that

$$\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} \frac{C_{\nu_n}(z)}{\frac{1}{\sqrt{z}}G_{\nu_n}\left(\frac{1}{\sqrt{z}}\right) - 1} = 1 \text{ uniformly in } n.$$

We have

$$\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} \frac{1}{\sqrt{z}}G_{\nu_n}\left(\frac{1}{\sqrt{z}}\right) = 1 \text{ uniformly in } n,$$

and when a complex number t tends to 1, $t - 1 \sim (\lambda t^2 + (1 - \lambda)t - 1)/(\lambda + 1)$, so it suffices to show that

$$\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} \frac{(\lambda + 1)C_{\nu_n}(z)}{\frac{H_{\nu_n}(z)}{z} - 1} = 1 \text{ uniformly in } n.$$

We have

$$\frac{H_{\nu_n}(z)}{z} - 1 = \frac{H_{\nu_n}(z)}{z} \left(1 - \frac{z}{H_{\nu_n}(z)} \right),$$

and we know, by lemma 1.3, that

$$\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} \frac{H_{\nu_n}(z)}{z} = 1 \text{ uniformly in } n. \quad (2.1)$$

So it suffices to show that

$$\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} \frac{(\lambda + 1)C_{\nu_n}(z)}{1 - \frac{z}{H_{\nu_n}(z)}} - 1 = 0 \text{ uniformly in } n.$$

We know, by proposition 1.1 and by theorem 1.2, that

$$\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} \frac{H_{\nu_n}^{-1}(z)}{z} = 1 \text{ uniformly in } n, \quad (2.2)$$

and the equivalent of $U(x)$ in the neighborhood of zero is $\frac{x}{\lambda+1}$. So, since $C_{\nu_n} = U\left(\frac{z}{H_{\nu_n}^{-1}(z)} - 1\right)$, it suffices to show that

$$\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} \frac{\frac{z}{H_{\nu_n}^{-1}(z)} - 1}{1 - \frac{z}{H_{\nu_n}(z)}} = 1 \text{ uniformly in } n.$$

Choose $\alpha' \in (\alpha, \pi)$. By theorem 1.2, there exists $\beta_1 > 0$ such that for all n , $H_{\nu_n}^{-1}$ is defined on $\Delta_{\alpha', 2\beta_1}$, and

$$H_{\nu_n}(\Delta_{\alpha, \beta_1}) \cup H_{\nu_n}^{-1}(\Delta_{\alpha, \beta_1}) \subset \Delta_{\alpha', 2\beta_1}.$$

We have, for $z \in \Delta_{\alpha, \beta_1}$,

$$\begin{aligned} \frac{\frac{z}{H_{\nu_n}^{-1}(z)} - 1}{1 - \frac{z}{H_{\nu_n}(z)}} - 1 &= \frac{\frac{zH_{\nu_n}(z)}{H_{\nu_n}^{-1}(z)} - H_{\nu_n}(z)}{H_{\nu_n}(z) - z} - 1 \\ &= \frac{1}{H_{\nu_n}(z) - z} \int_{[z, H_{\nu_n}(z)]} f'_n(\xi) d\xi, \end{aligned}$$

where f_n is the function defined by

$$f_n(\xi) = \frac{H^{-1}(\xi)H(z)}{H^{-1}(z)} - \xi.$$

But the lemma 2.4 of [BPB99] states that

$$\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha'}} (H_{\nu_n}^{-1})'(z) = 1 \text{ uniformly in } n.$$

Hence, using also (2.1) and (2.2) (which stay true if α is replaced by α'), we have

$$\lim_{\substack{\xi \rightarrow 0 \\ |\arg \xi - \pi| < \alpha'}} f'_n(\xi) = 0 \text{ uniformly in } n.$$

so

$$\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} \frac{\frac{z}{H_{\nu_n}^{-1}(z)} - 1}{1 - \frac{z}{H_{\nu_n}(z)}} = 1 \text{ uniformly in } n.$$

So we know that there exists a sequence (ν_n) of functions such that $(\lambda + 1)C_{\nu_n}(z) = \frac{H_{\nu_n}(z)}{z} - 1$ (which implies $C_{\nu_n}(z) = (\frac{1}{\sqrt{z}}G_{\nu_n}(\frac{1}{\sqrt{z}}) - 1)(1 + \nu_n(z))$) and $\lim_{z \rightarrow \infty} \nu_n(z) = 0$ uniformly in n . Hence we can choose β_2 such that $\forall n, \forall z \in \Delta_{\alpha, \beta_2}, |\nu_n(z)| \leq 1/2$.

Let us now prove (ii). First, note that since $\nu_n \rightarrow \delta_0$, the sequence (G_{ν_n}) converges uniformly to $G_{\delta_0} : z \mapsto 1/z$ on every compact of the upper half plane (see [A61] or section 3.1 of [HP00]), so, as in the proof of (i), it suffices to show that

$$\lim_{n \rightarrow \infty} \frac{(\lambda + 1)C_{\nu_n}(z)}{\frac{H_{\nu_n}(z)}{z} - 1} = 1.$$

The convergence of ν_n to δ_0 implies too that (H_{ν_n}) converges to $H_{\delta_0} : z \mapsto z$. So, since

$$\frac{H_{\nu_n}(z)}{z} - 1 = \frac{H_{\nu_n}(z)}{z} \left(1 - \frac{z}{H_{\nu_n}(z)}\right),$$

it suffices to prove that

$$\lim_{n \rightarrow \infty} \frac{(\lambda + 1)C_{\nu_n}(z)}{1 - \frac{z}{H_{\nu_n}(z)}} = 1.$$

Furthermore, by theorem 1.4, there exists $\beta_3 \leq \beta_2$ such that the sequence C_{ν_n} converges uniformly to $C_{\delta_0} = 0$ on every compact of Δ_{α, β_3} . So $z/H_{\nu_n}^{-1}(z) = (\lambda C_{\nu_n}(z) + 1)(C_{\nu_n}(z) + 1)$ converges uniformly to 1 on every compact of Δ_{α, β_3} . So, since $(\lambda + 1)U(z)$ is equivalent to z as z tends to zero and since $C_{\nu_n} = U(z/H_{\nu_n}^{-1}(z) - 1)$, it suffices to show that for all $z \in \Delta_{\alpha, \beta_3}$,

$$\lim_{n \rightarrow \infty} \frac{\frac{z}{H_{\nu_n}^{-1}(z)} - 1}{1 - \frac{z}{H_{\nu_n}(z)}} = 1.$$

As in the proof of (i), let us choose $\beta < \beta_3/2$ such that for all n , $H_{\nu_n}^{-1}$ is defined on $\Delta_{\alpha', \beta}$, $H_{\nu_n}(\Delta_{\alpha, \beta}) \cup H_{\nu_n}^{-1}(\Delta_{\alpha, \beta}) \subset \Delta_{\alpha', 2\beta}$, and

$$\lim_{n \rightarrow \infty} \frac{H_{\nu_n}^{-1}(z)}{z} = 1 \text{ uniformly on every compact of } \Delta_{\alpha', 2\beta}.$$

By analyticity of the $H_{\nu_n}^{-1}$'s, the last assertion implies that

$$\lim_{n \rightarrow \infty} (H_{\nu_n}^{-1})'(z) = 1 \text{ uniformly on every compact of } \Delta_{\alpha', 2\beta}.$$

We have, for $z \in \Delta_{\alpha,\beta}$,

$$\begin{aligned} \frac{\frac{z}{H_{\nu_n}^{-1}(z)} - 1}{1 - \frac{z}{H_{\nu_n}(z)}} - 1 &= \frac{\frac{zH_{\nu_n}(z)}{H_{\nu_n}^{-1}(z)} - H_{\nu_n}(z)}{H_{\nu_n}(z) - z} - 1 \\ &= \frac{1}{H_{\nu_n}(z) - z} \int_{[z, H_{\nu_n}(z)]} f'_n(\xi) d\xi, \end{aligned}$$

where f_n is still the function defined by

$$f_n(\xi) = \frac{H^{-1}(\xi)H(z)}{H^{-1}(z)} - \xi.$$

f'_n tends to zero as n tends to infinity, uniformly on every compact of $\Delta_{\alpha',2\beta}$, so

$$\frac{\frac{z}{H_{\nu_n}^{-1}(z)} - 1}{1 - \frac{z}{H_{\nu_n}(z)}} - 1$$

tends to zero when n tends to infinity, and the result is proved. \square

In the following, we shall refer to *weak convergence* for sequences of positive finite measures on the real line: that is convergence for which the test functions are the continuous bounded functions.

Theorem 2.5 (Lévy-Kinchine formula, part 1). *Let μ be a symmetric probability measure, (ν_n) be a sequence of symmetric probability measures and k_n a sequence of integers tending to infinity such that $\nu_n^{\boxplus k_n}$ converges weakly to μ . Then there exists a symmetric positive finite measure G such that*

- (1) *the sequence of positive finite measures $\left(k_n \frac{t^2}{1+t^2} d\nu_n(t)\right)$ converges weakly to G ,*
- (2) *the R -transform of μ has an analytic continuation to the complement of the real non-negative half line and is given by the formula*

$$C_\mu(z) = z \int_{\mathbb{R}} \frac{1+t^2}{1-zt^2} dG(t). \quad (2.3)$$

Moreover, G is symmetric and is the only positive finite measure F such that

$$C_\mu(z) = z \int_{\mathbb{R}} \frac{1+t^2}{1-zt^2} dF(t).$$

Proof.

- (1) The sequence (ν_n) converges weakly to δ_0 . Indeed, for every n , $C_{\nu_n^{\boxplus k_n}} = k_n C_{\nu_n}$, and by theorem 1.4, we have
 - (a) $\lim_{\substack{x \rightarrow 0 \\ x < 0}} k_n C_{\nu_n}(x) = 0$ uniformly in n ,
 - (b) There exists $\beta > 0$ such that the sequence $(k_n C_{\nu_n})$ converges pointwise on $(-\beta, 0)$.
 So
 - (a) $\lim_{\substack{x \rightarrow 0 \\ x < 0}} C_{\nu_n}(x) = 0$ uniformly in n ,
 - (b) there exists $\beta > 0$ such that the sequence (C_{ν_n}) converges pointwise to $0 = C_{\delta_0}$ on $(-\beta, 0)$.

