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**Spatial smoothness of the stationary solutions of the 3D
Navier–Stokes equations**

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Abstract: We consider stationary solutions of the three dimensional Navier–Stokes equations (NS3D) with periodic boundary conditions and driven by an external force which might have a deterministic and a random part. The random part of the force is white in time and very smooth in space. We investigate smoothness properties in space of the stationary solutions.

Classical technics for studying smoothness of stochastic PDEs do not seem to apply since global existence of strong solutions is not known. We use the Kolmogorov operator and Galerkin approximations. We first assume that the noise has spatial regularity of order p in the L^2 based Sobolev spaces, in other words that its paths are in H^p . Then we prove that at each fixed time the law of the stationary solutions is supported by H^{p+1} .

Then, using a totally different technic, we prove that if the noise has Gevrey regularity then at each fixed time, the law of a stationary solution is supported by a Gevrey space. Some informations on the Kolmogorov dissipation scale are deduced.

Key words: Stochastic three-dimensional Navier-Stokes equations, invariant measure, Gevrey spaces, Kolmogorov operator, Kolmogorov dissipation scale.

INTRODUCTION

We are concerned with the stochastic Navier–Stokes equations in dimension 3 (NS3D) with periodic boundary conditions and zero mean value. These equations describe the time evolution of an incompressible fluid and are given by

$$(0.1) \quad \left\{ \begin{array}{l} dX + \nu(-\Delta)X dt + (X, \nabla)X dt + \nabla p dt = \phi(X)dW + g(X)dt, \\ (\operatorname{div} X)(t, \xi) = 0, \quad \text{for } \xi \in D, \quad t > 0, \\ \int_D X(t, \xi)d\xi = 0, \quad \text{for } t > 0, \\ X(0, \xi) = x_0(\xi), \quad \text{for } \xi \in D, \end{array} \right.$$

where $D = (0, 2\pi)^3$. We have denoted by $X(t, \xi)$ the velocity and by $p(t, \xi)$ the pressure at time t and at the point $\xi \in D$, also ν denotes the viscosity. The external

force acting on the fluid is the sum of a random force of white noise type $\phi(X)dW$ and a determinist one $g(X)dt$.

As is well known, in the deterministic case, global existence of weak (in the PDE sense) solutions and uniqueness of strong solutions hold for the Navier-Stokes equations. In space dimension two, weak solutions are strong and global existence and uniqueness follows. Such a result is an open problem in dimension three. (See for instance [4], [7], [16], [20], [21], [22], [23], [28] and [32]).

In the stochastic case, the situation is similar. However due to the lack of uniqueness, we have to work with global weak (in the PDE sense) solutions of the martingale problem. (See for instance [1], [2], [3], [5], [6], [13], [14], [24], [30] and [31]). Roughly speaking, this means that in (0.1), we take X , p and W for unknown.

As is usual in the context of the incompressible Navier-Stokes equation, we get rid of the pressure thanks to the Leray projector. Let us denote by (X, W) a weak (in the PDE sense) stationary solution of the martingale problem (0.1) and by μ the law of $X(t)$, which is an invariant measure if we can prove that (0.1) defines a Markov evolution. In this article, we establish that μ admits a moment in spaces of smooth functions provided the external force is sufficiently smooth. We think that this is an interesting question to study. First, it can be seen that if we were able to prove that μ has a moment of sufficiently high order in a well chosen Sobolev norm (order 4 in H^1 or 2 in H^2 for instance) then this would imply global existence of strong solutions for μ almost every initial data.

Moreover, this result is an important ingredient if one tries to follow the method of [8] to construct a Markov transition semi-group in $H^p(D)$ under suitable conditions on ϕ and f . Since even uniqueness in law is not known for NS3D, such result might be important.

We first prove that if the external force is in $H^{p-1}(D)$ and the noise term has paths in $H^p(D)$ then μ admits a moment in the Sobolev space $H^{p+1}(D)$

Note that analogous results are well-known for the two dimensional Navier–Stokes equations (NS2D). Actually a stronger result is true for NS2D. Namely, for any square integrable x_0 , the unique solution of NS2D is continuous from $(0, \infty)$ into $H^p(D)$ and is square integrable from (t_0, t_1) into $H^{p+1}(D)$. It follows that μ admits moments of any orders in $H^p(D)$ and a moment of order 2 in $H^{p+1}(D)$. This stronger result is linked to the global existence of strong solutions for NS2D.

This kind of idea cannot be used for NS3D and we use a generalization of an idea used in [8] for the case $p = 1$. The method is based on the use of the Kolmogorov operator applied to suitable Lyapunov functional. These functionals have already been used in the deterministic case in [29], chapter 4.

