Infinitely divisible distributions for rectangular free convolution: classification and matricial interpretation

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INFINITELY DIVISIBLE DISTRIBUTIONS FOR RECTANGULAR FREE CONVOLUTION: CLASSIFICATION AND MATRICIAL INTERPRETATION

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Abstract. In a previous paper ([B-G1]), we defined the rectangular free convolution $\boxplus_\lambda$. Here, we investigate the related notion of infinite divisibility, which happens to be closely related the classical infinite divisibility: there exists a bijection between the set of classical symmetric infinitely divisible distributions and the set of $\boxplus_\lambda$-infinitely divisible distributions, which preserves limit theorems. We give an interpretation of this correspondence in terms of random matrices: we construct distributions on sets of complex rectangular matrices which give rise to random matrices with singular laws going from the symmetric classical infinitely divisible distributions to their $\boxplus_\lambda$-infinitely divisible correspondents when the dimensions go from one to infinity in a ratio λ.

Contents

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Introduction

In a previous paper ([B-G1]), we modeled the asymptotic behavior of rectangular random matrices with freeness with amalgamation. Therefore we defined, for each $\lambda \in [0,1]$, the rectangular free convolution with ratio λ, denoted by $\boxplus_\lambda$. It is a binary operation on the set of symmetric probability measures on the real line defined in the following way. Let us call the singular law of a matrix M the uniform law on its singular values, i.e. on the spectrum of its absolute value $|M| = (MM^*)^{1/2}$. Consider μ, ν symmetric probability measures on the real line, consider two sequences $q_1(n), q_2(n)$ of integers tending to $+\infty$ such that

$$\frac{q_1(n)}{q_2(n)} \underset{n\to\infty}{\longrightarrow} \lambda,$$

and consider, for each n, $M(n), N(n)$ independent $q_1(n)\times q_2(n)$ random matrices, one of them being biunitarily invariant (i.e. having a distribution invariant under the left and right actions of the unitary groups) such that the symmetrization of the singular law of $M(n)$ (resp. of $N(n)$) converges weakly in probability to μ (resp. ν). Then the symmetrization of the singular law of $M(n)+N(n)$ converges weakly in probability to a probability measure which depends only on μ, ν, and λ, denoted by $\mu\boxplus_\lambda\nu$, and called the *rectangular free convolution with ratio* λ of μ and ν.

In the present paper, we study the notion of infinite divisibility for $\boxplus_\lambda$, which leads to a Lévy-Kinchine formula for the rectangular R-transform (whose definition we shall recall in section 1): a symmetric probability measure μ is $\boxplus_\lambda$-infinitely divisible if and only if there exists a positive finite symmetric measure G (called its *Lévy measure*) such that the rectangular R-transform with ratio λ of μ is given by the formula:

$$C_\mu(z) = z\int_{\mathbb{R}} \frac{1+t^2}{1-zt^2}\mathrm{d}G(t).$$

Therefore we can define a bijection Λ_λ between the set of classical symmetric infinitely divisible distributions and the set of $\boxplus_\lambda$-infinitely divisible distributions: Λ_λ maps a symmetric $*$-infinitely divisible distribution to the $\boxplus_\lambda$-infinitely divisible distribution with the same Lévy measure. This bijection happens, like the one of Bercovici and Pata ([BPB99]) between $*$- and $\boxplus$-infinitely divisible distributions, to have deep properties. It is a semi-group morphism:

$$\Lambda_\lambda(\mu * \nu) = \Lambda_\lambda(\mu)\boxplus_\lambda\Lambda_\lambda(\nu),$$

and it preserves limit theorems: for all sequences (μ_n) of symmetric distributions and (k_n) of positive integers tending to infinity, we have, for all probability measures μ,

$$\mu_n^{*k_n} \underset{n\to\infty}{\longrightarrow} \mu \iff \mu_n^{\boxplus_\lambda k_n} \underset{n\to\infty}{\longrightarrow} \Lambda_\lambda(\mu).$$

Λ_λ will be called the rectangular Bercovici-Pata bijection with ratio λ.

In section 4, we give examples of $\boxplus_\lambda$-infinitely divisible distributions. First, in section 4.1 we give the density of the image, by the bijection Λ_λ, of the standard Gaussian distribution. An interesting interpretation of this result is made in a forthcoming paper ([B-G2]) where we construct analogues of Voiculescu's free entropy and free Fisher information for operators between different Hilbert spaces, and where the maximum of entropy and the minimum of Fisher information are realized for operators the absolute value of which has this (symmetrized) distribution. Another consequence of this result is a new formula for the moments of the Marchenko-Pastur distribution (which is closely related to this distribution): for all $a > 0$, for all $n \geq 1$, the n-th moment of the Marchenko-Pastur distribution with parameter a (see [HP00] p. 65) is equal to $\sum_\pi a^{o(\pi)}$, where the sum is taken over all noncrossing pairings of $[2n]$, and where $o(\pi)$ is the

number of blocks of a partition π the first element of which is odd. Then, in section 4.2, we give the densities of the images, by the above mentioned bijection, of the symmetric Cauchy laws. Cauchy laws are well known to be invariant under many transformations, but we are going to see that unless $\lambda = 1$, they are not invariant under this bijection. At last, in section 4.3, we characterize the images, by the same bijection, of the symmetric Poisson distributions. When $\lambda = 0$, we have a formula for the density.

In section 5, we shall construct a matricial model for the $\boxplus_\lambda$-infinitely divisible laws and present in a maybe more palpable way the Bercovici-Pata bijection with ratio λ (whereas the proofs of the other sections rely on integral transforms and complex analysis): we are going to construct, in the same way as in [B-G04] and in [C-D04], for each $d, d' \geq 1$, for each symmetric $*$-infinitely divisible distribution μ, an infinitely divisible distribution $\mathbb{P}^{\mu}_{d,d'}$ on the set of $d \times d'$ complex matrices such that for all μ, ν, $\mathbb{P}^{\mu}_{d,d'} * \mathbb{P}^{\nu}_{d,d'} = \mathbb{P}^{\mu * \nu}_{d,d'}$ and such that the symmetrization of the singular law of M (with M random matrix distributed according to $\mathbb{P}^{\mu}_{d,d'}$) goes from μ to its image by the rectangular Bercovici-Pata bijection with ratio λ when $d, d' \to \infty$, $\frac{d}{d'} \to \lambda$.

In the last section, we shall give a representation of the image of the symmetric Poisson distribution by the rectangular Bercovici-Pata bijection with ratio λ as the distribution of the absolute value of sums of rank-one matrices.

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1. Preliminaries

Until the end of section 4, λ is a fixed number of $[0, 1]$.

1.1. General introduction to the rectangular R-transform with ratio $\lambda \in [0, 1]$.

In this section, we shall recall definitions and basic results from [B-G1] about the rectangular R-transform C_μ of a symmetric probability measure μ.

Let us denote by $z \mapsto z^{1/2}$ (resp. $z \mapsto \sqrt{z}$) the analytic version of the square root on the complement of the real non positive (resp. non negative) half line such that $1^{1/2} = 1$ (resp. $\sqrt{-1} = i$). On the set of non null complex numbers, we will use the argument function which takes values in $[0, 2\pi)$. Let us define the analytic function on a neighborhood of zero $U(z) = \frac{-\lambda - 1 + \left[(\lambda+1)^2 + 4\lambda z\right]^{1/2}}{2\lambda}$ (when $\lambda = 0$, $U(z) = z$). Then one can summarize the different steps of the construction of the rectangular R-transform with ratio λ in the following chain

$$\underset{\substack{\text{sym. prob.}\\ \text{measure}}}{\mu} \longrightarrow \underbrace{G_\mu(z) = \int \frac{\mathrm{d}\mu(t)}{z - t}}_{\text{Cauchy transform}} \longrightarrow H_\mu(z) = \lambda G_\mu\left(\frac{1}{\sqrt{z}}\right)^2 + (1-\lambda)\sqrt{z} G_\mu\left(\frac{1}{\sqrt{z}}\right) \longrightarrow$$
$$\underset{\text{rect. } R\text{-transf. with ratio } \lambda}{C_\mu(z) = U\left(\frac{z}{H_\mu^{-1}(z)} - 1\right)},$$

where H_μ^{-1} is the inverse (for composition) of H_μ. Proposition 1.1 and theorem 1.2 bellow, which have first been established in [B-G1], prove that such an inverse exists, give its domain, and prove that for any tight set A of symmetric probability measures, the properties and the domains of the functions H_μ^{-1} ($\mu \in A$) are "uniform".

Proposition 1.1. *Let A be a set of symmetric probability measures on the real line. Then the following assertions are equivalent*

(i) *A is tight,*
(ii) *for every $0 < \theta < \pi$, $\lim_{\substack{z\to 0 \\ |\arg z - \pi| < \theta}} \frac{1}{z} H_\mu(z) = 1$ uniformly in $\mu \in A$,*
(iii) $\lim_{\substack{x\to 0 \\ x\in(-\infty,0)}} \frac{1}{x} H_\mu(x) = 1$ *uniformly in $\mu \in A$.*

Define, for $\alpha \in (0,\pi)$, $\beta > 0$, $\Delta_{\alpha,\beta}$ to be the set of complex numbers z such that $|\arg z - \pi| < \alpha$ and $|z| < \beta$.

Let $\mathcal{H}$ be the set of functions f which are analytic in a domain $\mathcal{D}_f$ such that for all $\alpha \in (0,\pi)$, there exists β positive such that

$$\Delta_{\alpha,\beta} \subset \mathcal{D}_f.$$

A family $(f_a)_{a\in A}$ of functions of $\mathcal{H}$ is said to be *uniform* if for all $\alpha \in (0,\pi)$, there exists β positive such that

$$\forall a \in A, \quad \Delta_{\alpha,\beta} \subset \mathcal{D}_{f_a}.$$

Theorem 1.2. *Let $(H_a)_{a\in A}$ be a uniform family of functions of $\mathcal{H}$ such that for every $\alpha \in (0,\pi)$,*

$$\lim_{\substack{z\to 0 \\ |\arg z - \pi| < \alpha}} \frac{H_a(z)}{z} = 1 \text{ uniformly in } a \in A.$$

Then there exists a uniform family $(F_a)_{a\in A}$ of functions of $\mathcal{H}$ such that for every $\alpha \in (0,\pi)$,

$$\lim_{\substack{z\to 0 \\ |\arg z - \pi| < \alpha}} \frac{F_a(z)}{z} = 1 \text{ uniformly in } a \in A,$$

and there exists β positive such that

$$\forall a \in A, \quad H_a \circ F_a = F_a \circ H_a = I_d \text{ on } \Delta_{\alpha,\beta}.$$

Moreover, the family $(F_a)_{a\in A}$ is unique in the following sense: if a family $(\tilde{F}_a)_{a\in A}$ of functions of $\mathcal{H}$ satisfies the same conditions, then for all $\alpha \in (0,\pi)$, there exists β positive such that

$$\forall a \in A, \quad F_a = \tilde{F}_a \text{ on } \Delta_{\alpha,\beta}.$$

Using the theory of cumulants in operator-valued free probability theory, we prove ([B-G1]) the *additivity of rectangular R-transform*:

Theorem 1.3. *For all μ, ν, symmetric probability measures, we have*

$$C_{\mu \boxplus_\lambda \nu} = C_\mu + C_\nu.$$

Note that it is also proved in [B-G1] that for all $\lambda \in [0,1]$, the rectangular R-transform with ratio λ is injective. The following remark gives a practical way to derive any symmetric probability measure μ on the real line from C_μ.

Remark 1.4 (How to compute μ when we know C_μ ?)**.** *Let us define the function $T(X) = (\lambda X + 1)(X + 1)$, such that $T(U(z)) = z + 1$. We have $z/H_\mu^{-1}(z) = T(C_\mu(z))$, for $z \in \mathbb{C}\backslash\mathbb{R}^+$ small enough. From this, we can compute $H_\mu(z)$ for $z \in \mathbb{C}\backslash\mathbb{R}^+$ small enough. Then we can use the equation, for $z \in \mathbb{C}\backslash\mathbb{R}^+$,*

$$\frac{1}{z} H_\mu(z) = \lambda \left(\frac{1}{\sqrt{z}} G_\mu \left(\frac{1}{\sqrt{z}} \right) \right)^2 + (1-\lambda) \frac{1}{\sqrt{z}} G_\mu \left(\frac{1}{\sqrt{z}} \right).$$

Moreover, when $z \in \mathbb{C}\backslash\mathbb{R}^+$ is small enough, $1/\sqrt{z}$ is large and in $\mathbb{C}^-$ (the set of complex numbers with negative imaginary part), so $\frac{1}{\sqrt{z}}G_\mu\left(\frac{1}{\sqrt{z}}\right)$ is closed to 1. $\frac{1}{z}H_\mu(z)$ is also closed to 1, and for h, g complex numbers closed to 1,

$$h = \lambda g^2 + (1-\lambda)g \Leftrightarrow g = V(h), \text{ with } V(z) = \frac{\lambda - 1 + ((\lambda-1)^2 + 4\lambda z)^{\frac{1}{2}}}{2\lambda} = U(z-1) + 1.$$

So one has, for $z \in \mathbb{C}\backslash\mathbb{R}^+$ small enough,

$$\frac{1}{\sqrt{z}}G_\mu\left(\frac{1}{\sqrt{z}}\right) = V\left(\frac{H_\mu(z)}{z}\right).$$

We shall mention here two other results, proved in [B-G1]. The second of them allows us to claim that $\boxplus_\lambda$ is continuous with respect to weak convergence.

Lemma 1.5 (Tightness and rectangular R-transform)**.** *Let A be a set of symmetric probability measures. Then we have equivalence between :*

(i) *A is tight,*
(ii) *for any $0 < \alpha < \pi$, $\lim\limits_{\substack{z\to 0 \\ |\arg z - \pi| < \alpha}} C_\mu(z) = 0$ uniformly in $\mu \in A$,*
(iii) $\lim\limits_{\substack{x\to 0 \\ x\in(-\infty,0)}} C_\mu(x) = 0$ *uniformly in $\mu \in A$.*

Theorem 1.6 (Paul Lévy's theorem for rectangular R-transform)**.** *Let (μ_n) be a sequence of symmetric probability measures. Then we have equivalence between:*

(i) *(μ_n) converges weakly to a symmetric probability measure;*
(ii) *there exists α, β such that*
  (a) $\lim\limits_{\substack{z\to 0 \\ |\arg z - \pi| < \alpha}} C_{\mu_n}(z) = 0$ *uniformly in n,*
  (b) *the sequence (C_{μ_n}) converges uniformly on every compact set of $\Delta_{\alpha,\beta}$ when $n \to \infty$;*
(iii) (a) $\lim\limits_{\substack{x\to 0 \\ x\in(-\infty,0)}} C_{\mu_n}(x) = 0$ *uniformly in n,*
  (b) *there exists $\beta > 0$ such that the sequence (C_{μ_n}) converges pointwise on $[-\beta, 0)$ when $n \to \infty$.*

Moreover, in this case, denoting by μ the weak limit of (μ_n), for every α, there exists β such that the sequence (C_{μ_n}) converges uniformly to C_μ on every compact set of $\Delta_{\alpha,\beta}$ when $n \to \infty$.

1.2. **The particular cases $\lambda = 0$ and $\lambda = 1$.** The results of this section are proved in [B-G1].

1.2.1. *Rectangular free convolution.* For μ, ν symmetric probability measures on the real line, the rectangular free convolution with ratio 1 of μ and ν is their free convolution (as defined in [BV93]), and their rectangular free convolution with ratio 0 is the unique symmetric probability measure on the real line whose push-forward by the function $t \to t^2$ is the free convolution of their push-forwards by the same function.

1.2.2. *Rectangular R-transform.* The rectangular R-transform with ratio 1 (resp. 0), for a symmetric distribution μ, is linked to the Voiculescu transform φ_μ of μ by the relation $C_\mu(z) = \sqrt{z}\varphi_\mu(1/\sqrt{z})$ (resp. $C_\mu(z) = z\varphi_\rho(1/z)$, where ρ is the push-forward of μ by the function $t \to t^2$) (see paragraph 5 of [BV93] for the construction of the Voiculescu transform).

2. Lévy-Kinchine Formula for $\boxplus_\lambda$-infinitely divisible distributions

$\boxplus_\lambda$-infinitely divisible distributions are defined in the same way as $*$- and $\boxplus$-infinitely divisible distributions:

Definition 2.1. *A symmetric probability measure ν is said to be $\boxplus_\lambda$-infinitely divisible if for each $n \in \mathbb{N}^*$, there exists a symmetric distribution ν_n such that $\nu_n^{\boxplus_\lambda n} = \nu$.*

As for $*$- and $\boxplus$-, we have the following characterization of $\boxplus_\lambda$-infinite divisibility.