- (2) The sequence of positive finite measures $\left(k_n \frac{t^2}{1+t^2} d\nu_n(t)\right)$ is tight. Indeed, for $y > 0$,

$$\begin{aligned} \int_{[-1/y, 1/y]^c} k_n \frac{t^2}{1+t^2} d\nu_n(t) &\leq 2 \int_{t \in \mathbb{R}} \frac{1+t^2}{y^{-2}+t^2} \cdot k_n \frac{t^2}{1+t^2} d\nu_n(t) \\ &= -2k_n((i/y)G_{\nu_n}(i/y) - 1). \end{aligned}$$

Let v_n be as in the previous lemma. For $y > 0$ small enough,

$$\int_{[-1/y, 1/y]^c} k_n \frac{t^2}{1+t^2} d\nu_n(t) \leq -\frac{2k_n C_{\nu_n}(-y^2)}{1+v_n(-y^2)} \leq 4|k_n C_{\nu_n}(-y^2)|,$$

which tends to zero uniformly in n when y tends to zero, by tightness of the sequence $(\nu_n^{\boxplus k_n})$.

- (3) The sequence of positive finite measures $\left(k_n \frac{t^2}{1+t^2} d\nu_n(t)\right)$ is bounded. Indeed, if $y > 0$ is such that $-y^2$ is in the domain of the v_n 's of the previous lemma and $\lim_{n \rightarrow \infty} k_n C_{\nu_n}(-y^2) = C_\mu(-y^2)$, we have, for each n ,

$$\begin{aligned} \int_{t \in \mathbb{R}} k_n \frac{t^2}{1+t^2} d\nu_n(t) &\leq y^{-2} \int_{t \in \mathbb{R}} k_n \frac{t^2}{y^{-2}+t^2} d\nu_n(t) \\ &= -y^{-2} k_n((i/y)G_{\nu_n}(i/y) - 1) \\ &= -y^{-2} k_n \frac{C_{\nu_n}(-y^2)}{1+v_n(-y^2)} \\ &\leq 4y^{-2} |k_n C_{\nu_n}(-y^2)|, \end{aligned}$$

which is bounded uniformly in n .

- (4) Let us now recall a few facts about the *Poisson integral* of positive measures on the real line which integrate $1/(1+t^2)$. If M is such a measure, for $y < 0$ and $x \in \mathbb{R}$, let us define

$$P_y(M)(x) = \int_{t \in \mathbb{R}} \frac{y}{y^2 + (x-t)^2} dM(t).$$

Then $(x+iy) \mapsto P_y(M)(x)$ is harmonic and determines the measure M ([D74], chapter II, theorem II).

Moreover, an easy computation shows that for each positive symmetric measure M on the real line that integrates $1/(1+t^2)$, the Poisson integral $P_y(M)(x)$ is the imaginary part of $\int_{\mathbb{R}} \frac{\sqrt{z}}{t^2 z - 1} dM(t)$ (with $z \notin [0, +\infty)$, $x+iy = 1/\sqrt{z}$, as it will be until the end of this proof). Indeed, since M is symmetric,

$$\begin{aligned} \int_{\mathbb{R}} \frac{\sqrt{z}}{t^2 z - 1} dM(t) &= \int_{\mathbb{R}} \frac{\sqrt{z}(t\sqrt{z}+1)}{t^2 z - 1} dM(t) = \int_{\mathbb{R}} \frac{\sqrt{z}(t\sqrt{z}+1)}{\sqrt{z}(t\sqrt{z}+1)(t-\frac{1}{\sqrt{z}})} dM(t) = \\ &= \int_{\mathbb{R}} \frac{dM(t)}{t - \frac{1}{\sqrt{z}}} = \int_{\mathbb{R}} \frac{(t-x) + iy}{(t-x)^2 + y^2} dM(t). \end{aligned}$$

Now let us compute the Poisson integral of the measures $k_n t^2 d\nu_n(t)$. Let $\alpha, \beta > 0$ and (v_n) be as in the previous lemma, $z \in \Delta_{\alpha, \beta}$. We have

$$P_y(k_n t^2 d\nu_n(t))(x) = \Im \left(\int_{\mathbb{R}} \frac{k_n \sqrt{z} t^2}{t^2 z - 1} d\nu_n(t) \right).$$

But since ν_n is symmetric, we have

$$\int_{\mathbb{R}} \frac{k_n \sqrt{z} t^2}{t^2 z - 1} d\nu_n(t) = k_n \int_{\mathbb{R}} \frac{t(1 + \sqrt{z}t)}{(t\sqrt{z} - 1)(t\sqrt{z} + 1)} d\nu_n(t) = k_n \int_{\mathbb{R}} \frac{t d\nu_n(t)}{t\sqrt{z} - 1},$$

which is equal, by an easy computation, to

$$-k_n \frac{(1/\sqrt{z})G_{\nu_n}(1/\sqrt{z}) - 1}{\sqrt{z}}.$$

So $P_y(k_n t^2 d\nu_n(t))(x) = -\Im \left(k_n \frac{C_{\nu_n}(z)}{\sqrt{z}(1 + v_n(z))} \right)$, which tends to the imaginary part of $-C_\mu(z)/\sqrt{z}$, because $\nu_n^{\boxplus k_n}$ converges weakly to μ and $\lim_{n \rightarrow \infty} v_n(z) = 0$.

The sequence $\left(k_n \frac{t^2}{1+t^2} d\nu_n(t) \right)$, bounded and tight, is relatively compact in the set of finite positive measures in the real line endowed with the topology of weak convergence (i.e. the topology defined by bounded continuous functions). If two measures G, H are the weak limit of a subsequences of $\left(k_n \frac{t^2}{1+t^2} d\nu_n(t) \right)$, then the measures $(1+t^2)dG(t)$ and $(1+t^2)dH(t)$ have the same Poisson integral on $\sqrt{\Delta_{\alpha,\beta}}$. Indeed, for $z \in \Delta_{\alpha,\beta}$,

$$P_y(k_n t^2 d\nu_n(t))(x) = \int_t \underbrace{\frac{y(1+t^2)}{y^2 + (x-t)^2}}_{\substack{\text{continuous} \\ \text{bounded fct of } t}} \frac{k_n t^2}{1+t^2} d\nu_n(t)$$

tends in the same time to $P_y((1+t^2)dG(t))(x)$, to $P_y((1+t^2)dH(t))(x)$, and to the imaginary part of $-C_\mu(z)/\sqrt{z}$. It implies, by harmonicity, that they have the same Poisson integral on the lower half plane, which implies $H = G$. So the sequence $\left(k_n \frac{t^2}{1+t^2} d\nu_n(t) \right)$ converges weakly to a measure G , such that the Poisson integral $P_y((1+t^2)dG(t))(x)$, is equal to the imaginary part of $-C_\mu(z)/\sqrt{z}$. Thus, the functions

$$C_\mu(z)/\sqrt{z} \quad \text{and} \quad \int_{\mathbb{R}} \frac{\sqrt{z}(t^2 + 1)}{1 - t^2 z} dG(t)$$

have the same imaginary part. For $z \in (-\infty, 0)$, it follows that

$$C_\mu(z) \quad \text{and} \quad z \int_{\mathbb{R}} \frac{t^2 + 1}{1 - t^2 z} dG(t)$$

have the same real part, so, by analyticity and since both tend to zero as z goes to zero, they are equal.

(5) If F is another positive finite measure such that $C_\mu(z) = z \int_{\mathbb{R}} \frac{t^2 + 1}{1 - t^2 z} dF(t)$, then

$$z \int_{\mathbb{R}} \frac{t^2 + 1}{1 - t^2 z} dG(t) = z \int_{\mathbb{R}} \frac{t^2 + 1}{1 - t^2 z} dF(t).$$

After division by $-\sqrt{z}$ and extraction of the imaginary part, this gives the equality of the Poisson integrals of $(1+t^2)dG(t)$ and of $(1+t^2)dF(t)$, which implies $G = F$.

□

The previous theorem implies that for all \boxplus_λ -infinitely divisible distribution μ , there exists a unique positive finite measure G such that C_μ is given by equation (2.3). G is symmetric (as limit of symmetric measures) and will be called the *Lévy measure* of μ . By injectivity of

the rectangular R -transform, two different probability measures cannot have the same Lévy measure.

Theorem 2.6 (Lévy-Kinchine formula, part 2). *Every symmetric positive finite measure on the real line is the Lévy measure of a \boxplus_λ -infinitely divisible distribution.*

Before the proof of the theorem, let us state a result about the way dilation of probability measures modify the rectangular R -transform. For $c > 0$, let us denote by $D_c : x \mapsto cx$. For any distribution μ , $D_c(\mu)$ is the push-forward of μ by D_c , i.e. $D_c(\mu) : B \mapsto \mu(c^{-1}B)$.

$$\begin{aligned}
 \text{We have} \quad G_{D_c(\mu)} &= \frac{1}{c} G_\mu\left(\frac{z}{c}\right), \\
 \text{so} \quad H_{D_c(\mu)}(z) &= \frac{\lambda}{c^2} \left(G_\mu\left(\frac{1}{c\sqrt{z}}\right) \right)^2 + \frac{(1-\lambda)c\sqrt{z}}{c^2} G_\mu\left(\frac{1}{c\sqrt{z}}\right) = \frac{1}{c^2} H_\mu(c^2 z), \\
 \text{i.e.} \quad H_{D_c(\mu)} &= D_{\frac{1}{c^2}} \circ H_\mu \circ D_{c^2}, \\
 H_{D_c(\mu)}^{-1} &= D_{\frac{1}{c^2}} \circ H_\mu^{-1} \circ D_{c^2}, \\
 \text{then} \quad C_{D_c(\mu)}(z) &= U\left(\frac{c^2 z}{H_\mu^{-1}(c^2 z)} - 1\right), \\
 \text{that is} \quad C_{D_c(\mu)}(z) &= C_\mu(c^2 z). \tag{2.4}
 \end{aligned}$$

Proof of the theorem. Let us denote by \mathcal{M} the set of symmetric positive finite measures G on the real line such that there exists a symmetric distribution μ whose rectangular R -transform is given by equation (2.3). We will show that \mathcal{M} is the set of symmetric positive finite measures, proving that $c\delta_0$ and $c(\delta_u + \delta_{-u}) \in \mathcal{M}$ for all $c, u > 0$, that \mathcal{M} is stable by addition, and that \mathcal{M} is closed under weak convergence. Note that once this result is proved, it will be clear that any symmetric probability measure with rectangular R -transform given by equation (2.3) will be \boxplus_λ -infinitely divisible. Indeed, denoting

$$C^{(G)}(z) = z \int_{\mathbb{R}} \frac{1+t^2}{1-zt^2} dG(t) \quad (G \in \mathcal{M}),$$

we have $C^{(G)} = nC^{(\frac{G}{n})}$.