Using a totally different method, we establish also that the invariant measure μ admits a moment in a Gevrey class of functions provided the external force is also in such a class. Gevrey regularity has been studied in the deterministic case in [15] and [17]. Our method is based on tools developed in [15]. In [27], these tools have been used to obtain an exponential moment for the invariant measure in Gevrey norms in the two dimensional case. The arguments used in [27] do not generalize to the three dimensional case since there strong existence and uniqueness is used. The three dimensional case NS3D requires substantial adaptations. We develop a framework which gives a control on a Gevrey norm by using a control of the $H^1(D)$ -norm of X only at fixed time.

In this way, we are able to generalize to NS3D the results of [27]. However, we do not have exponential moments. We deduce that the Kolmogorov dissipation scale is larger than $\nu^{-(4+\delta)}$. This is certainly not optimal since it is expected that the scale is of order $\nu^{-\frac{3}{4}}$. Note that our result is rigorous and does not use any heuristic argument.

1. NOTATIONS

For $m \in \mathbb{N}$, we denote by $\mathbb{H}_{\text{per}}^m(D)$ the space of functions which are restrictions of periodic functions in $H_{\text{loc}}^m(D; \mathbb{R}^3)$ and whose average on D is zero. We set

$$H = \{X \in \mathbb{H}_{\text{per}}^0(D) \mid \text{div } X = 0 \text{ on } D\},$$

and

$$V = H \cap \mathbb{H}_{\text{per}}^1(D).$$

Let π be the orthogonal projection in $L^2(D; \mathbb{R}^3)$ onto the space H . We set

$$A = \pi(-\Delta), \quad D(A) = V \cap \mathbb{H}_{\text{per}}^2(D) \quad \text{and} \quad B(u) = \pi((u, \nabla)u).$$

It is convenient to endow $\mathbb{H}_{\text{per}}^m(D)$ with the inner product $((\cdot, \cdot))_m = (A^{m\cdot}, \cdot)_{L^2(D; \mathbb{R}^3)}$. The corresponding norm is denoted by $\|\cdot\|_m$. It is classical that this defines a norm which is equivalent to the usual one. For $m = 0$ we write $|\cdot| = \|\cdot\|_0$ and for $m = 1$ we write $\|\cdot\| = \|\cdot\|_1$. Note that, since we work with functions whose average is zero on $(0, 2\pi)^3$, we have the following Poincaré type inequality

$$\|x\|_{m_1} \leq \|x\|_{m_2}, \quad m_1 \leq m_2, \quad x \in \mathbb{H}_{\text{per}}^{m_2}(D; \mathbb{R}^3).$$

We also use the spaces $L^p(D; \mathbb{R}^3)$ endowed with their usual norm denoted by $|\cdot|_p$. Moreover, given two Hilbert spaces K_1 and K_2 , $\mathcal{L}_2(K_1; K_2)$ is the space of Hilbert-Schmidt operators from K_1 to K_2 .

The noise is described by a cylindrical Wiener process W defined on a Hilbert space U and a mapping ϕ defined on H with values in $\mathcal{L}_2(U; H)$. We also consider a deterministic forcing term described by a mapping g from H into H . More precise assumptions on ϕ and g are made below.

Now, we can write problem (0.1) in the form

$$(1.1) \quad \begin{cases} dX + \nu AX dt + B(X) dt &= \phi(X) dW + g(X) dt, \\ X(0) &= x_0, \end{cases}$$

where W is a cylindrical Wiener process on a Hilbert space U .

In all the paper, we consider (X, W) a H -valued stationary solution of the martingale problem (1.1). Existence of such a solution has been proved in [13]. We denote by μ the law of $X(t)$. We do not consider any stationary solutions but only those which are limit in distribution of stationary solutions of Galerkin approximations of (1.1). More precisely, for any $N \in \mathbb{N}$, we denote by P_N the eigenprojector of A associated to the first N eigenvalues and consider the following approximation of (1.1)

$$(1.2) \quad \begin{cases} dX_N + \nu AX_N dt + P_N B(X_N) dt &= P_N \phi(X_N) dW + P_N g(X_N) dt, \\ X_N(0) &= P_N x_0. \end{cases}$$

It is easily shown that (1.2) has a stationary solution X_N . Proceeding as in [14], we can see that their laws are tight in a well chosen functional space and, for a subsequence, (X_N, W) converges in law to (X, W) a stationary solution of (1.1). We only consider stationary solutions constructed in that way.

All the computations made in this paper are derived formally and are not totally rigorous. A rigorous proof could easily be obtained by making all the computations on the Galerkin approximations and then letting N go to infinity in the final estimate. This type of argument is classical and, in order to lighten the redaction, we chose to write all the computations directly on (X, W) .