Theorem 2.2. *Let ν be a symmetric distribution. Then ν is $\boxplus_\lambda$-infinitely divisible if and only if there exists a sequence (ν_n) of symmetric probability measures such that $\nu_n^{\boxplus_\lambda n}$ converges weakly to ν.*

Proof. If ν is $\boxplus_\lambda$-infinitely divisible, it is clear. Assume the existence of a sequence (ν_n) such that $\nu_n^{\boxplus_\lambda n}$ converges weakly to ν. Consider $k \geq 1$. Let us show that there exists a symmetric probability measure σ such that $\sigma^{\boxplus_\lambda k} = \nu$. We have $\lim_{\substack{y\to 0\\ y<0}} nC_{\nu_n}(y) = 0$ uniformly in n, so $\lim_{\substack{y\to 0\\ y<0}} nC_{\nu_{kn}}(y) = 0$ uniformly in n. So by lemma 1.5, the sequence $\left(\nu_{kn}^{\boxplus_\lambda n}\right)$ is tight. If the symmetric distribution σ is the limit of one of its subsequences, we have

$$\sigma^{\boxplus_\lambda k} = \lim_{n\to\infty} \left(\nu_{kn}^{\boxplus_\lambda n}\right)^{\boxplus_\lambda k} = \lim_{n\to\infty} \nu_{kn}^{\boxplus_\lambda nk} = \nu.$$

□

Corollary 2.3. *The set of $\boxplus_\lambda$-infinitely divisible distributions is closed under weak convergence.*

Proof. If a sequence (μ_n) of $\boxplus_\lambda$-infinitely divisible distributions converges weakly to a distribution μ, then if for every n, $\nu_n^{\boxplus_\lambda n} = \mu_n$, the sequence $(\nu_n^{\boxplus_\lambda n})$ converges weakly to μ. □

To prove the Lévy-Kinchine formula for $\boxplus_\lambda$-infinitely divisible distributions, we need the following lemma, which is the analogue of propositions 2.6 and 2.7 of [BPB99]. Until the end of the paper, for f, g functions defined on a domain whose closure contains an element x_0, "$f(x) \sim g(x)$ in the neighborhood of x_0 (or as x tends to x_0)" will mean that $f(x)/g(x)$ tends to 1 as x tends to x_0.

Lemma 2.4. *Let (ν_n) be a sequence of symmetric probability measures that converges weakly to δ_0. Consider $\alpha \in (0, \pi)$. Then there exists $\beta > 0$ such that on $\Delta_{\alpha,\beta}$,*

$$C_{\nu_n}(z) = \left(\frac{1}{\sqrt{z}} G_{\nu_n}\left(\frac{1}{\sqrt{z}}\right) - 1\right)(1 + v_n(z))$$

where the functions v_n, defined on $\Delta_{\alpha,\beta}$, are such that

(i) $\forall n, \forall z, |v_n(z)| \leq 1/2$ *and* $\lim_{z\to 0} v_n(z) = 0$ *uniformly in n,*
(ii) $\forall z, \lim_{n\to\infty} v_n(z) = 0$.

Proof. First, note that unless $\nu_n = \delta_0$ (in which case $v_n = 0$ is suitable), for all z, $\frac{1}{\sqrt{z}} G_{\nu_n}\left(\frac{1}{\sqrt{z}}\right) - 1 \neq 0$, so there is a function v_n on the domain of C_{ν_n} such that

$$C_{\nu_n}(z) = \left(\frac{1}{\sqrt{z}} G_{\nu_n}\left(\frac{1}{\sqrt{z}}\right) - 1\right)(1 + v_n(z)). \tag{2.1}$$

To prove (i), we will only use the tightness of $\{\nu_n\,;\,n\in\mathbb{N}\}$. It suffices to show that

$$\lim_{\substack{z\to 0\\ |\arg z-\pi|<\alpha}} \frac{C_{\nu_n}(z)}{\frac{1}{\sqrt{z}}G_{\nu_n}\left(\frac{1}{\sqrt{z}}\right)-1}=1 \text{ uniformly in } n.$$

We have, by the paragraph following Proposition 5.1 in [BV93],

$$\lim_{\substack{z\to 0\\ |\arg z-\pi|<\alpha}} \frac{1}{\sqrt{z}}G_{\nu_n}\left(\frac{1}{\sqrt{z}}\right)=1 \text{ uniformly in } n,$$

and when a complex number t tends to 1, $t-1\sim(\lambda t^2+(1-\lambda)t-1)/(\lambda+1)$, so it suffices to show that

$$\lim_{\substack{z\to 0\\ |\arg z-\pi|<\alpha}} \frac{(\lambda+1)C_{\nu_n}(z)}{\frac{H_{\nu_n}(z)}{z}-1}=1 \text{ uniformly in } n.$$

We have

$$\frac{H_{\nu_n}(z)}{z}-1=\frac{H_{\nu_n}(z)}{z}\left(1-\frac{z}{H_{\nu_n}(z)}\right),$$

and we know, by proposition 1.1, that

$$\lim_{\substack{z\to 0\\ |\arg z-\pi|<\alpha}} \frac{H_{\nu_n}(z)}{z}=1 \text{ uniformly in } n. \tag{2.2}$$

So it suffices to show that

$$\lim_{\substack{z\to 0\\ |\arg z-\pi|<\alpha}} \frac{(\lambda+1)C_{\nu_n}(z)}{1-\frac{z}{H_{\nu_n}(z)}}-1=0 \text{ uniformly in } n.$$

We know, by proposition 1.1 and by theorem 1.2, that

$$\lim_{\substack{z\to 0\\ |\arg z-\pi|<\alpha}} \frac{H_{\nu_n}^{-1}(z)}{z}=1 \text{ uniformly in } n, \tag{2.3}$$

and the equivalent of $U(x)$ in a neighborhood of zero is $\frac{x}{\lambda+1}$. So, since $C_{\nu_n}(z)=U\left(\frac{z}{H_{\nu_n}^{-1}(z)}-1\right)$, it suffices to show that

$$\lim_{\substack{z\to 0\\ |\arg z-\pi|<\alpha}} \frac{\frac{z}{H_{\nu_n}^{-1}(z)}-1}{1-\frac{z}{H_{\nu_n}(z)}}=1 \text{ uniformly in } n.$$

Choose $\alpha'\in(\alpha,\pi)$. By theorem 1.2, there exists $\beta_1>0$ such that for all n, $H_{\nu_n}^{-1}$ is defined on $\Delta_{\alpha',2\beta_1}$, and

$$H_{\nu_n}(\Delta_{\alpha,\beta_1})\cup H_{\nu_n}^{-1}(\Delta_{\alpha,\beta_1})\subset\Delta_{\alpha',2\beta_1}.$$

We have, for $z\in\Delta_{\alpha,\beta_1}$,

$$\begin{aligned}\frac{\frac{z}{H_{\nu_n}^{-1}(z)}-1}{1-\frac{z}{H_{\nu_n}(z)}}-1 &= \frac{\frac{zH_{\nu_n}(z)}{H_{\nu_n}^{-1}(z)}-H_{\nu_n}(z)}{H_{\nu_n}(z)-z}-1\\ &= \frac{1}{H_{\nu_n}(z)-z}\int_{[z,H_{\nu_n}(z)]}f'_{n,z}(\xi)\mathrm{d}\xi,\end{aligned}$$

where $f_{n,z}$ is the function defined by

$$f_{n,z}(\xi)=\frac{H_{\nu_n}^{-1}(\xi)H_{\nu_n}(z)}{H_{\nu_n}^{-1}(z)}-\xi.$$

But the lemma 2.4 of [BPB99] states that

$$\lim_{\substack{z\to 0\\ |\arg z-\pi|<\alpha'}} (H_{\nu_n}^{-1})'(z) = 1 \text{ uniformly in } n.$$

Hence, using also (2.2) and (2.3) (which stay true if α is replaced by α'), we have

$$\lim_{\substack{z\to 0\\ |\arg z-\pi|<\alpha}} \sup\{\left|f'_{n,z}(\xi)\right| ; \xi \in [z, H_{\nu_n}(z)]\} = 0 \text{ uniformly in } n.$$

so

$$\lim_{\substack{z\to 0\\ |\arg z-\pi|<\alpha}} \frac{\frac{z}{H_{\nu_n}^{-1}(z)} - 1}{1 - \frac{z}{H_{\nu_n}(z)}} = 1 \text{ uniformly in } n.$$

So we know that the sequence (v_n) of functions satisfying (2.1) satisfies $\lim_{z\to\infty} v_n(z) = 0$ uniformly in n. Hence we can choose β_2 such that $\forall n, \forall z \in \Delta_{\alpha,\beta_2}, |v_n(z)| \leq 1/2$, and (i) is satisfied.

Let us now prove (ii). First, note that since $\nu_n \to \delta_0$, the sequence (G_{ν_n}) converges uniformly to $G_{\delta_0} : z \mapsto 1/z$ on every compact of the upper half plane (see [A61] or section 3.1 of [HP00]), so, as in the proof of (i), it suffices to show that

$$\lim_{n\to\infty} \frac{(\lambda+1)C_{\nu_n}(z)}{\frac{H_{\nu_n}(z)}{z} - 1} = 1.$$

The convergence of ν_n to δ_0 implies too that (H_{ν_n}) converges to $H_{\delta_0} : z \mapsto z$. So, since

$$\frac{H_{\nu_n}(z)}{z} - 1 = \frac{H_{\nu_n}(z)}{z}\left(1 - \frac{z}{H_{\nu_n}(z)}\right),$$

it suffices to prove that

$$\lim_{n\to\infty} \frac{(\lambda+1)C_{\nu_n}(z)}{1 - \frac{z}{H_{\nu_n}(z)}} = 1.$$

Furthermore, by theorem 1.6, there exists $\beta_3 \leq \beta_2$ such that the sequence C_{ν_n} converges uniformly to $C_{\delta_0} = 0$ on every compact of Δ_{α,β_3}. So $z/H_{\nu_n}^{-1}(z) = (\lambda C_{\nu_n}(z)+1)(C_{\nu_n}(z)+1)$ converges uniformly to 1 on every compact of Δ_{α,β_3}. So, since $(\lambda+1)U(z)$ is equivalent to z as z tends to zero and since $C_{\nu_n} = U\left(z/H_{\nu_n}^{-1}(z) - 1\right)$, it suffices to show that for all $z \in \Delta_{\alpha,\beta_3}$,

$$\lim_{n\to\infty} \frac{\frac{z}{H_{\nu_n}^{-1}(z)} - 1}{1 - \frac{z}{H_{\nu_n}(z)}} = 1.$$

As in the proof of (i), let us choose $\beta < \beta_3/2$ such that for all n, $H_{\nu_n}^{-1}$ is defined on $\Delta_{\alpha',\beta}$, $H_{\nu_n}(\Delta_{\alpha,\beta}) \cup H_{\nu_n}^{-1}(\Delta_{\alpha,\beta}) \subset \Delta_{\alpha',2\beta}$, and

$$\lim_{n\to\infty} \frac{H_{\nu_n}^{-1}(z)}{z} = 1 \text{ uniformly on every compact of } \Delta_{\alpha',2\beta}.$$

By analycity of the $H_{\nu_n}^{-1}$'s, the last assertion implies that

$$\lim_{n\to\infty} (H_{\nu_n}^{-1})'(z) = 1 \text{ uniformly on every compact of } \Delta_{\alpha',2\beta}.$$

We have, for $z \in \Delta_{\alpha,\beta}$,

$$\begin{aligned}\frac{\frac{z}{H_{\nu_n}^{-1}(z)} - 1}{1 - \frac{z}{H_{\nu_n}(z)}} - 1 &= \frac{\frac{zH_{\nu_n}(z)}{H_{\nu_n}^{-1}(z)} - H_{\nu_n}(z)}{H_{\nu_n}(z) - z} - 1 \\ &= \frac{1}{H_{\nu_n}(z) - z}\int_{[z,H_{\nu_n}(z)]} f'_n(\xi)\mathrm{d}\xi,\end{aligned}$$

where f_n is still the function defined by

$$f_n(\xi) = \frac{H_{\nu_n}^{-1}(\xi)H_{\nu_n}(z)}{H_{\nu_n}^{-1}(z)} - \xi.$$

f_n' tends to zero as n tends to infinity, uniformly on every compact of $\Delta_{\alpha',2\beta}$, so

$$\frac{\frac{z}{H_{\nu_n}^{-1}(z)} - 1}{1 - \frac{z}{H_{\nu_n}(z)}} - 1$$

tends to zero when n tends to infinity, and the result is proved. □

In the following, we shall refer to *weak convergence* for sequences of positive finite measures on the real line: it is the convergence for which the test functions are the continuous bounded functions.

Theorem 2.5 (Lévy-Kinchine formula, part 1)**.** *Let μ be a symmetric probability measure, (ν_n) be a sequence of symmetric probability measures and k_n a sequence of integers tending to infinity such that $\nu_n^{\boxplus_\lambda k_n}$ converges weakly to μ. Then there exists a symmetric positive finite measure G such that*

1. *the sequence of positive finite measures $\left(k_n \frac{t^2}{1+t^2}\mathrm{d}\nu_n(t)\right)$ converges weakly to G,*
2. *the rectangular R-transform of μ has an analytic continuation to the complement of the real nonnegative half line and is given by the formula*

$$C_\mu(z) = z\int_{\mathbb{R}} \frac{1+t^2}{1-zt^2}\mathrm{d}G(t). \tag{2.4}$$

Moreover, G is symmetric and is the only positive finite measure F such that

$$C_\mu(z) = z\int_{\mathbb{R}} \frac{1+t^2}{1-zt^2}\mathrm{d}F(t).$$

Proof.

1. The sequence (ν_n) converges weakly to δ_0. Indeed, for every n, $C_{\nu_n^{\boxplus_\lambda k_n}} = k_n C_{\nu_n}$, and by theorem 1.6, we have
   (a) $\lim\limits_{\substack{x\to 0\\ x<0}} k_n C_{\nu_n}(x) = 0$ uniformly in n,
   (b) There exists $\beta > 0$ such that the sequence $(k_n C_{\nu_n})$ converges pointwise on $(-\beta, 0)$.
   So
   (a) $\lim\limits_{\substack{x\to 0\\ x<0}} C_{\nu_n}(x) = 0$ uniformly in n,
   (b) there exists $\beta > 0$ such that the sequence (C_{ν_n}) converges pointwise to $0 = C_{\delta_0}$ on $(-\beta, 0)$.
2. The sequence of positive finite measures $\left(k_n \frac{t^2}{1+t^2}\mathrm{d}\nu_n(t)\right)$ is tight. Indeed, for $y > 0$,

$$\begin{aligned}\int_{[-1/y,1/y]^c} k_n \frac{t^2}{1+t^2}\mathrm{d}\nu_n(t) &\leq 2\int_{t\in\mathbb{R}} \frac{1+t^2}{y^{-2}+t^2}\cdot k_n\frac{t^2}{1+t^2}\mathrm{d}\nu_n(t)\\ &= -2k_n((i/y)G_{\nu_n}(i/y) - 1).\end{aligned}$$

We used the symmetry of ν_n in the second line. Let v_n be as in the previous lemma. For $y > 0$ small enough,

$$\int_{[-1/y,1/y]^c} k_n \frac{t^2}{1+t^2} \mathrm{d}\nu_n(t) \leq -\frac{2k_n C_{\nu_n}(-y^2)}{1+v_n(-y^2)} \leq 4\left|k_n C_{\nu_n}(-y^2)\right|,$$

which tends to zero uniformly in n when y tends to zero, by tightness of the sequence $\left(\nu_n^{\boxplus_\lambda k_n}\right)$.

(3) The sequence of positive finite measures $\left(k_n \frac{t^2}{1+t^2}\mathrm{d}\nu_n(t)\right)$ is bounded. Indeed, choose $y \in (0,1)$ is such that $-y^2$ is in the domain of the v_n's of the previous lemma and $\lim_{n\to\infty} k_n C_{\nu_n}(-y^2) = C_\mu(-y^2)$. Note that for all $t \in \mathbb{R}$, we have $\frac{t^2}{1+t^2} < 1 < y^{-2}$, so $\frac{y^{-2}+t^2}{1+t^2} < 2y^{-2}$, hence

$$\frac{t^2}{1+t^2} < 2y^{-2}\frac{t^2}{y^{-2}+t^2}.$$

So we have, for each n,

$$\begin{aligned}
\int_{t\in\mathbb{R}} k_n \frac{t^2}{1+t^2}\mathrm{d}\nu_n(t) &\leq 2y^{-2}\int_{t\in\mathbb{R}} k_n \frac{t^2}{y^{-2}+t^2}\mathrm{d}\nu_n(t) \\
&= -2y^{-2}k_n((i/y)G_{\nu_n}(i/y)-1) \\
&= -2y^{-2}k_n\frac{C_{\nu_n}(-y^2)}{1+v_n(-y^2)} \\
&\leq 4y^{-2}\left|k_n C_{\nu_n}(-y^2)\right|,
\end{aligned}$$

which is bounded uniformly in n.

(4) Let us now recall a few facts about the *Poisson integral* of positive measures on the real line which integrate $1/(1+t^2)$. If M is such a measure, for $y < 0$ and $x \in \mathbb{R}$, let us define

$$P_y(M)(x) = \int_{t\in\mathbb{R}} \frac{y}{y^2+(x-t)^2}\mathrm{d}M(t).$$

Then $(x+iy) \mapsto P_y(M)(x)$ is harmonic and determines the measure M ([D74], chapter II, theorem II).