- (1) For every $c > 0$, $c\delta_0 \in \mathcal{M}$. Indeed, by equation (2.4), if $C^{(\delta_0)} = C_\mu$, then for every $c > 0$, $C^{(c\delta_0)} = C_{\mu'}$, with $\mu' = D_{c^{1/2}}(\mu)$, so it suffices to show that there exists a symmetric distribution whose rectangular R -transform is $C^{(\delta_0)}$. This distribution will appear as the limit in the *rectangular free central limit theorem*: the sequence $D_{n^{-1/2}}((\delta_1 + \delta_{-1})/2)^{\boxplus_\lambda n}$ converges weakly to a distribution with rectangular R -transform $C^{(\delta_0)}$. The proof is an easy application of additivity of the rectangular R -transform, equation (2.4), and theorem 1.4. We will see in the following that it stays true if one replaces $(\delta_1 + \delta_{-1})/2$ by any symmetric probability measure with variance equal to 1.
- (2) For all $c, u > 0$, $c(\delta_u + \delta_{-u}) \in \mathcal{M}$. Indeed, we have

$$C^{(c(\delta_u + \delta_{-u}))}(z) = 2c \frac{z(1+u^2)}{1-u^2z} = 2 \frac{c(1+u^2)}{2u^2} \frac{(u^2z)(1+1^2)}{1-(u^2z)} = C^{(c'(\delta_1 + \delta_{-1}))}(u^2z),$$

where $c' = \frac{c(1+u^2)}{2u^2}$. So, by equation (2.4), it suffices to show that for all $c > 0$, there exists a distribution whose rectangular R -transform is $C^{(c(\delta_1 + \delta_{-1}))}$. It is the same to prove that there exists a distribution whose rectangular R -transform is $C^{(\frac{c}{4}(\delta_1 + \delta_{-1}))}$. This distribution will appear as the limit in the *rectangular free Poisson limit theorem*: the

sequence $\nu_n^{\boxplus \lambda^n}$, with $\nu_n = (1 - \frac{c}{n})\delta_0 + \frac{c}{2n}(\delta_1 + \delta_{-1})$, converges weakly to a distribution with rectangular R -transform is $C^{(\frac{c}{4}(\delta_1 + \delta_{-1}))}$.

Indeed, $G_{\nu_n}(z) = \frac{z^2 - 1 + c/n}{z(z^2 - 1)}$, so, if (v_n) is a sequence of functions on $\Delta_{\alpha, \beta}$ as in the lemma 2.4, we have

$$C_{\nu_n}(z) = \left(\frac{1}{\sqrt{z}} G_{\nu_n} \left(\frac{1}{\sqrt{z}} \right) - 1 \right) (1 + v_n(z)) = \frac{cz}{n(1-z)} (1 + v_n(z)),$$

so for $\mu_n = \underbrace{\nu_n \boxplus \lambda \cdots \boxplus \lambda \nu_n}_{n \text{ times}}$,

$$C_{\mu_n}(z) = n C_{\nu_n}(z) = \frac{cz}{1-z} (1 + v_n(z)).$$

So by hypothesis on the functions v_n , we have both

$$\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} C_{\mu_n}(z) = 0 \text{ uniformly on } n$$

and

$$\forall z \in \Delta_{\alpha, \beta}, \lim_{n \rightarrow \infty} C_{\mu_n}(z) = \frac{cz}{1-z} = C^{(\frac{c}{4}(\delta_1 + \delta_{-1}))}(z).$$

So, by theorem 1.4, we know that there exists a distribution whose rectangular R -transform is $C^{(\frac{c}{4}(\delta_1 + \delta_{-1}))}$.

- (3) \mathcal{M} is stable by addition because $C_\mu + C_\nu = C_{\mu \boxplus \lambda \nu}$.
- (4) \mathcal{M} is closed under weak convergence: let (G_n) be a sequence of \mathcal{M} that converges to a finite measure G . Then clearly, the sequence $(C^{(G_n)})$ converges pointwise to $C^{(G)}$. So, by theorem 1.4, to prove that $G \in \mathcal{M}$, it suffices to show that

$$\lim_{\substack{x \rightarrow 0 \\ x < 0}} C^{(G_n)}(x) = 0 \text{ uniformly in } n.$$

For each n and $x \in (0, 1)$, since G is symmetric, $C^{(G_n)}(-x^2) = - \int_{\mathbb{R}} \frac{x^2 + t^2 x^2}{1 + t^2 x^2} dG_n(t)$,

$$\forall t \in \mathbb{R}, \frac{x^2 + t^2 x^2}{1 + t^2 x^2} \leq \begin{cases} x(x+1) & \text{if } -1/x^{1/2} \leq t \leq 1/x^{1/2}, \\ 1 & \text{elseif.} \end{cases}$$

So

$$\left| C^{(G_n)}(x) \right| \leq x(x+1)G_n(\mathbb{R}) + G_n\left(\mathbb{R} - [-1/x^{1/2}, 1/x^{1/2}]\right),$$

which tends to zero uniformly in n when x tends to $-\infty$, by boundness and tightness of $\{G_n; n \in \mathbb{N}\}$.

□

Both previous theorems together allow us to state the following corollary.

Corollary 2.7. *A symmetric probability measure μ is \boxplus_λ -infinitely divisible if and only if there exists a sequence (ν_n) of symmetric probability measures and a sequence (k_n) of integers tending to infinity such that the sequence $(\nu_n^{\boxplus \lambda^{k_n}})$ tends to μ .*

3. RECTANGULAR BERCOVICI-PATA'S BIJECTION

In this section, we will show that the bijective correspondence between classical symmetric infinitely divisible distributions and rectangular free infinitely divisible distributions is an homeomorphism, and that there exists a correspondence between limit theorems for sums of independent symmetric random variables and sums of free rectangular random variables.

Let us recall a few facts about symmetric $*$ -infinitely divisible distributions, that can be found in [GK54] (or [F66], [P97] ...). A symmetric probability measure μ on the real line is $*$ -infinitely divisible if and only if there exists a finite positive symmetric measure G such that

$$\forall \xi \in \mathbb{R}, \int_{t \in \mathbb{R}} e^{it\xi} d\mu(t) = \exp \left(\int_{t \in \mathbb{R}} (\cos(t\xi) - 1) \frac{1+t^2}{t^2} dG(t) \right).$$

In this case, such a measure G is unique, and we will call it the *Lévy measure* of μ , and a sequence of symmetric $*$ -infinitely divisible distributions converges weakly if and only if the sequence of the corresponding Lévy measures converges weakly. Moreover, in this case, the Lévy measure of the limit will be the limit of the Lévy measures.

We can then define the *rectangular Bercovici-Pata bijection with ratio λ* , denoted by Λ_λ , from the set of symmetric $*$ -infinitely divisible distributions to the set of \boxplus_λ -infinitely divisible distributions, that maps a $*$ -infinitely divisible distribution to the \boxplus_λ -infinitely divisible distribution with the same Lévy measure. Let μ, ν be two $*$ -infinitely divisible distributions with Lévy measures G, H . Then the Lévy measures of $\mu * \nu$ and of $\Lambda_\lambda(\mu) \boxplus_\lambda \Lambda_\lambda(\nu)$ are both $G + H$, so we have

$$\Lambda_\lambda(\mu * \nu) = \Lambda_\lambda(\mu) \boxplus_\lambda \Lambda_\lambda(\nu).$$

Theorem 3.1. *The rectangular Bercovici-Pata bijection with ratio λ is an homeomorphism, which means that a sequence of \boxplus_λ -infinitely divisible distributions converges weakly if and only if the sequence of the corresponding Lévy measures converges weakly, and in this case, the Lévy measure of the limit is be the limit of the Lévy measures.*

Remark 3.2. *Note that, for G symmetric positive finite measure, the function $C^{(G)}(z)$ can also be written, by symmetry,*

$$C^{(G)}(z) = \int_{\mathbb{R}} \frac{z + t\sqrt{z}}{1 - t\sqrt{z}} dG(t).$$

Proof. Since the rectangular R -transform C_μ with ratio 1 of a symmetric distribution μ is linked to its Voiculescu transform φ_μ by the relation $C_\mu(z) = \sqrt{z}\varphi_\mu(1/\sqrt{z})$ (see paragraph 5 of [BV93] for the construction of the Voiculescu transform, and use the fact that for symmetric distributions, the Lévy measure is symmetric to obtain $C_\mu(z) = \sqrt{z}\varphi_\mu(1/\sqrt{z})$), the previous remark and theorem 5.10 of [BV93] shows that the map Λ_1 is the restriction of the “usual” Bercovici-Pata bijection to the set of symmetric distributions. It has been proved in [B-NT02] that the Bercovici-Pata bijection is an homeomorphism. So the theorem is proved in the case where $\lambda = 1$. But for every $*$ -infinitely divisible distribution μ , the formula of the rectangular R -transform with ratio λ of $\Lambda_\lambda(\mu)$ does not depend on λ , so theorem 1.4 allows us to claim that all Λ_λ 's are homeomorphisms. \square

The next theorem furthers the analogy between the free rectangular convolution and the classical convolution of symmetric measures. As recalled in Theorem 3.3 of [BPB99], It is proved in [GK54] that when (ν_n) is a sequence of symmetric probability measures on the real line and (k_n) is a sequence of integers tending to infinity, the sequence $(\nu_n^{*k_n})$ converges weakly to a $*$ -infinitely divisible distribution if and only if the sequence $\left(\frac{k_n t^2}{1+t^2} d\nu_n(t) \right)$ of positive finite measures

converges weakly to its Lévy measure. By the theorem 2.5, we know that it will be the case if the sequence $(\nu_n^{\boxplus \lambda^{k_n}})$ converges weakly to the image of the $*$ -infinitely divisible distribution by the rectangular Bercovici-Pata bijection. The following theorem states the reciprocal implication. So we have, for all $*$ -infinitely divisible distribution μ ,

$$(\nu_n^{*k_n}) \text{ converges to } \mu \iff (\nu_n^{\boxplus \lambda^{k_n}}) \text{ converges to } \Lambda_\lambda(\mu) \quad (3.1)$$

Theorem 3.3. *Let (ν_n) be a sequence of symmetric probability measures on the real line and (k_n) be a sequence of integers tending to infinity. The sequence $(\nu_n^{*k_n})$ converges weakly to an $*$ -infinitely divisible distribution if and only if the sequence $(\nu_n^{\boxplus \lambda^{k_n}})$ converges weakly to its image by the rectangular Bercovici-pata bijection with ratio λ .*

Proof. By what precedes, it suffices to prove that if the sequence $(\frac{k_n t^2}{1+t^2} d\nu_n(t))$ of positive finite measures converges weakly to a finite measure G , then the sequence $(\nu_n^{\boxplus \lambda^{k_n}})$ converges weakly to the \boxplus_λ -infinitely divisible distribution with Lévy measure G . Assume the sequence $(\frac{k_n t^2}{1+t^2} d\nu_n(t))$ of positive finite measures to converge weakly to a finite measure G .