Some of our results described properties of μ in Gevrey type spaces. According to the setting given in [15], we set for any $(\alpha, \beta) \in \mathbb{R}_*^+ \times (0, 1]$

$$\left\{ \begin{array}{l} (\cdot, \cdot)_{G(\alpha, \beta)} = \left(A^{\frac{1}{2}} e^{\alpha A^{\frac{\beta}{2}}} \cdot, A^{\frac{1}{2}} e^{\alpha A^{\frac{\beta}{2}}} \cdot \right), \quad \|\cdot\|_{G(\alpha, \beta)} = \left| A^{\frac{1}{2}} e^{\alpha A^{\frac{\beta}{2}}} \cdot \right|, \\ G(\alpha, \beta) = \left\{ X \in H \mid \|X\|_{G(\alpha, \beta)} < \infty \right\}. \end{array} \right.$$

Clearly, $(G(\alpha, \beta), (\cdot, \cdot)_{G(\alpha, \beta)})$ is a Hilbert space.

We are not interested in large viscosities and in all the article it is assumed that $\nu \leq 1$. We will use various constants which may depend on some parameter such as p, ν, \dots . When this dependance is important, we make it explicit.

2. $\mathbb{H}_{\text{PER}}^p(D)$ –REGULARITY

2.1. Statement of the result. Let $p \in \mathbb{N}$. We now make the following smoothness assumptions on the forcing terms.

Hypothesis 2.1. *The mapping ϕ (resp. g) takes values in $\mathcal{L}_2(U; H \cap \mathbb{H}_{\text{per}}^p(D))$ (resp. $H \cap \mathbb{H}_{\text{per}}^{p-1}(D)$) and $\phi : H \rightarrow \mathcal{L}_2(U; H \cap \mathbb{H}_{\text{per}}^p(D))$ and $g : H \rightarrow H \cap \mathbb{H}_{\text{per}}^{p-1}(D)$ are bounded.*

We set, when Hypothesis 2.1 holds,

$$B_p = \sup_H \left(\|\phi\|_{\mathcal{L}_2(U; \mathbb{H}_{\text{per}}^p(D))}^2 + \|g\|_{p-1}^2 \right).$$

It is also convenient to define

$$\bar{B}_p = \sup_H \|\phi\|_{\mathcal{L}_2(U; \mathbb{H}_{\text{per}}^p(D))}^2 + \left(\frac{2}{\nu} \sup_H \|g\|_{p-1}^2 \right) \wedge \left(\sup_H \|g\|_{p+1}^2 \right).$$

The aim of this section is to establish the following result.

Theorem 2.2. *Let μ be the invariant law of a stationary solution of the three dimensional Navier-Stokes equation and assume that Hypothesis 2.1 holds for some $p \geq 1$. There exists $c_{p, \nu}$ depending on p, ν and B_p such that for any $\nu \leq 1$*

$$\int_H \|x\|_{p+1}^{\frac{2}{2p+1}} d\mu(x) \leq c_{p, \nu}.$$

This result is proved in sections 2.2 to 2.5. We now wish to make few comments.

Note that it would be very important to obtain an estimate on $\int_H \|x\|_{p+1}^{\delta_p} d\mu(x)$ with $p\delta_p > 3$. Indeed, by Agmon inequality, we have

$$\int_H |x|_{\infty}^2 d\mu(x) \leq c \int_H |x|^{2-3/p} \|x\|_p^{\frac{3}{p}} d\mu(x)$$

and this would give an estimate on the right hand side. Since uniqueness is easily shown to hold for solutions in $L^2(0, T; L^{\infty}(D; \mathbb{R}^3))$, a classical argument could be used to deduce that for μ almost every initial data there exists a unique global weak solution. Combining with the result in [12], this would partially solve Leray’s conjecture.

Consider the case $g = 0$, $U = H$ and $\phi = A^{-s-\frac{3}{2}}$. Then Hypothesis 2.1 holds for any $p < s$ and the unique invariant measure of the three dimensional linear stochastic Stokes equation in H is in $\mathbb{H}_{\text{per}}^{r+1}(D)$ with probability zero if $r > s$. Therefore it seems that $\|\cdot\|_{p+1}$ is the strongest norm we can control under Hypothesis 2.1.

Remark that in the two dimensional case a much stronger result holds. Indeed, standard arguments imply that under Hypothesis 2.1 we have for any invariant measure μ and any $q \in \mathbb{N}^*$

$$\int_H \|x\|_p^{2q} d\mu(x) < \infty, \quad \int_H \|x\|_{p+1}^2 d\mu(x) < \infty.$$

In the proof, we use ideas developped in [29]. Similar but more refined techniques have been used in [17] to derive interesting properties on the decay of the Fourier spectrum of smooth solutions of the deterministic Navier-Stokes equations. Using such techniques does not seem to yield great improvement of our result. Indeed, trying to do so, we have been able to precise the estimate of Theorem 2.2 as follows

$$\nu \int_H \|x\|_{p+1}^{\frac{c_*}{p}} d\mu(x) \leq \bar{B}_p + c2^p(1 + \bar{B}_0),$$

where c and c_* are positive constants and c_* is close to 1.02. We have not been able to derive very interesting results from this improved estimate and therefore have preferred to give the simpler one which follows from easier arguments.