Moreover, an easy computation shows that for each positive symmetric measure M on the real line that integrates $1/(1+t^2)$, the Poisson integral $P_y(M)(x)$ is the imaginary part of $\int_{\mathbb{R}} \frac{\sqrt{z}}{t^2 z-1}\mathrm{d}M(t)$ (with $z \notin [0,+\infty)$, $x+iy = 1/\sqrt{z}$, as it will be until the end of this proof). Indeed, since M is symmetric,

$$\int_{\mathbb{R}} \frac{\sqrt{z}}{t^2z-1}\mathrm{d}M(t) = \int_{\mathbb{R}} \frac{\sqrt{z}(t\sqrt{z}+1)}{t^2z-1}\mathrm{d}M(t) = \int_{\mathbb{R}} \frac{\sqrt{z}(t\sqrt{z}+1)}{\sqrt{z}(t\sqrt{z}+1)(t-\frac{1}{\sqrt{z}})}\mathrm{d}M(t) =$$

$$= \int_{\mathbb{R}} \frac{\mathrm{d}M(t)}{t-\frac{1}{\sqrt{z}}} = \int_{\mathbb{R}} \frac{(t-x)+iy}{(t-x)^2+y^2}\mathrm{d}M(t).$$

Now let us compute the Poisson integral of the measures $k_n t^2 \mathrm{d}\nu_n(t)$. Let $\alpha, \beta > 0$ and (v_n) be as in the previous lemma, $z \in \Delta_{\alpha,\beta}$. We have

$$P_y(k_n t^2 \mathrm{d}\nu_n(t))(x) = \Im\left(\int_{\mathbb{R}} \frac{k_n\sqrt{z}t^2}{t^2z-1}\mathrm{d}\nu_n(t)\right).$$

But since ν_n is symmetric, we have

$$\int_{\mathbb{R}} \frac{k_n\sqrt{z}t^2}{t^2z-1}\mathrm{d}\nu_n(t) = k_n\int_{\mathbb{R}} \frac{t(1+\sqrt{z}t)}{(t\sqrt{z}-1)(t\sqrt{z}+1)}\mathrm{d}\nu_n(t) = k_n\int_{\mathbb{R}} \frac{t\mathrm{d}\nu_n(t)}{t\sqrt{z}-1},$$

which is equal, by an easy computation, to

$$-k_n\frac{(1/\sqrt{z})G_{\nu_n}(1/\sqrt{z})-1}{\sqrt{z}}.$$

So $P_y(k_nt^2\mathrm{d}\nu_n(t))(x) = -\Im\left(k_n\frac{C_{\nu_n}(z)}{\sqrt{z}(1+v_n(z))}\right)$, which tends to the imaginary part of $-C_\mu(z)/\sqrt{z}$, because $\nu_n^{\boxplus_\lambda k_n}$ converges weakly to μ and $\lim_{n\to\infty} v_n(z) = 0$.

The sequence $\left(k_n\frac{t^2}{1+t^2}\mathrm{d}\nu_n(t)\right)$, bounded and tight, is relatively compact in the set of finite positive measures in the real line endowed with the topology of weak convergence (i.e. the topology defined by bounded continuous functions). If two measures G, H are the weak limit of subsequences of $\left(k_n\frac{t^2}{1+t^2}\mathrm{d}\nu_n(t)\right)$, then the measures $(1+t^2)\mathrm{d}G(t)$ and $(1+t^2)\mathrm{d}H(t)$ have the same Poisson integral on $\sqrt{\Delta_{\alpha,\beta}}$. Indeed, for $z\in\Delta_{\alpha,\beta}$,

$$P_y(k_nt^2\mathrm{d}\nu_n(t))(x) = \int_{\mathbb{R}} \underbrace{\frac{y(1+t^2)}{y^2+(x-t)^2}}_{\substack{\text{continuous}\\ \text{bounded fct of } t}} \frac{k_nt^2}{1+t^2}\mathrm{d}\nu_n(t)$$

tends at the same time to $P_y((1+t^2)\mathrm{d}G(t))(x)$, to $P_y((1+t^2)\mathrm{d}H(t))(x)$, and to the imaginary part of $-C_\mu(z)/\sqrt{z}$. It implies, by harmonicity, that they have the same Poisson integral on the lower half plane, which implies $H = G$. So the sequence $\left(k_n\frac{t^2}{1+t^2}\mathrm{d}\nu_n(t)\right)$ converges weakly to a measure G, such that the Poisson integral $P_y((1+t^2)\mathrm{d}G(t))(x)$, is equal to the imaginary part of $-C_\mu(z)/\sqrt{z}$. Thus, the functions

$$C_\mu(z)/\sqrt{z} \text{ and } \int_{\mathbb{R}} \frac{\sqrt{z}(t^2+1)}{1-t^2z}\mathrm{d}G(t)$$

have the same imaginary part. For $z\in(-\infty,0)$, it follows that

$$C_\mu(z) \text{ and } z\int_{\mathbb{R}} \frac{t^2+1}{1-t^2z}\mathrm{d}G(t)$$

have the same real part, so, by analycity and since both tend to zero as z goes to zero, they are equal.

(5) If F is another positive finite measure such that $C_\mu(z) = z\int_{\mathbb{R}} \frac{t^2+1}{1-t^2z}\mathrm{d}F(t)$, then

$$z\int_{\mathbb{R}} \frac{t^2+1}{1-t^2z}\mathrm{d}G(t) = z\int_{\mathbb{R}} \frac{t^2+1}{1-t^2z}\mathrm{d}F(t).$$

After division by $-\sqrt{z}$ and extraction of the imaginary part, this gives the equality of the Poisson integrals of $(1+t^2)\mathrm{d}G(t)$ and of $(1+t^2)\mathrm{d}F(t)$, which implies $G = F$.

□

The previous theorem implies that for all $\boxplus_\lambda$-infinitely divisible distribution μ, there exists a unique positive finite measure G such that C_μ is given by equation (2.4). G is symmetric (as limit of symmetric measures) and will be called the *Lévy measure* of μ. By injectivity of

the rectangular R-transform, two different probability measures cannot have the same Lévy measure.

Theorem 2.6 (Lévy-Kinchine formula, part 2)**.** *Every symmetric positive finite measure on the real line is the Lévy measure of a $\boxplus_\lambda$-infinitely divisible distribution.*

Before the proof of the theorem, let us state two lemmas. The first one is about the rectangular R-transform of the symmetric Bernoulli distribution.

Lemma 2.7. *There exists a sequence $(\alpha_k)_{k\geq 2}$ such that the associated power series has a positive radius of convergence and such that the rectangular R-transform with ratio λ of $(\delta_1+\delta_{-1})/2$ is given by the formula*

$$C_{(\delta_1+\delta_{-1})/2}(z) = z + \sum_{k\geq 2} \alpha_k z^k.$$

Proof. By the subsection called "The case of compactly supported probability measures" of the section called "The rectangular R-transform" of [B-G1] applied to $\mu = (\delta_1+\delta_{-1})/2$, we know that

$$C_{(\delta_1+\delta_{-1})/2}(z) = c_2((\delta_1+\delta_{-1})/2)z + \sum_{k\geq 2} c_{2k}((\delta_1+\delta_{-1})/2)z^k,$$

where the power series has a positive radius of convergence. So it suffices to prove that $c_2((\delta_1+\delta_{-1})/2) = 1$, which follows from the equation (4.1) of the present paper. □

We will also need a result about the way dilation of probability measures modify the rectangular R-transform. For $c>0$, let us denote by $D_c : x \mapsto cx$. For any distribution μ, $D_c(\mu)$ is the push-forward of μ by D_c, i.e. $D_c(\mu) : B \mapsto \mu(c^{-1}B)$.

Lemma 2.8. *For all μ symmetric probability measure, for all $c>0$,*

$$C_{D_c(\mu)}(z) = C_\mu(c^2 z). \tag{2.5}$$

Proof. We have $G_{D_c(\mu)} = \frac{1}{c} G_\mu(\frac{z}{c})$,

$$\text{so} \qquad H_{D_c(\mu)}(z) = \frac{\lambda}{c^2}\left(G_\mu(\frac{1}{c\sqrt{z}})\right)^2 + \frac{(1-\lambda)c\sqrt{z}}{c^2} G_\mu(\frac{1}{c\sqrt{z}}) = \frac{1}{c^2} H_\mu\left(c^2 z\right),$$

$$\begin{aligned}
\text{i.e.} \qquad H_{D_c(\mu)} &= D_{\frac{1}{c^2}} \circ H_\mu \circ D_{c^2}, \\
H^{-1}_{D_c(\mu)} &= D_{\frac{1}{c^2}} \circ H^{-1}_\mu \circ D_{c^2}, \\
\text{then} \qquad C_{D_c(\mu)}(z) &= U\left(\frac{c^2 z}{H^{-1}_\mu(c^2 z)} - 1\right), \\
\text{that is} \qquad C_{D_c(\mu)}(z) &= C_\mu(c^2 z).
\end{aligned}$$

□

Proof of the theorem. Let us denote by $\mathcal{M}$ the set of symmetric positive finite measures G on the real line such that there exists a symmetric distribution μ whose rectangular R-transform is given by equation (2.4). We will show that $\mathcal{M}$ is the set of symmetric positive finite measures, proving that $c\delta_0$ and $c(\delta_u+\delta_{-u}) \in \mathcal{M}$ for all $c,u>0$, that $\mathcal{M}$ is stable under addition, and that M is closed under weak convergence. Note that once this result is proved, it will be clear that

any symmetric probability measure with rectangular R-transform given by equation (2.4) will be $\boxplus_\lambda$-infinitely divisible. Indeed, denoting

$$C^{(G)}(z) = z\int_{\mathbb{R}} \frac{1+t^2}{1-zt^2}\mathrm{d}G(t) \qquad (G \in \mathcal{M}),$$

we have $C^{(G)} = nC^{(\frac{G}{n})}$.

(1) For every $c > 0$, $c\delta_0 \in \mathcal{M}$. Indeed, by equation (2.5), if $C^{(\delta_0)} = C_\mu$, then for every $c > 0$, $C^{(c\delta_0)} = C_{\mu'}$, with $\mu' = D_{c^{1/2}}(\mu)$, so it suffices to show that there exists a symmetric distribution whose rectangular R-transform is $C^{(\delta_0)}$. This distribution will appear as the limit in the *rectangular free central limit theorem*: the sequence $D_{n^{-1/2}}\left((\delta_1+\delta_{-1})/2)^{\boxplus_\lambda n}\right)$ converges weakly to a distribution with rectangular R-transform $C^{(\delta_0)}$ (we will see in the following that it stays true if one replaces $(\delta_1+\delta_{-1})/2$ by any symmetric probability measure with variance equal to 1). Indeed, let C_n denote the rectangular R-transform of $D_{n^{-1/2}}\left((\delta_1+\delta_{-1})/2)^{\boxplus_\lambda n}\right)$. By theorem 1.6, we have to prove that
   (a) $\lim\limits_{\substack{x\to 0\\ x\in(-\infty,0)}} C_n(x) = 0$ uniformly in n,
   (b) there exists $\beta > 0$ such that for all $y \in (0,\beta]$, the sequence $(C_n(-y))$ converges to $-y$.

   Note that $C_n(z) = nC_{(\delta_1+\delta_{-1})/2}(z/n)$ (we used lemma 2.8 and the additivity of the rectangular R-transform (Theorem 1.3)). Hence lemma 2.7 allows to conclude.

(2) For all $c, u > 0$, $c(\delta_u + \delta_{-u}) \in \mathcal{M}$. Indeed, we have

$$C^{(c(\delta_u+\delta_{-u}))}(z) = 2c\frac{z(1+u^2)}{1-u^2z} = 2\frac{c(1+u^2)}{2u^2}\frac{(u^2z)(1+1^2)}{1-(u^2z)} = C^{(c'(\delta_1+\delta_{-1}))}(u^2z),$$

where $c' = \frac{c(1+u^2)}{2u^2}$. So, by equation (2.5), it suffices to show that for all $c > 0$, there exists a distribution whose rectangular R-transform is $C^{(c(\delta_1+\delta_{-1}))}$. It is the same to prove that there exists a distribution whose rectangular R-transform is $C^{(\frac{c}{4}(\delta_1+\delta_{-1}))}$. This distribution will appear as the limit in the *rectangular free Poisson limit theorem*: the sequence $\nu_n^{\boxplus_\lambda n}$, with $\nu_n = \left(1-\frac{c}{n}\right)\delta_0 + \frac{c}{2n}(\delta_1+\delta_{-1})$, converges weakly to a distribution with rectangular R-transform $C^{(\frac{c}{4}(\delta_1+\delta_{-1}))}$.

Indeed, $G_{\nu_n}(z) = \frac{z^2-1+c/n}{z(z^2-1)}$, so, if (v_n) is a sequence of functions on $\Delta_{\alpha,\beta}$ as in the lemma 2.4, we have

$$C_{\nu_n}(z) = \left(\frac{1}{\sqrt{z}}G_{\nu_n}\left(\frac{1}{\sqrt{z}}\right) - 1\right)(1+v_n(z)) = \frac{cz}{n(1-z)}(1+v_n(z)),$$

so for $\mu_n = \underbrace{\nu_n \boxplus_\lambda \cdots \boxplus_\lambda \nu_n}_{n \text{ times}}$,

$$C_{\mu_n}(z) = nC_{\nu_n}(z) = \frac{cz}{1-z}(1+v_n(z)).$$

So by the properties of the functions v_n, we have both

$$\lim_{\substack{z\to 0\\ |\arg z-\pi|<\alpha}} C_{\mu_n}(z) = 0 \text{ uniformly in } n$$

and

$$\forall z \in \Delta_{\alpha,\beta}, \lim_{n\to\infty} C_{\mu_n}(z) = \frac{cz}{1-z} = C^{(\frac{c}{4}(\delta_1+\delta_{-1}))}(z).$$

So, by theorem 1.6, we know that there exists a distribution whose rectangular R-transform is $C^{(\frac{c}{4}(\delta_1+\delta_{-1}))}$.

(3) $\mathcal{M}$ is stable under addition because $C_\mu + C_\nu = C_{\mu \boxplus_\lambda \nu}$.
(4) $\mathcal{M}$ is closed under weak convergence: let (G_n) be a sequence of $\mathcal{M}$ that converges to a finite measure G. Then clearly, the sequence $\left(C^{(G_n)}\right)$ converges pointwise to $C^{(G)}$. So, by theorem 1.6, to prove that $G \in \mathcal{M}$, it suffices to show that

$$\lim_{\substack{x \to 0 \\ x<0}} C^{(G_n)}(x) = 0 \text{ uniformly in } n.$$

For each n and $x \in (0,1)$, since G_n is symmetric, $C^{(G_n)}(-x^2) = -\int_{\mathbb{R}} \frac{x^2+t^2x^2}{1+t^2x^2} \mathrm{d}G_n(t)$,

$$\forall t \in \mathbb{R}, \frac{x^2+t^2x^2}{1+t^2x^2} \leq \begin{cases} x(x+1) & \text{if } -1/x^{1/2} \leq t \leq 1/x^{1/2}, \\ 1 & \text{otherwise.} \end{cases}$$

So

$$\left| C^{(G_n)}(x) \right| \leq x(x+1) G_n(\mathbb{R}) + G_n\left(\mathbb{R} - [-1/x^{1/2}, 1/x^{1/2}]\right),$$

which tends to zero uniformly in n when x tends to 0, by boundedness and tightness of $\{G_n\,;\, n \in \mathbb{N}\}$.

□

Both previous theorems together allow us to state the following corollary.

Corollary 2.9. *A symmetric probability measure μ is $\boxplus_\lambda$-infinitely divisible if and only if there exists a sequence (ν_n) of symmetric probability measures and a sequence (k_n) of integers tending to infinity such that the sequence $\left(\nu_n^{\boxplus_\lambda k_n}\right)$ tends to μ.*

3. Rectangular Bercovici-Pata bijection

In this section, we will show that the bijective correspondence between classical symmetric infinitely divisible distributions and rectangular free infinitely divisible distributions is a homeomorphism, and that there exists a correspondence between limit theorems for sums of independent symmetric random variables and sums of free rectangular random variables.

Let us recall a few facts about symmetric $*$-infinitely divisible distributions, that can be found in [GK54] (or [F66], [P97] ...). A symmetric probability measure μ on the real line is $*$-infinitely divisible if and only if there exists a finite positive symmetric measure G such that

$$\forall \xi \in \mathbb{R}, \int_{t \in \mathbb{R}} e^{it\xi} \mathrm{d}\mu(t) = \exp\left(\int_{t \in \mathbb{R}} (cos(t\xi) - 1) \frac{1+t^2}{t^2} \mathrm{d}G(t) \right).$$

In this case, such a measure G is unique, and we will call it the *Lévy measure* of μ, and a sequence of symmetric $*$-infinitely divisible distributions converges weakly if and only if the sequence of the corresponding Lévy measures converges weakly. Moreover, in this case, the Lévy measure of the limit will be the limit of the Lévy measures.