- (1) The sequence (ν_n) converges weakly to δ_0 :

Indeed, for all $\varepsilon > 0$, as the function $t \mapsto \frac{t^2}{1+t^2}$ is increasing on \mathbb{R}^+ , we have

$$\nu_n([- \varepsilon, \varepsilon]^c) \leq \frac{1 + \varepsilon^2}{\varepsilon^2} \int_{\mathbb{R}} \frac{t^2}{1 + t^2} d\nu_n(t),$$

which tends to zero as n tends to infinity, because the sequence $(\frac{k_n t^2}{1+t^2} d\nu_n(t))$ is bounded.

- (2) We have pointwise convergence of the rectangular R -transforms:

Let α, β and (v_n) be as in the lemma 2.4. On $\Delta_{\alpha, \beta}$, we have

$$C_{\nu_n^{\boxplus \lambda^{k_n}}}(z) = k_n C_{\nu_n}(z) = k_n \left(\frac{1}{\sqrt{z}} G_{\nu_n} \left(\frac{1}{\sqrt{z}} \right) - 1 \right) (1 + v_n(z)), \quad (3.2)$$

but we have seen in the proof of theorem 2.5 that

$$k_n \left(\frac{1}{\sqrt{z}} G_{\nu_n} \left(\frac{1}{\sqrt{z}} \right) - 1 \right) = z \int_{\mathbb{R}} \underbrace{\frac{t^2 + 1}{1 - t^2 z}}_{\substack{\text{continuous} \\ \text{bounded fct of } t}} \frac{k_n t^2}{1 + t^2} d\nu_n(t),$$

so, by pointwise convergence of the sequence (v_n) to zero, the rectangular R -transform of $\nu_n^{\boxplus \lambda^{k_n}}$ converges pointwise to $z \mapsto \int_{\mathbb{R}} \frac{t^2 + 1}{1 - t^2 z} dG(t)$ on the set $\Delta_{\alpha, \beta}$.

- (3) We have $\lim_{\substack{y \rightarrow 0 \\ y > 0}} C_{\nu_n^{\boxplus \lambda^{k_n}}}(-y^2) = 0$ uniformly in n :

By equation (3.2) and (i) of lemma 2.4, it suffices to prove that

$$\lim_{\substack{y \rightarrow 0 \\ y > 0}} k_n ((i/y) G_{\nu_n}(i/y) - 1) = 0 \quad \text{uniformly in } n,$$

that is, since ν_n is symmetric,

$$\lim_{\substack{y \rightarrow 0 \\ y > 0}} \int_{\mathbb{R}} \frac{y^2 + t^2 y^2}{1 + t^2 y^2} \frac{k_n t^2}{1 + t^2} d\nu_n(t) = 0 \quad \text{uniformly in } n.$$

When $y < 1$, $t \mapsto \frac{y^2+t^2y^2}{1+t^2y^2} \leq 1$ is increasing on $[0, \infty)$, so we have, for every $T > 0$,

$$\int_{\mathbb{R}} \frac{y^2 + t^2 y^2}{1 + t^2 y^2} \frac{k_n t^2}{1 + t^2} d\nu_n(t) \leq \int_{[-T, T]^c} \frac{k_n t^2}{1 + t^2} d\nu_n(t) + \frac{y^2 + T^2 y^2}{1 + T^2 y^2} \int_{\mathbb{R}} \frac{k_n t^2}{1 + t^2} d\nu_n(t).$$

Now fix $\varepsilon > 0$, choose $T > 0$ such that for all n , $\int_{[-T, T]^c} \frac{k_n t^2}{1 + t^2} d\nu_n(t) \leq \varepsilon$. For y large enough, $\frac{y^2 + T^2 y^2}{1 + T^2 y^2} \sup_n \int_{\mathbb{R}} \frac{k_n t^2}{1 + t^2} d\nu_n(t)$ is less than ε , which closes the proof.

□

The following corollary could have been proved with the equation (2.4), but the proof we give is shorter and does not use any computations.

Corollary 3.4. *The rectangular Bercovici-Pata bijection commutes with the dilations D_c , $c > 0$.*

Proof. Let μ be a $*$ -infinitely divisible distribution. Let, for each $n \geq 1$, ν_n be a symmetric distribution such that $\nu_n^{*n} = \mu$. We have

$$\begin{aligned} \Lambda_\lambda \circ D_c(\mu) &= \Lambda_\lambda \circ D_c \left(\lim_{n \rightarrow \infty} \nu_n^{*n} \right) \\ &= \Lambda_\lambda \left(\lim_{n \rightarrow \infty} D_c(\nu_n)^{*n} \right) \\ &= \lim_{n \rightarrow \infty} D_c(\nu_n)^{\boxplus \lambda^n}. \end{aligned}$$

But from equation (2.4) and additivity of the rectangular R -transform, we know that

$$\forall n \geq 1, D_c(\nu_n)^{\boxplus \lambda^n} = D_c \left(\nu_n^{\boxplus \lambda^n} \right),$$

so, by continuity of D_c ,

$$\Lambda_\lambda \circ D_c(\mu) = D_c \left(\lim_{n \rightarrow \infty} \nu_n^{\boxplus \lambda^n} \right)$$

which is $D_c \circ \Lambda_\lambda(\mu)$ by equation (3.1). □

So, if we define the \boxplus_λ -stable distributions to be the symmetric distributions whose orbit under the action of the group of the dilations is stable under \boxplus_λ , the rectangular Bercovici-Pata bijection exchanges symmetric $*$ -stable and \boxplus_λ -stable distributions. Moreover, the index of any $*$ -stable distribution μ (i.e. the unique $\alpha \in (0, 2]$ such that for all $n \geq 1$, $\mu^{*n} = D_{n^{-\frac{1}{\alpha}}}(\mu)$) is preserved, i.e. one has $\Lambda_\lambda(\mu)^{\boxplus \lambda^n} = D_{n^{-\frac{1}{\alpha}}}(\Lambda_\lambda(\mu))$.

Remark This remark could be called *missed appointment for the Cauchy distribution*. The Cauchy type, $\{\mathcal{C}_t = \frac{1}{\pi} \frac{tdx}{x^2+t^2}; t > 0\}$, is well known to be invariant under many transformations. For example, this set is the set of symmetric $*$ - and \boxplus -stable distributions with index 1 (\mathcal{C}_t has Lévy measure $t\mathcal{C}_1$). But unless $\lambda = 1$, the set of \boxplus_λ -stable distributions with index 1 is not the Cauchy type. Indeed, for $t > 0$, the rectangular R -transform with ratio λ of $\Lambda_\lambda(\mathcal{C}_t)$ is $it\sqrt{z}$. On the other hand, from the easy computation $G_{\mathcal{C}_t}(z) = 1/(z - it)$ on the lower half-plane, one has

$H_{\mathcal{C}_t}(z) = z \frac{1-(1-\lambda)it\sqrt{z}}{(1-it\sqrt{z})^2}$, and

$$\begin{aligned}
\mathcal{C}_t = \Lambda_\lambda(\mathcal{C}_t) &\Leftrightarrow it\sqrt{z} = U\left(\frac{z}{H_{\mathcal{C}_t}^{-1}(z)} - 1\right) \\
&\Leftrightarrow (\lambda it\sqrt{z} + 1)(it\sqrt{z} + 1) = \frac{z}{H_{\mathcal{C}_t}^{-1}(z)} \\
&\Leftrightarrow H_{\mathcal{C}_t}^{-1}(z) = \frac{z}{(\lambda it\sqrt{z} + 1)(it\sqrt{z} + 1)} \\
&\Leftrightarrow H_{\mathcal{C}_t}\left(\frac{z}{A(z)}\right) = z, \text{ where } A(z) = (\lambda it\sqrt{z} + 1)(it\sqrt{z} + 1) \\
&\Leftrightarrow 1 - (1 - \lambda)itV(z) = A(z)((1 - itV(z))^2, \text{ where } V(z) = \sqrt{z/A(z)} \\
&\Leftrightarrow itV(z)(2A(z) - 1 + \lambda) = A(z) - t^2z - 1 \\
&\Leftrightarrow t^2z(2A(z) - 1 + \lambda)^2 = A(z)(A(z) - t^2z - 1)^2,
\end{aligned}$$

which can be easily verified using a formal calculus program, and which happens to be false unless $\lambda = 1$.

Question. Inspired by what happens in the cases $\lambda = 0$ and $\lambda = 1$, we ask the following question: is there, for each λ in $[0, 1]$, a fonctionnal f_λ from the set of symmetric probability measures on \mathbb{R} to the set of probability measures on \mathbb{R} such that for all μ, ν symmetric probability measures, $\mu \boxplus_\lambda \nu$ is the only symmetric probability measure satisfying

$$f_\lambda(\mu \boxplus_\lambda \nu) = f_\lambda(\mu) \boxplus f_\lambda(\nu) \quad ?$$

Note that in the case $\lambda = 1$, the fonctionnal $f_\lambda(\mu) = \mu$ works, and in the case $\lambda = 0$, fonctionnal which maps a measure to its push-forward by the square function works.