2.2. Proof of Theorem 2.2.

The proof is based on the fact that for any suitable f

$$(2.1) \quad \int_H Lf(x) d\mu(x) = 0,$$

where L is the Kolmogorov operator associated to the stochastic Navier-Stokes equations (1.1)

$$Lf(x) = \frac{1}{2} \text{tr}(\phi(x)\phi^*(x)D^2f(x)) - (\nu Ax + B(x) - g(x), Df(x)).$$

As already mentionned, this is not fully rigorous. Indeed, (2.1) is valid only for the Galerkin approximations and for smooth and bounded functions. The sequence of Galerkin approximations described in section 1 should be used. Moreover, for a better understanding, we only consider the case $g = 0$. The generalization is easy.

Step 1: $p = 0$

Applying (2.1) to $f = |\cdot|^2$, we obtain

$$\int_H (\nu Ax + B(x), x) d\mu(x) = \frac{1}{2} \int_H \text{tr}(\phi(x)\phi^*(x)) d\mu(x),$$

which yields, since $(B(x), x) = 0$,

$$(2.2) \quad \nu \int_H \|x\|^2 d\mu(x) \leq \frac{1}{2} \bar{B}_0.$$

Step 2: Estimate of a quotient of norms.

We set

$$\varepsilon_p = \frac{1}{2p-1}$$

and

$$R'_p = \nu \int_H \frac{\|x\|_{p+1}^2}{(1 + \|x\|_p^2)^{1+\varepsilon_p}} d\mu(x),$$

$$R_p = \nu \int_H \frac{1 + \|x\|_{p+1}^2}{(1 + \|x\|_p^2)^{1+\varepsilon_p}} d\mu(x),$$

In order to establish Theorem 2.2 for $p \geq 1$, we prove now that there exists c_p such that

$$(2.3) \quad R_p \leq \bar{B}_p + c_p \bar{B}_0 + 1$$

Let us set

$$f = \frac{1}{(1 + \|\cdot\|_p^2)^{\varepsilon_p}}.$$

We have

$$Df(x) = -2\varepsilon_p \frac{A^p x}{(1 + \|x\|_p^2)^{1+\varepsilon_p}},$$

and

$$D^2 f(x) = -2\varepsilon_p \frac{A^p}{(1 + \|x\|_p^2)^{1+\varepsilon_p}} + 4\varepsilon_p(1 + \varepsilon_p) \frac{A^p x \otimes A^p x}{(1 + \|x\|_p^2)^{2+\varepsilon_p}},$$

which yields

$$(Lf)(x) = \frac{2\varepsilon_p}{(1 + \|x\|_p^2)^{1+\varepsilon_p}} \left(\nu \|x\|_{p+1}^2 + (A^p x, B(x)) - \frac{1}{2} \|\phi(x)\|_{\mathcal{L}_2(U; \mathbb{H}_{\text{per}}^p(D))}^2 \right)$$

$$+ \frac{2\varepsilon_p(1 + \varepsilon_p)}{(1 + \|x\|_p^2)^{2+\varepsilon_p}} \sum_n |(A^p x, \phi(x) \cdot e'_n)|^2,$$

where $(e'_n)_n$ is an orthonormal basis of U . Hence we have

$$2\nu \frac{\|x\|_{p+1}^2}{(1 + \|x\|_p^2)^{1+\varepsilon_p}} \leq \frac{1}{\varepsilon_p} Lf(x) + \bar{B}_p - \frac{2}{(1 + \|x\|_p^2)^{1+\varepsilon_p}} (A^p x, B(x)).$$

Integrating with respect to μ and applying (2.1), it follows

$$(2.4) \quad 2R'_p \leq \bar{B}_p - 2 \int_H \frac{(A^p x, B(x))}{(1 + \|x\|_p^2)^{1+\varepsilon_p}} d\mu(x).$$

It is proved in [29] (see equation (4.8) chapter 4) that there exists a constant $c_{p,\nu}$ such that

$$(2.5) \quad -(A^p x, B(x)) \leq c_{p,\nu} \|x\|^2 \|x\|_p^{4p/2p-1} + \frac{\nu}{2} \|x\|_{p+1}^2.$$

Then taking into account (2.2) and (2.4), (2.3) follows.