We can then define the *rectangular Bercovici-Pata bijection with ratio* λ, denoted by Λ_λ, from the set of symmetric $*$-infinitely divisible distributions to the set of $\boxplus_\lambda$-infinitely divisible distributions, that maps a $*$-infinitely divisible distribution to the $\boxplus_\lambda$-infinitely divisible distribution with the same Lévy measure. Let μ, ν be two $*$-infinitely divisible distributions with Lévy measures G, H. Then the Lévy measures of $\mu * \nu$ and of $\Lambda_\lambda(\mu) \boxplus_\lambda \Lambda_\lambda(\nu)$ are both $G+H$, so we have

$$\Lambda_\lambda(\mu * \nu) = \Lambda_\lambda(\mu) \boxplus_\lambda \Lambda_\lambda(\nu).$$

Theorem 3.1. *The rectangular Bercovici-Pata bijection with ratio λ is a homeomorphism, which means that a sequence of $\boxplus_\lambda$-infinitely divisible distributions converges weakly if and only if the sequence of the corresponding Lévy measures converges weakly, and in this case, the Lévy measure of the limit is the limit of the Lévy measures.*

Remark 3.2. *Note that, for G symmetric positive finite measure, the function $C^{(G)}(z)$ can also be written, by symmetry,*

$$C^{(G)}(z) = \int_{\mathbb{R}} \frac{z + t\sqrt{z}}{1 - t\sqrt{z}} \mathrm{d}G(t).$$

Proof. Since the rectangular R-transform C_μ with ratio 1 of a symmetric distribution μ is linked to its Voiculescu transform φ_μ by the relation $C_\mu(z) = \sqrt{z}\varphi_\mu(1/\sqrt{z})$ (see paragraph 5 of [BV93] for the construction of the Voiculescu transform, and use the fact that for symmetric distributions, the Lévy measure is symmetric to obtain $C_\mu(z) = \sqrt{z}\varphi_\mu(1/\sqrt{z})$), the previous remark and theorem 5.10 of [BV93] shows that the map Λ_1 is the restriction of the "usual" Bercovici-Pata bijection to the set of symmetric distributions. It has been proved in [B-NT02] that the Bercovici-Pata bijection is a homeomorphism. So the theorem is proved in the case where $\lambda = 1$. But for every $*$-infinitely divisible distribution μ, the formula of the rectangular R-transform with ratio λ of $\Lambda_\lambda(\mu)$ does not depend on λ, so theorem 1.6 allows us to claim that all Λ_λ's are homeomorphisms. □

The next theorem furthers the analogy between the free rectangular convolution and the classical convolution of symmetric measures. As recalled in Theorem 3.3 of [BPB99], it is proved in [GK54] that when (ν_n) is a sequence of symmetric probability measures on the real line and (k_n) is a sequence of integers tending to infinity, the sequence $\left(\nu_n^{*k_n}\right)$ converges weakly to a $*$-infinitely divisible distribution if and only if the sequence $\left(\frac{k_n t^2}{1+t^2}\mathrm{d}\nu_n(t)\right)$ of positive finite measures converges weakly to its Lévy measure. By the theorem 2.5, we know that it will be the case if the sequence $\left(\nu_n^{\boxplus_\lambda k_n}\right)$ converges weakly to the image of the $*$-infinitely divisible distribution by the rectangular Bercovici-Pata bijection. The following theorem states the converse implication. So we have, for all $*$-infinitely divisible distributions μ,

$$\left(\nu_n^{*k_n}\right) \text{ converges to } \mu \iff \left(\nu_n^{\boxplus_\lambda k_n}\right) \text{ converges to } \Lambda_\lambda(\mu) \tag{3.1}$$

Theorem 3.3. *Let (ν_n) be a sequence of symmetric probability measures on the real line and (k_n) be a sequence of integers tending to infinity. The sequence $\left(\nu_n^{*k_n}\right)$ converges weakly to an $*$-infinitely divisible distribution if and only if the sequence $\left(\nu_n^{\boxplus_\lambda k_n}\right)$ converges weakly to its image by the rectangular Bercovici-Pata bijection with ratio λ.*

Proof. By what precedes, it suffices to prove that if the sequence $\left(\frac{k_n t^2}{1+t^2}\mathrm{d}\nu_n(t)\right)$ of positive finite measures converges weakly to a finite measure G, then the sequence $\left(\nu_n^{\boxplus_\lambda n}\right)$ converges weakly to the $\boxplus_\lambda$-infinitely divisible distribution with Lévy measure G. Assume the sequence $\left(\frac{k_n t^2}{1+t^2}\mathrm{d}\nu_n(t)\right)$ of positive finite measures to converge weakly to a finite measure G.

(1) The sequence (ν_n) converges weakly to δ_0:
Indeed, for all $\varepsilon > 0$, as the function $t \mapsto \frac{t^2}{1+t^2}$ is increasing on $\mathbb{R}^+$, we have

$$\nu_n\left([-\varepsilon, \varepsilon]^c\right) \leq \frac{1+\varepsilon^2}{\varepsilon^2} \int_{\mathbb{R}} \frac{t^2}{1+t^2} \mathrm{d}\nu_n(t),$$

which tends to zero as n tends to infinity, because the sequence $\left(\frac{k_n t^2}{1+t^2}\mathrm{d}\nu_n(t)\right)$ is bounded.

(2) We have pointwise convergence of the rectangular R-transforms:
Let α, β and (v_n) be as in the lemma 2.4. On $\Delta_{\alpha,\beta}$, we have

$$C_{\nu_n^{\boxplus_\lambda k_n}}(z) = k_n C_{\nu_n}(z) = k_n\left(\frac{1}{\sqrt{z}}G_{\nu_n}\left(\frac{1}{\sqrt{z}}\right) - 1\right)(1 + v_n(z)), \tag{3.2}$$

but we have seen in the proof of theorem 2.5 that

$$k_n\left(\frac{1}{\sqrt{z}}G_{\nu_n}\left(\frac{1}{\sqrt{z}}\right) - 1\right) = z\int_{\mathbb{R}} \underbrace{\frac{t^2+1}{1-t^2 z}}_{\substack{\text{continuous}\\ \text{bounded fct of } t}} \frac{k_n t^2}{1+t^2}\mathrm{d}\nu_n(t),$$

so, by pointwise convergence of the sequence (v_n) to zero, the rectangular R-transform of $\nu_n^{\boxplus_\lambda k_n}$ converges pointwise to $z \mapsto z\int_{\mathbb{R}} \frac{t^2+1}{1-t^2 z}\mathrm{d}G(t)$ on the set $\Delta_{\alpha,\beta}$.

(3) We have $\lim\limits_{\substack{y\to 0\\ y>0}} C_{\nu_n^{\boxplus_\lambda k_n}}(-y^2) = 0$ uniformly in n:
By equation (3.2) and (i) of lemma 2.4, it suffices to prove that

$$\lim_{\substack{y\to 0\\ y>0}} k_n((i/y)G_{\nu_n}(i/y) - 1) = 0 \quad \text{uniformly in } n,$$

that is, since ν_n is symmetric,

$$\lim_{\substack{y\to 0\\ y>0}} \int_{\mathbb{R}} \frac{y^2 + t^2 y^2}{1 + t^2 y^2}\frac{k_n t^2}{1+t^2}\mathrm{d}\nu_n(t) = 0 \quad \text{uniformly in } n.$$

When $y < 1$, $t \mapsto \frac{y^2+t^2y^2}{1+t^2y^2}$ is ≤ 1 and is increasing on $[0,\infty)$, so we have, for every $T > 0$,

$$\int_{\mathbb{R}} \frac{y^2 + t^2 y^2}{1 + t^2 y^2}\frac{k_n t^2}{1+t^2}\mathrm{d}\nu_n(t) \leq \int_{[-T,T]^c} \frac{k_n t^2}{1+t^2}\mathrm{d}\nu_n(t) + \frac{y^2 + T^2 y^2}{1 + T^2 y^2}\int_{\mathbb{R}} \frac{k_n t^2}{1+t^2}\mathrm{d}\nu_n(t).$$

Now fix $\varepsilon > 0$, choose $T > 0$ such that for all n, $\int_{[-T,T]^c} \frac{k_n t^2}{1+t^2}\mathrm{d}\nu_n(t) \leq \varepsilon$. For y small enough, $\frac{y^2 + T^2 y^2}{1 + T^2 y^2}\sup\limits_n \int_{\mathbb{R}} \frac{k_n t^2}{1+t^2}\mathrm{d}\nu_n(t)$ is less than ε, which closes the proof.

□

The following corollary could have been proved with the equation (2.5), but the proof we give is shorter and does not use any computations.

Corollary 3.4. *The rectangular Bercovici-Pata bijection commutes with the dilations D_c, $c > 0$.*

Proof. Let μ be a $*$-infinitely divisible distribution. Let, for each $n \geq 1$, ν_n be a symmetric distribution such that $\nu_n^{*n} = \mu$. We have

$$\begin{aligned}\Lambda_\lambda \circ D_c(\mu) &= \Lambda_\lambda \circ D_c\left(\nu_n^{*n}\right)\\ &= \Lambda_\lambda\left(D_c(\nu_n)^{*n}\right)\\ &= \lim_{n\to\infty} D_c(\nu_n)^{\boxplus_\lambda n}.\end{aligned}$$

But from equation (2.5) and additivity of the rectangular R-transform, we know that

$$\forall n \geq 1, D_c(\nu_n)^{\boxplus_\lambda n} = D_c\left(\nu_n^{\boxplus_\lambda n}\right),$$

so, by continuity of D_c,

$$\Lambda_\lambda \circ D_c(\mu) = D_c\left(\lim_{n\to\infty} \nu_n^{\boxplus_\lambda n}\right)$$

which is $D_c \circ \Lambda_\lambda(\mu)$ by equivalence (3.1). □

Let us define the $\boxplus_\lambda$-*stable distributions* to be the symmetric distributions whose orbit under the action of the group of the dilations is stable under $\boxplus_\lambda$. The previous corollary allows us to give the following one.

Corollary 3.5. *The rectangular Bercovici-Pata bijection exchanges symmetric $*$-stable and $\boxplus_\lambda$-stable distributions. Moreover, the index of any $*$-stable distribution μ (i.e. the unique $\alpha \in (0,2]$ such that for all $n \geq 1$, $\mu^{*n} = D_{n^{\frac{1}{\alpha}}}(\mu)$) is preserved, i.e. one has $\Lambda_\lambda(\mu)^{\boxplus_\lambda n} = D_{n^{\frac{1}{\alpha}}}(\Lambda_\lambda(\mu))$.*

The theorem 3.3 has another surprising consequence, which concerns classical probability theory. It mights already be known by specialists of limit theorems in classical probability theory, but we since it can surprisingly be deduced from our results on Bercovici-Pata bijections, we state it and prove it here. In order to state it, we have to go further in the description of divisible distributions with respect to $\boxplus$ and $*$: we have to give the Lévy-Kinchine formulas for non symmetric infinitely divisible distributions. These distributions have been classified in [BV93] and [GK54]: a probability measure on the real line μ is infinitely divisible with respect to $\boxplus$ (resp. $*$) if and only if there exists a real number γ and a positive finite measure on the real line σ such that $\varphi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1+zt}{z-t} \mathrm{d}\sigma(t)$ (resp. the Fourier transform is $\hat{\mu}(t) = \exp\left[i\gamma t + \int_{\mathbb{R}} (e^{itx} - 1 - \frac{itx}{x^2+1}) \frac{x^2+1}{x^2} \mathrm{d}\sigma(x)\right]$). Moreover, in this case, such a pair (γ, σ) is unique, and we denote μ by $\nu_{\boxplus}^{\gamma,\sigma}$ (resp. $\nu_*^{\gamma,\sigma}$). Thus, one can define a bijection Λ, called the *Bercovici-Pata bijection*, from the set of $*$-infinitely divisible distributions to the set of $\boxplus$-infinitely divisible distributions by

$$\Lambda : \nu_*^{\gamma,\sigma} \mapsto \nu_{\boxplus}^{\gamma,\sigma}.$$

It is proved in [BPB99] that for all sequence (μ_n) of probability measures and for all sequence (k_n) of integers tending to $+\infty$, the sequence $\mu_n^{*k_n}$ tends weakly to a probability measure μ if and only if the sequence $\mu_n^{\boxplus k_n}$ tends weakly to $\Lambda(\mu)$. By section 1.2, the infinitely divisible distributions with respect to $\boxplus_1$ are the symmetric infinitely divisible distributions with respect to $\boxplus$ and the rectangular Bercovici-Pata bijection with ratio 1 is the restriction of the Bercovici-Pata bijection to the set of symmetric $*$-infinitely divisible distributions.

Corollary 3.6. *Let (ν_n) be a sequence of symmetric probability measures on the real line and (k_n) be a sequence of integers tending to infinity. Let, for all n, ρ_n be the push-forward of ν_n by the function $t \to t^2$. Then the sequence $\left(\nu_n^{*k_n}\right)$ converges weakly to a probability measure if and only if the sequence $\left(\rho_n^{*k_n}\right)$ converges weakly to a probability measure. Moreover, this case, if one denotes the Lévy measure of the limit of $\left(\nu_n^{*k_n}\right)$ by G (as the limit of such a sequence, the limit probability measure has actually got to be symmetric and $*$-infinitely divisible), then the limit of $\left(\rho_n^{*k_n}\right)$ is $\nu_*^{\gamma,\sigma}$, with*

$$\gamma = \int_{t\in\mathbb{R}} \frac{1+t^2}{1+t^4} \mathrm{d}G(t), \quad \sigma = \frac{t^2+t}{t^2+1} \mathrm{d}F(t),$$

where F is the push-forward, by $t \to t^2$, of G.

Proof. Let us first prove the equivalence. Recall that, as explained in section 1.2, for all n, the push-forward, by the function $t \to t^2$, of $\nu_n^{\boxplus_0 k_n}$ is $\rho_n^{\boxplus k_n}$. Hence for any symmetric probability measure μ, if

$$\nu_n^{*k_n} \underset{n\to\infty}{\longrightarrow} \mu, \text{ i.e. } \nu_n^{\boxplus_0 k_n} \underset{n\to\infty}{\longrightarrow} \Lambda_0(\mu),$$

then

$$\rho_n^{\boxplus k_n} \underset{n\to\infty}{\longrightarrow} \Lambda_0(\mu)^2,$$

where $\Lambda_0(\mu)^2$ denotes the push-forward, by the function $t \to t^2$, of $\Lambda_0(\mu)$. Hence

$$\rho_n^{*k_n} \underset{n\to\infty}{\longrightarrow} \Lambda^{-1}(\Lambda_0(\mu)^2).$$

Reciprocally, if there is a probability measure ρ on $[0+\infty)$ such that

$$\rho_n^{*k_n} \underset{n\to\infty}{\longrightarrow} \rho,$$

then

$$\rho_n^{\boxplus k_n} \underset{n\to\infty}{\longrightarrow} \Lambda(\rho),$$

i.e. $\nu_n^{\boxplus_0 k_n}$ converges weakly to the symmetric probability measure ν whose push-forward by the square function is $\Lambda(\rho)$, which implies that $\nu_n^{*k_n}$ converges weakly to a symmetric probability measure.

To prove the last part of the corollary, recall the fact from [GK54], which is also recalled in [BPB99], that for all sequence (η_n) of probability measures on the real line, for all sequence (k_n) of integers tending to $+\infty$, for all real number a and all positive measure finite H on the real line, we have the equivalence

$$\eta_n^{*k_n} \underset{n\to\infty}{\longrightarrow} \nu_*^{a,H} \Longleftrightarrow \int_{t\in\mathbb{R}} \frac{k_n t}{1+t^2}\mathrm{d}\eta_n(t) \underset{n\to\infty}{\longrightarrow} a, \text{ and } \frac{k_n t^2}{1+t^2}\mathrm{d}\eta_n(t) \underset{n\to\infty}{\longrightarrow} H,$$

where he convergences of measures are with respect to the weak topology, i.e. against all continuous bounded functions (note that this equivalence could have been a way to prove the result without reference to the Bercovici-Pata bijections). Suppose that $\left(\nu_n^{*k_n}\right)$ converges weakly to a probability measure. This measure has to be symmetric and $*$-infinitely divisible. Let us denote its Lévy measure by G. Then

$$\int_{t\in\mathbb{R}} \frac{k_n t}{1+t^2}\mathrm{d}\rho_n(t) = \int_{t\in\mathbb{R}} \frac{k_n t^2}{1+t^4}\mathrm{d}\nu_n(t) = \int_{t\in\mathbb{R}} \frac{1+t^2}{1+t^4}\frac{k_n t^2}{1+t^2}\mathrm{d}\nu_n(t) \underset{n\to\infty}{\longrightarrow} \int_{t\in\mathbb{R}} \frac{1+t^2}{1+t^4}\mathrm{d}G(t),$$

and for all continuous bounded function f,

$$\int_{t\in\mathbb{R}} f(t)\frac{k_n t^2}{1+t^2}\mathrm{d}\rho_n(t) = \int_{t\in\mathbb{R}} f(t^2)\frac{k_n t^4}{1+t^4}\mathrm{d}\nu_n(t) = \int_{t\in\mathbb{R}} f(t^2)\frac{t^2+t^4}{1+t^4}\frac{k_n t^2}{1+t^2}\mathrm{d}\nu_n(t),$$

which tends, when n goes to infinity, to

$$\int_{t\in\mathbb{R}} f(t^2)\frac{t^2+t^4}{1+t^4}\mathrm{d}G(t) = \int_{t\in\mathbb{R}} f(t)\frac{t+t^2}{1+t^2}\mathrm{d}F(t).$$

This concludes the proof. □

4. Examples

In this section, we give examples of symmetric $*$-infinitely divisible distributions whose images by the rectangular Bercovici-Pata bijections we are able to give. Unfortunately, there are as few examples as for the "classical" Bercovici-Pata bijection. But in the section 5, we shall give some matricial models for all $\boxplus_\lambda$-infinitely divisible distributions.