4. RECTANGULAR GAUSSIAN DISTRIBUTION AND MARCHENKO-PASTUR DISTRIBUTION

In this section, we will identify the *rectangular Gaussian distribution* ν , that is the image, by the rectangular Bercovici-Pata bijection, of the Gaussian distribution with mean zero and variance one. The corresponding Lévy measure is δ_0 , so the rectangular R -transform is z . We will show that unless $\lambda = 0$, in which case $\nu = (\delta_{-1} + \delta_1)/2$, ν is the symmetric distribution whose push forward by the function $x \rightarrow x^2$ has the density

$$\frac{[4\lambda - (x - 1 - \lambda)^2]^{1/2}}{2\pi\lambda x} \chi(x),$$

where χ stands for the characteristic function of the interval $[(1 - \lambda^{1/2})^2, (1 + \lambda^{1/2})^2]$, which means that for all $n \geq 1$, the $2n$ -th moment of ν is $1/\lambda$ times the n -th moment of the Marchenko-Pastur distribution with expectation λ (the Marchenko-Pastur distributions are presented page 65 of [HP00]).

Recall that the sequence $(c_{2n}(\mu))_{n \geq 1}$ of the *free cumulants with ratio* λ of a symmetric probability measure μ , defined in the very beginning of section 4.1 of [B-G1], are linked to the sequence $(m_n(\mu))_{n \geq 0}$ of its moments by the relation (proposition 3.5 of [B-G1]):

$$\forall n \geq 1, \quad m_{2n}(\mu) = \sum_{\pi \in \text{NC}'(2n)} \lambda^{e(\pi)} \prod_{V \in \pi} c_{|V|}(\mu), \quad (4.1)$$

where $\text{NC}'(2n)$ is the set of noncrossing partitions of $\{1, \dots, 2n\}$ in which all blocks have even cardinality, and where $e(\pi)$ denotes the number of blocks of π with even minimum.

The following lemma will be useful to study distributions coming from rectangular free probability theory. A function f defined on a conjugaison-stable subset of \mathbb{C} is said to be *commuting with the conjugaison* (abbreviated by *c.w.c.*) if $f(\bar{z}) = \overline{f(z)}$. Note that the function $z \rightarrow z^{1/2}$ is c.w.c., whereas $z \rightarrow \sqrt{z}$ is not.

Lemma 4.1. *If the rectangular R -transform of symmetric probability measure μ extends to an analytic c.w.c. function in a neighborhood $B(0, r)$ of zero in the complex plane and tends to zero at zero, then the probability measure has compact support, and the expansion of $C_\mu(z)$ for small z is given by the formula*

$$C_\mu(z) = \sum_{n=1}^{+\infty} c_{2n}(\mu) z^n. \quad (4.2)$$

Proof. Let us define $T(z) = (\lambda z + 1)(z + 1)$. Note that U is the inverse of $T - 1$. Since the extension of C_μ tends to zero at zero, $z/H_\mu^{-1}(z)$ extends to a neighborhood of zero such that we have, in this neighborhood,

$$\frac{z}{H_\mu^{-1}(z)} = T(C_\mu(z)),$$

and this function tends to 1 at zero. Thus $H_\mu^{-1}(z)$ is one to one in a neighborhood of zero, and H_μ extends to an analytic c.w.c. function in a neighborhood of zero such that

$$\lim_{z \rightarrow 0} \frac{H_\mu(z)}{z} = 1.$$

So the function $G_\mu(1/\sqrt{z})/\sqrt{z}$, which is equal to

$$\frac{\lambda - 1 + [(1 - \lambda)^2 + 4\lambda(H_\mu(z)/z)]^{1/2}}{2\lambda}$$

if $\lambda > 0$ and to $H_\mu(z)$ if $\lambda = 0$, extends to an analytic c.w.c. function in a neighborhood of zero. But since μ is symmetric, for all z in the complement of the real nonnegative half line,

$$\frac{G_\mu(1/\sqrt{z})}{\sqrt{z}} = \frac{1}{\sqrt{z}} \int_{\mathbb{R}} \frac{d\mu(t)}{\frac{1}{\sqrt{z}} - t} = \frac{1}{2\sqrt{z}} \int_{\mathbb{R}} \left[\frac{1}{\frac{1}{\sqrt{z}} - t} + \frac{1}{\frac{1}{\sqrt{z}} + t} \right] d\mu(t) = \int_{\mathbb{R}} \frac{d\mu(t)}{\frac{1}{z} - t^2} = G_\rho(1/z),$$

where ρ is the push forward of μ by the function $t \rightarrow t^2$. Hence the Cauchy transform of ρ extends to an analytic c.w.c. function in a neighborhood of infinity. Thus, by the Stieltjes inversion formula, ρ is compactly supported, which implies that μ has compact support too.

μ has now been proved to be compactly supported. Then the second part of the lemma, equation (4.2), has been established in section 4.1 of [B-G1]. \square

So ν has compact support, and for all $n \geq 1$, $c_{2n}(\nu) = \delta_{1,n}$.

Let us first treat the case where $\lambda = 0$. By (4.1), all even moment of ν are 1, so $\nu = (\delta_{-1} + \delta_1)/2$.

$\lambda > 0$, by (4.1), the moments of ν are given by

$$\forall n \geq 1, \quad m_{2n}(\nu) = \sum_{\pi} \lambda^{e(\pi)} = \lambda^n \sum_{\pi} \left(\frac{1}{\lambda} \right)^{o(\pi)},$$

where the sums are taken over noncrossing pairings of $\{1, \dots, 2n\}$ (a *noncrossing pairing* is a noncrossing partition where all classes have cardinality two, recall also that for π partition, $e(\pi)$ and $o(\pi)$ are respectively the number of classes of π with even and odd minimum).

Lemma 4.2. *Let $I = \{x_1 < \dots < x_n\}$ and $J = \{y_1 < z_1 < y_2 < z_2 < \dots < y_n < z_n\}$ be totally ordered sets. There is a bijection $\pi \rightarrow \tilde{\pi}$ from the set of noncrossing partitions of I to the set of noncrossing pairings of J such that for all π ,*

$$|\pi| = o(\tilde{\pi}).$$

Proof. Let us first construct the map $\pi \rightarrow \tilde{\pi}$ by the induction on n , using the following well known result : a partition π of a finite totally ordered set is noncrossing if and only if one of its class V is an interval and $\pi \setminus \{V\}$ is noncrossing (page 3 of [S98]). Consider a noncrossing partition π of I . If π has only one class, we define $\tilde{\pi}$ to be

$$\{\{y_1, z_n\}, \{z_1, y_2\}, \{z_2, y_3\}, \dots, \{z_{n-1}, y_n\}\}.$$

In the other case, a strict class V of π is an interval, $V = \{x_k, x_{k+1}, \dots, x_l\}$. Then we define $\tilde{\pi}$ to be

$$\tilde{\sigma} \cup \{\{y_k, z_l\}, \{z_k, y_{k+1}\}, \{z_{k+1}, y_{k+2}\}, \dots, \{z_{l-1}, y_l\}\},$$

where $\tilde{\sigma}$ is the image (defined by the induction hypothesis) of the partition

$$\sigma = \pi - \{V\}$$

of $I - V$ (it is easy to see that the result does not depend on the choice of the interval V).

The relation $|\pi| = o(\tilde{\pi})$ follows from the construction of $\pi \rightarrow \tilde{\pi}$.

Let us now prove, by induction on n , that $\pi \rightarrow \tilde{\pi}$ is a bijection. If $n = 1$, the result is obvious. Suppose the result to be proved to the ranks $1, \dots, n-1$, and consider a noncrossing pairing τ of J . Let us prove that there exists exactly one noncrossing partition π of I such that $\tilde{\pi} = \tau$. Consider $l \in [n]$ minimal such that there exists $k < l$ such that $\{y_k, z_l\}$ is a class of τ (such an l exists because it is the case of n). Then it is easy to see that $\{z_k, y_{k+1}\}, \{z_{k+1}, y_{k+2}\}, \dots, \{z_{l-1}, y_l\}$ are classes of τ , and any partition π of I such that $\tilde{\pi} = \tau$ must satisfy $V := \{x_k, x_{k+1}, \dots, x_l\} \in \pi$, and

$$\tilde{\sigma} = \tau - \{\{y_k, z_l\}, \{z_k, y_{k+1}\}, \{z_{k+1}, y_{k+2}\}, \dots, \{z_{l-1}, y_l\}\},$$

where $\sigma = \pi - \{V\}$ (partition of $I - V$). Thus, by the induction hypothesis, there exists exactly one noncrossing partition π of I such that $\tilde{\pi} = \tau$. \square

So the moments of ν are given by

$$\forall n \geq 1, \quad m_{2n}(\nu) = \lambda^n \sum_{\pi \in \text{NC}(n)} \left(\frac{1}{\lambda}\right)^{|\pi|}.$$

But for all $n \geq 1$, $\sum_{\pi \in \text{NC}(n)} (1/\lambda)^{|\pi|}$ is the n -th moment of a distribution with all free cumulants being equal to $1/\lambda$, i.e. of the Marchenko-Pastur distribution with parameter $1/\lambda$ (see [HP00] p. 65). Thus the push-forward of ν by $t \rightarrow t^2$ is the push-forward of the Marchenko-Pastur distribution with parameter $1/\lambda$ by the map $t \rightarrow \lambda t$, and has density

$$\frac{[4\lambda - (x - 1 - \lambda)^2]^{1/2}}{2\pi\lambda x} \chi(x),$$

where χ stands for the characteristic function of the interval $[(1 - \lambda^{1/2})^2, (1 + \lambda^{1/2})^2]$. Hence we have proved the following result:

Theorem 4.3. *The rectangular Gaussian distribution ν with ratio $\lambda > 0$ has cumulants given by*

$$\forall n \geq 1, c_{2n}(\nu) = \delta_{n,1},$$

and has density

$$\frac{[4\lambda - (x^2 - 1 - \lambda)^2]^{1/2}}{2\pi\lambda|x|}\chi(x^2).$$

Its support is $[-1 - \lambda^{1/2}, -1 + \lambda^{1/2}] \cup [1 - \lambda^{1/2}, 1 + \lambda^{1/2}]$.