Step 3: Estimate of $\int_H \|x\|_p^{2/2p-1} d\mu(x)$.

We estimate $A_p = \nu \int_H (1 + \|x\|_p^2)^{1/2p-1} d\mu(x)$ by induction. The case $p = 1$ has been treated in step 1.

Using Hölder inequality yields

$$\begin{aligned} & \nu \int_H (1 + \|x\|_{p+1}^2)^{1/2p+1} d\mu(x) \\ &= \nu \int_H \left(\frac{(1 + \|x\|_{p+1}^2)}{(1 + \|x\|_{p+1}^2)^{1+\varepsilon_p}} \right)^{1/2p+1} (1 + \|x\|_{p+1}^2)^{(1+\varepsilon_p)/2p+1} d\mu(x) \\ &\leq R_p^{1/2p+1} \left(\nu \int_H (1 + \|x\|_{p+1}^2)^{(1+\varepsilon_p)/2p} d\mu(x) \right)^{2p/2p+1}. \end{aligned}$$

Since $(1 + \varepsilon_p)/2p = 1/2p - 1$, we deduce

$$A_{p+1} \leq R_p^{1/2p+1} A_p^{2p/2p+1}.$$

The result follows.

3. GEVREY REGULARITY

3.1. Statement of the result. We prove in this section that if the external force is bounded in a Gevrey class of functions, then μ have support in another Gevrey class of function.

The main assumption in this section is the following

Hypothesis 3.1. *There exists $(\alpha, \beta) \in \mathbb{R}_+^* \times (0, 1]$ such that the mappings $g : H \rightarrow H \cap G(\alpha, \beta)$ and $\phi : H \rightarrow \mathcal{L}_2(U; H \cap G(\alpha, \beta))$ are bounded.*

We set

$$B'_0 = \sup_{x \in H} \|\phi(x)\|_{\mathcal{L}_2(U; G(\alpha, \beta))}^2 + \sup_{x \in H} \|g(x)\|_{G(\alpha, \beta)}^2.$$

The aim of this section is to establish the following results proved in the following subsections.

Theorem 3.2. *Assume that Hypothesis 3.1 holds. There exist a family $(C_\gamma)_\gamma$ only depending on (α, β, B'_0) and a family $(\alpha_\nu)_{\nu \in (0, 1)}$ of measurable mappings $H \rightarrow (0, \alpha)$ such that for any $\nu \in (0, 1)$*

$$(3.1) \quad \int_H \|x\|_{G(\nu\alpha_\nu(x), \beta)}^{2\gamma} d\mu(x) \leq C_\gamma \nu^{-\frac{3}{4}},$$

$$(3.2) \quad \int_H (\alpha_\nu(x))^{-\frac{7}{4}} d\mu(x) \leq C_\gamma \nu^{-\frac{3}{4}},$$

for any $\gamma \in (0, 1)$.

This result gives some informations on the Kolmogorov dissipation scale. Indeed, it follows from (3.1), (3.2) that

$$|\hat{x}(k)| \leq \|x\|_{G(\nu\alpha_\nu(x),\beta)} |k| e^{-\nu\alpha_\nu(x)|k|^\beta},$$

where $(\hat{x}(k))_{k \in \mathbb{Z}^3}$ are the Fourier coefficients of x .

Hence, if Hypothesis 3.1 holds with $\beta = 1$, then $|\hat{x}(k)|$ decreases faster than any powers of $|k|$ for $|k| \gg (\nu\alpha_\nu(x))^{-1}$. By (3.2), for any $\delta > 0$

$$\frac{1}{\alpha_\nu(x)} \leq c_{\delta,\nu}(x)\nu^{-3(1+\delta)} \quad \text{with} \quad \int c_{\delta,\nu}(x)^{\frac{1}{4(1+\delta)}} \mu(dx) \leq \Theta_\delta < \infty,$$

and Θ_δ not depending on ν . It follows that $|\hat{x}(k)|$ decreases faster than any powers of $|k|$ for $|k| \gg \nu^{-(4+3\delta)}$. This indicates that the Kolmogorov dissipation scale is larger than $\nu^{-(4+3\delta)}$. Note that by physical arguments it is expected that the Kolmogorov dissipation scale is of order of $\nu^{-\frac{3}{4}}$.

We can also control a moment of a fixed Gevrey norm.

Corollary 3.3. *Under the same assumptions, there exists a family $(C_{\gamma,\alpha',\beta',\nu})_{\gamma,\alpha',\beta',\nu}$ only depending on (α,β,B'_0,ν) such that*

$$(3.3) \quad \int \left(\ln^+ \|x\|_{G(\alpha',\beta')}^2 \right)^\gamma d\mu(x) \leq C_{\gamma,\alpha',\beta',\nu},$$

where $\ln^+ r = \max\{0, \ln r\}$ and provided $\alpha' > 0$ and $\beta', \gamma > 0$ verify

$$\beta' < \beta \quad \text{and} \quad 4\gamma < \frac{\beta}{\beta'} - 1.$$

3.2. Estimate of blow-up time in Gevrey spaces.

Before proving Theorem 3.2, we establish the following result which implies that the time of blow-up of the solution in Gevrey spaces admits a negative moment.