4.1. **Rectangular Gaussian distribution and Marchenko-Pastur distribution.** In this section, we will identify the *rectangular Gaussian distribution* ν, that is the image, by the rectangular Bercovici-Pata bijection, of the Gaussian distribution with mean zero and variance one. The corresponding Lévy measure is δ_0, so the rectangular R-transform is z. We will show that unless $\lambda = 0$, in which case $\nu = (\delta_{-1} + \delta_1)/2$, ν is the symmetric distribution whose push forward by the function $x \to x^2$ has the density

$$\frac{\left[4\lambda - (x-1-\lambda)^2\right]^{1/2}}{2\pi\lambda x}\chi(x),$$

where χ stands for the characteristic function of the interval $[(1-\lambda^{1/2})^2, (1+\lambda^{1/2})^2]$, which means that for all $n \geq 1$, the $2n$-th moment of ν is $1/\lambda$ times the n-th moment of the Marchenko-Pastur distribution with expectation λ (the Marchenko-Pastur distributions are presented in section 4.3.2).

Recall that the sequence $(c_{2n}(\mu))_{n\geq 1}$ of the *free cumulants with ratio* λ of a symmetric probability measure μ with moments of any order, defined in the subsection called "The case of compactly supported probability measures" of the section called "The rectangular R-transform" of [B-G1], are linked to the sequence $(m_n(\mu))_{n\geq 0}$ of its moments by the relation (see the proposition 3.5 of [B-G1]):

$$\forall n \geq 1, \quad m_{2n}(\mu) = \sum_{\pi \in \mathrm{NC'}(2n)} \lambda^{e(\pi)} \prod_{V\in\pi} c_{|V|}(\mu), \tag{4.1}$$

where NC'$(2n)$ is the set of noncrossing partitions of $\{1, \ldots, 2n\}$ in which all blocks have even cardinality, and where $e(\pi)$ denotes the number of blocks of π with even minimum.

The following lemma will be useful to study distributions coming from rectangular free probability theory. A function f defined on a conjugation-stable subset of $\mathbb{C}$ is said to be *commuting with the conjugation* (abbreviated by *c.w.c.*) if $f(\bar{z}) = \overline{f(z)}$. Note that the function $z \to z^{1/2}$ is c.w.c., whereas $z \to \sqrt{z}$ is not.

Lemma 4.1. *If the rectangular R-transform of a symmetric probability measure μ extends to an analytic c.w.c. function in a neighborhood $B(0,r)$ of zero in the complex plane and tends to zero at zero, then the probability measure has compact support, and the expansion of $C_\mu(z)$ for small z is given by the formula*

$$C_\mu(z) = \sum_{n=1}^{+\infty} c_{2n}(\mu) z^n. \tag{4.2}$$

Proof. Let us define $T(z) = (\lambda z + 1)(z + 1)$. Note that U is the inverse of $T - 1$. Since the extension of C_μ tends to zero at zero, $z/H_\mu^{-1}(z)$ extends to a neighborhood of zero such that we have, in this neighborhood,

$$\frac{z}{H_\mu^{-1}(z)} = T\left(C_\mu(z)\right),$$

and this function tends to 1 at zero. Thus $H_\mu^{-1}(z)$ is one to one in a neighborhood of zero, and H_μ extends to an analytic c.w.c. function in a neighborhood of zero such that

$$\lim_{z\to 0} \frac{H_\mu(z)}{z} = 1.$$

So the function $G_\mu(1/\sqrt{z})/\sqrt{z}$, which is equal to

$$\frac{\lambda - 1 + \left[(1-\lambda)^2 + 4\lambda(H_\mu(z)/z)\right]^{1/2}}{2\lambda}$$

if $\lambda > 0$ and to $H_\mu(z)$ if $\lambda = 0$, extends to an analytic c.w.c. function in a neighborhood of zero. But since μ is symmetric, for all z in the complement of the real nonnegative half line,

$$\frac{G_\mu(1/\sqrt{z})}{\sqrt{z}}=\frac{1}{\sqrt{z}}\int_{\mathbb{R}}\frac{\mathrm{d}\mu(t)}{\frac{1}{\sqrt{z}}-t}=\frac{1}{2\sqrt{z}}\int_{\mathbb{R}}\left[\frac{1}{\frac{1}{\sqrt{z}}-t}+\frac{1}{\frac{1}{\sqrt{z}}+t}\right]\mathrm{d}\mu(t)=\frac{1}{z}\int_{\mathbb{R}}\frac{\mathrm{d}\mu(t)}{\frac{1}{z}-t^2}=\frac{1}{z}G_\rho(1/z),$$

where ρ is the push forward of μ by the function $t \to t^2$. Hence the Cauchy transform of ρ extends to an analytic c.w.c. function in a neighborhood of infinity. Thus, by the Stieltjes inversion formula, ρ is compactly supported, which implies that μ has compact support too.

μ has now been proved to be compactly supported. Then the second part of the lemma, equation (4.2), has been established in the subsection called "The case of compactly supported probability measures" of the section called "The rectangular R-transform" of [B-G1]. □

So ν has compact support, and for all $n \geq 1$, $c_{2n}(\nu) = \delta_{1,n}$.

Let us first treat the case where $\lambda = 0$. By (4.1), all even moment of ν are 1, so $\nu = (\delta_{-1}+\delta_1)/2$.

Assume $\lambda > 0$. By (4.1), the moments of ν are given by

$$\forall n \geq 1, \;\; m_{2n}(\nu) = \sum_\pi \lambda^{e(\pi)} = \lambda^n \sum_\pi \left(\frac{1}{\lambda}\right)^{o(\pi)},$$

where the sums are taken over noncrossing pairings of $\{1, \ldots, 2n\}$ (a *noncrossing pairing* is a noncrossing partition where all classes have cardinality two, recall also that for a partition π, $e(\pi)$ and $o(\pi)$ are respectively the number of classes of π with even and odd minimum).

Lemma 4.2. *Let $I = \{x_1 < \cdots < x_n\}$ and $J = \{y_1 < z_1 < y_2 < z_2 < \cdots < y_n < z_n\}$ be totally ordered sets. There is a bijection $\pi \to \ddot{\pi}$ from the set of noncrossing partitions of I to the set of noncrossing pairings of J such that for all π,*

$$|\pi| = o(\ddot{\pi}).$$

Proof. Let us first construct the map $\pi \to \ddot{\pi}$ by induction on n, using the following well known result : a partition π of a finite totally ordered set is noncrossing if and only if one of its classes V is an interval and $\pi\backslash\{V\}$ is noncrossing (page 3 of [S98]). Consider a noncrossing partition π of I. If π has only one class, we define $\ddot{\pi}$ to be

$$\{\{y_1, z_n\}, \{z_1, y_2\}, \{z_2, y_3\}, \ldots, \{z_{n-1}, y_n\}\}.$$

In the other case, a strict class V of π is an interval, $V = \{x_k, x_{k+1}, \ldots, x_l\}$. Then we define $\ddot{\pi}$ to be

$$\ddot{\sigma} \cup \{\{y_k, z_l\}, \{z_k, y_{k+1}\}, \{z_{k+1}, y_{k+2}\}, \ldots, \{z_{l-1}, y_l\}\},$$

where $\ddot{\sigma}$ is the image (defined by the induction hypothesis) of the partition

$$\sigma = \pi - \{V\}$$

of $I - V$ (it is easy to see that the result does not depend on the choice of the interval V).

The relation $|\pi| = o(\ddot{\pi})$ follows from the construction of $\pi \to \ddot{\pi}$.

Let us now prove, by induction on n, that $\pi \to \ddot{\pi}$ is a bijection. If $n = 1$, the result is obvious. Suppose the result to be proved to the ranks $1, \ldots, n-1$, and consider a noncrossing pairing τ of J. Let us prove that there exists exactly one noncrossing partition π of I such that $\ddot{\pi} = \tau$. Consider $l \in [n]$ minimal such that there exists $k < l$ such that $\{y_k, z_l\}$ is a class of τ (such an l exists because it is the case of n). Then it is easy to see that $\{z_k, y_{k+1}\}, \{z_{k+1}, y_{k+2}\}, \ldots, \{z_{l-1}, y_l\}$ are

classes of τ, and any partition π of I such that $\ddot{\pi} = \tau$ must satisfy $V := \{x_k, x_{k+1}, \ldots, x_l\} \in \pi$, and

$$\ddot{\sigma} = \tau - \{\{y_k, z_l\}, \{z_k, y_{k+1}\}, \{z_{k+1}, y_{k+2}\}, \ldots, \{z_{l-1}, y_l\}\},$$

where $\sigma = \pi - \{V\}$ (partition of $I - V$). Thus, by the induction hypothesis, there exists exactly one noncrossing partition π of I such that $\ddot{\pi} = \tau$. $\square$

So the moments of ν are given by

$$\forall n \geq 1, \;\; m_{2n}(\nu) = \lambda^n \sum_{\pi \in \mathrm{NC}(n)} \left(\frac{1}{\lambda}\right)^{|\pi|}.$$

But for all $n \geq 1$, $\sum_{\pi \in \mathrm{NC}(n)} (1/\lambda)^{|\pi|}$ is the n-th moment of a distribution with all free cumulants being equal to $1/\lambda$, i.e. of the Marchenko-Pastur distribution with parameter $1/\lambda$ (see section 4.3.2). Thus the push-forward of ν by $t \to t^2$ is the push-forward of the Marchenko-Pastur distribution with parameter $1/\lambda$ by the map $t \to \lambda t$, and has density

$$\frac{\left[4\lambda - (x - 1 - \lambda)^2\right]^{1/2}}{2\pi\lambda x} \chi(x),$$

where χ stands for the characteristic function of the interval $[(1 - \lambda^{1/2})^2, (1 + \lambda^{1/2})^2]$. Hence we have proved the following result:

Theorem 4.3. *The rectangular Gaussian distribution ν with ratio λ has cumulants given by*

$$\forall n \geq 1, c_{2n}(\nu) = \delta_{n,1}.$$

When $\lambda = 0$, $\nu = (\delta_1 + \delta_{-1})/2$. When $\lambda > 0$, ν has density

$$\frac{\left[4\lambda - (x^2 - 1 - \lambda)^2\right]^{1/2}}{2\pi\lambda|x|} \chi(x^2).$$

Its support is $[-1 - \lambda^{1/2}, -1 + \lambda^{1/2}] \cup [1 - \lambda^{1/2}, 1 + \lambda^{1/2}]$.

Note that when $\lambda = 1$, it is the well-known semi-circle law with radius two.

Remark 4.4 (Moments of the Marchenko-Pastur distribution)**.** *Note that the previous lemma, used with the fact that the free cumulants of the Marchenko-Pastur distribution with parameter a are all equal to a (see* [HP00] *p. 65), gives us a formula for the n-th moment of the Marchenko-Pastur distribution with parameter a: it is equal to $\sum_\pi a^{o(\pi)}$, where the sum is taken over all noncrossing pairings of $[2n]$. This formula, proved using a random matrix approach, appeared already in an unpublished paper of Ferenc Oravecz and Dénes Petz.*

Remark 4.5 (Growth of the support in the related semigroup)**.** *Let us define, for a fixed $\lambda \in [0, 1]$ and for all $c > 0$, N_c to be the law of $c^{\frac{1}{2}} X$, when X is a random variable distributed according ν (i.e. N_c is the image of the symmetric Gaussian law with variance c by the rectangular Bercovici-Pata bijection). Then $(\{N_c\,;\, c > 0\}, \boxplus_\lambda)$ is an additive semigroup, whereas the size of the support of N_c is not linear in c but in $c^{\frac{1}{2}}$. This kind of phenomenon had already been observed in free probability.*

4.2. **Rectangular Cauchy distributions.** This section could be called *missed appointment for the Cauchy distribution*. The Cauchy type, $\{\mathcal{C}_t = \frac{1}{\pi} \frac{t\mathrm{d}x}{x^2 + t^2}\,;\, t > 0\}$, is well known to be invariant under many transformations. For example, this set is the set of symmetric $*$- and $\boxplus$-stable distributions with index 1 ($\mathcal{C}_t$ has Lévy measure $t\mathcal{C}_1$). But we are going to compute the

set of $\boxplus_\lambda$-stable distributions with index 1, and it will appear that unless $\lambda = 1$, it is not the Cauchy type.

So let us fix $\lambda \in [0,1]$, and let us denote the image $\Lambda_\lambda(\mathcal{C}_t)$ of the symmetric Cauchy law $\mathcal{C}_t$ with index t by the Bercovici-Pata bijection with ratio λ by ν_t. We know that the rectangular R-transform with ratio λ of ν_t admits an analytic extension to $\mathbb{C}\backslash\mathbb{R}^+$ given by the formula $it\sqrt{z}$. So, by the remark 1.4, $H_{\nu_t}^{-1}$ admits an analytic extension to $\mathbb{C}\backslash(\mathbb{R}^+ \cup \{-\frac{1}{t^2}, \frac{1}{t^2\lambda^2}\})$ given by the formula

$$H_{\nu_t}^{-1}(z) = \frac{z}{T(C_{\nu_t}(z))} = \frac{z}{(\lambda it\sqrt{z}+1)(it\sqrt{z}+1)},$$

where $T(X) = (\lambda X + 1)(X + 1)$.

In order to compute $H_{\nu_t}(x)$, we have to invert the previous formula and to remember that it is a bijection form a neighborhood of 0 in $\mathbb{C}\backslash\mathbb{R}^+$ to a neighborhood of zero in $\mathbb{C}\backslash\mathbb{R}^+$, equivalent to x in the neighborhood of zero. So let us fix $x \in \mathbb{C}\backslash\mathbb{R}^+$, and denote $H_{\nu_t}(x)$ by z. If x is closed enough from zero, we have $x = H_{\nu_t}^{-1}(z)$, hence

$$\begin{aligned} x(\lambda it\sqrt{z}+1)(it\sqrt{z}+1) &= z \\ -(\lambda x t^2+1)z + itx(\lambda+1)\sqrt{z} + x &= 0, \end{aligned}$$

Note that $(itx(\lambda+1))^2 - 4x(-(\lambda x t^2+1)) = 4x - t^2x^2(\lambda-1)^2$, so

$$\sqrt{z} = \frac{itx(\lambda+1) \pm \sqrt{4x - t^2x^2(\lambda-1)^2}}{2\lambda x t^2 + 2}.$$

But when x goes to zero in $\mathbb{C}\backslash\mathbb{R}^+$,

$$\sqrt{4x - t^2x^2(\lambda-1)^2} \sim 2\sqrt{x}, \text{ so } \frac{itx(\lambda+1) \pm \sqrt{4x - t^2x^2(\lambda-1)^2}}{2\lambda x t^2 + 2} \sim \pm\sqrt{x},$$

hence

$$\sqrt{z} = \frac{itx(\lambda+1) + \sqrt{4x - t^2x^2(\lambda-1)^2}}{2\lambda x t^2 + 2} = \frac{itx(\lambda+1) + \sqrt{x}(4 - t^2x(\lambda-1)^2)^{\frac{1}{2}}}{2\lambda x t^2 + 2},$$

hence

$$H_{\nu_t}(x) = \left(\frac{itx(\lambda+1) + \sqrt{x}(4 - t^2x(\lambda-1)^2)^{\frac{1}{2}}}{2\lambda x t^2 + 2}\right)^2 = \frac{x}{4}\left(\frac{it\sqrt{x}(\lambda+1) + (4 - t^2x(\lambda-1)^2)^{\frac{1}{2}}}{\lambda x t^2 + 1}\right)^2.$$

But by the remark 1.4,

$$\frac{1}{\sqrt{x}} G_{\nu_t}(\frac{1}{\sqrt{x}}) = V(\frac{H_{\nu_t}(x)}{x}) = \frac{\lambda - 1 + [(\lambda-1)^2 + 4\lambda H_{\nu_t}(x)/x]^{\frac{1}{2}}}{2\lambda}.$$

So we compute

$$\begin{aligned} &(\lambda-1)^2 + 4\lambda H_{\nu_t}(x)/x \\ &= (\lambda-1)^2 + \tfrac{\lambda}{(\lambda x t^2+1)^2}\{it\sqrt{x}(\lambda+1) + [4 - t^2x(\lambda-1)^2]^{\frac{1}{2}}\}^2 \\ &= (\lambda-1)^2 + \tfrac{\lambda}{(\lambda x t^2+1)^2}\{-t^2x(\lambda+1)^2 + 4 - t^2x(\lambda-1)^2 + 2it\sqrt{x}(\lambda+1)[4 - t^2x(\lambda-1)^2]^{\frac{1}{2}}\} \end{aligned}$$