Note that when $\lambda = 1$, it is the well-known semi-circle law with radius two.

Remark 4.4 (Moments of the Marchenko-Pastur distribution). *Note that the fact that the free cumulants of the Marchenko-Pastur distribution with parameter a (see [HP00] p. 65) are all a and the previous lemma give us a formula for its n -th moment: it is equal to $\sum_{\pi} a^{o(\pi)}$, where the sum is taken over all noncrossing pairings of $[2n]$. This formula, proved using a random matrix approach, appeared already in an unpublished paper of Ferenc Oravecz and Dénes Petz.*

5. A MATRICIAL MODEL FOR THE RECTANGULAR BERCOVICI-PATA BIJECTION

In the previous section, the proofs rely on integral transforms and complex analysis. We will construct, in this subsection, a matricial model for the \boxplus_{λ} -infinitely divisible laws and present in a maybe more palpable way the Bercovici-Pata bijection with ratio λ .

In this section, d, d' will represent dimensions of rectangular matrices, because n will be used to another role. For any distribution \mathbb{P} and any function f on a set of matrices, $\mathbb{E}_{\mathbb{P}}(f(M))$ denotes $\int f(M)d\mathbb{P}(M)$. Let us recall that the *singular law* of a matrix M designs the uniform distribution on the spectrum of $|M| := (MM^*)^{\frac{1}{2}}$. Let us define the *symmetrization* $\tilde{\mu}$ of a distribution μ on the real line: it is the distribution which maps a Borel set B to $(\mu(B) + \mu(-B))/2$. The symmetrization of the singular law of a matrix M will be denoted by $\tilde{\mu}_{|M|}$.

We are going to construct, in the same way as in [B-G04] and in [C-D04], for each $d, d' \geq 1$, for each symmetric $*$ -infinitely divisible distribution μ , an infinitely divisible distribution $\mathbb{P}_{d,d'}^{\mu}$ on the set of $d \times d'$ complex matrices such that for all μ, ν , $\mathbb{P}_{d,d'}^{\mu} * \mathbb{P}_{d,d'}^{\nu} = \mathbb{P}_{d,d'}^{\mu * \nu}$ and such that the symmetrization of the singular law of M (with M random matrix distributed according to $\mathbb{P}_{d,d'}^{\mu}$) goes from μ to its image by the rectangular Bercovici-Pata bijection with ratio λ when $d, d' \rightarrow \infty, \frac{d}{d'} \rightarrow \lambda$.

Let us introduce the heuristic argument that led us to choose the model we will present. Consider a symmetric $*$ -infinitely divisible distribution μ , and two sequences (ν_n) (symmetric probability measures), (k_n) (integers tending to infinity) such that $\nu_n^{*k_n}$ tends weakly to μ . Define, for each $1 \leq d \leq d'$ and each $n \geq 1$, $\mathbb{Q}_{d,d'}^{\nu_n}$ to be the law of the $d \times d'$ random matrix

$$U \left[X_{n,i} \delta_i^j \right]_{\substack{1 \leq i \leq d \\ 1 \leq j \leq d'}} V$$

where U (resp. V) is a uniform $d \times d$ (resp. $d' \times d'$) unitary random matrix, $X_{n,1}, \dots, X_{n,d}$ are distributed according to ν_n , and $U, V, X_{n,1} \dots$ are independent.

Then if one fixes n and lets d, d' go to infinity in such a way that $\frac{d}{d'} \rightarrow \lambda$, the symmetrization of the singular law of $M_1(\nu_n) + \dots + M_n(\nu_n)$ (with $M_1(\nu_n), \dots, M_n(\nu_n)$ independent and distributed according to $\mathbb{Q}_{d,d'}^{\nu_n}$) goes to $\nu_n^{\boxplus \lambda k_n}$.

Moreover, if one fixes d, d' and lets n go to infinity, the distribution $\left(\mathbb{Q}_{d,d'}^{\nu_n} \right)^{*k_n}$ of $M_1(\nu_n) + \dots + M_n(\nu_n)$ converges weakly to a distribution $\mathbb{P}_{d,d'}^{\mu}$ on the set of $d \times d'$ matrices, whose Fourier

transform is given by the following formula: for all $d \times d'$ matrix A

$$\mathbb{E}_{\mathbb{P}_{d,d'}^\mu} (\exp (i \Re (\text{Tr } A^* M))) = \exp (\mathbb{E} (d \times \psi_\mu (\Re (< u, Av >)))) \quad (5.1)$$

where ψ_μ is the *Lévy exponent* of μ , i.e. the unique continuous function f on \mathbb{R} such that $f(0) = 0$ and the Fourier transform of μ is $\exp \circ f$, $< \cdot, \cdot >$ is the canonical hermitian product of \mathbb{C}^d , and $u = (u_1, \dots, u_d)$, $v = (v_1, \dots, v_{d'})$ are independent random vectors, uniformly distributed on the unit sphere of respectively \mathbb{C}^d , $\mathbb{C}^{d'}$. The proof of this weak convergence, analogous to the one of theorem 3.1 of [B-G04], uses the polar decomposition of $d \times d'$ matrices and the bi-unitarily invariance of the distributions $\mathbb{Q}_{d,d'}^{\nu_n}$. Note that for all μ, ν , $\mathbb{P}_{d,d'}^\mu * \mathbb{P}_{d,d'}^\nu = \mathbb{P}_{d,d'}^{\mu * \nu}$, and that when $\mu = N(0, 1)$, $\mathbb{P}_{d,d'}^\mu$ is the distribution of a matrix $[M_{i,j}]$ with $(\Re M_{i,j}, \Im M_{i,j})_{\substack{1 \leq i \leq d \\ 1 \leq j \leq d'}}$ i.i.d. random variables $N(0, \frac{1}{2d'})$ -distributed i.i.d..

So the convergence of the symmetrization of the singular law of a $\mathbb{P}_{d,d'}^\mu$ -random matrix is the expression of the commutativity of the following diagram:

$$\begin{array}{ccc} M_1(\nu_n) + \dots + M_n(\nu_n) & \xrightarrow{n \rightarrow \infty} & \mathbb{P}_{d,d'}^\mu \\ \downarrow \begin{array}{c} d, d' \text{ go to } \infty \\ d/d' \simeq \lambda \end{array} & & \downarrow \begin{array}{c} d, d' \text{ go to } \infty \\ d/d' \simeq \lambda \end{array} \\ \text{symmetrized} & & \text{symmetrized} \\ \text{singular law:} & \xrightarrow{n \rightarrow \infty} & \text{singular law:} \\ \nu_n^{\boxplus \lambda^{k_n}} & & \Lambda_\lambda(\mu) \end{array}$$

To prove this result, we need a preliminary result about cumulants of \boxplus_λ -infinitely divisible laws with compactly supported Lévy measure. First, note that by lemma 4.1, such laws are compactly supported. Recall that free cumulants with ratio λ have been defined in the beginning of section 4 by (4.1).

Theorem 5.1. *Let μ be a $*$ -infinitely divisible distribution with compactly supported Lévy measure, and let, for n integer, μ_n be a probability measure such that $\mu_n^{*n} = \mu$. Then for each $k \geq 1$, the sequence $(n \times m_{2k}(\mu_n))_n$ tends to the $2k$ -th classical cumulant $c_{2k}^*(\mu)$ of μ , which is equal to $c_{2k}(\Lambda_\lambda(\mu))$.*

Proof. Recall that classical cumulants of a probability measure ν (whose moments of all orders are defined) linearize the classical convolution and are linked to its moments by the formula

$$\forall k \geq 1, m_k(\nu) = \sum_{\pi \in \text{Part}(k)} \prod_{V \in \pi} c_{|V|}^*(\nu).$$

So one has

$$n \times m_{2k}(\mu_n) = n \sum_{\pi \in \text{Part}(2k)} \prod_{V \in \pi} \underbrace{c_{|V|}^*(\mu_n)}_{n^{-1} c_{|V|}^*(\mu)} = \sum_{\pi \in \text{Part}(2k)} n^{1-|\pi|} c_\pi^*(\mu) = c_{2k}^*(\mu) + o(1).$$

Let us denote $\nu_n = \mu_n^{\boxplus \lambda^n}$. By the line above, for all k , $m_{2k}(\mu_n) = O(n^{-1})$, so $c_{2k}(\mu_n) = O(n^{-1})$, so $c_{2k}(\nu_n) = O(1)$, and $m_{2k}(\nu_n) = O(1)$. Moreover, by equation (3.1), the sequence (ν_n) converges weakly to $\Lambda_\lambda(\mu)$. So the moments of ν_n tend to the moments of $\Lambda_\lambda(\mu)$ (cf [B68]). But

thanks to proposition 3.5 of [B-G1],

$$\begin{aligned} n \times m_{2k}(\mu_n) &= n \sum_{\pi \in \text{NC}'(2k)} \lambda^{e(\pi)} \prod_{V \in \pi} \underbrace{c_{|V|}(\mu_n)}_{n^{-1}c_{|V|}(\nu_n)} \\ &= \sum_{\pi \in \text{NC}'(2k)} \lambda^{e(\pi)} n^{1-|\pi|} \prod_{V \in \pi} c_{|V|}(\nu_n), \end{aligned}$$

which tends to

$$\sum_{\pi \in \text{NC}'(2k)} \lambda^{e(\pi)} \delta_1^{|\pi|} \prod_{V \in \pi} c_{|V|}(\Lambda_\lambda(\mu)) = c_{2k}(\Lambda_\lambda(\mu)).$$

□

We will first prove the result when μ has a compactly supported Lévy measure. We will work with a sequence $(d'_d)_d$ such that $1 \leq d \leq d'_d$, and d/d'_d tends to $\lambda \in (0, 1]$ (even though the proof can be adapted to the case $\lambda = 0$, we assume that $\lambda > 0$ in order to simplify). To simplify notations, d' will stand for d'_d .