Lemma 3.4. *Assume that Hypothesis 3.1 holds. For any stationary solution of the Navier-Stokes equations and any $\nu \in (0, 1)$, there exist K only depending on (α,β,B'_0) and a stopping time $\tau > 0$ such that*

$$(3.4) \quad \mathbb{E} \left(\sup_{t \in (0,\tau)} \|X(t)\|_{G(\nu t,\beta)}^2 \right) \leq 4(\bar{B}_0 + 1),$$

$$(3.5) \quad \mathbb{P}(\tau \leq t) \leq K t^{\frac{1}{4}} \nu^{-\frac{3}{4}}.$$

This result is a refinement of the result developed by Foias and Temam in [15] and is strongly based on the tools developed in this latter article. Let us set

$$(3.6) \quad \tau = \inf \left\{ t \geq 0 \mid \|X(t)\|_{G(\nu t,\beta)}^2 > 4 \left(\|X(0)\|^2 + 1 \right) \right\}.$$

Clearly

$$\mathbb{E} \left(\sup_{t \in (0,\tau)} \|X(t)\|_{G(\nu t,\beta)}^2 \right) \leq 4\mathbb{E} \left(\|X(0)\|^2 + 1 \right)$$

and (3.4) follows from (2.2). It remains to prove (3.5).

We apply Ito Formula to $\|X(t)\|_{G(\nu t,\beta)}^2$ for $t \in (0, \alpha)$

$$(3.7) \quad d \|X(t)\|_{G(\nu t,\beta)}^2 + 2\nu \left\| A^{\frac{1}{2}} X(t) \right\|_{G(\nu t,\beta)}^2 dt = \nu \left\| A^{\frac{\beta}{4}} X(t) \right\|_{G(\nu t,\beta)}^2 dt + dM(t) + I(t)dt,$$

where

$$\begin{cases} I(t) &= 2I_g(t) + 2I_B(t) + I_\phi(t), & I_B(t) &= -(X(t), B(X(t)))_{G(\nu t, \beta)}, \\ I_\phi(t) &= \|\phi(X(t))\|_{\mathcal{L}_2(U; G(\nu t, \beta))}^2, & I_g(t) &= (g(X(t)), X(t))_{G(\nu t, \beta)}, \\ M(t) &= 2 \int_0^t (X(s), \phi(X(s)) dW(s))_{G(\nu t, \beta)}. \end{cases}$$

The following inequality is proved in [15] for $\beta \leq 1$

$$(3.8) \quad 2I_B(t) \leq \nu \left\| A^{\frac{1}{2}} X(t) \right\|_{G(\nu t, \beta)}^2 + \frac{c}{\nu^3} \|X(t)\|_{G(\nu t, \beta)}^6.$$

By Hypothesis 3.1 we have

$$(3.9) \quad I_\phi(t) + 2I_g(t) \leq \|X(t)\|_{G(\nu t, \beta)}^6 + B'_0 + 1.$$

Combining (3.7), (3.8) and (3.9), we obtain since $\beta, \nu \leq 1$

$$(3.10) \quad d\|X(t)\|_{G(\nu t, \beta)}^2 \leq dM(t) + \frac{c}{\nu^3} \|X(t)\|_{G(\nu t, \beta)}^6 dt + (B'_0 + 1)dt.$$

Applying Ito formula to $\left(1 + \|X(t)\|_{G(\nu t, \beta)}^2\right)^{-2}$, we then deduce from (3.10) and from Hypothesis 3.1 that for any $t \in (0, \alpha)$ and any $\nu \leq 1$

$$(3.11) \quad -d\left(1 + \|X(t)\|_{G(\nu t, \beta)}^2\right)^{-2} \leq d\mathcal{M}(t) + C_0\nu^{-3}dt,$$

where $C_0 = c(B'_0 + 1)$ and

$$\mathcal{M}(t) = 4 \int_0^t \left(1 + \|X(s)\|_{G(\nu t, \beta)}^2\right)^{-3} (X(s), \phi(X(s)) dW(s))_{G(\nu t, \beta)}.$$