Hence

$$\begin{aligned}
&(\lambda x t^2+1)^2[(\lambda-1)^2+4\lambda H_{\nu_t}(x)/x]\\
&=(\lambda-1)^2+2\lambda x t^2(\lambda-1)^2+\lambda^2x^2t^4(\lambda-1)^2-2\lambda xt^2(\lambda^2+1)+4\lambda+2it\lambda\sqrt{x}(\lambda+1)[4-t^2x(\lambda-1)^2]^{\frac12}\\
&=(\lambda-1)^2+4\lambda+2\lambda xt^2[(\lambda-1)^2-\lambda^2-1]+\lambda^2x^2t^4(\lambda-1)^2+2it\lambda\sqrt{x}(\lambda+1)[4-t^2x(\lambda-1)^2]^{\frac12}\\
&=(\lambda+1)^2-4\lambda^2xt^2+\lambda^2x^2t^4(\lambda-1)^2+2it\lambda\sqrt{x}(\lambda+1)[4-t^2x(\lambda-1)^2]^{\frac12}\\
&=\{(\lambda+1)+it\lambda\sqrt{x}[4-t^2x(\lambda-1)^2]^{\frac12}\}^2
\end{aligned}$$

Hence for all $x\in\mathbb{C}\backslash\mathbb{R}^+$ closed enough to zero,

$$\begin{aligned}
G_{\nu_t}(\frac{1}{\sqrt{x}}) &= \frac{\sqrt{x}}{2\lambda}\{\lambda-1+\underbrace{[(\lambda-1)^2+4\lambda H_{\nu_t}(x)/x]^{\frac12}}_{\frac{(\lambda+1)+it\lambda\sqrt{x}[4-t^2x(\lambda-1)^2]^{\frac12}}{\lambda xt^2+1}}\}\\
&= \frac{\sqrt{x}}{2\lambda}\frac{(\lambda-1)(\lambda xt^2+1)+\lambda+1+it\lambda\sqrt{x}[4-t^2x(\lambda-1)^2]^{\frac12}}{\lambda xt^2+1}\\
&= \sqrt{x}\left\{\frac{2+(\lambda-1)xt^2+2it\sqrt{x}[1-t^2x(\lambda-1)^2/4]^{\frac12}}{2(\lambda xt^2+1)}\right\}.
\end{aligned}$$

So for all $z\in\mathbb{C}^-$ faraway enough from zero (in a non tangential way), for $x\in\mathbb{C}\backslash\mathbb{R}^+$ such that $z=1/\sqrt{x}$, i.e. $x=1/z^2$,

$$\begin{aligned}
G_{\nu_t}(z)=G_{\nu_t}\left(\frac{1}{\sqrt{x}}\right)&=\sqrt{x}\left\{\frac{2+(\lambda-1)xt^2+2it\sqrt{x}[1-t^2x(\lambda-1)^2/4]^{\frac12}}{2(\lambda xt^2+1)}\right\}\\
&=\frac1z\frac{1}{\frac{2\lambda t^2}{z^2}+2}\left\{2+(\lambda-1)\frac{t^2}{z^2}+\frac{2it}{z}\left[1-\frac{t^2(\lambda-1)^2}{4z^2}\right]^{\frac12}\right\}\\
&=\frac{z}{\lambda t^2+z^2}\left\{1+(\lambda-1)\frac{t^2}{2z^2}+\frac{it}{z}\left[1-\frac{t^2(\lambda-1)^2}{4z^2}\right]^{\frac12}\right\}
\end{aligned}$$

Note that for $\rho\in\mathbb{R}^+$, for all $z\in\mathbb{C}\backslash\{0\}$,

$$1-\frac{t^2(\lambda-1)^2}{4z^2}=-\rho\Leftrightarrow z^2=\frac{t^2(1-\lambda)^2}{4(1+\rho)}\Leftrightarrow z=\pm\frac{t(1-\lambda)}{2(1+\rho)^{\frac12}}\Rightarrow z\in[-\frac{t(1-\lambda)}{2},\frac{t(1-\lambda)}{2}].$$

So the function $z\mapsto\left[1-\frac{t^2(\lambda-1)^2}{4z^2}\right]^{\frac12}$ extends analytically to $\mathbb{C}\backslash[-\frac{t(1-\lambda)}{2},\frac{t(1-\lambda)}{2}]$ with the same formula. So, by analycity, we have

$$\forall z\in\mathbb{C}^-\backslash\{i\lambda^{\frac12}t\},\qquad G_{\nu_t}(z)=\frac{z}{\lambda t^2+z^2}\left\{1+(\lambda-1)\frac{t^2}{2z^2}+\frac{it}{z}\left[1-\frac{t^2(\lambda-1)^2}{4z^2}\right]^{\frac12}\right\}.\qquad(4.3)$$

Remark 4.6. *Note that $G_{\nu_t}(z)$ has to be analytic at $-i\lambda^{\frac12}t$. If the formula we give had no analytic extension at $-i\lambda^{\frac12}t$, we would have made a mistake. Hopefully, one can check that the pole of $z\mapsto\frac{z}{\lambda t^2+z^2}$ at $-i\lambda^{\frac12}t$ is simple, and that the function*

$$z\mapsto 1+(\lambda-1)\frac{t^2}{2z^2}+\frac{it}{z}\left[1-\frac{t^2(\lambda-1)^2}{4z^2}\right]^{\frac12}$$

has a zero at $-i\lambda^{\frac12}t$.

In order to compute ν_t, we are going to use the following lemma.

Lemma 4.7. *Let ν be probability measure on the real line such that the restriction of G_ν to $\mathbb{C}^-$ extends analytically to an open set containing $\mathbb{C}^- \cup I$, where I is an open interval. Then the restriction of μ to I admits an analytic density: $x \in I \mapsto \frac{1}{\pi}\Im G_\nu(x)$.*

Proof. Let us define, for $t \geq 0$,

$$\rho_t : x \in \mathbb{R} \mapsto \begin{cases} \frac{1}{\pi}\Im G_\nu(x - it) & \text{if } x \in I, \\ 0 & \text{in the other case.} \end{cases}$$

Then for all $t > 0$, ρ_t is well known to be the restriction, to I, of the density of $\nu * \mathcal{C}_t$. Moreover, $\nu * \mathcal{C}_t$ converges weakly (i.e. against any continuous bounded function) to ν as t tends to zero. So it suffices to prove that for all f compactly supported continuous function on I, $\int f(x)\rho_t(x)\mathrm{d}x$ tends to $\int f(x)\rho_0(x)\mathrm{d}x$ when t goes to zero, which is an easy application of the dominated convergence theorem. □

This lemma allows us to claim that the restriction of ν_t to $\mathbb{R}\backslash[-\frac{t(1-\lambda)}{2}, \frac{t(1-\lambda)}{2}]$ has an analytic density given by the function

$$x \mapsto \frac{t}{\pi(\lambda t^2 + x^2)}\left[1 - \frac{t^2(\lambda-1)^2}{4x^2}\right]^{\frac{1}{2}} \qquad (x \in \mathbb{R}\backslash[-\frac{t(1-\lambda)}{2}, \frac{t(1-\lambda)}{2}]). \tag{4.4}$$

In order to prove that ν_t is carried by $\mathbb{R}\backslash[-\frac{t(1-\lambda)}{2}, \frac{t(1-\lambda)}{2}]$ and has this density, it suffices to prove that there is no mass out of $\mathbb{R}\backslash[-\frac{t(1-\lambda)}{2}, \frac{t(1-\lambda)}{2}]$, i.e. that $I = 1$, with

$$I := \int_{\mathbb{R}\backslash[-\frac{t(1-\lambda)}{2}, \frac{t(1-\lambda)}{2}]} \frac{t}{\pi(\lambda t^2 + x^2)}\left[1 - \frac{t^2(\lambda-1)^2}{4x^2}\right]^{\frac{1}{2}} \mathrm{d}x.$$

We have

$$\begin{aligned}
I &= \frac{2t}{\pi}\int_{\frac{t(1-\lambda)}{2}}^{+\infty} \frac{1}{\lambda t^2 + x^2}\left[1 - \frac{t^2(\lambda-1)^2}{4x^2}\right]^{\frac{1}{2}} \mathrm{d}x \\
&= \frac{-2t}{\pi}\int_{\frac{t(1-\lambda)}{2}}^{+\infty} \frac{x^2}{\lambda t^2 + x^2}\left[1 - \frac{t^2(\lambda-1)^2}{4x^2}\right]^{\frac{1}{2}} \mathrm{d}\left(\frac{1}{x}\right) \\
&= \frac{-2t}{\pi}\int_{\frac{t(1-\lambda)}{2}}^{+\infty} \frac{1}{\frac{\lambda t^2}{x^2} + 1}\left[1 - \frac{t^2(\lambda-1)^2}{4x^2}\right]^{\frac{1}{2}} \mathrm{d}\left(\frac{1}{x}\right) \\
&= \frac{2t}{\pi}\int_0^{\frac{2}{t(1-\lambda)}} \frac{1}{\lambda t^2 u^2 + 1}\left[1 - \frac{t^2(\lambda-1)^2u^2}{4}\right]^{\frac{1}{2}} \mathrm{d}u \\
&= \frac{2t}{\pi}\frac{2}{(1-\lambda)t}\int_0^{\frac{2}{t(1-\lambda)}} \frac{1}{\frac{4\lambda}{(1-\lambda)^2}\left(\frac{(1-\lambda)tu}{2}\right)^2 + 1}\left[1 - \frac{t^2(\lambda-1)^2u^2}{4}\right]^{\frac{1}{2}} \mathrm{d}\left(\frac{(1-\lambda)tu}{2}\right) \\
&= \frac{4(1-\lambda)}{\pi}\int_0^1 \frac{1}{4\lambda v^2 + (1-\lambda)^2}\left[1 - v^2\right]^{\frac{1}{2}} \mathrm{d}v
\end{aligned}$$

Let us define $y = \frac{v}{(1-v^2)^{\frac{1}{2}}}$. When v goes from 0 to 1, y goes increasingly from 0 to $+\infty$. One has $\mathrm{d}y = \frac{\mathrm{d}v}{(1-v^2)^{\frac{3}{2}}}$, and $v^2 = \frac{y^2}{1+y^2}$, hence

$$\left[1-v^2\right]^{\frac{1}{2}}\mathrm{d}v = \left[1-v^2\right]^2\mathrm{d}y = \frac{\mathrm{d}y}{(1+y^2)^2}$$

and

$$\frac{1}{4\lambda v^2+(1-\lambda)^2} = \frac{1}{4\lambda\frac{y^2}{1+y^2}+(1-\lambda)^2} = \frac{1+y^2}{4\lambda y^2+(1-\lambda)^2(1+y^2)} = \frac{1+y^2}{(1+\lambda)^2y^2+(1-\lambda)^2}.$$

So

$$\begin{aligned} I &= 2(1-\lambda)\int_{\mathbb{R}} \frac{1}{(1+\lambda)^2y^2+(1-\lambda)^2}\frac{\mathrm{d}y}{\pi(1+y^2)} \\ &= \frac{2(1-\lambda)}{(1+\lambda)^2}\int_{\mathbb{R}} g(y)\frac{\mathrm{d}y}{\pi(1+y^2)}, \end{aligned}$$

where $g(y) = \frac{1}{y^2+a^2}$, with $a = \frac{1-\lambda}{1+\lambda}$. Note that

$$g(y) = \frac{1}{(y+ia)(y-ia)} = \frac{1}{2ia}\left(\frac{1}{y-ia}-\frac{1}{y+ia}\right),$$

so the well known formula of the Cauchy transform of the Cauchy distribution with parameter 1 gives us

$$\begin{aligned} I &= \frac{2(1-\lambda)}{(1+\lambda)^2}\cdot\frac{1}{2ia}\left(\frac{1}{-i-ia}-\frac{1}{i+ia}\right) = \frac{2(1-\lambda)}{(1+\lambda)^2}\cdot\frac{1}{2ia}\cdot\frac{2i}{a+1} \\ &= \frac{2(1-\lambda)}{(1+\lambda)^2}\cdot\frac{1+\lambda}{1-\lambda}\cdot\frac{1}{a+1} = \frac{2}{(1+\lambda)a+(1+\lambda)} = 1. \end{aligned}$$

So we have proved the following result :

Proposition 4.8. *For all $\lambda \in [0,1]$, for all $t>0$, the image of the symmetric Cauchy law with parameter t by the Bercovici-Pata bijection with ratio λ is*

$$1_{\mathbb{R}\setminus[-\frac{t(1-\lambda)}{2},\frac{t(1-\lambda)}{2}]}(x)\frac{t}{\pi(\lambda t^2+x^2)}\left[1-\frac{t^2(\lambda-1)^2}{4x^2}\right]^{\frac{1}{2}}\mathrm{d}x.$$

Its support is $\mathbb{R}\setminus\left(-\frac{t(1-\lambda)}{2},\frac{t(1-\lambda)}{2}\right)$

The Cauchy type is well known to be invariant by the push-forward by the function $t\to 1/t$. In the following corollary, we are going to see that again, unless $\lambda=1$, things append differently for $\boxplus_\lambda$-stable laws with index 1.

Corollary 4.9 (Push-forward by the function $x\mapsto 1/x$)**.** - *For $\lambda=1$, the push-forward, by the function $x\mapsto 1/x$, of the measure presented in the proposition 4.8 is the symmetric Cauchy law with parameter $1/t$.*

- *For $\lambda\in[0,1)$, it is the measure carried by $\left[-\frac{2}{t(1-\lambda)},\frac{2}{t(1-\lambda)}\right]$ with density*

$$x\mapsto\frac{1}{\pi(\lambda t^2x^2+1)}\left[1-\frac{x^2t^2(\lambda-1)^2}{4}\right]^{\frac{1}{2}} \qquad \left(x\in\left[-\frac{2}{t(1-\lambda)},\frac{2}{t(1-\lambda)}\right]\right). \tag{4.5}$$

(a) When $\lambda=0$, it is equal to

$$x\mapsto\frac{t}{2\pi}\left[\frac{4}{t^2}-x^2\right]^{\frac{1}{2}} \qquad \left(x\in\left[-\frac{2}{t},-\frac{2}{t}\right]\right), \tag{4.6}$$

hence it is the symmetric semi-circle law with radius $2/t$ *and variance* $1/t^2$.
(b) When $\lambda \in (0,1)$*, it is equal to*

$$x \mapsto \frac{1-\lambda}{2\pi t\lambda}\frac{1}{\frac{1}{\lambda t^2}+x^2}\left[\left(\frac{2}{t(1-\lambda)}\right)^2 - x^2\right]^{\frac{1}{2}} \qquad \left(x \in \left[-\frac{2}{t(1-\lambda)}, \frac{2}{t(1-\lambda)}\right]\right). \tag{4.7}$$

This density is the one of a symmetric semi-circle law with radius $\frac{2}{t(1-\lambda)}$ *and variance* $\frac{1}{t^2(1-\lambda)^2}$ *times the density of a Cauchy law with parameter* $\frac{1}{t\lambda^{\frac{1}{2}}}$ *times* $\frac{\pi}{(1-\lambda)t^2\lambda^{\frac{1}{2}}}$.

Proof. Let ρ be the density of a probability on a Borel set I. Then for all bounded Borel function f,

$$\int_I f\left(\frac{1}{x}\right)\rho(x)\mathrm{d}x = -\int_I f\left(\frac{1}{x}\right)\rho\left(\frac{1}{1/x}\right)\frac{1}{(1/x)^2}\mathrm{d}\left(\frac{1}{x}\right) = \int_J f(y)\frac{\rho(1/y)}{y^2}\mathrm{d}y,$$

where $J = \{1/x\,;\, x \in I\}$. Hence the push-forward of $\rho(x)\mathrm{d}x$ by $x \mapsto 1/x$ is carried by J and has density $\frac{\rho(1/y)}{y^2}$. This proves the (well known) result for $\lambda = 1$, and (4.5). (4.6) and (4.7) follow easily. To recognize the products of semi-circle and Cauchy densities, just remember that for all $r > 0$, the semi-circle law with radius r has variance $r^2/4$ and density

$$x \in [-r, r] \mapsto \frac{2}{\pi r^2}\left[r^2 - x^2\right]^{\frac{1}{2}},$$

and that the Cauchy law with parameter $c > 0$ is $\mathcal{C}_c = \frac{1}{\pi}\frac{c\mathrm{d}x}{x^2+c^2}$. □

Question. Inspired by what happens in the cases $\lambda = 0$ and $\lambda = 1$, we ask the following question, the answer of which could have spared us the long previous calculus: is there, for each λ in $[0,1]$, a functional f_λ from the set of symmetric probability measures on $\mathbb{R}$ into the set of probability measures on $\mathbb{R}$ such that for all μ, ν symmetric probability measures, $\mu \boxplus_\lambda \nu$ is the only symmetric probability measure satisfying

$$f_\lambda(\mu \boxplus_\lambda \nu) = f_\lambda(\mu) \boxplus f_\lambda(\nu) \quad ?$$

Note that in the case $\lambda = 1$, the functional $f_\lambda(\mu) = \mu$ works, and in the case $\lambda = 0$, the functional which maps a measure to its push-forward by the square function works.