Proposition 5.2. *Let μ be a symmetric $*$ -infinitely divisible distribution with compactly supported Lévy measure. Then for all integer k ,*

$$(a) \lim_{d \rightarrow \infty} \mathbb{E}_{\mathbb{P}_{d,d'}^\mu} (m_k(\tilde{\mu}_{|M|})) = m_k(\Lambda_\lambda(\mu)).$$

(b) *The variance, under $\mathbb{P}_{d,d'}^\mu$, of $m_k(\tilde{\mu}_{|M|})$ tends to zero as d goes to infinity.*

Proof. For n integer, $[n]$ will denote $\{1, \dots, n\}$, and $\text{NC}(n)$ will denote the set of noncrossing partitions of $[n]$. Recall that $\text{NC}'(n)$ denotes the set of noncrossing partitions of $[n]$ in which all blocks have even cardinality.

(a) First, for every complex $d \times d$ matrix M , for all integer k , $m_k(\tilde{\mu}_{|M|})$ is null if k is odd and is equal to $\text{tr}(MM^*)^{\frac{k}{2}}$ (tr denotes normalized trace) if k is even. $\Lambda_\lambda(\mu)$ is symmetric, so it suffices to show that for all $k \in \mathbb{N}^*$,

$$\lim_{d \rightarrow \infty} \mathbb{E}_{\mathbb{P}_{d,d'}^\mu} \left(\text{tr}(MM^*)^k \right) = m_{2k}(\Lambda_\lambda(\mu)).$$

Let, for $n \in \mathbb{N}^*$, μ_n be the probability measure such that $\mu_n^{*n} = \mu$. Consider, for $d \geq 1$ and $n \geq 1$, $(M_{d,n}^{(i)})_{1 \leq i \leq n}$ i.i.d. random matrices with distribution $\mathbb{Q}_{d,d'}^{\mu_n}$. By definition, for every $d \geq 1$, the sum of the $M_{d,n}^{(i)}$'s ($i = 1 \dots n$) converges in distribution to $\mathbb{P}_{d,d'}^\mu$ when n goes to ∞ . We know, by theorem 5.1 that for all $k \in \mathbb{N}^*$, the sequence $(n \times m_k(\mu_n))_n$ is bounded, and so (see [B68]) for all $k, d \in \mathbb{N}^*$,

$$\mathbb{E}_{\mathbb{P}_{d,d'}^\mu} (m_{2k}(\tilde{\mu}_{|M|})) = \lim_{n \rightarrow \infty} \mathbb{E} \left(\text{tr} \left(\left(\sum_{i=1}^n M_{d,n}^{(i)} \right) \left(\sum_{i=1}^n M_{d,n}^{(i)*} \right) \right)^k \right). \quad (5.2)$$

Let us fix $k \in \mathbb{N}^*$.

We are going to use the formula (5.2).

Let, for $d, n \geq 1$,

$$b_{d,n} = \mathbb{E} \left(\text{tr} \left(\left(\sum_{i=1}^n M_{d,n}^{(i)} \right) \left(\sum_{i=1}^n M_{d,n}^{(i)*} \right) \right)^k \right).$$

From now on, we do not write anymore the index d in $M_{d,n}^{(i)}$. We denote, for l, n non-negative integers, by A_n^l the number of one-to-one maps from $[l]$ to $[n]$, i.e. $n(n-1)\cdots(n-l+1)$. For π partition of $[2k]$, for $1 \leq l \leq 2k$, we denote by $\pi(l)$ the index of the class of l , after having ordered the classes according to the order of their first element (for example, $\pi(1) = 1$; $\pi(2) = 1$ if $1 \tilde{\sim} 2$ and $\pi(2) = 2$ if $1 \not\sim 2$). Then we have

$$\begin{aligned} b_{d,n} &= \operatorname{tr} \left(\mathbb{E} \left(\sum_{f \in \{1, \dots, n\}^{2k}} M_n^{(f(1))} M_n^{(f(2))^*} \dots M_n^{(f(2k))^*} \right) \right) \\ &= \operatorname{tr} \left(\mathbb{E} \left(\sum_{\pi \in \operatorname{Part}(2k)} A_n^{|\pi|} M_n^{(\pi(1))} M_n^{(\pi(2))^*} M_n^{(\pi(3))} \dots M_n^{(\pi(2k))^*} \right) \right). \end{aligned}$$

$$\begin{aligned} \text{But} \quad \mathbb{E} \left(\underbrace{M_n^{(1)*} M_n^{(1)} \dots M_n^{(1)*}}_{2l+1 \text{ alterned factors}} \right) &= \mathbb{E} \left(\underbrace{M_n^{(1)} M_n^{(1)*} \dots M_n^{(1)}}_{2l+1 \text{ alterned factors}} \right) = 0, \\ \mathbb{E} \left(\underbrace{M_n^{(1)*} M_n^{(1)} \dots M_n^{(1)}}_{2l \text{ alterned factors}} \right) &= \frac{d}{d'} m_{2l}(\mu_n) I_{d'}, \\ \mathbb{E} \left(\underbrace{M_n^{(1)} M_n^{(1)*} \dots M_n^{(1)*}}_{2l \text{ alterned factors}} \right) &= m_{2l}(\mu_n) I_d. \end{aligned}$$

So, using many times the fact that a partition π of a finite totally ordered set is noncrossing if and only if one of its class V is an interval and $\pi \setminus \{V\}$ is noncrossing (page 3 of [S98]) and integrating successively with respect to the different independent random matrices, one has

$$\pi \in \operatorname{NC}'(2k) \Rightarrow \operatorname{tr} \mathbb{E} \left(M_n^{(\pi(1))} M_n^{(\pi(2))^*} \dots M_n^{(\pi(2k))^*} \right) = \left(\frac{d}{d'} \right)^{e(\pi)} \underbrace{\prod_{B \in \pi} m_{|B|}(\mu_n)}_{:= m_\pi(\mu_n)},$$

$$\pi \in \operatorname{NC}(2k) \setminus \operatorname{NC}'(2k) \Rightarrow \operatorname{tr} \mathbb{E} \left(M_n^{(\pi(1))} M_n^{(\pi(2))^*} \dots M_n^{(\pi(2k))^*} \right) = 0.$$

But $A_n^{|\pi|} \sim n^{|\pi|}$, so, by the preceding theorem, the limit, when n goes to infinity, of

$$\operatorname{tr} \left(\mathbb{E} \left(\sum_{\pi \in \operatorname{Part}(2k)} A_n^{|\pi|} M_n^{(\pi(1))} M_n^{(\pi(2))^*} \dots M_n^{(\pi(2k))^*} \right) \right)$$

is $\sum_{\pi \in \operatorname{NC}'(2k)} \left(\frac{d}{d'} \right)^{e(\pi)} m_\pi(\mu)$, which tends to $\sum_{\pi \in \operatorname{NC}'(2k)} \lambda^{e(\pi)} m_\pi(\mu) = m_{2k}(\Lambda_\lambda(\mu))$ when d goes to infinity.

So it suffices to prove that

$$b'_{d,n} := \sum_{\substack{\pi \in \operatorname{Part}(2k) \\ \pi \notin \operatorname{NC}(2k)}} A_n^{|\pi|} \operatorname{tr} \mathbb{E} \left\{ M_n^{(\pi(1))} M_n^{(\pi(2))^*} M_n^{(\pi(3))} \dots M_n^{(\pi(2k))^*} \right\}$$

vanishes when n , and then d , go to infinity. Let us expand the trace: $b'_{n,d}$ is equal to

$$\frac{1}{d} \sum_{\substack{\pi \in \text{Part}(k) \\ \pi \notin \text{NC}(k)}} \sum_{\substack{j \in [d']^{2k} \\ \forall r \text{ odd}, j_r \leq d}} A_n^{|\pi|} \mathbb{E} \left\{ \left[M_n^{(\pi(1))} \right]_{j_1, j_2} \left[M_n^{(\pi(2))^*} \right]_{j_2, j_3} \cdots \left[M_n^{(\pi(2k))^*} \right]_{j_{2k}, j_1} \right\}.$$

Using the fact that $(M_{d,n}^{(i)})_{1 \leq i \leq n}$ are independent copies of a matrix with distribution $\mathbb{Q}_{d,d'}^{\mu_n}$, we deduce (with the notation $j_{2k+1} = j_1$)

$$\begin{aligned} b'_{d,n} &= \frac{1}{d} \sum_{\substack{\pi \in \text{Part}(k) \\ \pi \notin \text{NC}(k)}} A_n^{|\pi|} \sum_{\substack{j \in [d']^{2k} \\ \forall r \text{ odd}, j_r \leq d}} \prod_{B \in \pi} \mathbb{E}_{\mathbb{Q}_{d,d'}^{\mu_n}} \left\{ \prod_{\substack{r \in B \\ r \text{ odd}}} M_{j_r, j_{r+1}} \prod_{\substack{r \in B \\ r \text{ even}}} M_{j_r, j_{r+1}}^* \right\} \\ &= \frac{1}{d} \sum_{\substack{\pi \in \text{Part}(k) \\ \pi \notin \text{NC}(k)}} \frac{A_n^{|\pi|}}{n^{|\pi|}} \sum_{\substack{j \in [d']^{2k} \\ \forall r \text{ odd}, j_r \leq d}} \\ &\quad \prod_{B \in \pi} n \mathbb{E} \left\{ \sum_{l \in [d]^B} \prod_{\substack{r \in B \\ r \text{ odd}}} u_{j_r, l_r} v_{l_r, j_{r+1}} X_{n, l_r} \prod_{\substack{r \in B \\ r \text{ even}}} \bar{v}_{l_r, j_r} X_{n, l_r} \bar{u}_{j_{r+1}, l_r} \right\}, \end{aligned}$$

where $U, V, X_{n,1}, \dots, X_{n,d}$ are independent, with respective distribution the Haar measure on the group of $d \times d$ unitary matrices, the Haar measure on the group of $d' \times d'$ unitary matrices, and μ_n .