Setting

$$\begin{cases} \tau_0 &= \inf \left\{ t \in (0, \alpha) \mid \mathcal{M}(t) > \frac{1}{4(1 + \|X(0)\|^2)^2} \right\}, \\ \tau_1 &= \tau_0 \wedge \left(\frac{\nu^3}{4C_0(1 + \|X(0)\|^2)^2} \right), \end{cases}$$

we obtain by integration of (3.11) on $[0, t]$ for $t \in (0, \tau_1)$

$$\|X(t)\|_{G(\nu t, \beta)}^2 \leq 4 \left(1 + \|X(0)\|^2\right).$$

We deduce that $\tau \geq \tau_1$ and

$$(3.12) \quad \mathbb{P}(\tau \leq t) \leq \mathbb{P}(\tau_0 \leq t) + \mathbb{P}\left(\left(1 + \|X(0)\|^2\right)^2 \geq \frac{\nu^3}{4C_0 t}\right).$$

Since μ is the law of $X(0)$, we have

$$\mathbb{P}\left(\left(1 + \|X(0)\|^2\right)^2 \geq \frac{\nu^3}{4C_0 t}\right) = \mu\left(x \mid \sqrt{1 + \|x\|^2} \geq \frac{\nu^{\frac{3}{4}}}{(4C_0 t)^{\frac{1}{4}}}\right).$$

Applying Chebyshev and Schwartz inequalities, we deduce from (2.2)

$$\begin{aligned}
 (3.13) \quad & \mathbb{P} \left(\left(1 + \|X(0)\|^2 \right)^2 \geq \frac{\nu^3}{4C_0 t} \right) \\
 & \leq 2\nu^{-\frac{3}{4}} (C_0 t)^{\frac{1}{4}} \sqrt{\int_H (1 + \|x\|^2) d\mu(x)} \\
 & \leq 2\nu^{-\frac{3}{4}} \sqrt{1 + \bar{B}_0} (C_0 t)^{\frac{1}{4}}.
 \end{aligned}$$

Moreover

$$\begin{aligned}
 \mathbb{P}(\tau_0 \leq t) &= \mathbb{P} \left(4 \left(\left(1 + \|X(0)\|^2 \right)^2 \sup_{s \in [0, t]} \mathcal{M}(s) \right) > 1 \right) \\
 &\leq 2\mathbb{E} \left(\left(1 + \|X(0)\|^2 \right)^2 \sup_{s \in [0, t]} |\mathcal{M}(s)|^{\frac{1}{2}} \right).
 \end{aligned}$$

Conditionning by \mathcal{F}_0 the σ -algebra generate by $X(0)$, it follows

$$\mathbb{P}(\tau_0 \leq t) \leq 2\mathbb{E} \left(\left(1 + \|X(0)\|^2 \right)^2 \mathbb{E} \left(\sup_{s \in [0, t]} |\mathcal{M}(s)|^{\frac{1}{2}} \middle| \mathcal{F}_0 \right) \right).$$

By Burkholder-Davis-Gundi inequality (see Theorem 3.28 page 166 in [19]) we obtain

$$\mathbb{E} \left(\sup_{s \in [0, t]} |\mathcal{M}(s)|^{\frac{1}{2}} \middle| \mathcal{F}_0 \right) \leq c \mathbb{E} \left(\langle \mathcal{M} \rangle^{\frac{1}{4}}(t) \middle| \mathcal{F}_0 \right),$$

and

$$(3.14) \quad \mathbb{P}(\tau_0 \leq t) \leq 2\mathbb{E} \left(\left(1 + \|X(0)\|_{G(\nu t, \beta)}^2 \right)^2 \mathbb{E} \left(\langle \mathcal{M} \rangle^{\frac{1}{4}}(t) \middle| \mathcal{F}_0 \right) \right).$$

We have

$$\langle \mathcal{M} \rangle(t) = 4 \int_0^t \left(1 + \|X(s)\|_{G(\nu t, \beta)}^2 \right)^{-6} \left| \left(A^{\frac{1}{2}} e^{\nu t A^{\frac{\beta}{2}}} \phi(X(s)) \right)^* \left(A^{\frac{1}{2}} e^{\nu t A^{\frac{\beta}{2}}} X(s) \right) \right|_U^2 ds.$$

Therefore

$$\langle \mathcal{M} \rangle(t) \leq 4 \int_0^t \left(1 + \|X(s)\|_{G(\nu t, \beta)}^2 \right)^{-6} \|\phi(X(s))\|_{\mathcal{L}(U; G(\nu t, \beta))}^2 \|X(s)\|_{G(\nu t, \beta)}^2 ds \leq 4B'_0 t.$$

Hence we infer from (3.14) that

$$\mathbb{P}(\tau_0 \leq t) \leq c(B'_0 t)^{\frac{1}{4}} \nu \int_H (1 + \|x\|^2) d\mu(x),$$

which yields by (2.2)

$$(3.15) \quad \mathbb{P}(\tau_0 \leq t) \leq c(1 + \bar{B}_0)(B'_0)^{1/4} t^{\frac{1}{4}}.$$

Combining (3.12), (3.13) and (3.15), we deduce (3.5) from $\bar{B}_0 \leq B'_0$.