4.3. **Rectangular analogues of symmetrized Poisson distributions.**

4.3.1. *The general case* $\lambda \in [0,1]$. Let us define the *symmetric Poisson distribution with parameter* $c > 0$ to be the $*$-infinitely divisible distribution with Lévy measure $\frac{c}{4}(\delta_1 + \delta_{-1})$. It can also be seen as the law of $X - Y$, where X, Y are independent random variables with (unsymmetric) Poisson law with parameter $c/2$, or as the weak limit of

$$\left(\left(1 - \frac{c}{n}\right)\delta_0 + \frac{c}{2n}(\delta_{-1} + \delta_1)\right)^{*n}.$$

The rectangular R-transform of its image P_c by the rectangular Bercovici-Pata bijection with ratio λ is

$$\frac{cz}{1-z}.$$

Hence

$$H_{P_c}^{-1}(z) = \frac{z}{T(C_{P_c}(z))} = \frac{z(1-z)^2}{[(\lambda c - 1)z + 1][(c-1)z + 1]},$$

whose inversion would be very heavy because of a third degree equation. So we know the rectangular R-transform of this law, but we don't give the law (except when $\lambda = 0$, see below).

However, we know that for all $c, c' > 0$, $P_c \boxplus_\lambda P_{c'} = P_{c+c'}$, and in the section 6, we shall give a nice matricial model for this distributions.

4.3.2. *The particular case* $\lambda = 0$. Let us recall the definition, for $c > 0$, of the *Marchenko-Pastur law* with parameter c, also called *free Poisson law* with parameter c. It is the probability measure on $[0, +\infty)$

$$M_c = \begin{cases} \frac{(4c-(x-1-c)^2)^{\frac{1}{2}}}{2\pi x}\chi_c(x)\mathrm{d}x & \text{if } c \geq 1, \\ (1-c)\delta_0 + \frac{(4c-(x-1-c)^2)^{\frac{1}{2}}}{2\pi x}\chi_c(x)\mathrm{d}x & \text{if } 0 < c < 1, \end{cases}$$

where χ_c is the characteristic function of $[(1-c^{\frac{1}{2}})^2, (1+c^{\frac{1}{2}})^2]$. It is well known (see [HP00]) that its Voiculescu transform is $cz/(z-1)$.

Let us consider the symmetric law whose push-forward by $t \to t^2$ is M_c. By section 1.2, its rectangular R-transform with ratio 0 is given b the formula $cz/(1-z)$. Hence for $\lambda = 0$, the distribution P_c introduced in the previous section 4.3.1 is this distribution, hence (still for $\lambda = 0$),

$$P_c = \begin{cases} \frac{(4c-(x^2-1-c)^2)^{\frac{1}{2}}}{2\pi}\chi_c(x^2)\mathrm{d}x & \text{if } c \geq 1, \\ (1-c)\delta_0 + \frac{(4c-(x^2-1-c)^2)^{\frac{1}{2}}}{2\pi}\chi_c(x^2)\mathrm{d}x & \text{if } 0 < c < 1. \end{cases}$$

Remark 4.10 (Growth of the support in the semigroup)**.** *Note that we observe the same kind of phenomenon as in the remark 4.5: in the additive semigroup* $(\{P_c\,;\, c > 0\}, \boxplus_0)$*, the size of the support of* P_c *is not linear in* c*, but in* $c^{\frac{1}{2}}$*.*

5. A matricial model for the rectangular Bercovici-Pata bijection

In the previous sections, the proofs rely on integral transforms and complex analysis. We will construct, in this subsection, a matricial model for the $\boxplus_\lambda$-infinitely divisible laws and present in a maybe more palpable way the Bercovici-Pata bijection with ratio λ.

In this section, d, d' will represent dimensions of rectangular matrices, because n will be used to another role. For any distribution $\mathbb{P}$ and any function f on a set of matrices, $\mathbb{E}_{\mathbb{P}}(f(M))$ denotes $\int f(M)\mathrm{d}\mathbb{P}(M)$. Let us recall that the *singular law* of a matrix M designates the uniform distribution on the spectrum of $|M| := (MM^*)^{\frac{1}{2}}$. Let us define the *symmetrization* $\tilde{\mu}$ of a distribution μ on the real line: it is the distribution which maps a Borel set B to $(\mu(B) + \mu(-B))/2$. The symmetrization of the singular law of a matrix M will be denoted by $\tilde{\mu}_{|M|}$.

We are going to construct, in the same way as in [B-G04] and in [C-D04], for each $d, d' \geq 1$, for each symmetric $*$-infinitely divisible distribution μ, an infinitely divisible distribution $\mathbb{P}^{\mu}_{d,d'}$ on the set of $d \times d'$ complex matrices such that for all μ, ν, $\mathbb{P}^{\mu}_{d,d'} * \mathbb{P}^{\nu}_{d,d'} = \mathbb{P}^{\mu*\nu}_{d,d'}$ and such that the symmetrization of the singular law of M (with M random matrix distributed according to $\mathbb{P}^{\mu}_{d,d'}$) goes from μ to its image by the rectangular Bercovici-Pata bijection with ratio λ when $d, d' \to \infty$, $\frac{d}{d'} \to \lambda$.

Let us introduce the heuristic argument that led us to choose the model we will present. Consider a symmetric $*$-infinitely divisible distribution μ, and two sequences (ν_n) (symmetric probability measures), (k_n) (integers tending to infinity) such that $\nu_n^{*k_n}$ tends weakly to μ. Define, for all $1 \leq d \leq d'$ and each $n \geq 1$, $\mathbb{Q}^{\nu_n}_{d,d'}$ to be the law of the $d \times d'$ random matrix

$$U \left[X_{n,i}\delta_i^j \right]_{\substack{1 \leq i \leq d \\ 1 \leq j \leq d'}} V$$

where U (resp. V) is a uniform $d\times d$ (resp. $d'\times d'$) unitary random matrix, $X_{n,1},\dots,X_{n,d}$ are distributed according to ν_n, and $U,V,X_{n,1}\dots$ are independent.

Then if one fixes n and lets d,d' go to infinity in such a way that $\frac{d}{d'}\to\lambda$, the symmetrization of the singular law of $M_1(\nu_n)+\cdots+M_{k_n}(\nu_n)$ (with $M_1(\nu_n),\dots,M_{k_n}(\nu_n)$ independent and distributed according to $\mathbb{Q}_{d,d'}^{\nu_n}$) goes to $\nu_n^{\boxplus_\lambda k_n}$.

Moreover, if one fixes d,d' and lets n go to infinity, the distribution $\left(\mathbb{Q}_{d,d'}^{\nu_n}\right)^{*k_n}$ of $M_1(\nu_n)+\cdots+M_{k_n}(\nu_n)$ converges weakly to a distribution $\mathbb{P}_{d,d'}^{\mu}$ on the set of $d\times d'$ matrices, whose Fourier transform is given by the following formula: for any $d\times d'$ matrix A

$$\mathbb{E}_{\mathbb{P}_{d,d'}^{\mu}}\left(\exp\left(i\Re\left(\operatorname{Tr}A^*M\right)\right)\right)=\exp\left(\mathbb{E}\left(d\times\psi_\mu\left(\Re\left(<u,Av>\right)\right)\right)\right) \tag{5.1}$$

where ψ_μ is the *Lévy exponent* of μ, i.e. the unique continuous function f on $\mathbb{R}$ such that $f(0)=0$ and the Fourier transform of μ is $\exp\circ f$, $<.,.>$ is the canonical hermitian product of $\mathbb{C}^d$, and $u=(u_1,\dots,u_d)$, $v=(v_1,\dots,v_{d'})$ are independent random vectors, uniformly distributed on the unit sphere of respectively $\mathbb{C}^d$, $\mathbb{C}^{d'}$. The proof of this weak convergence, analogous to the one of theorem 3.1 of [B-G04], uses the polar decomposition of $d\times d'$ matrices and the bi-unitarily invariance of the distributions $\mathbb{Q}_{d,d'}^{\nu_n}$. Note that for all μ,ν, $\mathbb{P}_{d,d'}^{\mu}*\mathbb{P}_{d,d'}^{\nu}=\mathbb{P}_{d,d'}^{\mu*\nu}$, and that when $\mu=N(0,1)$, $\mathbb{P}_{d,d'}^{\mu}$ is the distribution of a matrix $[M_{i,j}]$ with $(\Re M_{i,j},\Im M_{i,j})_{\substack{1\le i\le d\\1\le j\le d'}}$ i.i.d. random variables $N(0,\frac{1}{2d'})$-distributed.

So the convergence of the symmetrization of the singular law of a $\mathbb{P}_{d,d'}^{\mu}$ -random matrix is the expression of the commutativity of the following diagram:

$$\begin{array}{ccc}
M_1(\nu_n)+\cdots+M_{k_n}(\nu_n) & \xrightarrow{n\to\infty} & \mathbb{P}_{d,d'}^{\mu}\\
\Big|\\
d,d'\text{ go to }\infty\\
d/d'\simeq\lambda\\
\downarrow & & \Big|\\
& & d,d'\text{ go to }\infty\\
& & d/d'\simeq\lambda\\
& & \downarrow\\
\text{symmetrized}\\
\text{singular law:}\\
\nu_n^{\boxplus_\lambda k_n} & \xrightarrow{n\to\infty} & \text{symmetrized}\\
& & \text{singular law:}\\
& & \Lambda_\lambda(\mu)
\end{array}$$

To prove this result, we need a preliminary result about cumulants of $\boxplus_\lambda$-infinitely divisible laws with compactly supported Lévy measure. First, note that by lemma 4.1, such laws are compactly supported. Recall that free cumulants with ratio λ have been defined in the beginning of section 4.1 by (4.1). For ν probability measure ν whose moments of all orders are defined, let us denote by $c_n^*(\nu)$ $(n\ge1)$ its classical cumulants. Recall that classical cumulants linearize the classical convolution and are linked to the moments by the formula

$$\forall k\ge1,\ m_k(\nu)=\sum_{\pi\in\mathrm{Part}(k)}\prod_{V\in\pi}c^*_{|V|}(\nu). \tag{5.2}$$

Theorem 5.1. *Let μ be a symmetric $*$-infinitely divisible distribution with compactly supported Lévy measure, and let, for each integer n, μ_n be a probability measure such that $\mu_n^{*n}=\mu$. Then for each $k\ge1$, the sequence $(n\times m_{2k}(\mu_n))_n$ tends to the $2k$-th classical cumulant $c_{2k}^*(\mu)$ of μ, which is equal to $c_{2k}(\Lambda_\lambda(\mu))$.*

Proof. By (5.2), for all n, one has

$$n\times m_{2k}(\mu_n)=n\sum_{\pi\in\mathrm{Part}(2k)}\prod_{V\in\pi}\underbrace{c^*_{|V|}(\mu_n)}_{n^{-1}c^*_{|V|}(\mu)}=\sum_{\pi\in\mathrm{Part}(2k)}n^{1-|\pi|}c_\pi^*(\mu)=c_{2k}^*(\mu)+o(1).$$

Let us denote $\nu_n = \mu_n^{\boxplus_\lambda n}$. By the line above, for all k, $m_{2k}(\mu_n) = O(n^{-1})$, so, by an easy induction on k based on equation (4.1), one gets $c_{2k}(\mu_n) = O(n^{-1})$. Hence $c_{2k}(\nu_n) = O(1)$, and $m_{2k}(\nu_n) = O(1)$. Moreover, by the equivalence (3.1), the sequence (ν_n) converges weakly to $\Lambda_\lambda(\mu)$. So the moments of ν_n tend to the moments of $\Lambda_\lambda(\mu)$ (cf [B68]). But thanks to (4.1),

$$\begin{aligned} n\times m_{2k}(\mu_n) &= n \sum_{\pi\in \mathrm{NC'}(2k)} \lambda^{e(\pi)} \prod_{V\in\pi} \underbrace{c_{|V|}(\mu_n)}_{n^{-1}c_{|V|}(\nu_n)} \\ &= \sum_{\pi\in \mathrm{NC'}(2k)} \lambda^{e(\pi)} n^{1-|\pi|} \prod_{V\in\pi} c_{|V|}(\nu_n). \end{aligned}$$

It has already been proved just above that the left hand term of the previous equation tends to $c^*_{2k}(\mu)$, whereas the right hand term tends to

$$\sum_{\pi\in \mathrm{NC'}(2k)} \lambda^{e(\pi)} \delta_1^{|\pi|} \prod_{V\in\pi} c_{|V|}(\Lambda_\lambda(\mu)) = c_{2k}(\Lambda_\lambda(\mu)).$$

It allows us to conclude. □

We will first prove the result when μ has a compactly supported Lévy measure. We will work with a sequence $(d'_d)_d$ such that $1 \le d \le d'_d$, and d/d'_d tends to $\lambda \in (0,1]$ (even though the proof can be adapted to the case $\lambda = 0$, we assume that $\lambda > 0$ in order to simplify). To simplify notations, d' will stand for d'_d.

Proposition 5.2. *Let μ be a symmetric $*$-infinitely divisible distribution with compactly supported Lévy measure. Then for all integers k,*

(a) $\lim_{d\to\infty} \mathbb{E}_{\mathbb{P}^\mu_{d,d'}} \left(m_k(\tilde{\mu}_{|M|})\right) = m_k\left(\Lambda_\lambda(\mu)\right).$

(b) The variance, under $\mathbb{P}^\mu_{d,d'}$, of $m_k(\tilde{\mu}_{|M|})$ tends to zero as d goes to infinity.

Proof. For an integer n, $[n]$ will denote $\{1,\dots,n\}$, and $\mathrm{NC}(n)$ will denote the set of noncrosing partitions of $[n]$. Recall that NC'(n) denotes the set of noncrosing partitions of $[n]$ in which all blocks have even cardinality.

(a) First, for every complex $d\times d'$ matrix M, for all integer k, $m_k\left(\tilde{\mu}_{|M|}\right)$ is null if k is odd and is equal to $\mathrm{tr}\left(MM^*\right)^{\frac{k}{2}}$ (tr denotes normalized trace) if k is even. $\Lambda_\lambda(\mu)$ is symmetric, so it suffices to show that for all $k \in \mathbb{N}^*$,

$$\lim_{d\to\infty} \mathbb{E}_{\mathbb{P}^\mu_{d,d'}} \left(\mathrm{tr}\left(MM^*\right)^k\right) = m_{2k}\left(\Lambda_\lambda(\mu)\right).$$

Let, for $n \in \mathbb{N}^*$, μ_n be the probability measure such that $\mu_n^{*n} = \mu$. Consider, for $d \ge 1$ and $n \ge 1$, $\left(M^{(i)}_{d,n}\right)_{1\le i\le n}$ i.i.d. random matrices with distribution $\mathbb{Q}^{\mu_n}_{d,d'}$. By definition, for every $d \ge 1$, the sum of the $M^{(i)}_{d,n}$'s $(i = 1\dots n)$ converges in distribution to $\mathbb{P}^\mu_{d,d'}$ when n goes to ∞. We know, by theorem 5.1 that for all $k \in \mathbb{N}^*$, the sequence $(n\times m_k(\mu_n))_n$ is bounded, and so (see [B68]) for all $k, d \in \mathbb{N}^*$,

$$\mathbb{E}_{\mathbb{P}^\mu_{d,d'}} \left(m_{2k}\left(\tilde{\mu}_{|M|}\right)\right) = \lim_{n\to\infty} \mathbb{E}\left(\mathrm{tr}\left(\left(\sum_{i=1}^n M^{(i)}_{d,n}\right)\left(\sum_{i=1}^n M^{(i)*}_{d,n}\right)\right)^k\right). \tag{5.3}$$

Let us fix $\mathbf{k} \in \mathbb{N}^*$.
We are going to use the formula (5.3).