For all $B \subset [2k]$, for all $j \in [d']^{2k}$ such that for all r odd, $j_r \leq d$, summing over the partition generated by l , one has

$$\begin{aligned} &n \mathbb{E} \left\{ \sum_{l \in [d]^B} \prod_{\substack{r \in B \\ r \text{ odd}}} u_{j_r, l_r} v_{l_r, j_{r+1}} X_{n, l_r} \prod_{\substack{r \in B \\ r \text{ even}}} \bar{v}_{l_r, j_r} X_{n, l_r} \bar{u}_{j_{r+1}, l_r} \right\} \\ &= \sum_{\sigma \in \text{Part}(B)} \sum_{l \in [d]^\sigma} n^{1-|\sigma|} \left[\prod_{B \in \sigma} n m_{|B|}(\mu_n) \right] \mathbb{E} \left\{ \prod_{\substack{r \in B \\ r \text{ odd}}} u_{j_r, l_r} v_{l_r, j_{r+1}} \prod_{\substack{r \in B \\ r \text{ even}}} \bar{v}_{l_r, j_r} \bar{u}_{j_{r+1}, l_r} \right\} \end{aligned}$$

which tends, according to theorem 5.1, to

$$\sum_{l=1}^d c_{|B|}(\mu) \mathbb{E} \left\{ \prod_{\substack{r \in B \\ r \text{ odd}}} u_{j_r, l} v_{l, j_{r+1}} \prod_{\substack{r \in B \\ r \text{ even}}} \bar{v}_{j_r, l} \bar{u}_{j_{r+1}, l} \right\}.$$

Which is equal, by invariance of uniform distributions on unitary groups by permutation of rows and columns, to

$$d \times c_{|B|}(\mu) \mathbb{E} \left\{ \prod_{\substack{r \in B \\ r \text{ odd}}} u_{j_r} v_{j_{r+1}} \prod_{\substack{r \in B \\ r \text{ even}}} \bar{v}_{j_r} \bar{u}_{j_{r+1}} \right\},$$

where u, v are independent uniform random vectors on the unit spheres of $\mathbb{C}^d, \mathbb{C}^{d'}$.

So the limit, when n goes to infinity, of $b'_{d,n}$ is

$$\frac{1}{d} \sum_{\substack{\pi \in \text{Part}(k) \\ \pi \notin \text{NC}(k)}} \sum_{\substack{j \in [d']^{2k} \\ \forall r \text{ odd}, j_r \leq d}} \prod_{B \in \pi} d^{\times c|B|}(\mu) \mathbb{E} \left\{ \prod_{\substack{r \in B \\ r \text{ odd}}} u_{j_r} v_{j_{r+1}} \prod_{\substack{r \in B \\ r \text{ even}}} \bar{v}_{j_r} \bar{u}_{j_{r+1}} \right\},$$

whose absolute value is less or equal, by invariance of the distributions of u and v under permutation of coordinates, to

$$\frac{1}{d} \sum_{\substack{\pi, \tau \in \text{Part}(k) \\ \pi \notin \text{NC}(k)}} d^{|\pi|+|\tau|} c_\pi(\mu) \prod_{B \in \pi} \left| \mathbb{E} \left\{ \prod_{\substack{r \in B \\ r \text{ odd}}} u_{j_r} v_{j_{r+1}} \prod_{\substack{r \in B \\ r \text{ even}}} \bar{v}_{j_r} \bar{u}_{j_{r+1}} \right\} \right|.$$

Moreover, by invariance of the distribution of u under the action of unitary diagonal matrices, for every pair (π, τ) of partitions of $[2k]$, if

$$\prod_{B \in \pi} \mathbb{E} \left(\prod_{\substack{r \in B \\ r \text{ odd}}} u_{\tau(r)} v_{\tau(r+1)} \prod_{\substack{r \in B \\ r \text{ even}}} \bar{u}_{\tau(r+1)} \bar{v}_{\tau(r)} \right)$$

is non zero, then for every class B of π , there exists ϕ , permutation of B , which maps odd numbers to even numbers and vice versa, such that for all $r \in B$, $\tau(r) = \tau(\phi(r) + 1)$. It implies, by lemma 4.4 of [B-G04], that $|\tau| + |\pi| \leq 2k$. So one has, using the Hölder inequality and equation (4.2.11) of [HP00],

$$\lim_{n \rightarrow \infty} b'_{n,d} = O(d^{-1}),$$

which closes the proof of (a).

One notes that the proof of (a) is a very closed adaptation of the proof of Proposition 4.1 of [B-G04], by adaptation of the arguments to the context of non hermitian and non square matrices. Using again the same adaptation, the proof of (b) is along the same lines as the proof of Proposition 5.1 of [B-G04]. \square

To conclude this section, we have to state its main theorem. The proof of the theorem is similar to the one of Theorem 7.6 of [B-G04], based on the previous proposition and on an approximation by compound Poisson laws. The only modification is to work with products of the type MM^* rather than M^*M .

Theorem 5.3. *Let μ be a symmetric $*$ -infinitely divisible distribution. Let, for $d \geq 1$, M_d be a random matrix with distribution $\mathbb{P}_{d,d'}^\mu$.*

Then the symmetrization $\tilde{\mu}_{|M_d|}$ of the singular law of M_d converges in probability to $\Psi(\mu)$.

Remark 5.4. *In the case where μ is a normal law, one recovers the well known result about asymptotic spectral distribution of Wishart random matrices.*

6. RECTANGULAR SYMMETRIC POISSON DISTRIBUTIONS AS LIMITS OF SUMS OF RANK-ONE MATRICES

Let us define the *symmetric Poisson distribution with parameter $a > 0$* to be the $*$ -infinitely divisible distribution with Lévy measure $\frac{c}{4}(\delta_1 + \delta_{-1})$. It can also be seen as the law of $X - Y$,

where X, Y are independent random variables with (unsymmetric) Poisson law with parameter $c/2$, or as the weak limit of

$$\left(\left(1 - \frac{c}{n} \right) \delta_0 + \frac{c}{2n} (\delta_{-1} + \delta_1) \right)^{*n}.$$

Note that rectangular analogues of Gaussian distributions have been identified to subsection 4, but rectangular analogues of symmetric Poisson distributions are still not characterized, whereas free analogues of (unsymmetric) Poisson distributions are Marchenko-Pastur laws. But the computations for symmetric Poisson laws are harder than for the unsymmetric ones (even for free analogues of symmetric Poisson distributions, the densities are not possible to express). Nevertheless, we have the following characterization of the rectangular analogue of symmetric Poisson distributions.

Proposition 6.1. *Consider $\lambda \in (0, 1]$, and $c > 0$. Then the image, by the Bercovici-Pata bijection with ratio λ , of the symmetric Poisson distribution with parameter c is the limit, for convergence in probability, of the symmetrization of the singular law of the random matrix*

$$M(d, d', d'') := \sum_{k=1}^{d''} u_d(k) v_{d'}(k)^*$$

when

$$d \rightarrow \infty, \quad \frac{d}{d'} \rightarrow \lambda, \quad \frac{d''}{d} \rightarrow c, \quad (6.1)$$

where $u_d(k), v_{d'}(k)$ ($k \geq 1$) are independent uniform random vectors of the unit spheres of $\mathbb{C}^d, \mathbb{C}^{d'}$ (considered as column matrices).

Remark 6.2. *Note that when $\lambda = 1$, the image, by the Bercovici-Pata bijection with ratio λ , of the symmetric Poisson distribution with parameter c is $\tau_+ \boxplus \tau_-$, where τ_+ is the Marchenko-Pastur distribution with parameter $c/2$, and τ_- is the push-forward, by the function $t \rightarrow -t$, of τ_+ .*

Proof. Let us denote by μ the symmetric Poisson distribution with parameter c , and by σ the push-forward by $t \rightarrow t^2$ of its image by the Bercovici-Pata bijection with ratio λ . As explained in the beginning of the proof of theorem 4.6 of [B-G1], it suffices to prove that for each $\varepsilon > 0$, the probability of the event

$$\left\{ \sup_{\Im z \geq 1} \left| \frac{1}{d} \operatorname{Tr} \Re_z (A(d, d', d'')) - G_\sigma(z) \right| > \varepsilon \right\}$$

tends to zero as d, d', d'' tend to infinity as in (6.1), where

$$A(d, d', d'') = M(d, d', d'') M(d, d', d'')^*.$$

Fix $\varepsilon > 0$. It can easily be seen that $\mathbb{P}_{d, d'}^\mu$ is the distribution of

$$N(d, d') := \sum_{k=1}^{X(cd)} u_d(k) v_{d'}(k)^*,$$

where $X(cd)$ is a random variable distributed according to an (unsymmetric) Poisson distribution with parameter cd , and $X(cd)$ is independent of the $u_d(k)$'s and of the $v_{d'}(k)$'s. Thus by the previous theorem, with the notation $B(d, d') = N(d, d') N(d, d')^*$, the probability of the event

$$\left\{ \sup_{\Im z \geq 1} \left| \frac{1}{d} \operatorname{Tr} \Re_z (B(d, d')) - G_\sigma(z) \right| > \varepsilon \right\}$$

tends to zero. Thus it suffices to prove that the probability of the event

$$\left\{ \sup_{\Im z \geq 1} \left| \frac{1}{d} \operatorname{Tr} (\Re_z (B(d, d')) - \Re_z (A(d, d', d''))) \right| > \varepsilon \right\}$$

tends to zero. But for all hermitian $d \times d$ matrices A, B , for all z such that $\Im z \geq 1$, we have

$$\Re_z(B) - \Re_z(A) = -\Re_z(B)(B - A)\Re_z(A),$$

whose normalized trace is not more than its norm times its rank divided by d . Moreover, $\|\Re_z(B) - \Re_z(A)\| \leq 2$, and the rank is not more than the one of $B - A$. So it suffices to prove that

$$\frac{1}{d} \operatorname{rg} (B(d, d') - A(d, d', d''))$$

converges in probability to zero. $B(d, d') - A(d, d', d'')$ can be put in the form

$$(\dots)(N(d, d') - M(d, d', d''))^* + (N(d, d') - M(d, d', d''))(\dots),$$

so

$$\frac{1}{d} \operatorname{rg} (B(d, d') - A(d, d', d'')) \leq \frac{2}{d} \operatorname{rg}(N(d, d') - M(d, d', d'')) \leq \frac{2}{d} |X(cd) - d''|,$$

which converges in probability to zero, by the weak law of great numbers. \square

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