3.3. Proof of Theorem 3.2.

Setting

$$\alpha_\nu(x) = \inf \left\{ s \geq 0 \mid \|x\|_{G(\nu s, \beta)}^2 > \frac{4}{s^{\frac{1}{4}}} (\bar{B}_0 + 1) \right\},$$

it follows that for any $\gamma \in (0, 1)$

$$(3.16) \quad \int \|x\|_{G(\nu \alpha_\nu(x), \beta)}^{2\gamma} d\mu(x) \leq 4^\gamma (\bar{B}_0 + 1)^\gamma \int (\alpha_\nu(x))^{-\frac{\gamma}{4}} d\mu(x).$$

Hence (3.1) is consequence of (3.2) and of $\bar{B}_0 \leq B'_0$. Then in order to establish Theorem 3.2, it is sufficient to prove (3.2).

Clearly

$$\mathbb{P} \left(\|X(t)\|_{G(\nu t, \beta)}^2 > \frac{4}{t^{\frac{1}{4}}} (\bar{B}_0 + 1) \right) \leq \mathbb{P} \left(\sup_{s \in [0, \tau]} \|X(s)\|_{G(\nu s, \beta)}^2 > \frac{4}{t^{\frac{1}{4}}} (\bar{B}_0 + 1) \right) + \mathbb{P}(\tau < t),$$

where τ is defined in section 3.2. Applying Chebyshev inequality, we infer from Lemma 3.4 that for any $t > 0$

$$(3.17) \quad \mathbb{P} \left(\|X(t)\|_{G(\nu t, \beta)}^2 > \frac{4}{t^{\frac{1}{4}}} (\bar{B}_0 + 1) \right) \leq (K + 1)t^{\frac{1}{4}}\nu^{-\frac{3}{4}}.$$

Since μ is the law of $X(t)$, we have

$$\mu(x \mid \alpha_\nu(x) \leq t) = \mathbb{P}(\alpha_\nu(X(t)) \leq t),$$

which yields

$$\mu(x \mid \alpha_\nu(x) \leq t) = \mathbb{P} \left(\|X(t)\|_{G(\nu t, \beta)}^2 > \frac{4}{t^{\frac{1}{4}}} (\bar{B}_0 + 1) \right).$$

Hence we deduce from (3.17) that for any $t > 0$

$$(3.18) \quad \mu(x \mid \alpha_\nu(x) \leq t) \leq (K + 1)t^{\frac{1}{4}}\nu^{-\frac{3}{4}}.$$

It is well-known that (3.18) for any $t > 0$ implies (3.2), which yields Theorem 3.2.

3.4. Proof of Corollary 3.3.

To deduce Corollary 3.3 from Theorem 3.2, it is sufficient to prove that for any $(\alpha', \alpha, \beta', \beta) \in (0, \infty)^2 \times (0, 1]^2$ such that $\beta' < \beta$, we have

$$(3.19) \quad \|x\|_{G(\alpha', \beta')} \leq \exp \left(c(\beta, \beta') (\alpha')^{\frac{\beta}{\beta - \beta'}} (\alpha)^{-\frac{\beta'}{\beta - \beta'}} \right) \|x\|_{G(\alpha, \beta)}.$$

Indeed, (3.19) implies that for any $\gamma \in \mathbb{R}_*^+$

$$\left(\ln^+ \|x\|_{G(\alpha', \beta')}^2 \right)^\gamma \leq c_\gamma \left(c(\beta, \beta') + (\alpha')^{\frac{\gamma\beta}{\beta - \beta'}} (\nu\alpha_\nu(x))^{-\frac{\gamma\beta'}{\beta - \beta'}} + \left(\ln^+ \|x\|_{G(\nu\alpha_\nu(x), \beta)}^2 \right)^\gamma \right),$$

which yields Corollary 3.3 provided Theorem 3.2 is true.

We now establish (3.19). It follows from arithmetico-geometric inequality that for any $k \in \mathbb{Z}^3$

$$(3.20) \quad \alpha' |k|^{\beta'} \leq c(\beta, \beta') (\alpha')^{\frac{\beta}{\beta - \beta'}} (\alpha)^{-\frac{\beta'}{\beta - \beta'}} + \alpha |k|^\beta.$$

Recalling that

$$\|x\|_{G(\alpha', \beta')}^2 = \sum_{k \in \mathbb{Z}^3} |k|^2 \exp \left(2\alpha' |k|^{\beta'} \right) |\hat{x}(k)|^2,$$

we infer (3.19) from (3.20).

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