Let, for $d, n \geq 1$,

$$b_{d,n} = \mathbb{E}\left(\mathrm{tr}\left(\left(\sum_{i=1}^{n} M_{d,n}^{(i)}\right)\left(\sum_{i=1}^{n} M_{d,n}^{(i)*}\right)\right)^{k}\right).$$

From now on, we do not write anymore the index d in $M_{d,n}^{(i)}$. We denote, for l, n non-negative integers, by A_n^l the number of one-to-one maps from $[l]$ to $[n]$, i.e. $n(n-1)\cdots(n-l+1)$. For a partition π of $[2k]$, for $1 \leq l \leq 2k$, we denote by $\pi(l)$ the index of the class of l, after having ordered the classes according to the order of their first element (for example, $\pi(1) = 1$; $\pi(2) = 1$ if $1 \overset{\pi}{\sim} 2$ and $\pi(2) = 2$ if $1 \overset{\pi}{\not\sim} 2$). Then we have

$$\begin{aligned} b_{d,n} &= \mathrm{tr}\left(\mathbb{E}\left(\sum_{f\in\{1,\ldots,n\}^{2k}} M_n^{(f(1))} M_n^{(f(2))*} \cdots M_n^{(f(2k))*}\right)\right) \\ &= \mathrm{tr}\left(\mathbb{E}\left(\sum_{\pi\in\mathrm{Part}(2k)} A_n^{|\pi|} M_n^{(\pi(1))} M_n^{(\pi(2))*} M_n^{(\pi(3))} \cdots M_n^{(\pi(2k))*}\right)\right). \end{aligned}$$

$$\text{But} \qquad \mathbb{E}\left(\underbrace{M_n^{(1)*} M_n^{(1)} \cdots M_n^{(1)*}}_{2l+1 \text{ alterned factors}}\right) = \mathbb{E}\left(\underbrace{M_n^{(1)} M_n^{(1)*} \cdots M_n^{(1)}}_{2l+1 \text{ alterned factors}}\right) = 0,$$

$$\mathbb{E}\left(\underbrace{M_n^{(1)*} M_n^{(1)} \cdots M_n^{(1)}}_{2l \text{ alterned factors}}\right) = \frac{d}{d'} m_{2l}(\mu_n) I_{d'},$$

$$\mathbb{E}\left(\underbrace{M_n^{(1)} M_n^{(1)*} \cdots M_n^{(1)*}}_{2l \text{ alterned factors}}\right) = m_{2l}(\mu_n) I_d.$$

So, using many times the fact that a partition π of a finite totally ordered set is noncrossing if and only if one of its class V is an interval and $\pi\backslash\{V\}$ is noncrossing (page 3 of [S98]) and integrating successively with respect to the different independent random matrices, one has

$$\pi \in \mathrm{NC'}(2k) \Rightarrow \mathrm{tr}\,\mathbb{E}\left(M_n^{(\pi(1))} M_n^{(\pi(2))*} \cdots M_n^{(\pi(2k))*}\right) = \left(\frac{d}{d'}\right)^{e(\pi)} \prod_{B\in\pi} m_{|B|}(\mu_n),$$

$$\pi \in \mathrm{NC}(2k) \backslash \mathrm{NC'}(2k) \Rightarrow \mathrm{tr}\,\mathbb{E}\left(M_n^{(\pi(1))} M_n^{(\pi(2))*} \cdots M_n^{(\pi(2k))*}\right) = 0.$$

But $A_n^{|\pi|} \sim n^{|\pi|}$, so, by the preceding theorem, the limit, when n goes to infinity, of

$$\mathrm{tr}\left(\mathbb{E}\left(\sum_{\pi\in\mathrm{Part}(2k)} A_n^{|\pi|} M_n^{(\pi(1))} M_n^{(\pi(2))*} \cdots M_n^{(\pi(2k))*}\right)\right)$$

is

$$\sum_{\pi\in\mathrm{NC'}(2k)} \left(\frac{d}{d'}\right)^{e(\pi)} \prod_{V\in\pi} c_{|V|}(\Lambda_\lambda(\mu)),$$

which tends, when d goes to infinity, to

$$\sum_{\pi\in\mathrm{NC'}(2k)} \lambda^{e(\pi)} \prod_{V\in\pi} c_{|V|}(\Lambda_\lambda(\mu)),$$

which is equal to $m_{2k}(\Lambda_\lambda(\mu))$ by (4.1).

So it suffices to prove that

$$b'_{d,n} := \sum_{\substack{\pi\in\mathrm{Part}(2k)\\ \pi\notin\mathrm{NC}(2k)}} A_n^{|\pi|}\,\mathrm{tr}\,\mathbb{E}\left\{M_n^{(\pi(1))}M_n^{(\pi(2))*}M_n^{(\pi(3))}\cdots M_n^{(\pi(2k))*}\right\}$$

vanishes when n, and then d, go to infinity. Let us expand the trace: $b'_{n,d}$ is equal to

$$\frac{1}{d}\sum_{\substack{\pi\in\mathrm{Part}(2k)\\ \pi\notin\mathrm{NC}(2k)}}\sum_{\substack{j\in[d']^{2k}\\ \forall r\text{ odd},\ j_r\le d}} A_n^{|\pi|}\mathbb{E}\left\{\left[M_n^{(\pi(1))}\right]_{j_1,j_2}\left[M_n^{(\pi(2))*}\right]_{j_2,j_3}\cdots\left[M_n^{(\pi(2k))*}\right]_{j_{2k},j_1}\right\}.$$

Using the fact that $\left(M_{d,n}^{(i)}\right)_{1\le i\le n}$ are independent copies of a matrix with distribution $\mathbb{Q}_{d,d'}^{\mu_n}$, we deduce (with the notation $j_{2k+1}=j_1$)

$$\begin{aligned} b'_{d,n} &= \frac{1}{d}\sum_{\substack{\pi\in\mathrm{Part}(2k)\\ \pi\notin\mathrm{NC}(2k)}} A_n^{|\pi|}\sum_{\substack{j\in[d']^{2k}\\ \forall r\text{ odd},\ j_r\le d}}\prod_{B\in\pi}\mathbb{E}_{\mathbb{Q}_{d,d'}^{\mu_n}}\left\{\prod_{\substack{r\in B\\ r\text{ odd}}}M_{j_r,j_{r+1}}\prod_{\substack{r\in B\\ r\text{ even}}}M^*_{j_r,j_{r+1}}\right\}\\ &= \frac{1}{d}\sum_{\substack{\pi\in\mathrm{Part}(2k)\\ \pi\notin\mathrm{NC}(2k)}}\frac{A_n^{|\pi|}}{n^{|\pi|}}\sum_{\substack{j\in[d']^{2k}\\ \forall r\text{ odd},\ j_r\le d}}\\ &\qquad\prod_{B\in\pi} n\mathbb{E}\left\{\sum_{l\in[d]^B}\prod_{\substack{r\in B\\ r\text{ odd}}}u_{j_r,l_r}v_{l_r,j_{r+1}}X_{n,l_r}\prod_{\substack{r\in B\\ r\text{ even}}}\bar v_{l_r,j_r}X_{n,l_r}\bar u_{j_{r+1},l_r}\right\},\end{aligned}$$

where $U,V,X_{n,1},\dots,X_{n,d}$ are independent, with respective distribution the Haar measure on the group of $d\times d$ unitary matrices, the Haar measure on the group of $d'\times d'$ unitary matrices, and μ_n.

For all $B\subset[2k]$, for all $j\in[d']^{2k}$ such that for all r odd, $j_r\le d$, summing over the partition generated by l, one has

$$\begin{aligned} &n\mathbb{E}\left\{\sum_{l\in[d]^B}\prod_{\substack{r\in B\\ r\text{ odd}}}u_{j_r,l_r}v_{l_r,j_{r+1}}X_{n,l_r}\prod_{\substack{r\in B\\ r\text{ even}}}\bar v_{l_r,j_r}X_{n,l_r}\bar u_{j_{r+1},l_r}\right\}\\ =\ &\sum_{\sigma\in\mathrm{Part}(B)}\sum_{l\in[d]^\sigma} n^{1-|\sigma|}\left[\prod_{W\in\sigma}nm_{|W|}(\mu_n)\right]\mathbb{E}\left\{\prod_{\substack{r\in B\\ r\text{ odd}}}u_{j_r,l_r}v_{l_r,j_{r+1}}\prod_{\substack{r\in B\\ r\text{ even}}}\bar v_{l_r,j_r}\bar u_{j_{r+1},l_r}\right\}.\end{aligned}$$

The measures μ_n are symmetric, so all their moments of odd order are null. Hence, according to theorem 5.1, the quantity of the previous equation tends, as n goes to infinity, to

$$\begin{cases}\displaystyle\sum_{l=1}^{d}c^*_{|B|}(\mu)\mathbb{E}\left\{\prod_{\substack{r\in B\\ r\text{ odd}}}u_{j_r,l}v_{l,j_{r+1}}\prod_{\substack{r\in B\\ r\text{ even}}}\bar v_{l,j_r}\bar u_{j_{r+1},l}\right\} & \text{if } |B| \text{ is even,}\\ 0 & \text{otherwise.}\end{cases}$$

Note that equation (5.2) implies, by an easy induction, that the classical cumulants with of even order of a symmetric probability measure are null. It allows us to avoid considering two cases on the parity of $|B|$ in the previous limit, and to claim that the first formula is valid whenever

$|B|$ is odd. This limit is equal, by invariance of uniform distributions on unitary groups by permutation of rows and columns, to

$$d\times c^*_{|B|}(\mu)\mathbb{E}\left\{\prod_{\substack{r\in B\\ r \text{ odd}}} u_{j_r}v_{j_{r+1}}\prod_{\substack{r\in B\\ r \text{ even}}}\bar{v}_{j_r}\bar{u}_{j_{r+1}}\right\},$$

where u, v are independent uniform random vectors on the unit spheres of $\mathbb{C}^d, \mathbb{C}^{d'}$.

So the limit, when n goes to infinity, of $b'_{d,n}$ is

$$\frac{1}{d}\sum_{\substack{\pi\in\mathrm{Part}(2k)\\ \pi\notin\mathrm{NC}(2k)}}\sum_{\substack{j\in[d']^{2k}\\ \forall r \text{ odd},\ j_r\leq d}}\prod_{B\in\pi} d\times c^*_{|B|}(\mu)\mathbb{E}\left\{\prod_{\substack{r\in B\\ r \text{ odd}}} u_{j_r}v_{j_{r+1}}\prod_{\substack{r\in B\\ r \text{ even}}}\bar{v}_{j_r}\bar{u}_{j_{r+1}}\right\},$$

the absolute value of which is less or equal, by invariance of the distributions of u and v under permutation of coordinates, to

$$\frac{1}{d}\sum_{\substack{\pi,\tau\in\mathrm{Part}(2k)\\ \pi\notin\mathrm{NC}(2k)}} d'^{|\pi|+|\tau|}\prod_{B\in\pi}\left|c^*_{|B|}(\mu)\mathbb{E}\left(\prod_{\substack{r\in B\\ r\ odd}} u_{\tau(r)}v_{\tau(r+1)}\prod_{\substack{r\in B\\ r\ even}}\overline{u}_{\tau(r+1)}\overline{v}_{\tau(r)}\right)\right|.$$

Moreover, by invariance of the distribution of u under the action of unitary diagonal matrices, for every pair (π,τ) of partitions of $[2k]$, if

$$\prod_{B\in\pi}\mathbb{E}\left(\prod_{\substack{r\in B\\ r\ odd}} u_{\tau(r)}v_{\tau(r+1)}\prod_{\substack{r\in B\\ r\ even}}\overline{u}_{\tau(r+1)}\overline{v}_{\tau(r)}\right)$$

is non zero, then for every class B of π, there exists ϕ, permutation of B, which maps odd numbers to even numbers and vice versa, such that for all $r\in B$, $\tau(r)=\tau(\phi(r)+1)$. It implies, by lemma 4.4 of [B-G04], that $|\tau|+|\pi|\leq 2k$. So one has, using the Hölder inequality and equation (4.2.11) of [HP00],

$$\lim_{n\to\infty} b'_{n,d} = O(d^{-1}),$$

which closes the proof of (a).

One notes that the proof of (a) is a very closed adaptation of the proof of Proposition 4.1 of [B-G04], by adaptation of the arguments to the context of non hermitian and non square matrices. Using again the same adaptation, the proof of (b) is along the same lines as the proof of Proposition 5.1 of [B-G04]. □

To conclude this section, we have to state its main theorem. Recall that the *convergence in probability* of a sequence X_n of random variables in a metrizable topological space $\mathcal{X}$ to a constant $l\in\mathcal{X}$ is the convergence of the probability of the event $\{d(X_n,l)<\varepsilon\}$ to 1 for all positive ε, where d is any distance which defines the topology (it does not depend on the choice of such a distance). In the following theorem, we shall refer to the convergence in probability in the set of probability measures on the real line, endowed with the metrizable topology of weak convergence. The proof of the theorem is similar to the one of Theorem 7.6 of [B-G04], based on the previous proposition and on an approximation by compound Poisson laws. The only modification is to work with products of the type MM^* rather than M^*M. Recall that d' is in fact d'_d (i.e. d' depends on d) and that d/d' tends to λ as d tends to infinity.

Theorem 5.3. *Let μ be a symmetric $*$-infinitely divisible distribution. Let, for $d \geq 1$, M_d be a random matrix with distribution $\mathbb{P}^{\mu}_{d,d'}$.*
Then as d goes to infinity, the symmetrization $\tilde{\mu}_{|M_d|}$ of the singular law of M_d converges in probability to $\Lambda_\lambda(\mu)$.

Remark 5.4. *In the case where μ is a normal law, one recovers the well known result about asymptotic spectral distribution of Wishart random matrices.*

6. Rectangular symmetric Poisson distributions as limits of sums of rank-one matrices

The symmetric Poisson distribution with parameter $c > 0$ has been introduced in section 4.3. The free analogues of unsymmetric Poisson distributions are Marchenko-Pastur laws. But as we said it in section 4.3, unless $\lambda = 0$, the computations for symmetric Poisson laws are harder than for the unsymmetric ones (even when $\lambda = 1$, the densities have not been expressed). Nevertheless, we have the following characterization of the rectangular analogues of symmetric Poisson distributions.

Proposition 6.1. *Consider $\lambda \in (0, 1]$, and $c > 0$. Then the image, by the Bercovici-Pata bijection with ratio λ, of the symmetric Poisson distribution with parameter c is the limit, for convergence in probability, of the symmetrization of the singular law of the random matrix*

$$M(d, d', d'') := \sum_{k=1}^{d''} u_d(k) v_{d'}(k)^*$$

when

$$d \to \infty, \quad \frac{d}{d'} \to \lambda, \quad \frac{d''}{d} \to c, \tag{6.1}$$

where $u_d(k), v_{d'}(k)$ $(k \geq 1)$ are independent uniform random vectors of the unit spheres of $\mathbb{C}^d, \mathbb{C}^{d'}$ (considered as column matrices).

Remark 6.2. *Note that when $\lambda = 1$, the image, by the Bercovici-Pata bijection with ratio λ, of the symmetric Poisson distribution with parameter c is $\tau_+ \boxplus \tau_-$, where τ_+ is the Marchenko-Pastur distribution with parameter $c/2$, and τ_- is the push-forward, by the function $t \to -t$, of τ_+.*

Proof. Let us denote by μ the symmetric Poisson distribution with parameter c, and by σ the push-forward by $t \to t^2$ of its image by the Bercovici-Pata bijection with ratio λ. As explained in the beginning of the proof of the last theorem of the section called "the rectangular R-transform" of [B-G1], it suffices to prove that for each $\varepsilon > 0$, the probability of the event

$$\left\{ \sup_{\Im z \geq 1} \left| \frac{1}{d} \operatorname{Tr} \mathfrak{R}_z \left(A(d, d', d'') \right) - G_\sigma(z) \right| > \varepsilon \right\}$$

tends to zero as d, d', d'' tend to infinity as in (6.1), where

$$A(d, d', d'') = M(d, d', d'') M(d, d', d'')^*,$$

and for M hermitian matrix and z complex non real number, $\mathfrak{R}_z(M) = (z - M)^{-1}$.

Fix $\varepsilon > 0$. It can easily be seen that $\mathbb{P}^{\mu}_{d,d'}$ is the distribution of

$$N(d, d') := \sum_{k=1}^{X(cd)} u_d(k) v_{d'}(k)^*,$$

where $X(cd)$ is a random variable distributed according to an (unsymmetric) Poisson distribution with parameter cd and $X(cd)$ is independent of the $u_d(k)$'s and of the $v_{d'}(k)$'s. Thus by the previous theorem, with the notation $B(d,d') = N(d,d')N(d,d')^*$, the probability of the event

$$\left\{ \sup_{\Im z \geq 1} \left| \frac{1}{d} \operatorname{Tr} \mathfrak{R}_z \left(B(d,d') \right) - G_\sigma(z) \right| > \varepsilon \right\}$$

tends to zero. Thus it suffices to prove that the probability of the event

$$\left\{ \sup_{\Im z \geq 1} \left| \frac{1}{d} \operatorname{Tr} \left(\mathfrak{R}_z \left(B(d,d') \right) - \mathfrak{R}_z \left(A(d,d',d'') \right) \right) \right| > \varepsilon \right\}$$

tends to zero. But for all hermitian $d \times d$ matrices A, B, for all z such that $\Im z \geq 1$, we have

$$\mathfrak{R}_z(B) - \mathfrak{R}_z(A) = -\mathfrak{R}_z(B)(B-A)\mathfrak{R}_z(A),$$

whose normalized trace is not more than its norm times its rank divided by d. Moreover, $||\mathfrak{R}_z(B) - \mathfrak{R}_z(A)|| \leq 2$, and the rank is not more than the one of $B - A$. So it suffices to prove that

$$\frac{1}{d} \operatorname{rg} \left(B(d,d') - A(d,d',d'') \right)$$

converges in probability to zero. $B(d,d') - A(d,d',d'')$ can be put in the form

$$(\ldots)(N(d,d') - M(d,d',d''))^* + (N(d,d') - M(d,d',d''))(\ldots),$$

so

$$\frac{1}{d} \operatorname{rg} \left(B(d,d') - A(d,d',d'') \right) \leq \frac{2}{d} \operatorname{rg}(N(d,d') - M(d,d',d'')) \leq \frac{2}{d} |X(cd) - d''|,$$

which converges in probability to zero, by the weak law of great numbers. □

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